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Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ to all orders

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Abstract. We present the main ideas and techniques of the proof that the duality-covariant four-dimensional noncommutative ϕ^4-model is renormalisable to all orders. This includes the reformulation as a dynamical matrix model, the solution of the free theory by orthogonal polynomials as well as the renormalisation by flow equations involving power-counting theorems for ribbon graphs drawn on Riemann surfaces.

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1 Introduction

In recent years there has been considerable interest in quantum field theories on the Moyal plane characterised by the $\star$-product (in D dimensions)

$$(a\star b)(x) := \int d^Dy \frac{d^Dk}{(2\pi)^D} a(x+\tfrac{1}{2}\theta\cdot k) b(x+y)\, \mathrm{e}^{\mathrm{i}ky}\;, \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R}\;. \tag{1}$$

The interest was to a large extent motivated by the observation that this kind of field theories arise in the zero-slope limit of open string theory in presence of a magnetic background field [1]. A few months later it was discovered [2] (first for scalar models) that these noncommutative field theories are not renormalisable beyond a certain loop order. The argument is that non-planar graphs are finite but their amplitude grows beyond any bound when the external momenta become exceptional. When inserted as subgraphs into bigger graphs, these exceptional momenta are attained in the loop integration and result in divergences for any number of external legs. This problem is called UV/IR-mixing. A more rigorous explanation was given in [3] where the problem was traced back to divergences in some of the Hepp sectors which correspond to *disconnected* ribbon subgraphs wrapping the same handle of a Riemann surface. Hepp sectors which correspond to connected non-planar subgraphs are always finite.

The UV/IR-problem was found in all UV-divergent field theories on the Moyal plane. Models with at most logarithmic UV-divergences (such as two-dimensional and supersymmetric theories) can be defined at any loop order, but their amplitudes are still unbounded at exceptional momenta.

The UV/IR-mixing contains a clear message: If we make the world noncommutative at very short distances, we must—whether we like it or not—at the same time modify the physics at large distances. The required modification is, to the best of our knowledge, unique: It is given by an harmonic oscillator potential for the free field action. In fact, we can prove the following

Theorem 1 *The quantum field theory associated with the action*

$$S=\int d^4x\Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi+\frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi)+\frac{\mu_0^2}{2}\phi\star\phi+\frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x)\;, \tag{2}$$

for $\tilde{x}_\mu:=2(\theta^{-1})_{\mu\nu}\,x^\nu$*,* ϕ*-real, Euclidean metric, is perturbatively renormalisable to all orders in* λ*.*

Our proof given in [4] and [5] is very long so that there is some need of an introductory presentation of its main ideas and techniques.

First, we remark that the action is covariant with respect to a *duality* between position space and momentum space [6]: Under the exchange of position and momentum (i.e. not the Fourier transformation),

$$p_\mu\leftrightarrow\tilde{x}_\mu\;,\qquad \hat{\phi}(p)\leftrightarrow\pi^2\sqrt{|\det\theta|}\;\phi(x)\;, \tag{3}$$

together with $\hat{\phi}(p_a)=\int d^4x\;\mathrm{e}^{(-1)^a\mathrm{i}p_{a,\mu}x_a^\mu}\phi(x_a)$ for a being a cyclic label, one has

$$S\big[\phi;\mu_0,\lambda,\Omega\big]\mapsto\Omega^2 S\big[\phi;\frac{\mu_0}{\Omega},\frac{\lambda}{\Omega^2},\frac{1}{\Omega}\big]\;. \tag{4}$$

2 Reformulation as a dynamical matrix model

It is clear from the explicit x-dependence that for quantisation we cannot proceed in momentum space. Fortunately, the Moyal plane has a very convenient matrix base.

We choose a coordinate frame where $\theta{=}\theta_{12}{=}{-}\theta_{21}{=}\theta_{34}{=}{-}\theta_{43}$ are the only non-vanishing θ-components. We expanding the fields according to $\phi(x)=\sum_{m^1,m^2,n^1,n^2\in\mathbb{N}}\phi_{\substack{m^1\\m^2}\substack{n^1\\n^2}}b_{\substack{m^1\\m^2}\substack{n^1\\n^2}}(x)$ where $b_{\substack{m^1\\m^2}\substack{n^1\\n^2}}(x)=f_{m^1n^1}(x_1,x_2)f_{m^2n^2}(x_3,x_4)$ with

$$f_{m^1n^1}(x_1,x_2)=\frac{(x_1{-}\mathrm{i}x_2)^{\star m^1}}{\sqrt{m^1!(2\theta)^{m^1}}}\star\Big(2\mathrm{e}^{-\frac{1}{\theta}(x_1^2+x_2^2)}\Big)\star\frac{(x_1{+}\mathrm{i}x_2)^{\star n^1}}{\sqrt{n^1!(2\theta)^{n^1}}}\;, \tag{5}$$

$$(b_{mn}\star b_{kl})(x)=\delta_{nk}b_{ml}(x)\;,\qquad \int d^4x\,b_{mn}(x)=(2\pi\theta)^2\,\delta_{mn}\;. \tag{6}$$

Due to (6) the non-local $\star$-product interaction becomes a simple matrix product, at the price of rather complicated kinetic terms and propagators. We obtain for the action (2)

$$S=(2\pi\theta)^2\sum_{m,n,k,l\in\mathbb{N}^2}\Big(\frac{1}{2}\phi_{mn}G_{mn;kl}\phi_{kl}+\frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big)\;, \tag{7}$$

$$
\begin{aligned}
G_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} &= \big(\mu_0^2+\tfrac{2+2\Omega^2}{\theta}(m^1+n^1+m^2+n^2+2)\big)\delta_{n^1k^1}\delta_{m^1l^1}\delta_{n^2k^2}\delta_{m^2l^2} \\
&\quad - \tfrac{2-2\Omega^2}{\theta}\big(\sqrt{k^1l^1}\,\delta_{n^1+1,k^1}\delta_{m^1+1,l^1} + \sqrt{m^1n^1}\,\delta_{n^1-1,k^1}\delta_{m^1-1,l^1}\big)\delta_{n^2k^2}\delta_{m^2l^2} \\
&\quad - \tfrac{2-2\Omega^2}{\theta}\big(\sqrt{k^2l^2}\,\delta_{n^2+1,k^2}\delta_{m^2+1,l^2} + \sqrt{m^2n^2}\,\delta_{n^2-1,k^2}\delta_{m^2-1,l^2}\big)\delta_{n^1k^1}\delta_{m^1l^1}\;. \qquad (8)
\end{aligned}
$$

The conservation of the angular momentum relative to the $SO(2)\times SO(2)$ symmetry of the action implies that $G_{mn;kl}=0$ unless $m+k=n+l$.

We are interested in a perturbative solution of the quantum field theory about the free theory, the solution of which is given by the propagator $\Delta_{mn;kl}$, i.e. the inverse of $G_{mn;kl}$. In a first step we diagonalise the kinetic matrix:

$$
G_{\substack{m^1\\m^2}\substack{m^1+\alpha^1\\m^2+\alpha^2};\substack{l^1+\alpha^1\\l^2+\alpha^2}\substack{l^1\\l^2}} = \sum_{y^1,y^2=0}^{\infty} U^{(\alpha^1)}_{m^1y^1}U^{(\alpha^2)}_{m^2y^2}\big(\mu_0^2+\tfrac{4\Omega}{\theta}(2y^1+2y^2+\alpha^1+\alpha^2+2)\big)U^{(\alpha^1)}_{y^1l^1}U^{(\alpha^2)}_{y^2l^2}\;, \qquad (9)
$$

$$
U^{(\alpha)}_{ny} = \sqrt{\binom{\alpha+n}{n}\binom{\alpha+y}{y}}\,\Big(\frac{1-\Omega}{1+\Omega}\Big)^{n+y}\Big(\frac{2\sqrt{\Omega}}{1+\Omega}\Big)^{\alpha+1}\,{}_2F_1\Big(\begin{matrix}-n,-y\\1+\alpha\end{matrix}\Big|\frac{4\Omega}{(1+\Omega)^2}\Big)\;.
$$

For fixed α, the kinetic matrix is in both components a Jacobi matrix (a certain tridiagonal band matrix). The diagonalisation of that band matrix yields the recursion relation for (orthogonal) Meixner polynomials $M_n(y;\beta,c) = {}_2F_1\big(\begin{smallmatrix}-n,-y\\ \beta\end{smallmatrix}\big|1-c\big)$. The corresponding equidistant eigenvalues are those of the harmonic oscillator. To compute the propagator we have to invert the eigenvalues $\big(\mu_0^2+\frac{4\Omega}{\theta}(2y^1+2y^2+\alpha^1+\alpha^2+2)\big)$ in (9). Using the identity

$$
\begin{aligned}
&\sum_{y=0}^{\infty}\frac{(\alpha+y)!}{y!\alpha!}\,a^y\,{}_2F_1\Big(\begin{matrix}-m,-y\\1+\alpha\end{matrix}\Big|b\Big)\,{}_2F_1\Big(\begin{matrix}-l,-y\\1+\alpha\end{matrix}\Big|b\Big) \\
&\qquad = \frac{(1-(1-b)a)^{m+l}}{(1-a)^{\alpha+m+l+1}}\,{}_2F_1\Big(\begin{matrix}-m\,,\;-l\\1+\alpha\end{matrix}\Big|\frac{ab^2}{(1-(1-b)a)^2}\Big)\;, \qquad |a|<1\;, \qquad (10)
\end{aligned}
$$

which can be regarded as the heart of the renormalisation proof, we arrive at

$$
\begin{aligned}
\Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}} &= \frac{\theta}{2(1+\Omega)^2}\sum_{v^1=\frac{|m^1-l^1|}{2}}^{\frac{m^1+l^1}{2}}\;\sum_{v^2=\frac{|m^2-l^2|}{2}}^{\frac{m^2+l^2}{2}} B\big(1+\tfrac{\mu_0^2\theta}{8\Omega}+\tfrac{1}{2}(m^1+k^1+m^2+k^2)-v^1-v^2,1+2v^1+2v^2\big) \\
&\quad\times {}_2F_1\Bigg(\begin{matrix}1+2v^1+2v^2\,,\;\frac{\mu_0^2\theta}{8\Omega}-\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2\\ 2+\frac{\mu_0^2\theta}{8\Omega}+\frac{1}{2}(m^1+k^1+m^2+k^2)+v^1+v^2\end{matrix}\Bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Bigg)\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v^1+2v^2} \\
&\quad\times\prod_{i=1}^{2}\delta_{m^i+k^i,n^i+l^i}\sqrt{\binom{n^i}{v^i+\frac{n^i-k^i}{2}}\binom{k^i}{v^i+\frac{k^i-n^i}{2}}\binom{m^i}{v^i+\frac{m^i-l^i}{2}}\binom{l^i}{v^i+\frac{l^i-m^i}{2}}}\;. \qquad (11)
\end{aligned}
$$

One should appreciate here that the sum in (11) is finite, i.e. we succeeded to solve the free theory with respect to the preferred base of the interaction. The explicit solution enables a fast numerical evaluation of the propagator, which is necessary to determine the asymptotic behaviour of the propagator for large indices. In few cases we can evaluate the sum exactly:

- $0 \leq \Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\mu_0) < \Delta_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(0)$

  This means that we can ignore the mass μ_0 in our estimations for $\Omega > 0$.

- $\Delta_{\substack{m\\0}\substack{m\\0};\substack{m\\0}\substack{m\\0}}(0) = \frac{\theta}{2(1+\Omega)^2(m+1)}\,{}_2F_1\Big(\begin{matrix}1,-m\\ m+2\end{matrix}\Big|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Big) \sim \frac{\theta/8}{\Omega(m+1)+\sqrt{\frac{4}{\pi}(m+1)}}$

  There is a discontinuity in the asymptotic behaviour of the propagator at $\Omega = 0$. For $\Omega = 0$ there is a long-range correlation which decays only very slowly with $\frac{1}{\sqrt{m}}$. This is the origin of the UV/IR-mixing. For $\Omega > 0$ the correlation decays with $\frac{1}{m}$ which guarantees a good power-counting behaviour of the model with $\Omega > 0$. The asymptotic behaviour provides the easy part of the renormalisation proof.

- $\Delta_{\substack{m^1\\m^2}\substack{m^1\\m^2};\substack{0\\0}\substack{0\\0}}(0) = \frac{\theta}{2(1+\Omega)^2(m^1+m^2+1)}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{m^1+m^2}$

  This property controls the non-locality. This means that there is a correlation $\Delta_{mn;kl} \neq 0$ for arbitrarily large $\|m-l\|$ which, however, is exponentially suppressed, preserving some sort of quasi-locality. This provides the tricky part of the renormalisation proof.

3 The Polchinski equation

It is, in principle, possible to proceed with the discussion of Feynman graphs built with the propagator (11) according to Zimmermann's forest formula. But the complexity of the arising graphs (compare (11) with the simple $\frac{1}{k^2+m^2}$ of commutative field theories) requires a more sophisticated approach: the renormalisation by flow equations. The idea goes back to Wilson [7] and was further developed by Polchinski to an efficient renormalisation proof of commutative ϕ^4-theory [8].

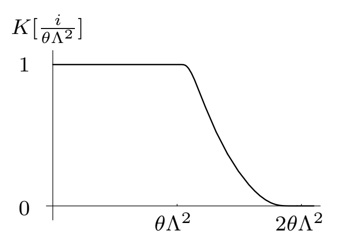
The starting point is to define the quantum field theory by the cut-off partition function

$$Z[J,\Lambda] = \int \Big(\prod_{a,b} d\phi_{ab}\Big)\,\exp\big(-S[\phi,J,\Lambda]\big)\,, \tag{12}$$

$$\begin{aligned} S[\phi,J,\Lambda] = (2\pi\theta)^2\Big(&\sum_{m,n,k,l}\frac{1}{2}\phi_{mn}G^K_{mn;kl}(\Lambda)\,\phi_{kl} + L[\phi,\Lambda] + C[\Lambda] \\ &+ \sum_{m,n,k,l}\phi_{mn}F_{mn;kl}[\Lambda]J_{kl} + \sum_{m,n,k,l}\frac{1}{2}J_{mn}E_{mn;kl}[\Lambda]J_{kl}\Big)\,. \end{aligned} \tag{13}$$

The most important pieces here are the cut-off kinetic term

$$G^K_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\Lambda) := \prod_{\substack{i\,\in m^1,m^2,n^1,n^2,\\ k^1,k^2,l^1,l^2}} K^{-1}\big(\tfrac{i}{\theta\Lambda^2}\big)\;G_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}\,, \tag{14}$$



where the weight of the matrix indices is altered according to a *smooth* cut-off function[1] K, and the *effective action* $L[\phi,\Lambda]$ which compensates the effect of the cut-off. We are interested in the limit $\Lambda \to \infty$, where the cut-off goes away, $\lim_{\Lambda\to\infty} K[\frac{i}{\theta\Lambda^2}] = 1$. Thus, we would formally obtain the original model for $\Lambda = \infty$ and $L[\phi,\infty] = \frac{\lambda}{4!}\sum_{m,n,k,l} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}$, $C[\infty] = 0$, $E_{mn;kl}[\infty] = 0$, $F_{mn;kl}[\infty] = \delta_{nk}\delta_{ml}$. However, $\Lambda = \infty$ is difficult to obtain due to the appearance of divergences, which require compensating counterterms in $L[\phi]$.

The genial idea of the renormalisation group approach is to require instead the *independence* of the partition function on the cut-off, $\Lambda\frac{\partial}{\partial\Lambda} Z[J,\Lambda] = 0$. Working out the details one arrives, in particular, at the Polchinski equation for matrix models

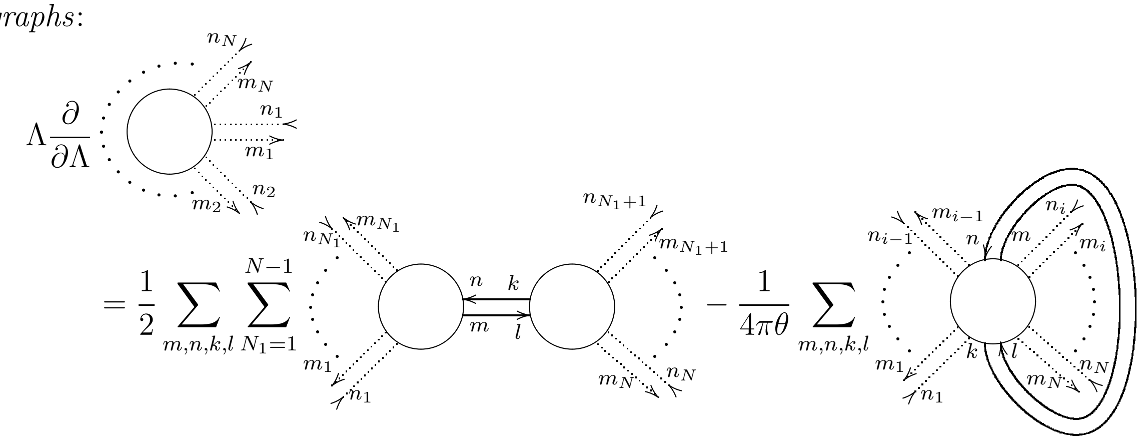
$$\Lambda \frac{\partial L[\phi,\Lambda]}{\partial\Lambda} = \sum_{m,n,k,l} \frac{1}{2}\Lambda \frac{\partial \Delta^K_{nm;lk}(\Lambda)}{\partial\Lambda}\Big(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}} - \frac{1}{(2\pi\theta)^2}\frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{mn}\,\partial\phi_{kl}}\Big)\,, \tag{15}$$

where $\Delta^K_{nm;lk}(\Lambda) := \prod_{i\in m^1,m^2,\dots,l^1,l^2} K(\frac{i}{\theta\Lambda^2})\Delta_{nm;lk}$. To obtain (15) it was, of course, important to realise finite matrices via a smooth function K. There are other differential equations for the functions C, E, F in (13) which, however, are trivial to integrate. The true difficulties are contained in the non-linear differential equation (15).

The Polchinski equation has a non-perturbative meaning, but to solve it we need, for the time being, a power series ansatz:

$$L[\phi,\Lambda] = \sum_{V=1}^{\infty} \lambda^V \sum_{N=2}^{2V+2} \frac{(2\pi\theta)^{\frac{N}{2}-2}}{N!} \sum_{m_1,n_i\in\mathbb{N}^2} A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\phi_{m_1n_1}\cdots\phi_{m_Nn_N}\,. \tag{16}$$

Then, the differential equation (15) provides an explicit recursive solution for the coefficients $A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ which, because the fields ϕ_{mn} carry two indices, is represented by *ribbon graphs*:

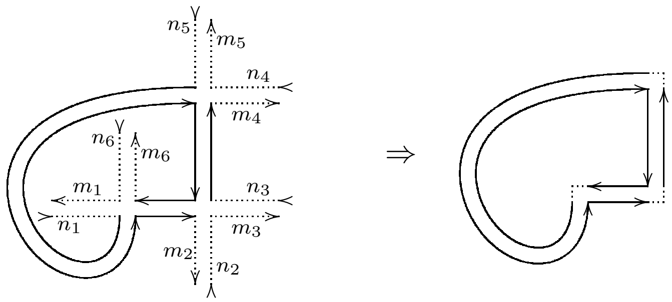


(17)

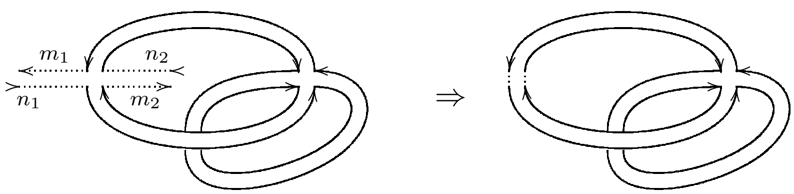
An internal double line symbolises the propagator $Q_{mn;kl}(\Lambda) := \frac{1}{2\pi\theta}\Lambda\frac{\partial}{\partial\Lambda}\Delta^K_{mn;kl}(\Lambda) = \underset{m\quad l}{\overset{n\quad k}{\leftrightarrows}}$.

[1]We understand the cut-off as a limiting process $\epsilon \to 0$ in $K^{-1}(\frac{i}{\theta\Lambda^2}) = \frac{1}{\epsilon}$ for $i \geq 2\theta\Lambda^2$. In the limit, the partition function (12) vanishes unless $\phi_{\substack{m^1\\m^2}\substack{n^1\\n^2}} = 0$ if $\max(m^1,m^2,n^1,n^2) \geq 2\theta\Lambda^2$, thus implementing a cut-off of the measure $\prod_{a,b} d\phi_{ab}$ in (12). All other formulae involve $K(\frac{i}{\theta\Lambda^2})$.

Clearly, in this way we produce very complicated ribbon graphs which cannot be drawn any more in a plane. Ribbon graphs define a Riemann surface on which they can be drawn. The Riemann surface is characterised by its genus g computable via the Euler characteristic of the graph, $g = 1 - \frac{1}{2}(L - I + V)$, and the number B of holes. Here, L is the number of single-line loops if we close the external lines of the graph, I is the number of double-line propagators and V the number of vertices. The number B of holes coincides with the number of single-line cycles which carry external legs. A few examples might help to understand the closure of external lines and the resulting topological data:



$$\begin{array}{ll} L=2 & g=0 \\ I=3 & B=2 \\ V=3 & N=6 \end{array} \qquad (18)$$



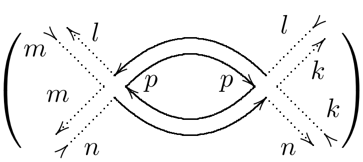
$$\begin{array}{ll} L=1 & g=1 \\ I=3 & B=1 \\ V=2 & N=2 \end{array} \qquad (19)$$

According to the topology we label the expansion coefficients of the effective action by $A^{(V,B,g)}_{m_1 n_1;\dots;m_N n_N}$.

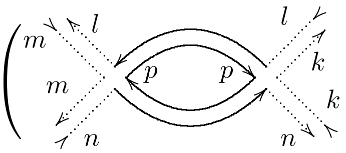
4 Integration procedure of the Polchinski equation

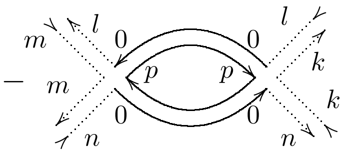
The integration procedure of the Polchinski equation involves the entire magic of renormalisation. We consider the example of the planar one-particle irreducible four-point function with two vertices, $A^{(2,1,0)1\mathrm{PI}}_{m_1 n_1;\dots;m_N n_N}$. The Polchinski equation (17) provides the Λ-derivative of that function:

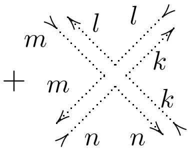
$$\Lambda \frac{\partial}{\partial \Lambda} A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda] = \sum_{p \in \mathbb{N}^2} \Bigg(\quad \Bigg)(\Lambda) + \text{ permutations} . \qquad (20)$$

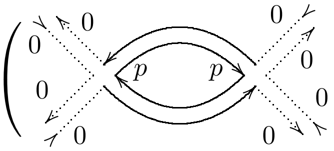


We consider the special case with constant indices on the trajectories. Performing the Λ-integration of (20) from some initial scale Λ_0 (sent to ∞ at the end) down to Λ, we obtain $A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda] \sim \ln \frac{\Lambda_0}{\Lambda}$, which diverges for $\Lambda_0 \to \infty$. Renormalisation can be understood as the change of the boundary condition for the integration. Thus, integrating (20) from a renormalisation scale Λ_R up to Λ, we have $A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda] \sim \ln \frac{\Lambda}{\Lambda_R}$, and there would be no problem for $\Lambda_0 \to \infty$. However, since there is an *infinite number* of matrix indices and there is no symmetry which could relate the amplitudes, that integration procedure entails an infinite number of initial conditions $A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda_R]$. To have a renormalisable model, we can only afford a finite number of integrations from Λ_R up to Λ. Thus, the correct choice is

$$A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda]$$
$$= -\int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$


$$-$$


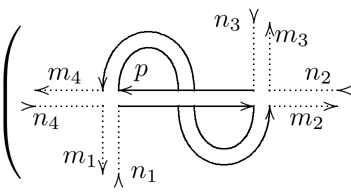
$$\Bigg)[\Lambda']$$
$$+$$


$$\Bigg[\int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$


$$\Bigg)[\Lambda'] + A^{(2,1,0)1\mathrm{PI}}_{00;00;00;00}[\Lambda_R]\Bigg] \;. \qquad (21)$$

The second graph in the first line on the rhs and the graph in brackets in the last line are identical, because only the indices on the propagators determine the value of the graph. Moreover, the vertex in the last line in front of the bracket equals 1. Thus, differentiating (21) with respect to Λ we obtain indeed (20). As a further check one can consider (21) for $m = n = k = l = 0$. Finally, the independence of $A^{(2,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}[\Lambda_0]$ on the indices m, n, k, l is built-in. This property is, for $\Lambda_0 \to \infty$, dynamically generated by the model.

There is a similar Λ_0-Λ_R-mixed integration procedure for the planar 1PI two-point functions $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}}$ and all other $A^{(V,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}$. These involve in total four different sub-integrations from Λ_R up to Λ. We refer to [4] for details. All other graphs are integrated from Λ_0 down to Λ, e.g.

$$A^{(2,2,0)1\mathrm{PI}}_{m_1n_1;\dots;m_4n_4}[\Lambda] = -\int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$


$$\Bigg)[\Lambda'] \;. \qquad (22)$$

5 The power-counting theorem

Theorem 2 *The previous integration procedure yields*

$$\begin{aligned}&\big|A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \\ &\leq \big(\sqrt{\theta}\Lambda\big)^{(4-N)+4(1-B-2g)}\; P^{4V-N}\Big[\frac{\max(\|m_1\|,\|n_1\|,\dots\|n_N\|)}{\theta\Lambda^2}\Big]\, P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]\;,\end{aligned} \qquad (23)$$

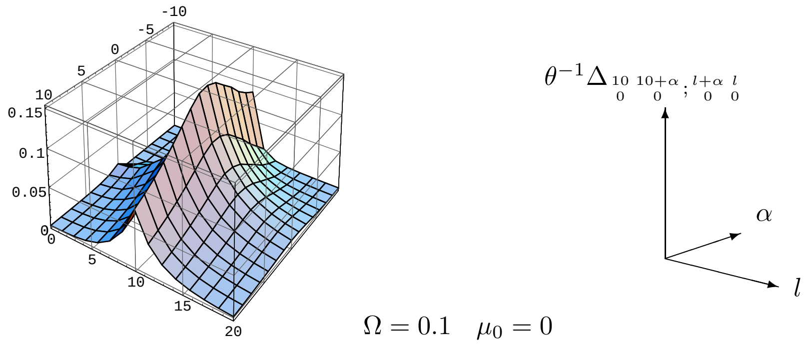
where $P^q[X]$ stands for a polynomial of degree q in X.

Idea of the proof. The cut-off propagator $Q_{mn;kl}(\Lambda)$ contains both an UV and an IR cut-off, $Q_{\substack{m_1\\m_2}\substack{n_1\\n_2};\substack{k_1\\k_2}\substack{l_1\\l_2}}(\Lambda) \neq 0$ only for $\theta\Lambda^2 < \max(m_1,\dots,l_2) < 2\theta\Lambda^2$. The UV cut-off limits the volume of the support of $Q_{mn;kl}(\Lambda)$ with respect to a single index to $4\theta^2\Lambda^4$. The IR cut-off results in the asymptotic behaviour

$$|Q_{mn;kl}(\Lambda)| < \frac{C_0}{\Omega\theta\Lambda^2}\delta_{m+k,n+l}\;. \qquad (24)$$

Thus, the propagator and the volume of a loop summation have the same power-counting dimensions as a commutative ϕ^4-model in momentum space, giving the total power-counting degree $4-N$ for an N-point function.

This is (more or less, see below) correct for planar graphs. The scaling behaviour of non-planar graphs is considerably improved by the *anisotropy* (or quasi-locality) of the propagator:



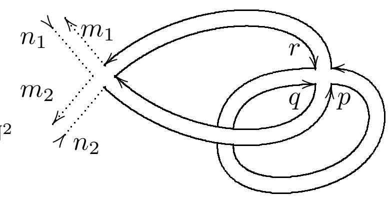
(25)

As a consequence, for given index m of the propagator $Q_{mn;kl}(\Lambda)=\underset{m\quad l}{\overset{n\quad k}{\rightleftarrows}}$, the contribution to a graph is strongly suppressed unless the other index l on the trajectory through m is close to m. Thus, the sum over l for given m converges and does not alter (apart from a factor Ω^{-1}) the power-counting behaviour of (24):

$$\sum_{l\in\mathbb{N}^2}\Big(\max_{n,k}|Q_{mn;kl}(\Lambda)|\Big)<\frac{C_1}{\theta\Omega^2\Lambda^2}\;. \tag{26}$$

In a non-planar graph like the one in (22), the index n_3—fixed as an external index—localises the summation index $p\approx n_3$. Thus, we save one volume factor $\theta^2\Lambda^4$ compared with a true loop summation as in (21). In general, each hole in the Riemann surface saves one volume factor, and each handle even saves two: In the genus-1 graph

$$\sum_{p,q,r\in\mathbb{N}^2}$$



(27)

n_2 is fixed as an external index, and the quasi-locality (25) implies $n_2\approx p\approx q\approx r$. Thus, instead of the two loops of a corresponding line graph, the non-planar ribbon graph (27) does not require any volume factor in the power-counting estimation.

A more careful analysis of (11) shows that also planar graphs get suppressed with $\big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\Lambda)\big|<\frac{C_2}{\Omega\theta\Lambda^2}\prod_{i=1}^2\Big(\frac{\max(m^i,l^i)+1}{\theta\Lambda^2}\Big)^{\frac{|m^i-l^i|}{2}}$, for $m^i\leq n^i$, if the index along a trajectory jumps. This leaves the functions $A^{(V,1,0)1\mathrm{PI}}_{mn;nk;kl;lm}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$, $A^{(V,1,0)1\mathrm{PI}}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$ and

$A^{(V,1,0)\mathrm{1PI}}_{\substack{m^1 \; n^1 ;\, n^1 \; m^1 \\ m^2+1 \; n^2+1 ;\, n^2 \; m^2}}$ as the only relevant or marginal ones. In these functions one has to use a discrete version of the Taylor expansion,

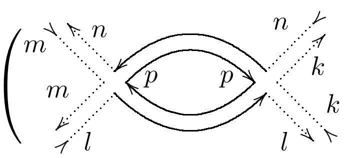
$$\Big| Q_{\substack{m^1 \, n^1 ; n^1 \, m^1 \\ m^2 \, n^2 ; n^2 \, m^2}}(\Lambda) - Q_{\substack{0 \, n^1 ; n^1 \, 0 \\ 0 \, n^2 ; n^2 \, 0}}(\Lambda) \Big| < \frac{C_3}{\Omega\theta\Lambda^2} \Big(\frac{\max(m^1, m^2)}{\theta\Lambda^2} \Big) , \tag{28}$$

$$\begin{aligned} \Big| Q_{\substack{m^1 \, n^1 ; n^1 \, m^1 \\ m^2 \, n^2 ; n^2 \, m^2}}(\Lambda) - Q_{\substack{0 \, n^1 ; n^1 \, 0 \\ 0 \, n^2 ; n^2 \, 0}}(\Lambda) - m^1 \big(Q_{\substack{1 \, n^1 ; n^1 \, 1 \\ 0 \, n^2 ; n^2 \, 0}}(\Lambda) - Q_{\substack{0 \, n^1 ; n^1 \, 0 \\ 0 \, n^2 ; n^2 \, 0}}(\Lambda) \big) & \\ - m^2 \big(Q_{\substack{0 \, n^1 ; n^1 \, 0 \\ 1 \, n^2 ; n^2 \, 1}}(\Lambda) - Q_{\substack{0 \, n^1 ; n^1 \, 0 \\ 0 \, n^2 ; n^2 \, 0}}(\Lambda) \big) \Big| &< \frac{C_4}{\Omega\theta\Lambda^2} \Big(\frac{\max(m^1, m^2)}{\theta\Lambda^2} \Big)^2 , \end{aligned} \tag{29}$$

$$\Big| Q_{\substack{m^1+1 \, n^1+1 ; n^1 \, m^1 \\ m^2 \, n^2 ; n^2 \, m^2}}(\Lambda) - \sqrt{m^1+1} Q_{\substack{1 \, n^1+1 ; n^1 \, 0 \\ 0 \, n^2 ; n^2 \, 0}}(\Lambda) \Big| < \frac{C_5}{\Omega\theta\Lambda^2} \Big(\frac{\max(m^1, m^2)}{\theta\Lambda^2} \Big)^{\frac{3}{2}} . \tag{30}$$

These estimations are traced back to the Meixner polynomials. The factor $\sqrt{m^1+1}$ in (30) is particularly remarkable. Any other Taylor subtraction (e.g. with pre-factors $\sqrt{m^1}$ or $\sqrt{m^1+2}$) would kill the renormalisation proof.

These discrete Taylor subtractions are used in the integration from Λ_0 down to Λ in prescriptions like (21):

$$- \int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \sum_{p \in \mathbb{N}^2} \Bigg(\quad$$

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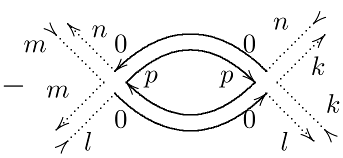
$$\Bigg) [\Lambda']$$

$$\begin{aligned} &= \int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \int_{\Lambda'}^{\Lambda_0} \frac{d\Lambda''}{\Lambda''} \sum_{p \in \mathbb{N}^2} \Big((Q_{np;pn} - Q_{0p;p0})(\Lambda') Q_{lp;pl}(\Lambda'') \\ &\qquad + Q_{0p;p0}(\Lambda')(Q_{lp;pl} - Q_{0p;p0})(\Lambda'') \Big) \sim \frac{C(\|n\| + \|l\|)}{\theta\Omega^2\Lambda^2} . \end{aligned} \tag{31}$$

This explains the polynomial in fractions like $\frac{\|m\|}{\theta\Lambda^2}$ in (23). □

Thus, replacing (similar as in the BPHZ subtraction) in planar 2- and 4-point functions the propagators by reference propagators at zero-indices and an irrelevant part, we have

$$\begin{aligned} A^{(V,1,0)}_{\substack{m^1 \, n^1 ; k^1 \, l^1 \\ m^2 \, n^2 ; k^2 \, l^2}} = \Big\{ & A^{(V,1,0)}_{\substack{0\,0;0\,0 \\ 0\,0;0\,0}} + m^1 \Big(A^{(V,1,0)}_{\substack{1\,0;0\,1 \\ 0\,0;0\,0}} - A^{(V,1,0)}_{\substack{0\,0;0\,0 \\ 0\,0;0\,0}} \Big) + n^1 \Big(A^{(V,1,0)}_{\substack{0\,1;1\,0 \\ 0\,0;0\,0}} - A^{(V,1,0)}_{\substack{0\,0;0\,0 \\ 0\,0;0\,0}} \Big) \\ & + m^2 \Big(A^{(V,1,0)}_{\substack{0\,0;0\,0 \\ 1\,0;0\,1}} - A^{(V,1,0)}_{\substack{0\,0;0\,0 \\ 0\,0;0\,0}} \Big) + n^2 \Big(A^{(V,1,0)}_{\substack{0\,0;0\,0 \\ 0\,1;1\,0}} - A^{(V,1,0)}_{\substack{0\,0;0\,0 \\ 0\,0;0\,0}} \Big) \Big\} \delta_{m^1 l^1} \delta_{n^1 k^1} \delta_{m^2 l^2} \delta_{n^2 k^2} \\ & + A^{(V,1,0)}_{\substack{1\,1;0\,0 \\ 0\,0;0\,0}} \big(\sqrt{k^1 l^1} \delta_{m^1+1,l^1} \delta_{n^1+1,k^1} \delta_{m^2 l^2} \delta_{n^2 k^2} + \sqrt{m^1 n^1} \delta_{m^1-1,l^1} \delta_{n^1-1,k^1} \delta_{m^2 l^2} \delta_{n^2 k^2} \big) \\ & + A^{(V,1,0)}_{\substack{0\,0;0\,0 \\ 1\,1;0\,0}} \big(\sqrt{k^2 l^2} \delta_{m^2+1,l^2} \delta_{n^2+1,k^2} \delta_{m^1 l^1} \delta_{n^1 k^1} + \sqrt{m^2 n^2} \delta_{m^2-1,2^1} \delta_{n^2-1,k^2} \delta_{m^1 l^1} \delta_{n^1 k^1} \big) \\ & + \text{ irrelevant part} , \end{aligned} \tag{32}$$

$$A^{(V,1,0)}_{m_1 n_1; \dots; m_4 n_4} = A^{(V,1,0)}_{00;\dots;00} \Big(\tfrac{1}{6} \delta_{n_1 m_2} \delta_{n_2 m_3} \delta_{n_3 m_4} \delta_{n_4 m_1} + 5 \text{ perms} \Big) + \text{ irrelevant part} . \tag{33}$$

We conclude that there are four independent relevant/marginal interaction coefficients:

$$\rho_1 = A^{(V,1,0)}_{\substack{00;00\\00;00}} , \qquad \rho_2 = A^{(V,1,0)}_{\substack{10;01\\00;00}} - A^{(V,1,0)}_{\substack{00;00\\00;00}} , \qquad \rho_3 = A^{(V,1,0)}_{\substack{11;00\\00;00}} , \qquad \rho_4 = A^{(V,1,0)}_{\substack{00;00;00;00\\00;00;00;00}} . \tag{34}$$

At $\Lambda = \Lambda_0$ we recover the same index structure as in the initial action (7), (8), identifying $\rho_a[\Lambda_0] \equiv \rho_a^0$ as functions of the coefficients $\mu_0, \theta, \Omega, \lambda$. This is a first indication that our model will be renormalisable. However, we have to remove the cut-off by sending $\Lambda_0 \to \infty$.

6 Removal of the cut-off

For given data Λ_0, ρ_a^0, the integration of the Polchinski equation yields the coefficients $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda, \Lambda_0, \rho_a^0]$ and thus, via (34), $\rho_b[\Lambda, \Lambda_0, \rho_a^0]$. Now, according to Section 4, in particular (21), we keep $\rho_b[\Lambda_R, \Lambda_0, \rho_a^0]$ *constant* when varying Λ_0. This leads to the identity

$$L[\phi, \Lambda_R, \Lambda_0', \rho^0[\Lambda_0']] - L[\phi, \Lambda_R, \Lambda_0'', \rho^0[\Lambda_0'']] = \int_{\Lambda_0''}^{\Lambda_0'} \frac{d\Lambda_0}{\Lambda_0}\, R[\phi, \Lambda_R, \Lambda_0, \rho^0[\Lambda_0]] , \tag{35}$$

$$R[\phi, \Lambda, \Lambda_0, \rho^0] := \Lambda_0 \frac{\partial L[\phi, \Lambda, \Lambda_0, \rho^0]}{\partial \Lambda_0} - \sum_{b=1}^{4} H^b[\Lambda, \Lambda_0, \rho^0]\, \Lambda_0 \frac{\partial \rho_b[\Lambda, \Lambda_0, \rho^0]}{\partial \Lambda_0} , \tag{36}$$

$$H^b[\Lambda, \Lambda_0, \rho^0] := \sum_{a=1}^{4} \frac{\partial L[\phi, \Lambda, \Lambda_0, \rho^0]}{\partial \rho_a^0} \frac{\partial \rho_a^0}{\partial \rho_b[\Lambda, \Lambda_0, \rho^0]} . \tag{37}$$

From (15) one derives Polchinski-like differential equations for the coefficients of R and H^a:

$$\Lambda \frac{\partial R}{\partial \Lambda} = M[L, R] - \sum_{a=1}^{4} H^a M_a[L, R] , \qquad \Lambda \frac{\partial H^a}{\partial \Lambda} = M[L, H^a] - \sum_{b=1}^{4} H^b M_b[L, H^a] , \tag{38}$$

for certain functions M, M_a which are linear in the second argument. We only have initial conditions at Λ_0 for these coefficients, thus the integration must always be performed from Λ_0 down to Λ. Fortunately, there are (by construction) remarkable cancellations in the rhs of (38) so that relevant contributions never appear. One proves

Proposition 3

$$\left| H^{a(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda, \Lambda_0, \rho_0] \right| \leq \left(\sqrt{\theta}\Lambda\right)^{(4-N-2\delta^{a1})+4(1-B-2g)} P^{2V+1+\delta^{a4}-\frac{N}{2}} \left[\ln \frac{\Lambda_0}{\Lambda_R} \right] , \tag{39}$$

$$\left| R^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda, \Lambda_0, \rho_0] \right| \leq \left(\frac{\Lambda^2}{\Lambda_0^2}\right) \left(\sqrt{\theta}\Lambda\right)^{(4-N)+4(1-B-2g)} P^{2V-\frac{N}{2}} \left[\ln \frac{\Lambda_0}{\Lambda_R} \right] . \tag{40}$$

We give the main ideas of the proof of (40). First, $R^{(1,1,0)}_{m_1n_1;\dots;m_4n_4} \equiv 0$, because the ϕ^4-vertex is scale-independent, which leads to a vanishing coefficient according to (36). Then, as $R^{(1,1,0)}_{m_1n_1;\dots;m_4n_4}$ appears in each term on the rhs of the first differential equation (38) for the 2-vertex six-point function and the 1-vertex two-point function, the coefficients $R^{(2,1,0)}_{m_1n_1;\dots;m_6n_6}$, $R^{(1,1,0)}_{m_1n_1;m_2n_2}$ and $R^{(1,2,0)}_{m_1n_1;m_2n_2}$ are Λ-independent. Next, one derives e.g.

$$R^{(2,1,0)}_{m_1n_1;\dots;m_6n_6}[\Lambda_0,\Lambda_0,\rho^0] = -\Big(\Lambda\frac{\partial}{\partial\Lambda}A^{(2,1,0)}_{m_1n_1;\dots;m_6n_6}[\Lambda,\Lambda_0,\rho^0]\Big)_{\Lambda=\Lambda_0} \sim \frac{C}{\theta\Lambda_0^2}\,, \tag{41}$$

where the scaling behaviour follows from (23). Since the first differential equation (38) is linear in R and relevant coefficients are projected away, the relative factor $\frac{\Lambda^2}{\Lambda_0^2}$ between $|A[\Lambda]|$ and $|R[\Lambda]|$ which first appears in (41) and similarly in $R^{(1,1,0)}_{m_1n_1;m_2n_2}, R^{(1,2,0)}_{m_1n_1;m_2n_2}$ survives to all R-coefficients. By integration of (35) we thus obtain

Theorem 4 *The duality-covariant noncommutative ϕ^4-model is (order by order in the coupling constant) renormalisable*

- *by an adjustment of the initial coefficients $\rho^0_a[\Lambda_0]$ to give renormalised constant couplings $\rho^R_a = \rho_a[\Lambda_R,\Lambda_0,\rho^0_b[\Lambda_0]]$, and*
- *by the corresponding integration of the flow equations.*

The limit $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] := \lim_{\Lambda_0\to\infty} A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]$ of the expansion coefficients of the effective action $L[\phi,\Lambda_R,\Lambda_0,\rho^0[\Lambda_0]]$ exists and satisfies

$$\begin{aligned}\Big|(2\pi\theta)^{\frac{N}{2}-2}A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] - (2\pi\theta)^{\frac{N}{2}-2}A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\Lambda_0,\rho^0]\Big| \\ \leq \frac{\Lambda_R^{6-N}}{\Lambda_0^2}\Big(\frac{1}{\theta^2\Lambda_R^4}\Big)^{B+2g-1} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,.\end{aligned} \tag{42}$$

This shows that non-planar graphs are regulated by the noncommutativity $\theta \neq 0$ so that the limit $\theta \to 0$ cannot be smooth. See the conclusion of [9] for a detailed discussion.

7 Renormalisation group equation

Knowing the relevant/marginal couplings, we can compute Feynman graphs with sharp matrix cut-off $\mathcal{N}$. The most important question concerns the β-function appearing in the renormalisation group equation which describes the cut-off dependence of the expansion coefficients $\Gamma_{m_1n_1;\dots;m_Nn_N}$ of the effective action when imposing normalisation conditions for the relevant and marginal couplings. We have [9]

$$\lim_{\mathcal{N}\to\infty}\Big(\mathcal{N}\frac{\partial}{\partial\mathcal{N}} + N\gamma + \mu_0^2\beta_{\mu_0}\frac{\partial}{\partial\mu_0^2} + \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_\Omega\frac{\partial}{\partial\Omega}\Big)\Gamma_{m_1n_1;\dots;m_Nn_N}[\mu_0,\lambda,\Omega,\mathcal{N}] = 0\,, \tag{43}$$

where

$$\beta_\lambda = \mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\lambda[\mu_{\text{phys}},\lambda_{\text{phys}},\Omega_{\text{phys}},\mathcal{N}]\Big)\,, \qquad \beta_\Omega = \mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\Omega[\mu_{\text{phys}},\lambda_{\text{phys}},\Omega_{\text{phys}},\mathcal{N}]\Big)\,, \tag{44}$$

$$\beta_{\mu_0} = \frac{\mathcal{N}}{\mu_0^2}\frac{\partial}{\partial\mathcal{N}}\Big(\mu_0^2[\mu_{\text{phys}},\lambda_{\text{phys}},\Omega_{\text{phys}},\mathcal{N}]\Big)\,, \qquad \gamma = \mathcal{N}\frac{\partial}{\partial\mathcal{N}}\Big(\ln\mathcal{Z}[\mu_{\text{phys}},\lambda_{\text{phys}},\Omega_{\text{phys}},\mathcal{N}]\Big)\,. \tag{45}$$

Here, $\mathcal{Z}$ is the wavefunction renormalisation. To one-loop order we find [9]

$$\beta_\lambda = \frac{\lambda_{\text{phys}}^2}{48\pi^2}\frac{(1-\Omega_{\text{phys}}^2)}{(1+\Omega_{\text{phys}}^2)^3}\,, \qquad \beta_\Omega = \frac{\lambda_{\text{phys}}\Omega_{\text{phys}}}{96\pi^2}\frac{(1-\Omega_{\text{phys}}^2)}{(1+\Omega_{\text{phys}}^2)^3}\,, \tag{46}$$

$$\beta_{\mu_0} = -\frac{\lambda_{\text{phys}}\Big(4\mathcal{N}\ln(2) + \frac{(8+\theta\mu_{\text{phys}}^2)\Omega_{\text{phys}}^2}{(1+\Omega_{\text{phys}}^2)^2}\Big)}{48\pi^2\theta\mu_{\text{phys}}^2(1+\Omega_{\text{phys}}^2)}\,, \qquad \gamma = \frac{\lambda_{\text{phys}}}{96\pi^2}\frac{\Omega_{\text{phys}}^2}{(1+\Omega_{\text{phys}}^2)^3}\,. \tag{47}$$

From (44) and (46) one finds that $\frac{\lambda}{\Omega^2}$ remains constant under the renormalisation flow. The integration of the resulting differential equation shows that, starting from given small values for Ω_R, λ_R at $\mathcal{N}_R$, the frequency grows in a small region around $\ln\frac{\mathcal{N}}{\mathcal{N}_R} = \frac{48\pi^2}{\lambda_R}$ to $\Omega \approx 1$. The coupling constant approaches $\lambda_\infty = \frac{\lambda_R}{\Omega_R^2}$, which can be made small for sufficiently small λ_R. This leaves the chance of a non-perturbative construction [10] of the model.

The one-loop renormalisation flow has a non-trivial UV fixed point given by the self-dual model $\Omega = 1$, see (4), where $\beta_\Omega = 0$ to all orders. We conjecture $\beta_\lambda = 0$ to all orders due to the resemblance of the duality-invariant theory with the exactly solvable models in [11].

Moreover, at one-loop order we have $\beta_\Omega \to 0$ for $\Omega \to 0$, which defines the standard noncommutative ϕ^4-theory. This means that UV/IR-mixing is not a problem at one-loop.

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