hep-th/0205094

Seiberg-Witten map for noncommutative super Yang-Mills theory

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Abstract. In this letter we derive the Seiberg-Witten map for noncommutative super Yang-Mills theory in Wess-Zumino gauge. Following (and using results of) `hep-th/0108045` we split the observer Lorentz transformations into a covariant particle Lorentz transformation and a remainder which gives directly the Seiberg-Witten differential equations. These differential equations lead to a θ-expansion of the noncommutative super Yang-Mills action which is invariant under commutative gauge transformations and commutative observer Lorentz transformation, but not invariant under commutative supersymmetry transformations: The θ-expansion of noncommutative supersymmetry leads to a θ-dependent symmetry transformation. For this reason the Seiberg-Witten map of super Yang-Mills theory cannot be expressed in terms of superfields.

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1 Introduction

The simplest model for noncommutative space-time is the so-called noncommutative $\mathbb{R}^4$ characterized by a constant antisymmetric tensor θ. Field theories on such a deformed space-time became recently very popular, mainly due to their relation to string theory [1] and the possibility to perform similar calculations of Feynman graphs as on usual Minkowski space. It turned out that field theories which are renormalizable on Minkowski space are in general not renormalizable (at any loop order) on noncommutative $\mathbb{R}^4$, see [2].

A remarkable result of Seiberg and Witten [1] was that *gauge theory* on noncommutative $\mathbb{R}^4$ is gauge-equivalent to a gauge theory on Minkowski space coupled to a constant external field θ. This equivalence can be traced back [3] to a deeper discussion of Lorentz transformations [4]: In presence of θ one has to distinguish between 'observer Lorentz transformations', which transform θ as a Lorentz two-tensor, and 'particle Lorentz transformations', which leave θ invariant. It turns out that observer Lorentz transformations are symmetries of the theory whereas particle Lorentz symmetry is broken. Being (in principle) an observable, the breaking of particle Lorentz symmetry must be gauge-invariant [3]. This is not automatically the case and demands a covariant redefinition of the splitting of the observer Lorentz transformation into particle Lorentz transformation plus θ-transformation, which is governed by the Seiberg-Witten differential equations.

This letter is an extension of [3] to the components of a noncommutative super vector field in Wess-Zumino gauge. We derive the Seiberg-Witten differential equations of super Yang-Mills theory via a covariant splitting of the observer Lorentz transformations into particle Lorentz transformations and a remainder, using the splitting for the gauge field derived in [3] as the starting point. The Seiberg-Witten differential equations lead to a θ-expansion of the noncommutative super Yang-Mills action in terms of fields living on commutative space-time. This θ-expanded action is automatically invariant under commutative gauge transformations and commutative Lorentz transformations. It is however not invariant under commutative supersymmetry transformations. Instead, the θ-expansion of the noncommutative supersymmetry transformation yields a symmetry transformation of the θ-expanded action which extends the usual supersymmetry transformations by terms of order $n \geq 1$ in θ. This result implies that the Seiberg-Witten map for super Yang-Mills theory cannot be expressed in terms of superfields.

2 The noncommutative super Yang-Mills action and its symmetries

The noncommutative $\mathcal{N}$=1 super Yang-Mills action is in the component formulation defined by

$$\Gamma = \int d^4x \, \mathrm{tr}\Big(-\frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} + \mathrm{i}\hat{\lambda}^a \sigma^\mu_{a\dot{a}} \hat{D}_\mu \hat{\bar{\lambda}}^{\dot{a}} + \frac{1}{2}\hat{D}^2\Big) , \tag{1}$$

where

$$\hat{F}_{\mu\nu} := \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - \mathrm{i}[\hat{A}_\mu, \hat{A}_\nu]_\star , \tag{2}$$

$$\hat{D}_\mu \hat{\bar{\lambda}}^{\dot{a}} := \partial_\mu \hat{\bar{\lambda}}^{\dot{a}} - \mathrm{i}[\hat{A}_\mu, \hat{\bar{\lambda}}^{\dot{a}}]_\star . \tag{3}$$

Some useful properties of objects carrying spinor indices $a, \dot{a} \in \{1,2\}$ are listed in the appendix. The $\star$-(anti)commutators of matrix-valued Schwartz class functions f, g are defined by

$$[f,g]_\star = g\star f - f\star g\ , \qquad\qquad \{f,g\}_\star = g\star f + f\star g\ , \tag{4}$$

where the $\star$-product is defined by

$$(f\star g)(x) = \int d^4y \int \frac{d^4k}{(2\pi)^4}\, f(x+\tfrac{1}{2}\theta{\cdot}k)\, g(x+y)\, \mathrm{e}^{\mathrm{i}k\cdot y}\,, \tag{5}$$

with $(\theta{\cdot}k)^\mu := \theta^{\mu\nu}k_\nu$, $k\cdot y := k_\mu y^\mu$ and $\theta^{\mu\nu} = -\theta^{\nu\mu} \in M_4(\mathbb{R})$. We consider $\theta^{\mu\nu}$ as the components of a translation-invariant tensor field.

The action (1) is invariant under gauge transformations

$$W^G_{\hat\omega} = \int d^4x\ \mathrm{tr}\Big(\hat D_\mu\hat\omega\,\frac{\delta}{\delta\hat A_\mu} - \mathrm{i}[\hat{\bar\lambda}^{\dot a},\hat\omega]_\star\frac{\delta}{\delta\hat{\bar\lambda}^{\dot a}} - \mathrm{i}[\hat\lambda^a,\hat\omega]_\star\frac{\delta}{\delta\hat\lambda^a} - \mathrm{i}[\hat D,\hat\omega]_\star\frac{\delta}{\delta\hat D}\Big)\ , \tag{6}$$

observer Lorentz transformations

$$W^T_\tau = \int d^4x\ \mathrm{tr}\Big(\partial_\tau\hat A_\mu\,\frac{\delta}{\delta\hat A_\mu} + \partial_\tau\hat\lambda^a\frac{\delta}{\delta\hat\lambda^a} + \partial_\tau\hat{\bar\lambda}^{\dot a}\frac{\delta}{\delta\hat{\bar\lambda}^{\dot a}} + \partial_\tau\hat D\frac{\delta}{\delta\hat D}\Big)\ , \tag{7}$$

$$\begin{aligned}
W^R_{\alpha\beta} := \int d^4x\ \mathrm{tr}\Big(&\Big(\frac{1}{2}\{x_\alpha,\partial_\beta\hat A_\mu\}_\star - \frac{1}{2}\{x_\beta,\partial_\alpha\hat A_\mu\}_\star + g_{\mu\alpha}\hat A_\beta - g_{\mu\beta}\hat A_\alpha\Big)\frac{\delta}{\delta\hat A_\mu}\\
&+\Big(\frac{1}{2}\{x_\alpha,\partial_\beta\hat\lambda^a\}_\star - \frac{1}{2}\{x_\beta,\partial_\alpha\hat\lambda^a\}_\star + \frac{\mathrm{i}}{2}\hat\lambda^b\sigma_{\alpha\beta\,b}{}^a\Big)\frac{\delta}{\delta\hat\lambda^a}\\
&+\Big(\frac{1}{2}\{x_\alpha,\partial_\beta\hat{\bar\lambda}^{\dot a}\}_\star - \frac{1}{2}\{x_\beta,\partial_\alpha\hat{\bar\lambda}^{\dot a}\}_\star - \frac{\mathrm{i}}{2}\bar\sigma^{\dot a}_{\alpha\beta\,\dot b}\hat{\bar\lambda}^{\dot b}\Big)\frac{\delta}{\delta\hat{\bar\lambda}^{\dot a}}\\
&+\Big(\frac{1}{2}\{x_\alpha,\partial_\beta\hat D\}_\star - \frac{1}{2}\{x_\beta,\partial_\alpha\hat D\}_\star\Big)\frac{\delta}{\delta\hat D}\Big)\\
+\Big(\delta^\mu_\alpha\theta_\beta{}^\nu - \delta^\mu_\beta\theta_\alpha{}^\nu &+ \delta^\nu_\alpha\theta^\mu{}_\beta - \delta^\nu_\beta\theta^\mu{}_\alpha\Big)\frac{\partial}{\partial\theta^{\mu\nu}}\ ,
\end{aligned} \tag{8}$$

$$\begin{aligned}
W^D := \int d^4x\ \mathrm{tr}\Big(&\Big(\frac{1}{2}\{x^\delta,\partial_\delta\hat A_\mu\}_\star + \hat A_\mu\Big)\frac{\delta}{\delta\hat A_\mu} + \Big(2\hat D + \frac{1}{2}\{x^\delta,\partial_\delta\hat D\}_\star\Big)\frac{\delta}{\delta\hat D}\\
&+\Big(\frac{3}{2}\hat\lambda^a + \frac{1}{2}\{x^\delta,\partial_\delta\hat\lambda^a\}_\star\Big)\frac{\delta}{\delta\hat\lambda^a} + \Big(\frac{3}{2}\hat{\bar\lambda}^{\dot a} + \frac{1}{2}\{x^\delta,\partial_\delta\hat{\bar\lambda}^{\dot a}\}_\star\Big)\frac{\delta}{\delta\hat{\bar\lambda}^{\dot a}}\Big)\\
-2\theta^{\mu\nu}\frac{\partial}{\partial\theta^{\mu\nu}}\ ,&
\end{aligned} \tag{9}$$

and supersymmetry transformations [5]

$$W^S_a = \int d^4x\ \mathrm{tr}\Big(\sigma_{\mu a\dot a}\hat{\bar\lambda}^{\dot a}\frac{\delta}{\delta\hat A_\mu} + \big(\delta^b_a\hat D + \frac{1}{2}\sigma^{\mu\nu}_a{}^b\hat F_{\mu\nu}\big)\frac{\delta}{\delta\hat\lambda^b} - \mathrm{i}\sigma^\mu_{a\dot a}\hat D_\mu\hat{\bar\lambda}^{\dot a}\frac{\delta}{\delta\hat D}\Big)\ , \tag{10}$$

$$W^{\bar S}_{\dot a} = \int d^4x\ \mathrm{tr}\Big(\hat\lambda^a\sigma_{\mu a\dot a}\frac{\delta}{\delta\hat A_\mu} + \big(\delta^{\dot b}_{\dot a}\hat D - \frac{1}{2}\bar\sigma^{\mu\nu\,\dot b}{}_{\dot a}\hat F_{\mu\nu}\big)\frac{\delta}{\delta\hat{\bar\lambda}^{\dot b}} - \mathrm{i}\hat D_\mu\hat\lambda^a\sigma^\mu_{a\dot a}\frac{\delta}{\delta\hat D}\Big)\ . \tag{11}$$

The partial derivative with respect to $\theta^{\mu\nu}$ has the property

$$\frac{\partial(\hat{U}\star\hat{V})}{\partial\theta^{\mu\nu}} = \frac{\partial\hat{U}}{\partial\theta^{\mu\nu}}\star\hat{V} + \hat{U}\star\frac{\partial\hat{V}}{\partial\theta^{\mu\nu}} + \frac{\mathrm{i}}{2}(\partial_\mu\hat{U})\star(\partial_\nu\hat{V}) \;, \tag{12}$$

where the fields $\hat{A}_\mu, \hat{\lambda}^a, \hat{\bar{\lambda}}^{\dot{a}}, \hat{D}$ must be assumed to be independent of θ.

3 Seiberg-Witten differential equations

As in (non-supersymmetric) noncommutative Yang-Mills theory [3] we derive the Seiberg-Witten differential equations via a splitting of the observer Lorentz transformation $W^R_{\alpha\beta}$ into the covariant particle Lorentz transformation $\tilde{W}^R_{\hat{\phi};\alpha\beta}$ and a remaining piece $\tilde{W}^R_{\theta;\alpha\beta}$ involving the Seiberg-Witten differential equation:

$$W^R_{\alpha\beta} \equiv \tilde{W}^R_{\hat{\phi};\alpha\beta} + \tilde{W}^R_{\theta;\alpha\beta} \;, \tag{13}$$

$$\tilde{W}^R_{\hat{\phi};\alpha\beta}(\theta^{\mu\nu}) = 0 \;, \tag{14}$$

$$[\tilde{W}^R_{\hat{\phi};\alpha\beta}, W^G_{\hat{\omega}}] = W^G_{\hat{\omega}'_{\alpha\beta}} \;, \qquad [\tilde{W}^R_{\theta;\alpha\beta}, W^G_{\hat{\omega}}] = W^G_{\hat{\omega}''_{\alpha\beta}} \;. \tag{15}$$

The motivation for this ansatz is the following. The commutator of an *observer Lorentz rotation* (8) with a gauge transformation (6) is again a gauge transformation,

$$[W^R_{\alpha\beta}, W^G_{\hat{\omega}}] = W^G_{\hat{\omega}_{\alpha\beta}} \;, \tag{16}$$

for some infinitesimal gauge parameter $\hat{\omega}_{\alpha\beta}[\hat{\omega}]$. A *particle Lorentz transformation* is defined as the part of an observer Lorentz transformation which does not transform the field $\theta^{\mu\nu}$, see (14). However, one should require that a particle Lorentz transformation transforms a gauge-invariant quantity into another gauge-invariant quantity, otherwise the particle Lorentz transformation cannot be considered as well-defined [3]. It is sufficient to demand (15) in order to achieve this property.

To find the sought for splitting we first apply the ansatz of [3] for the Yang-Mills field $\hat{A}_\mu$:

$$\tilde{W}^R_{\hat{\phi};\alpha\beta}\hat{A}_\mu = \hat{D}_\mu\hat{\chi}_{\alpha\beta} + \Big(\frac{1}{2}\{\hat{X}_\alpha, \hat{F}_{\beta\mu}\}_\star - \frac{1}{2}\{\hat{X}_\beta, \hat{F}_{\alpha\mu}\}_\star - W^R_{\alpha\beta}(\theta^{\rho\sigma})\hat{\Omega}_{\rho\sigma\mu}\Big) \;, \tag{17}$$

where $\hat{X}^\mu = x^\mu + \theta^{\mu\nu}\hat{A}_\nu$ are the covariant coordinates [6] and $\hat{\Omega}_{\rho\sigma\mu}$ is a polynomial in covariant quantities such as $\theta^{\alpha\beta}, \hat{F}_{\kappa\lambda}, \hat{D}_{\mu_1}\dots\hat{D}_{\mu_n}\hat{F}_{\kappa\lambda}$, antisymmetric in ρ,σ, of power-counting dimension 3, and expresses the freedom in the splitting. In the following we set $\hat{\Omega}_{\rho\sigma\mu} = 0$. The parameter $\hat{\chi}_{\alpha\beta}$ is unchanged and given by [3]

$$\hat{\chi}_{\alpha\beta} = \frac{1}{4}\{2x_\alpha + \theta_\alpha{}^\rho\hat{A}_\rho, \hat{A}_\beta\}_\star - \frac{1}{4}\{2x_\beta + \theta_\beta{}^\rho\hat{A}_\rho, \hat{A}_\alpha\}_\star \;. \tag{18}$$

Comparing (17) with the $\hat{A}_\mu$-part of (8) and extending this covariantization to the remaining fields $\hat{\lambda}^a, \hat{\bar{\lambda}}^{\dot{a}}, \hat{D}$ we obtain from (8)

$$\begin{aligned}\tilde{W}^R_{\hat{\phi};\alpha\beta} = W^G_{\hat{\chi}_{\alpha\beta}} + \int d^4x \,\mathrm{tr}\Big(&\Big(\frac{1}{2}\{\hat{X}_\alpha, \hat{F}_{\beta\mu}\}_\star - \frac{1}{2}\{\hat{X}_\beta, \hat{F}_{\alpha\mu}\}_\star\Big)\frac{\delta}{\delta\hat{A}_\mu} \\ &+ \Big(\frac{1}{2}\{\hat{X}_\alpha, \hat{D}_\beta\hat{\lambda}^a\}_\star - \frac{1}{2}\{\hat{X}_\beta, \hat{D}_\alpha\hat{\lambda}^a\}_\star + \frac{\mathrm{i}}{2}\hat{\lambda}^b\sigma_{\alpha\beta\, b}{}^a\Big)\frac{\delta}{\delta\hat{\lambda}^a} \\ &+ \Big(\frac{1}{2}\{\hat{X}_\alpha, \hat{D}_\beta\hat{\bar{\lambda}}^{\dot{a}}\}_\star - \frac{1}{2}\{\hat{X}_\beta, \hat{D}_\alpha\hat{\bar{\lambda}}^{\dot{a}}\}_\star - \frac{\mathrm{i}}{2}\bar{\sigma}^{\dot{a}}_{\alpha\beta\,\dot{b}}\hat{\bar{\lambda}}^{\dot{b}}\Big)\frac{\delta}{\delta\hat{\bar{\lambda}}^{\dot{a}}} \\ &+ \Big(\frac{1}{2}\{\hat{X}_\alpha, \hat{D}_\beta\hat{D}\}_\star - \frac{1}{2}\{\hat{X}_\beta, \hat{D}_\alpha\hat{D}\}_\star\Big)\frac{\delta}{\delta\hat{D}}\;, \end{aligned} \tag{19}$$

Now it is straightforward to evaluate

$$\tilde{W}^R_{\theta;\alpha\beta} = W^R_{\alpha\beta} - \tilde{W}^R_{\hat{\phi};\alpha\beta} = W^R_{\alpha\beta}(\theta^{\rho\sigma})\frac{d}{d\theta^{\rho\sigma}}\;, \tag{20}$$

with

$$\frac{d}{d\theta^{\rho\sigma}} = \frac{\partial}{\partial\theta^{\rho\sigma}} + \int d^4x \,\mathrm{tr}\Big(\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}\frac{\delta}{\delta\hat{A}_\mu} + \frac{d\hat{\lambda}^a}{d\theta^{\rho\sigma}}\frac{\delta}{\delta\hat{\lambda}^a} + \frac{d\hat{\bar{\lambda}}^{\dot{a}}}{d\theta^{\rho\sigma}}\frac{\delta}{\delta\hat{\bar{\lambda}}^{\dot{a}}} + \frac{d\hat{D}}{d\theta^{\rho\sigma}}\frac{\delta}{\delta\hat{D}}\Big)\;, \tag{21}$$

which yields the Seiberg-Witten differential equations

$$\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} = -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma\hat{A}_\mu + \hat{F}_{\sigma\mu}\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho\hat{A}_\mu + \hat{F}_{\rho\mu}\big\}_\star\;, \tag{22}$$

$$\frac{d\hat{\lambda}^a}{d\theta^{\rho\sigma}} = -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma\hat{\lambda}^a + \hat{D}_\sigma\hat{\lambda}^a\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho\hat{\lambda}^a + \hat{D}_\rho\hat{\lambda}^a\big\}_\star\;, \tag{23}$$

$$\frac{d\hat{\bar{\lambda}}^{\dot{a}}}{d\theta^{\rho\sigma}} = -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma\hat{\bar{\lambda}}^{\dot{a}} + \hat{D}_\sigma\hat{\bar{\lambda}}^{\dot{a}}\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho\hat{\bar{\lambda}}^{\dot{a}} + \hat{D}_\rho\hat{\bar{\lambda}}^{\dot{a}}\big\}_\star\;, \tag{24}$$

$$\frac{d\hat{D}}{d\theta^{\rho\sigma}} = -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma\hat{D} + \hat{D}_\sigma\hat{D}\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho\hat{D} + \hat{D}_\rho\hat{D}\big\}_\star\;. \tag{25}$$

The differential equation (22) was first found in [1].

4 θ-expansion of the action

The differential equations (22)–(25) are now taken as the starting point for a θ-expansion of the action,

$$\Gamma^{(n)} := \sum_{j=0}^{n}\frac{1}{j!}\theta^{\rho_1\sigma_1}\cdots\theta^{\rho_j\sigma_j}\Big(\frac{d^j\Gamma}{d\theta^{\rho_1\sigma_1}\ldots d\theta^{\rho_j\sigma_j}}\Big)_{\theta=0}\;. \tag{26}$$

It follows from the the second identity in (15) that the θ-expansion (26) of the action (1) is invariant under commutative gauge transformations. One also checks the identity

$$\Big[W^{\{T,R,D\}}, \theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}}\Big] = 0 \tag{27}$$

for super Yang-Mills theory, which means that the θ-expansion of the fields leads to a commutative action invariant under commutative rotations and translations and with commutative dilatational symmetry.

The θ-expansion of (1) yields an action which is *not invariant* under commutative supersymmetry transformations. Indeed, the commutator of a supersymmetry transformation and a θ-differentiation is given by[1]

$$
\begin{aligned}
\Big[\frac{d}{d\theta^{\rho\sigma}}, W^S_a\Big] &= \tilde{W}^G_{\frac{1}{8}\sigma_{\rho\,a\dot a}\{\hat A_\sigma,\hat{\bar\lambda}^{\dot a}\}_\star - \frac{1}{8}\sigma_{\sigma\,a\dot a}\{\hat A_\rho,\hat{\bar\lambda}^{\dot a}\}_\star} \\
&\quad + \int d^4x\ \mathrm{tr}\Big(\Big(\frac{1}{4}\sigma_{\rho\,a\dot a}\{\hat F_{\sigma\mu},\hat{\bar\lambda}^{\dot a}\}_\star - \frac{1}{4}\sigma_{\sigma\,a\dot a}\{\hat F_{\rho\mu},\hat{\bar\lambda}^{\dot a}\}_\star\Big)\frac{\delta}{\delta\hat A_\mu} \\
&\qquad + \Big(\frac{1}{4}\sigma^{\mu\nu\,b}_a\{\hat F_{\mu\rho},\hat F_{\nu\sigma}\}_\star + \frac{1}{4}\sigma_{\rho\,a\dot a}[\hat{\bar\lambda}^{\dot a},\hat D_\sigma\hat\lambda^b]_\star - \frac{1}{4}\sigma_{\sigma\,a\dot a}[\hat{\bar\lambda}^{\dot a},\hat D_\rho\hat\lambda^b]_\star\Big)\frac{\delta}{\delta\hat\lambda^b} \\
&\qquad + \Big(\frac{1}{4}\sigma_{\rho\,a\dot a}[\hat{\bar\lambda}^{\dot a},\hat D_\sigma\hat{\bar\lambda}^{\dot b}]_\star - \frac{1}{4}\sigma_{\sigma\,a\dot a}[\hat{\bar\lambda}^{\dot a},\hat D_\rho\hat{\bar\lambda}^{\dot b}]_\star\Big)\frac{\delta}{\delta\hat{\bar\lambda}^{\dot b}} \\
&\qquad + \Big(\frac{\mathrm{i}}{4}\sigma^\mu_{a\dot a}\{\hat F_{\sigma\mu},\hat D_\rho\hat{\bar\lambda}^{\dot a}\}_\star - \frac{\mathrm{i}}{4}\sigma^\mu_{a\dot a}\{\hat F_{\rho\mu},\hat D_\sigma\hat{\bar\lambda}^{\dot a}\}_\star \\
&\qquad\qquad + \frac{1}{4}\sigma_{\rho\,a\dot a}\{\hat{\bar\lambda}^{\dot a},\hat D_\sigma\hat D\}_\star - \frac{1}{4}\sigma_{\sigma\,a\dot a}\{\hat{\bar\lambda}^{\dot a},\hat D_\rho\hat D\}_\star\Big)\frac{\delta}{\delta\hat D}\ ,
\end{aligned}
\tag{28}
$$

where the gauge transformation with respect to a fermionic parameter $\tilde\omega$ is defined by

$$
\tilde W^G_{\tilde\omega} = \int d^4x\ \mathrm{tr}\Big(\hat D_\mu\hat\omega\,\frac{\delta}{\delta\hat A_\mu} + \mathrm{i}\{\hat{\bar\lambda}^{\dot a},\tilde\omega\}_\star\frac{\delta}{\delta\hat{\bar\lambda}^{\dot a}} + \mathrm{i}\{\hat\lambda^a,\tilde\omega\}_\star\frac{\delta}{\delta\hat\lambda^a} - \mathrm{i}[\hat D,\tilde\omega]_\star\frac{\delta}{\delta\hat D}\Big)\ . \tag{29}
$$

The action (1) is invariant under the transformation (29). It follows now from (26) that the θ-expansion of (1) is invariant under the transformation

$$
W^{S,comm}_a = \big(W^S_a\big)_{\theta=0} + \sum_{n=1}^{\infty}\frac{1}{n!}\theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}\Big(\Big[\frac{d}{d\theta^{\rho_1\sigma_1}},\Big[\dots\Big[\frac{d}{d\theta^{\rho_n\sigma_n}},W^S_a\Big]\dots\Big]\Big]\Big)_{\theta=0}\ , \tag{30}
$$

which due to $[\frac{d}{d\theta},W^S_a]\neq 0$ is different from the commutative supersymmetry transformation $\big(W^S_a\big)_{\theta=0}$. The first terms of (30) read

$$
\begin{aligned}
W^{S,comm}_a &= \int d^4x\ \mathrm{tr}\Big(\Big(\sigma_{\mu a\dot a}\bar\lambda^{\dot a} + \frac{1}{2}\theta^{\rho\sigma}\sigma_{\rho\,a\dot a}\{F_{\sigma\mu},\bar\lambda^{\dot a}\}\Big)\frac{\delta}{\delta A_\mu} + \Big(\frac{1}{2}\theta^{\rho\sigma}\sigma_{\rho\,a\dot a}[\bar\lambda^{\dot a},D_\sigma\bar\lambda^{\dot b}]\Big)\frac{\delta}{\delta\bar\lambda^{\dot b}} \\
&\quad + \Big(\delta^b_a D + \frac{1}{2}\sigma^{\mu\nu\,b}_a F_{\mu\nu} + \frac{1}{4}\theta^{\rho\sigma}\sigma^{\mu\nu\,b}_a\{F_{\mu\rho},F_{\nu\sigma}\} + \frac{1}{2}\theta^{\rho\sigma}\sigma_{\rho\,a\dot a}\{F_{\sigma\mu},\bar\lambda^{\dot a}\}\Big)\frac{\delta}{\delta\lambda^b} \\
&\quad + \Big(-\mathrm{i}\sigma^\mu_{a\dot a}D_\mu\bar\lambda^{\dot a} + \frac{\mathrm{i}}{2}\theta^{\rho\sigma}\sigma^\mu_{a\dot a}\{F_{\sigma\mu},D_\rho\bar\lambda^{\dot a}\} + \frac{1}{2}\theta^{\rho\sigma}\sigma_{\rho\,a\dot a}\{\bar\lambda^{\dot a},D_\sigma D\}\Big)\frac{\delta}{\delta D}\Big) + \mathcal{O}(\theta^2)\ .
\end{aligned}
\tag{31}
$$

Similar formulae exist for the anti-supersymmetry transformation $W^{\bar S}_{\dot a}$.

[1] There is of course a freedom in the differential equations (22)–(25) given by the Ω-terms in (17) and similarly for the other fields. This freedom is not sufficient to obtain a vanishing right hand side of (28).

At order $n = 0$ in θ the expansion of (1) is obviously the standard super Yang-Mills action

$$\Gamma^{(0)} = \int d^4x \ \mathrm{tr}\Big(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \mathrm{i}\lambda^a\sigma^\mu_{a\dot{a}}D_\mu\bar{\lambda}^{\dot{a}} + \frac{1}{2}D^2\Big) , \tag{32}$$

where $\phi = \hat{\phi}|_{\theta=0}$ for $\phi \in \{A_\mu, \lambda^a, \bar{\lambda}^{\dot{a}}, D\}$. At first order in θ one finds

$$\begin{aligned}\Gamma^{(1)} = \Gamma^{(0)} - \frac{1}{2}\int d^4x \ \mathrm{tr}\Big(\theta^{\rho\sigma}F_{\rho\sigma}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mathrm{i}}{2}\bar{\lambda}^{\dot{a}}\bar{\sigma}^\mu_{\dot{a}a}D_\mu\lambda^a + \frac{\mathrm{i}}{2}\lambda^a\sigma^\mu_{a\dot{a}}D_\mu\bar{\lambda}^{\dot{a}} + \frac{1}{2}D^2\Big) \\ + \theta^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}F^{\mu\nu} + \theta^{\rho\sigma}F_{\mu\rho}\big(\mathrm{i}\bar{\lambda}^{\dot{a}}\bar{\sigma}^\mu_{\dot{a}a}D_\sigma\lambda^a + \mathrm{i}\lambda^a\sigma^\mu_{a\dot{a}}D_\sigma\bar{\lambda}^{\dot{a}}\big)\Big) .\end{aligned} \tag{33}$$

The θ-expanded action (33) could be further analysed, for instance with respect to new decay channels of supersymmetric particles—in a similar manner as investigations of models without supersymmetry, see *e.g.* [7].

5 Remarks on the superspace formalism

The most compact way to formulate supersymmetric theories is to use the superfield formalism. The above considered fields $\hat{A}_\mu, \hat{\lambda}^a, \hat{\bar{\lambda}}^{\dot{a}}, \hat{D}$ of super Yang-Mills theory can be regarded as components of the superfield

$$\begin{aligned}\hat{\phi} = \hat{C} + \hat{\chi}^a\theta_a + \bar{\theta}_{\dot{a}}\hat{\bar{\chi}}^{\dot{a}} + \theta^a\theta_a\hat{M} + \bar{\theta}_{\dot{a}}\bar{\theta}^{\dot{a}}\hat{\bar{M}} \\ - 2\theta^a\sigma^\mu_{a\dot{a}}\bar{\theta}^{\dot{a}}\hat{A}_\mu - 2\bar{\theta}_{\dot{a}}\hat{\bar{\lambda}}^{\dot{a}}\theta^a\theta_a - 2\hat{\lambda}^a\theta_a\bar{\theta}_{\dot{a}}\bar{\theta}^{\dot{a}} - \theta^a\theta_a\bar{\theta}_{\dot{a}}\bar{\theta}^{\dot{a}}\hat{D} .\end{aligned} \tag{34}$$

The anticommuting variables $\theta^a, \bar{\theta}^{\dot{a}}$ should not be confused with the noncommutativity parameter $\theta^{\mu\nu}$. The Wess-Zumino gauge consists in setting the components $\hat{C}, \hat{\chi}^a, \hat{\bar{\chi}}^{\dot{a}}, \hat{M}, \hat{\bar{M}}$ equal to zero. One has $\hat{\phi}\star\hat{\phi}\star\hat{\phi} = 0$ in this gauge. For details about the superfield formalism we refer to [8].

Due to $[\frac{d}{d\theta}, W^S_a] \neq 0$, see (28), a Seiberg-Witten map in terms of superfields cannot exit. All one can do is to write the previous formulae in a more compact form, in which the super vector field is understood to be in Wess-Zumino gauge. The gauge transformations and observer Lorentz transformations can be written in the compact form

$$W^G_{\hat{\omega}} = \int d^4x \ \Big(-2\theta^a\sigma^\mu_{a\dot{a}}\bar{\theta}^{\dot{a}}\partial_\mu\hat{\omega} - \mathrm{i}[\hat{\phi}, \hat{\omega}]_\star\Big)\frac{\delta}{\delta\hat{\phi}} , \tag{35}$$

$$W^T_\tau := \int d^4x \ \mathrm{tr}\Big(\partial_\tau\hat{\phi}\ \frac{\delta}{\delta\hat{\phi}}\Big), \tag{36}$$

$$\begin{aligned}W^R_{\alpha\beta} := \int d^4x \ \mathrm{tr}\Big(\Big(\frac{1}{2}\big\{x_\alpha, \partial_\beta\hat{\phi}\big\}_\star - \frac{1}{2}\big\{x_\beta, \partial_\alpha\hat{\phi}\big\}_\star + \Sigma_{\alpha\beta}\hat{\phi}\Big)\frac{\delta}{\delta\hat{\phi}}\Big) \\ + \Big(\delta^\mu_\alpha\theta_\beta{}^\nu - \delta^\mu_\beta\theta_\alpha{}^\nu + \delta^\nu_\alpha\theta^\mu{}_\beta - \delta^\nu_\beta\theta^\mu{}_\alpha\Big)\frac{\partial}{\partial\theta^{\mu\nu}} ,\end{aligned} \tag{37}$$

$$W^D = \int d^4x \ \mathrm{tr}\Big(\frac{1}{2}\big\{x^\delta, \partial_\delta\hat{\phi}\big\}_\star\frac{\delta}{\delta\hat{\phi}}\Big) - 2\theta^{\mu\nu}\frac{\partial}{\partial\theta^{\mu\nu}} . \tag{38}$$

Here $\Sigma_{\alpha\beta} = -\frac{\mathrm{i}}{2}\theta^a \sigma_{\alpha\beta\,a}{}^b \frac{\partial}{\partial\theta^b} + \frac{\mathrm{i}}{2}\bar\theta_{\dot a}\bar\sigma^{\dot a}_{\alpha\beta\,\dot b}\frac{\partial}{\partial\bar\theta_{\dot b}}$ is the spin operator for the superfield. The covariant particle Lorentz rotation reads

$$\tilde W^R_{\hat\phi;\alpha\beta} := W^G_{\hat\chi_{\alpha\beta}} + \int d^4x \; \mathrm{tr}\Big(\Big(\frac{1}{2}\big\{\hat X_\alpha, \hat F_\beta\big\}_\star - \frac{1}{2}\big\{\hat X_\beta, \hat F_\alpha\big\}_\star + \Sigma_{\alpha\beta}\big(\hat\phi + 2\theta^a\sigma^\mu_{a\dot a}\bar\theta^{\dot a}\hat A_\mu\big)\Big)\frac{\delta}{\delta\hat\phi}\Big)\;, \tag{39}$$

where $\hat\chi_{\alpha\beta}$ is given by (18) and

$$\hat F_\sigma := \partial_\sigma\hat\phi + 2\theta^a\sigma^\mu_{a\dot a}\bar\theta^{\dot a}\partial_\mu\hat A_\sigma - \mathrm{i}[\hat A_\sigma, \hat\phi]_\star\;. \tag{40}$$

This object, resembling the usual field strength tensor $F_{\mu\nu}$, transforms covariantly under supergauge transformations (35). The calculation of the Seiberg-Witten expansion is straightforward and yields

$$\frac{d\hat\phi}{d\theta^{\rho\sigma}} = -\frac{1}{8}\big\{\hat A_\rho, \partial_\sigma\hat\phi + \hat F_\sigma\big\}_\star + \frac{1}{8}\big\{\hat A_\sigma, \partial_\rho\hat\phi + \hat F_\rho\big\}_\star\;. \tag{41}$$

6 Conclusion

Following the ideas of [3, 9] we have derived the Seiberg-Witten map for noncommutative super Yang-Mills theory in Wess-Zumino gauge via the splitting of the observer Lorentz transformation into a covariant particle Lorentz transformation and a remainder, which directly leads to the Seiberg-Witten differential equations. We have also computed the θ-expansion of the noncommutative super Yang-Mills action, up to first order in θ. The θ-expanded action is invariant under a transformation which *differs* from the commutative supersymmetry transformations by terms of order $n \geq 1$ in θ. For this reason the Seiberg-Witten map cannot be expressed in terms of superfields.

A Useful formulae

Spinor indices $a, \dot a \in \{1,2\}$ are shifted by the antisymmetric metric $\varepsilon^{ab} = -\varepsilon^{ba}, \varepsilon^{\dot a\dot b} = -\varepsilon^{\dot b\dot a}$ according to

$$\chi_a = \varepsilon_{ab}\chi^b\;, \qquad\qquad \bar\chi^{\dot a} = \varepsilon^{\dot a\dot b}\bar\chi_{\dot b}\;. \tag{A.1}$$

Note that spinors are anticommuting,

$$\chi^a\eta_a = -\chi_a\eta^a = \eta^a\chi_a = -\eta_a\chi^a\;, \qquad \bar\chi_{\dot a}\bar\eta^{\dot a} = -\bar\chi^{\dot a}\bar\eta_{\dot a} = \bar\eta_{\dot a}\bar\chi^{\dot a} = -\bar\chi^{\dot a}\bar\eta_{\dot a}\;. \tag{A.2}$$

The $2{\times}2$ σ-matrices are given by

$$\sigma^\mu_{a\dot a} = \big(1, \vec\sigma\big)_{a\dot a}\;, \qquad \bar\sigma^{\mu\,\dot a a} = \big(1, -\vec\sigma\big)^{\dot a a}\;, \qquad \sigma^\mu_{a\dot a} = \bar\sigma^\mu_{\dot a a}\;, \tag{A.3}$$

where $\vec\sigma$ denotes the three Pauli matrices. The σ-matrices satisfy

$$\sigma^\mu_{a\dot a}\bar\sigma^{\nu\,\dot a b} = g^{\mu\nu}\delta_a{}^b - \mathrm{i}\sigma^{\mu\nu}_a{}^b\;, \tag{A.4}$$

$$\bar\sigma^{\mu\,\dot a a}\sigma^\nu_{a\dot b} = g^{\mu\nu}\delta^{\dot a}{}_{\dot b} - \mathrm{i}\bar\sigma^{\mu\nu\,\dot a}{}_{\dot b}\;, \tag{A.5}$$

$$\sigma^\mu_{a\dot a}\bar\sigma^{\nu\,\dot a b}\sigma^\rho_{b\dot b} = g^{\mu\nu}\sigma^\rho_{a\dot b} + g^{\nu\rho}\sigma^\mu_{a\dot b} - g^{\rho\mu}\sigma^\nu_{a\dot b} - \mathrm{i}\epsilon^{\mu\nu\rho\lambda}\sigma_{\lambda\,a\dot b}\;, \tag{A.6}$$

$$\bar\sigma^{\mu\,\dot a a}\sigma^\nu_{a\dot b}\bar\sigma^{\rho\,\dot b b} = g^{\mu\nu}\bar\sigma^{\rho\,\dot a b} + g^{\nu\rho}\bar\sigma^{\mu\,\dot a b} - g^{\rho\mu}\bar\sigma^{\nu\,\dot a b} + \mathrm{i}\epsilon^{\mu\nu\rho\lambda}\bar\sigma^{\dot a b}_\lambda\;, \tag{A.7}$$

$$\sigma^\mu_{a\dot a}\sigma_{\mu\,b\dot b} = 2\varepsilon_{ab}\varepsilon_{\dot a\dot b}\;, \tag{A.8}$$

with $\sigma^{\mu\nu}_a{}^b = -\sigma^{\nu\mu}_a{}^b$ and $\bar\sigma^{\mu\nu\,\dot a}{}_{\dot b} = -\bar\sigma^{\nu\mu\,\dot a}{}_{\dot b}$.

References

[1] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909** (1999) 032 [hep-th/9908142].

[2] I. Chepelev and R. Roiban, "Convergence theorem for non-commutative Feynman graphs and renormalization," JHEP **0103** (2001) 001 [arXiv:hep-th/0008090].

[3] A. A. Bichl, J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, "Noncommutative Lorentz symmetry and the origin of the Seiberg-Witten map," hep-th/0108045.

[4] D. Colladay and V. A. Kostelecky, "Lorentz-violating extension of the standard model," Phys. Rev. D **58** (1998) 116002 [hep-ph/9809521].

[5] O. Piguet, "Supersymmetry, supercurrent, and scale invariance," arXiv:hep-th/9611003.

[6] J. Madore, S. Schraml, P. Schupp and J. Wess, "Gauge theory on noncommutative spaces," Eur. Phys. J. C **16** (2000) 161 [hep-th/0001203].

[7] W. Behr, N. G. Deshpande, G. Duplancic, P. Schupp, J. Trampetic and J. Wess, "The $Z \to \gamma\gamma$, gg decays in the noncommutative standard model," arXiv:hep-ph/0202121.

[8] O. Piguet and K. Sibold, "Renormalized Supersymmetry. The Perturbation Theory Of $N = 1$ Supersymmetric Theories In Flat Space-Time," *Boston, USA: Birkhaeuser (1986) 346 P. (Progress In Physics, 12).*

[9] J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, "Noncommutative spin-1/2 representations," arXiv:hep-th/0110231.