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Noncommutative spin-$\frac{1}{2}$ representations

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Abstract. In this letter we apply the methods of our previous paper `hep-th/0108045` to noncommutative fermions. We show that the fermions form a spin-$\frac{1}{2}$ representation of the Lorentz algebra. The covariant splitting of the conformal transformations into a field-dependent part and a θ-part implies the Seiberg-Witten differential equations for the fermions.

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1 Introduction

This letter is an extension of ideas of our previous paper [1] to noncommutative fermion fields. We define a noncommutative version of (infinitesimal, rigid) conformal transformations and show that they leave the noncommutative Dirac action invariant. The conformal operators and noncommutative gauge transformations form a Lie algebra (a semidirect product). However, since gauge transformations involve the $\star$-product (and thereby the noncommutative parameter θ) one immediately sees that the individual conformal operators (field- and θ-transformations) commuted with gauge transformations do not close in the above Lie algebra. There exists a certain splitting of the combined conformal operators into new individual components so that the commutator of gauge transformations with them is again a gauge transformation. From this new splitting we derive the Seiberg-Witten differential equations for the fermion fields.

Finally a comment on Lorentz transformations. In general one should distinguish between *observer* and *particle* Lorentz (or more general, conformal) transformations, which are inequivalent when one considers background fields equipped with Lorentz indices. In the following we solely refer to 'observer' Lorentz transformations. Please refer to [1] and references therein for further details.

2 Commutative case

We recall from our previous paper [1] the commutative Ward identity operators of primitive conformal[1] transformations of the gauge field A_μ:

$$W^T_{A;\tau} := \int d^4x\, \mathrm{tr}\Big(\partial_\tau A_\mu \frac{\delta}{\delta A_\mu}\Big), \tag{1}$$

$$W^R_{A;\alpha\beta} := \int d^4x\, \mathrm{tr}\Big(\big(g_{\mu\alpha}A_\beta - g_{\mu\beta}A_\alpha + x_\alpha\partial_\beta A_\mu - x_\beta\partial_\alpha A_\mu\big)\frac{\delta}{\delta A_\mu}\Big), \tag{2}$$

$$W^D_A := \int d^4x\, \mathrm{tr}\Big(\big(A_\mu + x^\delta\partial_\delta A_\mu\big)\frac{\delta}{\delta A_\mu}\Big). \tag{3}$$

The commutative (primitive) conformal transformations of fermions ψ and $\bar\psi = \psi^\dagger\gamma^0$ are given by

$$W^T_{\psi;\tau} = \int d^4x\,\Big(\Big\langle\frac{\overleftarrow{\delta}}{\delta\psi}\,\partial_\tau\psi\Big\rangle + \Big\langle\partial_\tau\bar\psi\,\frac{\overrightarrow{\delta}}{\delta\bar\psi}\Big\rangle\Big)\,, \tag{4}$$

$$\begin{aligned} W^R_{\psi;\alpha\beta} = \int d^4x\,\Big(&\Big\langle\frac{\overleftarrow{\delta}}{\delta\psi}\Big(x_\alpha\partial_\beta\psi - x_\beta\partial_\alpha\psi + \frac{1}{4}[\gamma_\alpha,\gamma_\beta]\psi\Big)\Big\rangle \\ &+ \Big\langle\Big(x_\alpha\partial_\beta\bar\psi - x_\beta\partial_\alpha\bar\psi - \frac{1}{4}\bar\psi[\gamma_\alpha,\gamma_\beta]\Big)\frac{\overrightarrow{\delta}}{\delta\bar\psi}\Big\rangle\Big)\,, \end{aligned} \tag{5}$$

$$W^D_\psi = \int d^4x\,\Big(\Big\langle\frac{\overleftarrow{\delta}}{\delta\psi}\Big(\frac{3}{2}\psi + x^\delta\partial_\delta\psi\Big)\Big\rangle + \Big\langle\Big(\frac{3}{2}\bar\psi + x^\delta\partial_\delta\bar\psi\Big)\frac{\overrightarrow{\delta}}{\delta\bar\psi}\Big\rangle\Big)\,, \tag{6}$$

[1] In the following the term 'conformal' will always refer to the rigid transformations as introduced in [1].

where the bracket $\langle\, , \rangle$ indicates the invariant product in spinor space and the trace in colour space.

The Dirac action

$$\Sigma_D = \int d^4x \,\langle \bar\psi\big(\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}A_\mu) - m\big)\psi\rangle \ , \tag{7}$$

is invariant under a gauge transformation $W^G_{A+\psi;\lambda}$ with

$$\begin{aligned} W^G_{A+\psi;\lambda} &= W^G_{A;\lambda} + W^G_{\psi;\lambda} \ , \\ W^G_{A;\lambda} &= \int d^4x \,\mathrm{tr}\Big(\big(\partial_\mu\lambda - \mathrm{i}[A_\mu,\lambda]\big)\,\frac{\delta}{\delta A_\mu}\Big) \ , \\ W^G_{\psi;\lambda} &= \int d^4x \Big(\Big\langle(-\mathrm{i}\bar\psi\lambda)\,\frac{\overrightarrow{\delta}}{\delta\bar\psi}\Big\rangle + \Big\langle\frac{\overleftarrow{\delta}}{\delta\psi}\,(\mathrm{i}\lambda\psi)\Big\rangle\Big) \ . \end{aligned} \tag{8}$$

Furthermore, we have invariance under translation $W^T_{A+\psi;\tau}$ and rotation $W^R_{A+\psi;\alpha\beta}$ and additionally, for $m=0$, under dilatation $W^D_{A+\psi}$, where

$$W^T_{A+\psi;\tau} = W^T_{A;\tau} + W^T_{\psi;\tau} \ , \qquad W^R_{A+\psi;\alpha\beta} = W^R_{A;\alpha\beta} + W^R_{\psi;\alpha\beta} \ , \qquad W^D_{A+\psi} = W^D_A + W^D_\psi \ . \tag{9}$$

3 Noncommutative case

The noncommutative generalization is obvious. The noncommutative conformal transformations of the gauge field [1] and fermions are given by

$$W^T_{\hat A;\tau} := \int d^4x \,\mathrm{tr}\Big(\partial_\tau\hat A_\mu\,\frac{\delta}{\delta\hat A_\mu}\Big) \ , \tag{10}$$

$$W^R_{\hat A;\alpha\beta} := \int d^4x \,\mathrm{tr}\Big(\Big(\frac12\{x_\alpha,\partial_\beta\hat A_\mu\}_\star - \frac12\{x_\beta,\partial_\alpha\hat A_\mu\}_\star + g_{\mu\alpha}\hat A_\beta - g_{\mu\beta}\hat A_\alpha\Big)\,\frac{\delta}{\delta\hat A_\mu}\Big) \ , \tag{11}$$

$$W^D_{\hat A} := \int d^4x \,\mathrm{tr}\Big(\Big(\frac12\{x^\delta,\partial_\delta\hat A_\mu\}_\star + \hat A_\mu\Big)\,\frac{\delta}{\delta\hat A_\mu}\Big) \ , \tag{12}$$

$$W^T_{\hat\psi;\tau} := \int d^4x \,\Big(\Big\langle\frac{\overleftarrow{\delta}}{\delta\hat\psi}\,\partial_\tau\hat\psi\Big\rangle + \Big\langle\partial_\tau\hat{\bar\psi}\,\frac{\overrightarrow{\delta}}{\delta\hat{\bar\psi}}\Big\rangle\Big) \ , \tag{13}$$

$$\begin{aligned} W^R_{\hat\psi;\alpha\beta} := \int d^4x \,\Big(&\Big\langle\frac{\overleftarrow{\delta}}{\delta\hat\psi}\Big(x_\alpha\star\partial_\beta\hat\psi - x_\beta\star\partial_\alpha\hat\psi - \frac{\mathrm{i}}{2}\theta_\alpha{}^\rho\partial_\rho\partial_\beta\hat\psi + \frac{\mathrm{i}}{2}\theta_\beta{}^\rho\partial_\rho\partial_\alpha\hat\psi + \frac14[\gamma_\alpha,\gamma_\beta]\hat\psi\Big)\Big\rangle \\ &+ \Big\langle\Big(\partial_\beta\hat{\bar\psi}\star x_\alpha - \partial_\alpha\hat{\bar\psi}\star x_\beta + \frac{\mathrm{i}}{2}\theta_\alpha{}^\rho\partial_\rho\partial_\beta\hat{\bar\psi} - \frac{\mathrm{i}}{2}\theta_\beta{}^\rho\partial_\rho\partial_\alpha\hat{\bar\psi} - \frac14\hat{\bar\psi}[\gamma_\alpha,\gamma_\beta]\Big)\frac{\overrightarrow{\delta}}{\delta\hat{\bar\psi}}\Big\rangle\Big) \ , \end{aligned} \tag{14}$$

$$W^D_{\hat\psi} := \int d^4x \,\Big(\Big\langle\frac{\overleftarrow{\delta}}{\delta\hat\psi}\Big(\frac32\hat\psi + x^\delta\star\partial_\delta\hat\psi\Big)\Big\rangle + \Big\langle\Big(\frac32\hat{\bar\psi} + \partial_\delta\hat{\bar\psi}\star x^\delta\Big)\frac{\overrightarrow{\delta}}{\delta\hat{\bar\psi}}\Big\rangle\Big) \ . \tag{15}$$

The noncommutative Dirac action is defined by

$$\hat\Sigma_D = \int d^4x \,\langle\hat{\bar\psi}\star\big(\mathrm{i}\gamma^\mu\hat{\mathrm{D}}_\mu - m\big)\hat\psi\rangle \ , \tag{16}$$

where

$$\hat{\mathrm{D}}_\mu\hat{\psi} = \partial_\mu\hat{\psi} - \mathrm{i}\hat{A}_\mu \star \hat{\psi} \ , \qquad\qquad \hat{\bar{\mathrm{D}}}_\mu\hat{\bar{\psi}} = \partial_\mu\hat{\bar{\psi}} + \mathrm{i}\hat{\bar{\psi}} \star \hat{A}_\mu \tag{17}$$

are the noncommutative covariant derivatives in the (anti)fundamental representation. The action (16) is invariant under noncommutative gauge transformations

$$\begin{aligned} W^G_{\hat{A}+\hat{\psi};\hat{\lambda}} &= W^G_{\hat{A};\hat{\lambda}} + W^G_{\hat{\psi};\hat{\lambda}} \ , \\ W^G_{\hat{A};\hat{\lambda}} &= \int d^4x \,\mathrm{tr}\Big(\hat{D}_\mu\hat{\lambda}\,\frac{\delta}{\delta\hat{A}_\mu}\Big) \ , \\ W^G_{\hat{\psi};\hat{\lambda}} &= \int d^4x \Big(\Big\langle(-\mathrm{i}\hat{\bar{\psi}}\star\hat{\lambda})\,\frac{\overrightarrow{\delta}}{\delta\hat{\bar{\psi}}}\Big\rangle + \Big\langle\frac{\overleftarrow{\delta}}{\delta\hat{\psi}}\,(\mathrm{i}\hat{\lambda}\star\hat{\psi})\Big\rangle\Big) \ , \end{aligned} \tag{18}$$

where $\hat{D}_\mu\bullet = \partial_\mu \bullet - \mathrm{i}[\hat{A}_\mu, \bullet]_\star$ is the noncommutative covariant derivative in the adjoint representation.

Let us first compute the rotational transformation of the action (16). We find:

$$\begin{aligned} (W^R_{\hat{A};\alpha\beta} + W^R_{\hat{\psi};\alpha\beta})\hat{\Sigma}_D = \int d^4x \,\Big(&\theta_\alpha{}^\rho\Big(-\frac{\mathrm{i}}{2}\langle\hat{\bar{\psi}}\star\gamma^\mu\partial_\rho\hat{A}_\mu\star\partial_\beta\hat{\psi}\rangle + \frac{\mathrm{i}}{2}\langle\hat{\bar{\psi}}\star\gamma^\mu\partial_\beta\hat{A}_\mu\star\partial_\rho\hat{\psi}\rangle\Big) \\ &+ \theta_\beta{}^\rho\Big(\frac{\mathrm{i}}{2}\langle\hat{\bar{\psi}}\star\gamma^\mu\partial_\rho\hat{A}_\mu\star\partial_\alpha\hat{\psi}\rangle - \frac{\mathrm{i}}{2}\langle\hat{\bar{\psi}}\star\gamma^\mu\partial_\alpha\hat{A}_\mu\star\partial_\rho\hat{\psi}\rangle\Big)\Big) \ . \end{aligned}$$

We must also take the rotational transformation of $\theta^{\mu\nu}$ into account,

$$W^T_{\theta;\tau}\theta^{\mu\nu} := 0 \ , \quad W^R_{\theta;\alpha\beta}\theta^{\mu\nu} := \delta^\mu_\alpha\theta_\beta{}^\nu - \delta^\mu_\beta\theta_\alpha{}^\nu + \delta^\nu_\alpha\theta^\mu{}_\beta - \delta^\nu_\beta\theta^\mu{}_\alpha \ , \quad W^D_\theta\theta^{\mu\nu} := -2\theta^{\mu\nu} \ , \tag{19}$$

which acts according to

$$\begin{aligned} &W^?_\theta(U\star V) = \big(W^?_\theta U\big)\star V + U\star\big(W^?_\theta V\big) + \frac{\mathrm{i}}{2}\,\big(W^?_\theta\theta^{\mu\nu}\big)\,(\partial_\mu U)\star(\partial_\nu V) \\ &W^?_\theta \in \big\{W^T_{\theta;\tau}, W^R_{\theta;\alpha\beta}, W^D_\theta\big\} \end{aligned} \tag{20}$$

on the $\star$-product in the gluon-fermion vertex $\int d^4x\,\langle\hat{\bar{\psi}}\star\gamma^\mu\hat{A}_\mu\star\hat{\psi}\rangle$. This yields the invariance of the noncommutative Dirac action under complete rotational transformations,

$$(W^R_{\hat{A};\alpha\beta} + W^R_{\hat{\psi};\alpha\beta} + W^R_{\theta;\alpha\beta})\hat{\Sigma}_D = 0 \ , \tag{21}$$

under the assumption

$$W^?_\theta\,\Big\{\hat{A}_\mu, \hat{\psi}, \hat{\bar{\psi}}\Big\} = 0 \ . \tag{22}$$

The complete dilatational transformation of the action (16) is given by

$$(W^D_{\hat{A}} + W^D_{\hat{\psi}} + W^D_\theta)\hat{\Sigma}_D = \int d^4x \, m\Big\langle\hat{\bar{\psi}}\star\hat{\psi}\Big\rangle \ . \tag{23}$$

In summary we have

$$W^T_{\hat{A}+\hat{\psi}+\theta;\tau}\hat{\Sigma}_D = 0 \ , \qquad W^R_{\hat{A}+\hat{\psi}+\theta;\alpha\beta}\hat{\Sigma}_D = 0 \ , \qquad W^D_{\hat{A}+\hat{\psi}+\theta}\hat{\Sigma}_D = -m\frac{\partial\hat{\Sigma}_D}{\partial m} \ , \tag{24}$$

where

$$\begin{aligned} W^T_{\hat{A}+\hat{\psi}+\theta;\tau} &= W^T_{\hat{A};\tau} + W^T_{\hat{\psi};\tau} + W^T_{\theta;\tau} \ , \qquad W^R_{\hat{A}+\hat{\psi}+\theta;\alpha\beta} = W^R_{\hat{A};\alpha\beta} + W^R_{\hat{\psi};\alpha\beta} + W^R_{\theta;\alpha\beta} \ , \\ W^D_{\hat{A}+\hat{\psi}+\theta} &= W^D_{\hat{A}} + W^D_{\hat{\psi}} + W^D_{\theta} \ . \end{aligned} \tag{25}$$

The Casimir operators (mass and spin) related to the representation (13), (14) are

$$m^2\hat{\psi} = -g^{\tau\sigma}W^T_{\hat{\psi};\tau}W^T_{\hat{\psi};\sigma}\hat{\psi} \ , \qquad s(s+1)m^2\hat{\psi} = -g_{\tau\sigma}W^{PL;\tau}_{\hat{\psi}}W^{PL;\sigma}_{\hat{\psi}}\hat{\psi} \ , \tag{26}$$

where

$$W^{PL;\sigma}_{\hat{\psi}}\hat{\psi} = -\frac{1}{2}\epsilon^{\sigma\tau\alpha\beta}W^T_{\hat{\psi};\tau}W^R_{\hat{\psi};\alpha\beta} = -\frac{1}{8}\epsilon^{\sigma\tau\alpha\beta}[\gamma_\alpha,\gamma_\beta]\partial_\tau\psi \tag{27}$$

is the Pauli-Ljubanski vector. This yields

$$m^2\hat{\psi} = -\partial^\tau\partial_\tau\hat{\psi} \ , \tag{28}$$

$$s(s+1)m^2\hat{\psi} = \frac{1}{32}(g^{\tau\sigma}g^{\alpha\gamma}g^{\beta\delta} + 2g^{\tau\gamma}g^{\alpha\delta}g^{\beta\sigma})[\gamma_\alpha,\gamma_\beta][\gamma_\gamma,\gamma_\delta]\partial_\sigma\partial_\tau\hat{\psi} = -\frac{3}{4}\partial^\tau\partial_\tau\hat{\psi} \ , \tag{29}$$

showing that (13), (14) is a spin-$\frac{1}{2}$ representation.

3.1 Seiberg-Witten differential equations

As in the bosonic case we derive the Seiberg-Witten differential equations via a covariant splitting of $W^?_{\hat{A}+\hat{\psi}+\theta}$

$$W^?_{\hat{A}+\hat{\psi}+\theta} \equiv W^?_{\hat{A}+\hat{\psi}} + W^?_{\theta} = \tilde{W}^?_{\hat{A}+\hat{\psi}} + \tilde{W}^?_{\theta} \ , \tag{30}$$

$$[\tilde{W}^?_{\hat{A}+\hat{\psi}}, W^G_{\hat{A}+\hat{\psi};\hat{\lambda}}] = W^G_{\hat{A}+\hat{\psi};\hat{\lambda}^?_{\hat{A}+\hat{\psi}}} \ , \qquad [\tilde{W}^?_{\theta}, W^G_{\hat{A};\hat{\lambda}}] = W^G_{\hat{A};\hat{\lambda}^?_{\theta}} \ , \tag{31}$$

whereby we require (31) to be valid on both $\hat{A}_\mu$ *and* $\hat{\psi}$. To find the sought for splitting we first apply the ansatz of [1]:

$$\tilde{W}^T_{\hat{A};\tau} = W^G_{\hat{A}+\hat{\psi};\hat{\lambda}^T_\tau} + \int d^4x\,\mathrm{tr}\Big(\hat{F}_{\tau\mu}\frac{\delta}{\delta\hat{A}_\mu}\Big) \ , \tag{32}$$

$$\tilde{W}^R_{\hat{A};\alpha\beta} = W^G_{\hat{A}+\hat{\psi};\hat{\lambda}^R_{\alpha\beta}} + \int d^4x\,\mathrm{tr}\Big(\Big(\frac{1}{2}\{\hat{X}_\alpha,\hat{F}_{\beta\mu}\}_\star - \frac{1}{2}\{\hat{X}_\beta,\hat{F}_{\alpha\mu}\}_\star - W^R_{\theta;\alpha\beta}(\theta^{\rho\sigma})\hat{\Omega}_{\rho\sigma\mu}\Big)\frac{\delta}{\delta\hat{A}_\mu}\Big) \ , \tag{33}$$

$$\tilde{W}^D_{\hat{A}} = W^G_{\hat{A}+\hat{\psi};\hat{\lambda}^D} + \int d^4x\,\mathrm{tr}\Big(\Big(\frac{1}{2}\{\hat{X}^\delta,\hat{F}_{\delta\mu}\}_\star - W^D_{\theta}(\theta^{\rho\sigma})\hat{\Omega}_{\rho\sigma\mu}\Big)\frac{\delta}{\delta\hat{A}_\mu}\Big) \ , \tag{34}$$

where we replaced $W^G_{\hat{A};\hat{\lambda}^?}$ by $W^G_{\hat{A}+\hat{\psi};\hat{\lambda}^?}$. $\hat{\Omega}_{\rho\sigma\mu}$ is a polynomial in the covariant quantities $\theta, \hat{F}, \hat{D}\dots\hat{D}\hat{F}$, antisymmetric in ρ, σ, of power-counting dimension 3, and expresses the freedom in the splitting. In the following we set $\hat{\Omega}_{\rho\sigma\mu} = 0$. The parameters $\hat{\lambda}^?$ are unchanged and given by [1]

$$\hat{\lambda}^T_\tau = \hat{A}_\tau \ , \quad \hat{\lambda}^R_{\alpha\beta} = \frac{1}{4}\{2x_\alpha + \theta_\alpha{}^\rho \hat{A}_\rho, \hat{A}_\beta\}_\star - \frac{1}{4}\{2x_\beta + \theta_\beta{}^\rho \hat{A}_\rho, \hat{A}_\alpha\}_\star \ , \quad \hat{\lambda}^D = \frac{1}{2}\{x^\delta, \hat{A}_\delta\}_\star \ . \tag{35}$$

We write down a covariantization of (13)–(15),

$$\tilde{W}^T_{\hat{\psi};\tau} = \int d^4x \left(\Big\langle \frac{\overleftarrow{\delta}}{\delta\hat{\psi}} \, \hat{\mathrm{D}}_\tau \hat{\psi} \Big\rangle + \Big\langle \hat{\bar{\mathrm{D}}}_\tau \hat{\bar{\psi}} \, \frac{\overrightarrow{\delta}}{\delta\hat{\bar{\psi}}} \Big\rangle \right) , \tag{36}$$

$$\begin{aligned}\tilde{W}^R_{\hat{\psi};\alpha\beta} = \int d^4x \bigg(\Big\langle \frac{\overleftarrow{\delta}}{\delta\hat{\psi}} \Big(\frac{1}{4}[\gamma_\alpha, \gamma_\beta]\hat{\psi} + \hat{X}_\alpha \star \hat{\mathrm{D}}_\beta \hat{\psi} - \frac{\mathrm{i}}{2}\theta_\alpha{}^\rho \hat{\mathrm{D}}_\rho \hat{\mathrm{D}}_\beta \hat{\psi} - \hat{X}_\beta \star \hat{\mathrm{D}}_\alpha \hat{\psi} + \frac{\mathrm{i}}{2}\theta_\beta{}^\rho \hat{\mathrm{D}}_\rho \hat{\mathrm{D}}_\alpha \hat{\psi} \Big) \Big\rangle \\ + \Big\langle \Big(-\frac{1}{4}\hat{\bar{\psi}}[\gamma_\alpha, \gamma_\beta] + \hat{\bar{\mathrm{D}}}_\beta \hat{\bar{\psi}} \star \hat{X}_\alpha + \frac{\mathrm{i}}{2}\theta_\alpha{}^\rho \hat{\bar{\mathrm{D}}}_\rho \hat{\bar{\mathrm{D}}}_\beta \hat{\bar{\psi}} - \hat{\bar{\mathrm{D}}}_\alpha \hat{\bar{\psi}} \star \hat{X}_\beta - \frac{\mathrm{i}}{2}\theta_\beta{}^\rho \hat{\bar{\mathrm{D}}}_\rho \hat{\bar{\mathrm{D}}}_\alpha \hat{\bar{\psi}} \Big) \frac{\overrightarrow{\delta}}{\delta\hat{\bar{\psi}}} \Big\rangle \bigg) ,\end{aligned} \tag{37}$$

$$\begin{aligned}\tilde{W}^D_{\hat{\psi}} = \int d^4x \bigg(\Big\langle \frac{\overleftarrow{\delta}}{\delta\hat{\psi}} \Big(\frac{3}{2}\hat{\psi} + \hat{X}^\delta \star \hat{\mathrm{D}}_\delta \hat{\psi} + \frac{1}{4}\theta^{\rho\sigma}\hat{F}_{\rho\sigma} \star \hat{\psi} \Big) \Big\rangle \\ + \Big\langle \Big(\frac{3}{2}\hat{\bar{\psi}} + \hat{\bar{\mathrm{D}}}_\delta \hat{\bar{\psi}} \star x^\delta - \frac{1}{4}\theta^{\rho\sigma}\hat{\bar{\psi}} \star \hat{F}_{\rho\sigma} \Big) \frac{\overrightarrow{\delta}}{\delta\hat{\bar{\psi}}} \Big\rangle \bigg) \ .\end{aligned} \tag{38}$$

where the covariant coordinates [2] are defined by $\hat{X}^\mu = x^\mu + \theta^{\mu\nu}\hat{A}_\nu$. We define $W^?_{\hat{A}+\hat{\psi}}$ as the sum of (32)–(34) with (36)–(38). Now it is easy to evaluate $\tilde{W}^?_\theta = W^?_{\hat{A}+\hat{\psi}+\theta} - \tilde{W}^?_{\hat{A}+\hat{\psi}} = W^?_\theta(\theta^{\rho\sigma})\frac{d}{d\theta^{\rho\sigma}}$, with

$$\frac{d}{d\theta^{\rho\sigma}} = \frac{\partial}{\partial\theta^{\rho\sigma}} + \int d^4x \left(\mathrm{tr}\Big(\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} \frac{\delta}{\delta\hat{A}_\mu} \Big) + \Big\langle \frac{\overleftarrow{\delta}}{\delta\hat{\psi}} \frac{d\hat{\psi}}{d\theta^{\rho\sigma}} \Big\rangle + \Big\langle \frac{d\hat{\bar{\psi}}}{d\theta^{\rho\sigma}} \frac{\overrightarrow{\delta}}{\delta\hat{\bar{\psi}}} \Big\rangle \right) , \tag{39}$$

which yields the Seiberg-Witten differential equations

$$\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} = -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma \hat{A}_\mu + \hat{F}_{\sigma\mu}\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho \hat{A}_\mu + \hat{F}_{\rho\mu}\big\}_\star \ , \tag{40}$$

$$\begin{aligned}\frac{d\hat{\psi}}{d\theta^{\rho\sigma}} &= -\frac{1}{4}\hat{A}_\rho \star \partial_\sigma \hat{\psi} + \frac{1}{4}\hat{A}_\sigma \star \partial_\rho \hat{\psi} + \frac{\mathrm{i}}{8}[\hat{A}_\rho, \hat{A}_\sigma]_\star \star \hat{\psi} \\ &\equiv -\frac{1}{8}\hat{A}_\rho \star \big(\partial_\sigma \hat{\psi} + \hat{\mathrm{D}}_\sigma \hat{\psi}\big) + \frac{1}{8}\hat{A}_\sigma \star \big(\partial_\rho \hat{\psi} + \hat{\mathrm{D}}_\rho \hat{\psi}\big) \ ,\end{aligned} \tag{41}$$

$$\begin{aligned}\frac{d\hat{\bar{\psi}}}{d\theta^{\rho\sigma}} &= -\frac{1}{4}\partial_\sigma \hat{\bar{\psi}} \star \hat{A}_\rho + \frac{1}{4}\partial_\rho \hat{\bar{\psi}} \star \hat{A}_\sigma + \frac{\mathrm{i}}{8}\hat{\bar{\psi}} \star [\hat{A}_\rho, \hat{A}_\sigma]_\star \\ &\equiv -\frac{1}{8}\big(\partial_\sigma \hat{\bar{\psi}} + \hat{\bar{\mathrm{D}}}_\sigma \hat{\psi}\big) \star \hat{A}_\rho + \frac{1}{8}\big(\partial_\rho \hat{\bar{\psi}} + \hat{\bar{\mathrm{D}}}_\rho \hat{\psi}\big) \star \hat{A}_\sigma \ .\end{aligned} \tag{42}$$

The differential equation (40) was first found in [3]. The equation (41) was for noncommutative QED to lowest order in θ first obtained in [4]. It follows from the algebra given in [1] (extended to include fermions) that $\tilde{W}^?_\theta$ satisfies automatically the second identity in (31), $[\tilde{W}^?_\theta, W^G_{\hat{A}+\hat{\psi};\hat{\lambda}}] = W^G_{\hat{A}+\hat{\psi};\hat{\lambda}^?_\theta}$ so that the θ-expansion of the action (16) is invariant under commutative gauge transformations. One checks the identity

$$\left[W^?_{\hat{A}+\hat{\psi}+\theta}, \theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}}\right] = 0 \tag{43}$$

for the theory enlarged by fermions, which means that the θ-expansion based on (40) and (41) leads to a commutative action invariant under commutative rotations and translations and with commutative dilational symmetry broken by the mass term.

4 Conclusion

Following the ideas of [1] we have constructed a representation of the infinitesimal rigid conformal transformations for noncommutative fermion fields. We have shown that the requirement that the individual operators of θ- and $(\hat{A}, \hat{\psi})$-transformations commute with gauge transformations up to another gauge transformation leads directly to the θ-dependency of the fermion fields first found in [4].

References

[1] A. A. Bichl, J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, "Noncommutative Lorentz symmetry and the origin of the Seiberg-Witten map," hep-th/0108045.

[2] J. Madore, S. Schraml, P. Schupp and J. Wess, "Gauge theory on noncommutative spaces," Eur. Phys. J. C **16** (2000) 161 [hep-th/0001203].

[3] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909** (1999) 032 [hep-th/9908142].

[4] A. A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda and R. Wulkenhaar, "Deformed QED via Seiberg-Witten map," hep-th/0102103.