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Noncommutative Lorentz Symmetry and the Origin of the Seiberg-Witten Map

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Abstract. We show that the noncommutative Yang-Mills field forms an irreducible representation of the (undeformed) Lie algebra of rigid translations, rotations and dilatations. The noncommutative Yang-Mills action is invariant under combined conformal transformations of the Yang-Mills field and of the noncommutativity parameter θ. The Seiberg-Witten differential equation results from a covariant splitting of the combined conformal transformations and can be computed as the missing piece to complete a covariant conformal transformation to an invariance of the action.

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1 Introduction

In noncommutative field theory one of the greatest surprises is the existence of the so-called Seiberg-Witten map [1]. The Seiberg-Witten map was originally deduced from the observation that different regularization schemes (point-splitting vs. Pauli-Villars) in the field theory limit of string theory lead either to a commutative or a noncommutative field theory and thus suggest an equivalence between them.

A particular application of the Seiberg-Witten map is the construction of the noncommutative analogue of gauge theories with arbitrary gauge group, which automatically leads to enveloping algebra-valued fields involving infinitely many degrees of freedom [2]. The Seiberg-Witten map solves this problem in an almost miraculous manner by mapping the enveloping algebra-valued noncommutative gauge field to a commutative gauge field with finitely many degrees of freedom.

The renormalization of noncommutative Yang-Mills (NCYM) theories is an open puzzle: Loop calculations [3] and power-counting analysis [4] show the existence of a new type of infrared divergences. The circumvention of the infrared problem by application of the Seiberg-Witten map leads to a power-counting non-renormalizable theory with infinitely many vertices. In an earlier work [5] we have proven the two-point function of θ-expanded noncommutative Maxwell theory to be renormalizable to all orders. However, to show renormalizability of all N-point functions one cannot proceed without strong symmetries that limit the number of possible counterterms. In particular, one needs to find a symmetry that fixes the special θ-structure of the θ-expanded theory.

The intuition that the symmetry searched for is related to space-time symmetries leads us to an investigation of rigid conformal symmetries (translation, rotation, dilatation) for NCYM theory characterized by a constant field $\theta^{\mu\nu}$. The term rigid means that the factor Ω in the conformal transformation $(ds')^2 = \Omega^2 ds^2$ of the line element is constant. The reason for this restriction is that θ has to be constant in all reference frames.

We show in this paper that the noncommutative Yang-Mills field $\hat{A}$ forms an irreducible spin-1 representation of the *undeformed* Lie algebra of conformal transformations. We also prove that the noncommutative Yang-Mills (NCYM) action is invariant under the sum of the conformal transformations of $\hat{A}$ and of θ. This result can either be regarded as an exact invariance (compatible with gauge transformations) with respect to *observer Lorentz transformations* or as the quantitative amount of symmetry breaking under *particle Lorentz transformations*, see also Section 3.

Regarding the combined conformal transformations of $\hat{A}$ and θ, one can consider various splittings into individual transformations. There is one (up to gauge transformations) distinguished splitting for which both individual components are compatible (covariant) with gauge transformations, i.e. the commutator of these components with a gauge transformation is again a gauge transformation. Whereas the θ-part of this covariant splitting cannot be computed, the $\hat{A}$-part is easily constructed by a covariance ansatz involving *covariant coordinates* [6, 7]. This covariance ansatz generalizes the gauge-covariant conformal transformations which in its commutative form were first investigated by Jackiw [8, 9]. These transformations are loosely related to the improvements allowing to pass from the canonical energy-momentum tensor to the symmetric and traceless one. Now, the covariant θ-complement of the covariant transformation of $\hat{A}$ can easily be computed as the missing piece to achieve invariance of the NCYM action. The result is the Seiberg-Witten differential equation [1].

Almost all splittings of the combined conformal transformation of $\hat{A}$ and θ lead to a first-order differential equation for $\hat{A}$ which can be used to express the noncommutative fields in terms of initial values living on commutative space-time. The covariant splitting (which leads to the Seiberg-Witten differential equation) has the distinguished property that the resulting θ-expansion of a gauge-invariant noncommutative action is invariant under commutative gauge transformations. This was the original motivation for the Seiberg-Witten map. We would like to point out, however, that the original gauge-equivalence condition [1] is more restrictive than the approach of this paper—a fact made transparent by our investigation of noncommutative conformal symmetries. Moreover, we prove that the θ-expansion of the noncommutative conformal symmetries reduces to the commutative conformal symmetries.

All this means that there are two quantum field theories associated with the NCYM action. The first one is obtained by a direct gauge-fixing of the NCYM action and the other one by gauge-fixing of the θ-expanded NCYM action. The second approach was adopted in [10, 5]: Take the Seiberg-Witten expansion of the NCYM action as a very special type of an action for a commutative gauge field A_μ coupled to a constant external field $\theta^{\mu\nu}$ and quantize it in the ordinary way (with the linear gauge-fixing in [10]). It is not completely clear in which sense this is equivalent to the first approach of a direct quantization of the noncommutative Yang-Mills action. The infrared problem found in noncommutative quantum field theory [3, 4] and its absence in the approach of [10] shows the inequivalence at least on a perturbative level. For interesting physical consequences of the Seiberg-Witten expanded action in noncommutative QED see [11].

The paper is organized as follows: First we recall in Section 2 necessary information about noncommutative field theory and covariant coordinates. In Section 3 we distinguish between observer and particle Lorentz transformations. After a review of rigid conformal symmetries in the commutative setting in Section 4 we extend these structures in Section 5 to noncommutative Yang-Mills theory, deriving in particular the Seiberg-Witten differential equation and the θ-expansion of the noncommutative conformal and gauge symmetries. In Section 6 we comment on quantization and Section 7 contains the summary. Longer but important calculations are delegated to the Appendix.

2 Noncommutative geometry and covariant coordinates

In this section we give a short introduction to noncommutative field theory and the concept of covariant coordinates. We consider a noncommutative geometry characterized by the algebra

$$[\mathsf{x}^\mu, \mathsf{x}^\nu] = \mathrm{i}\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is an antisymmetric constant tensor. The noncommutative algebra may be represented on a commutative manifold by the $\star$-product

$$(f \star g)(x) = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}(k_\mu+p_\mu)x^\mu}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu}k_\mu p_\nu}\, \tilde{f}(k)\, \tilde{g}(p)\ , \tag{2}$$

where $f(x)$ and $g(x)$ are ordinary functions on Minkowski space and $\tilde{f}(p)$ and $\tilde{g}(p)$ their Fourier transforms. Denoting the ordinary (commutative) coordinates by x we have

$$[x^\mu, x^\nu]_\star \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\theta^{\mu\nu}. \tag{3}$$

Let us now consider an infinitesimal gauge transformation δ^G of a field $\Phi(x)$,

$$\delta^G \Phi(x) = \mathrm{i}\epsilon(x) \star \Phi(x) \ , \tag{4}$$

with $\epsilon(x)$ being an infinitesimal gauge parameter. As usual one chooses the coordinates to be invariant under gauge transformations, $\delta^G x = 0$. However, with this construction one finds that multiplication by x does not lead to a covariant object:

$$\delta^G \big(x^\mu \star \Phi(x)\big) \neq \mathrm{i}\epsilon(x) \star \big(x^\mu \star \Phi(x)\big) \ . \tag{5}$$

The solution of this problem, which was given in [7], is to introduce *covariant coordinates* [6]

$$\hat{X}^\mu \equiv x^\mu \mathbf{1} + \theta^{\mu\nu} \hat{A}_\nu \ , \tag{6}$$

where the transformation of the field $\hat{A}(x)$ is defined by the requirement

$$\delta^G \big(\hat{X}^\mu \star \Phi(x)\big) = \mathrm{i}\epsilon(x) \star \big(\hat{X}^\mu \star \Phi(x)\big) \ . \tag{7}$$

The relation (7) leads to the transformation rule for the field $\hat{A}(x)$

$$\delta^G \hat{A}_\mu(x) = \partial_\mu \epsilon(x) - \mathrm{i}[\hat{A}_\mu(x), \epsilon(x)]_\star \equiv \hat{D}_\mu \epsilon(x) \ , \tag{8}$$

and $\hat{A}(x)$ is interpreted as a noncommutative gauge field. In this way gauge theory is seen to be intimately related to the noncommutative structure (3) of space and time. The covariant coordinates fulfill

$$\big[\hat{X}^\mu, \hat{X}^\nu\big]_\star = \mathrm{i}\theta^{\mu\nu} + \mathrm{i}\theta^{\mu\alpha}\theta^{\nu\beta}\hat{F}_{\alpha\beta} \ , \tag{9}$$

where $\hat{F}_{\alpha\beta} = \partial_\alpha \hat{A}_\beta - \partial_\beta \hat{A}_\alpha - \mathrm{i}\big[\hat{A}_\alpha, \hat{A}_\beta\big]_\star$ is the noncommutative field strength.

3 Observer versus particle Lorentz transformations

In general one should distinguish between two kinds of Lorentz (or more general, conformal) transformations (see [12] and references therein). Lorentz transformations in special relativity relate physical observations made in two inertial reference frames characterized by different velocities and orientations. These transformations can be implemented as coordinate changes, known as *observer Lorentz transformations*. Alternatively one considers transformations which relate physical properties of two particles with different helicities or momenta within one specific inertial frame. These are known as *particle Lorentz transformations*. Usually (without background) these two approaches are equivalent. However, in the presence of a background tensor field this equivalence fails, because the background field will transform as a tensor under observer Lorentz transformation and as a set of scalars under particle Lorentz transformations.

Thirdly, having a background tensor field one may consider the transformations of *all* fields within a specific inertial frame simultaneously, including the background field. These transformations are known as *(inverse) active Lorentz transformations* and are equivalent to observer Lorentz transformations.

What kind of 'field' is $\theta^{\alpha\beta}$? Since we are considering the case of a constant θ, it certainly is a background field. Therefore, all results of this paper refer to 'observer' transformations. This also matches the setting of noncommutative field theory appearing in string theory. Here θ is related to the inverse of a 'magnetic field' (mostly taken to be constant). In this sense, Lorentz invariance of the action means that its value is the same for observers in different inertial reference frames. Since invariance of the action always involves the sum of conformal transformations of $\hat{A}$ and θ, see Section 5.1, one can however take the 'particle' point of view and regard our 'observer' invariance as the quantitative amount of 'particle' symmetry breaking due to the presence of θ.

However, we find it desirable to extend the general analysis to the case of a non-constant θ. In this case one could choose to view θ as a dynamical field which also transforms under 'particle' transformations.

In the rest of the paper we will simply refer to conformal transformations, leaving out the 'observer' prefix.

4 Rigid conformal symmetries: commutative case

The Lie algebra of the rigid conformal transformations is generated by $\{P_\tau, M_{\alpha\beta}, D\}$ and the following commutation relations:

$$\begin{aligned} [P_\tau, P_\sigma] &= 0 \;, & [D, D] &= 0 \;, \\ [P_\tau, M_{\alpha\beta}] &= g_{\tau\beta} P_\alpha - g_{\tau\alpha} P_\beta \;, & [P_\tau, D] &= -P_\tau \;, \\ [M_{\alpha\beta}, M_{\gamma\delta}] &= g_{\alpha\gamma} M_{\beta\delta} - g_{\beta\gamma} M_{\alpha\delta} - g_{\alpha\delta} M_{\beta\gamma} + g_{\beta\delta} M_{\alpha\gamma} \;, & [M_{\alpha\beta}, D] &= 0 \;. \end{aligned} \tag{10}$$

A particular representation is given by infinitesimal rigid conformal transformations of the coordinates x^μ,

$$(x^\mu)^T = (1 + a^\tau \rho_x(P_\tau)) x^\mu + \mathcal{O}(a^2) \;, \qquad \rho_x(P_\tau) = \partial_\tau \qquad \text{(translation)}, \tag{11}$$

$$(x^\mu)^R = (1 + \omega^{\alpha\beta} \rho_x(M_{\alpha\beta})) x^\mu + \mathcal{O}(\omega^2) \;, \quad \rho_x(M_{\alpha\beta}) = x_\beta \partial_\alpha - x_\alpha \partial_\beta \quad \text{(rotation)}, \tag{12}$$

$$(x^\mu)^D = (1 + \epsilon \rho_x(D)) x^\mu + \mathcal{O}(\epsilon^2) \;, \qquad \rho_x(D) = -x^\delta \partial_\delta \qquad \text{(dilatation)}, \tag{13}$$

for constant parameters $a^\tau, \omega^{\alpha\beta}, \epsilon$.

A field is by definition an irreducible representation of the Lie algebra (10). In view of the noncommutative generalization we are interested in the Yang-Mills field A_μ and the constant antisymmetric two-tensor field $\theta^{\mu\nu}$ whose representations are given by

$$\rho_1(P_\tau) A_\mu = W^T_{A;\tau} A_\mu \;, \qquad W^T_{A;\tau} := \int d^4x \,\mathrm{tr}\Big(\partial_\tau A_\mu \frac{\delta}{\delta A_\mu}\Big), \tag{14}$$

$$\rho_1(M_{\alpha\beta}) A_\mu = W^R_{A;\alpha\beta} A_\mu \;, \quad W^R_{A;\alpha\beta} := \int d^4x \,\mathrm{tr}\Big(\big(g_{\mu\alpha} A_\beta - g_{\mu\beta} A_\alpha + x_\alpha \partial_\beta A_\mu - x_\beta \partial_\alpha A_\mu\big) \frac{\delta}{\delta A_\mu}\Big), \tag{15}$$

$$\rho_1(D) A_\mu = W^D_A A_\mu \;, \qquad W^D_A := \int d^4x \,\mathrm{tr}\Big(\big(A_\mu + x^\delta \partial_\delta A_\mu\big) \frac{\delta}{\delta A_\mu}\Big), \tag{16}$$

and[1]

$$\rho_{-2}(P_\tau)\theta^{\mu\nu} = W^T_{\theta;\tau}\theta^{\mu\nu} \qquad W^T_{\theta;\tau}\theta^{\mu\nu} := 0 \;, \tag{17}$$

$$\rho_{-2}(M_{\alpha\beta})\theta^{\mu\nu} = W^R_{\theta;\alpha\beta}\theta^{\mu\nu} \;, \qquad W^R_{\theta;\alpha\beta}\theta^{\mu\nu} := \delta^\mu_\alpha\theta_\beta{}^\nu - \delta^\mu_\beta\theta_\alpha{}^\nu + \delta^\nu_\alpha\theta^\mu{}_\beta - \delta^\nu_\beta\theta^\mu{}_\alpha \;, \tag{18}$$

$$\rho_{-2}(D)\theta^{\mu\nu} = W^D_\theta\theta^{\mu\nu} \;, \qquad W^D_\theta\theta^{\mu\nu} := -2\theta^{\mu\nu} \;. \tag{19}$$

Throughout this paper we use the following differentiation rule for an antisymmetric two-tensor field:

$$\frac{\partial\theta^{\mu\nu}}{\partial\theta^{\rho\sigma}} := \frac{1}{2}\Big(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho\Big) \;. \tag{20}$$

The factor $\frac{1}{2}$ in (20) ensures the same rotational behaviour of the spin indices in (15) and (18). The Yang-Mills action

$$\Sigma = -\frac{1}{4g^2}\int d^4x\,\mathrm{tr}\big(F_{\mu\nu}F^{\mu\nu}\big) \;, \tag{21}$$

for $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu]$ being the Yang-Mills field strength and g a coupling constant, is invariant under (14)–(16). Moreover the action (21) is invariant under gauge transformations

$$W^G_{A;\lambda} = \int d^4x\,\mathrm{tr}\Big(D_\mu\lambda\,\frac{\delta}{\delta A_\mu}\Big) \;, \qquad D_\mu\bullet = \partial_\mu\bullet - \mathrm{i}[A_\mu, \bullet] \;, \tag{22}$$

with a possibly field-dependent transformation parameter λ.

5 Rigid conformal symmetries: noncommutative case

In this section we show that the noncommutative gauge field forms an irreducible representation of *the same undeformed* Lie algebra of rigid conformal transformations. To obtain the representation one has to take the symmetric product when going to the noncommutative realm: $AB \to \frac{1}{2}\{A,B\}_\star$. Compatibility with gauge transformations implies that only the sum of the conformal transformations of gauge field $\hat{A}$ and θ has a meaning. A covariant splitting of this sum allows a θ-expansion into a commutative gauge theory.

5.1 Conformal transformations of the noncommutative gauge field

We generalize the (rigid) conformal transformations (14)–(16) to noncommutative Yang-Mills theory, i.e. a gauge theory for the field $\hat{A}_\mu$ transforming according to (8):

$$W^T_{\hat{A};\tau} := \int d^4x\,\mathrm{tr}\Big(\partial_\tau\hat{A}_\mu\,\frac{\delta}{\delta\hat{A}_\mu}\Big) \;, \tag{23}$$

[1] The translation invariance $\rho_{-2}(P_\tau)\theta^{\mu\nu} = 0$ qualifies $\theta^{\mu\nu}$ as a constant field. It takes however different (constant!) values in different reference frames. The necessity to have a constant field in the model forces us to restrict ourselves to rigid conformal transformations. Local conformal transformations as in [13] are incompatible with constant fields. In particular, the special conformal transformations K_σ are excluded because the commutator $[K_\sigma, P_\tau] = 2(g_{\sigma\tau}D - M_{\sigma\tau})$ cannot be represented.

$$W^R_{\hat{A};\alpha\beta} := \int d^4x \,\mathrm{tr}\Big(\Big(\frac{1}{2}\big\{x_\alpha, \partial_\beta \hat{A}_\mu\big\}_\star - \frac{1}{2}\big\{x_\beta, \partial_\alpha \hat{A}_\mu\big\}_\star + g_{\mu\alpha}\hat{A}_\beta - g_{\mu\beta}\hat{A}_\alpha\Big) \frac{\delta}{\delta \hat{A}_\mu}\Big) \;, \tag{24}$$

$$W^D_{\hat{A}} := \int d^4x \,\mathrm{tr}\Big(\Big(\frac{1}{2}\big\{x^\delta, \partial_\delta \hat{A}_\mu\big\}_\star + \hat{A}_\mu\Big) \frac{\delta}{\delta \hat{A}_\mu}\Big) \;, \tag{25}$$

where $\big\{U,V\big\}_\star := U\star V + V\star U$ is the $\star$-anticommutator. It is important to take the symmetric product in the "quantization" $x_\alpha\partial_\beta A_\mu \mapsto \frac{1}{2}\{x_\alpha, \partial_\beta \hat{A}_\mu\}_\star$. Let us introduce the convenient abbreviation $W^?_{\hat{A}}$ standing for one of the operators $\{W^T_{\hat{A};\tau}, W^R_{\hat{A};\alpha\beta}, W^D_{\hat{A}}\}$ and similarly for $W^?_\theta$ in (17)–(19).

Applying $W^R_{\hat{A};\alpha\beta}$ to the noncommutative Yang-Mills field strength $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - \mathrm{i}[\hat{A}_\mu, \hat{A}_\nu]_\star$ one obtains

$$\begin{aligned} W^R_{\hat{A};\alpha\beta}\hat{F}_{\mu\nu} = {} & \tfrac{1}{2}\{x_\alpha, \partial_\beta \hat{F}_{\mu\nu}\}_\star - \tfrac{1}{2}\{x_\beta, \partial_\alpha \hat{F}_{\mu\nu}\}_\star + g_{\mu\alpha}\hat{F}_{\beta\nu} - g_{\mu\beta}\hat{F}_{\alpha\nu} + g_{\nu\alpha}\hat{F}_{\mu\beta} - g_{\nu\beta}\hat{F}_{\mu\alpha} \\ & - \tfrac{1}{2}\theta_\alpha{}^\rho\{\partial_\rho \hat{A}_\mu, \partial_\beta \hat{A}_\nu\}_\star + \tfrac{1}{2}\theta_\beta{}^\rho\{\partial_\rho \hat{A}_\mu, \partial_\alpha \hat{A}_\nu\}_\star \\ & + \tfrac{1}{2}\theta_\alpha{}^\rho\{\partial_\rho \hat{A}_\nu, \partial_\beta \hat{A}_\mu\}_\star - \tfrac{1}{2}\theta_\beta{}^\rho\{\partial_\rho \hat{A}_\nu, \partial_\alpha \hat{A}_\mu\}_\star \;, \end{aligned} \tag{26}$$

which is not the expected Lorentz transformation of the field strength. However, we must also take the θ-transformation (17)–(19) into account, which acts on the $\star$-product in the $\hat{A}$-bilinear part of $\hat{F}_{\mu\nu}$. Using the differentiation rule for the $\star$-product

$$W^?_\theta(U\star V) = \big(W^?_\theta U\big)\star V + U \star \big(W^?_\theta V\big) + \frac{\mathrm{i}}{2}\big(W^?_\theta \theta^{\mu\nu}\big)\,(\partial_\mu U)\star(\partial_\nu V) \;, \tag{27}$$

which is a consequence of (2) and (20), together with

$$W^?_\theta \hat{A}_\mu = 0 \;, \tag{28}$$

one finds that $W^R_{\theta;\alpha\beta}\hat{F}_{\mu\nu}$ cancels exactly the last two lines in (26):

$$\begin{aligned} (W^R_{\hat{A};\alpha\beta} + W^R_{\theta;\alpha\beta})\hat{F}_{\mu\nu} = {} & \tfrac{1}{2}\{x_\alpha, \partial_\beta \hat{F}_{\mu\nu}\}_\star - \tfrac{1}{2}\{x_\beta, \partial_\alpha \hat{F}_{\mu\nu}\}_\star \\ & + g_{\mu\alpha}\hat{F}_{\beta\nu} - g_{\mu\beta}\hat{F}_{\alpha\nu} + g_{\nu\alpha}\hat{F}_{\mu\beta} - g_{\nu\beta}\hat{F}_{\mu\alpha} \;. \end{aligned} \tag{29}$$

In the same way one finds

$$(W^D_{\hat{A}} + W^D_\theta)\hat{F}_{\mu\nu} = \tfrac{1}{2}\big\{x^\delta, \partial_\delta \hat{F}_{\mu\nu}\big\}_\star + 2\hat{F}_{\mu\nu} \;. \tag{30}$$

It is then easy to verify that the noncommutative Yang-Mills (NCYM) action

$$\hat{\Sigma} = -\frac{1}{4g^2}\int d^4x \,\mathrm{tr}(\hat{F}^{\mu\nu}\star \hat{F}_{\mu\nu}) \tag{31}$$

is invariant under noncommutative translations, rotations and dilatations[2]:

$$W^T_{\hat{A}+\theta;\tau}\hat{\Sigma} = 0 \;, \qquad W^R_{\hat{A}+\theta;\alpha\beta}\hat{\Sigma} = 0 \;, \qquad W^D_{\hat{A}+\theta}\hat{\Sigma} = 0 \;, \tag{32}$$

[2] In [14] we have shown that an identity like $W^D_\phi\hat{\Sigma} - 2\theta^{\mu\nu}(\partial\hat{\Sigma}/\partial\theta^{\mu\nu}) = 0$ exists for dilatation in the case of noncommutative ϕ^4 theory.

with the general notation

$$W^{?}_{A;C} + W^{?}_{B;C} = W^{?}_{A+B;C} \ . \tag{33}$$

Computing the various commutators between $W^{?}_{\hat{A}}$ given in (23)–(25) one convinces oneself that the noncommutative gauge field $\hat{A}_\mu$ forms an irreducible representation of the conformal Lie algebra (10). For convenience we list these commutators (for $W^{?}_{\hat{A}+\theta}$, which makes no difference to $W^{?}_{\hat{A}}$ when applied to $\hat{A}_\mu$) below in (41). It is remarkable that the conformal group remains the same and should not be deformed when passing from a commutative space to a noncommutative one whereas the gauge groups are very different in both cases. This shows that the fundamentals of quantum field theory—Lorentz covariance, locality, unitarity—have good chances to survive in the noncommutative framework.

In particular, the Wigner theorem [15] that a field is classified by mass and spin holds. The conformal Lie algebra is of rank 2, hence its irreducible representations ρ are (in non-degenerate cases) classified by two Casimir operators,

$$m^2 = -g^{\tau\sigma}\rho(P_\tau)\rho(P_\sigma) \ , \qquad s(s+1)m^2 = -g_{\mu\nu}W^{PL;\mu}W^{PL;\nu} \ , \tag{34}$$

where

$$W^{PL;\mu} = -\frac{1}{2}\epsilon^{\mu\tau\alpha\beta}\rho(P_\tau)\rho(M_{\alpha\beta}) \tag{35}$$

is the Pauli-Ljubanski vector and m and s mass and spin of the particle, respectively. In our case where $\rho(?)$ is given by the action of $W^{?}_{\hat{A}+\theta}$ on $\hat{A}_\mu$ we find

$$m^2\hat{A}_\mu = -\partial^\tau\partial_\tau\hat{A}_\mu \ , \qquad g_{\rho\sigma}W^{PL;\rho}_{\hat{A}}W^{PL;\sigma}_{\hat{A}}\hat{A}_\mu = 2(g_{\mu\tau}\partial^\sigma\partial_\sigma - \partial_\mu\partial_\tau)\hat{A}^\tau + 0\,\partial_\mu\partial_\tau\hat{A}^\tau \ , \tag{36}$$

which means that the transverse components of $\hat{A}_\mu$ have spin $s = 1$ and the longitudinal component spin $s = 0$.

5.2 Compatibility with gauge symmetry

The NCYM action (31) is additionally invariant under noncommutative gauge transformations

$$W^{G}_{\hat{A};\hat{\lambda}} = \int d^4x \,\mathrm{tr}\Big(\big(\partial_\mu\lambda - \mathrm{i}\,[\hat{A}_\mu,\hat{\lambda}]_\star\big)\frac{\delta}{\delta\hat{A}_\mu}\Big) \ , \tag{37}$$

where $\hat{\lambda}$ is a possibly $\hat{A}$-dependent gauge parameter. This means that the symmetry algebra of the NCYM action is at least[3] given by the Lie algebra

$$\mathcal{L} = \mathcal{G} \rtimes \mathcal{C} \tag{38}$$

[3]Renormlizability seems to require that the symmetry algebra of the NCYM action is actually bigger than $\mathcal{L}$.

of Ward identity operators, which is the semidirect product of the Lie algebra $\mathcal{G}$ of possibly field-dependent gauge transformations $W^G_{\hat{A};\hat{\lambda}}$ with the Lie algebra $\mathcal{C}$ of rigid conformal transformations $W^{\{T,R,D\}}_{\hat{A}+\theta}$. The commutator relations of $\mathcal{L}$ are computed to

$$[W^G_{\hat{A};\hat{\lambda}_1}, W^G_{\hat{A};\hat{\lambda}_2}] = -\mathrm{i}W^G_{\hat{A};[\hat{\lambda}_1,\hat{\lambda}_2]_\star + \mathrm{i}W^G_{\hat{A};\hat{\lambda}_1}\hat{\lambda}_2 - \mathrm{i}W^G_{\hat{A};\hat{\lambda}_2}\hat{\lambda}_1} , \tag{39}$$

$$\begin{aligned}
[W^T_{\hat{A}+\theta;\tau}, W^G_{\hat{A};\hat{\lambda}}] &= W^G_{\hat{A};-\partial_\tau\hat{\lambda}+W^T_{\hat{A}+\theta;\tau}\hat{\lambda}} , \\
[W^R_{\hat{A}+\theta;\alpha\beta}, W^G_{\hat{A};\hat{\lambda}}] &= W^G_{\hat{A};-\frac{1}{2}\{x_\alpha,\partial_\beta\hat{\lambda}\}_\star+\frac{1}{2}\{x_\beta,\partial_\alpha\hat{\lambda}\}_\star+W^R_{\hat{A}+\theta;\alpha\beta}\hat{\lambda}} , \\
[W^D_{\hat{A}+\theta}, W^G_{\hat{A};\hat{\lambda}}] &= W^G_{\hat{A};-\frac{1}{2}\{x^\delta,\partial_\delta\hat{\lambda}\}_\star+W^D_{\hat{A}+\theta}\hat{\lambda}} ,
\end{aligned} \tag{40}$$

$$\begin{aligned}
[W^T_{\hat{A}+\theta;\tau}, W^T_{\hat{A}+\theta;\sigma}] &= 0 , \\
[W^T_{\hat{A}+\theta;\tau}, W^R_{\hat{A}+\theta;\alpha\beta}] &= g_{\tau\beta}W^T_{\hat{A}+\theta;\alpha} - g_{\tau\alpha}W^T_{\hat{A}+\theta;\beta} , \\
[W^T_{\hat{A}+\theta;\tau}, W^D_{\hat{A}+\theta}] &= -W^T_{\hat{A}+\theta;\tau} , \\
[W^R_{\hat{A}+\theta;\alpha\beta}, W^R_{\hat{A}+\theta;\gamma\delta}] &= g_{\alpha\gamma}W^R_{\hat{A}+\theta;\beta\delta} - g_{\beta\gamma}W^R_{\hat{A}+\theta;\alpha\delta} - g_{\alpha\delta}W^R_{\hat{A}+\theta;\beta\gamma} + g_{\beta\delta}W^R_{\hat{A}+\theta;\alpha\gamma} , \\
[W^R_{\hat{A}+\theta;\alpha\beta}, W^D_{\hat{A}+\theta}] &= 0 , \\
[W^D_{\hat{A}+\theta}, W^D_{\hat{A}+\theta}] &= 0 .
\end{aligned} \tag{41}$$

It is crucial to use the sum of the individual transformations $W^{\{R,D\}}_{\hat{A}}$ and $W^{\{R,D\}}_\theta$ because the individual commutators do not preserve the Lie algebra $\mathcal{L}$:

$$\begin{aligned}
[W^G_{\hat{A};\hat{\lambda}}, W^R_{\theta;\alpha\beta}]\hat{A}_\mu &= W^G_{\hat{A};-W^R_{\theta;\alpha\beta}\hat{\lambda}}\hat{A}_\mu - \tfrac{1}{2}\theta_\beta{}^\rho\{\partial_\alpha\hat{A}_\mu, \partial_\rho\hat{\lambda}\}_\star + \tfrac{1}{2}\theta_\alpha{}^\rho\{\partial_\beta\hat{A}_\mu, \partial_\rho\hat{\lambda}\}_\star \\
&\quad + \tfrac{1}{2}\theta_\beta{}^\rho\{\partial_\rho\hat{A}_\mu, \partial_\alpha\hat{\lambda}\}_\star - \tfrac{1}{2}\theta_\alpha{}^\rho\{\partial_\rho\hat{A}_\mu, \partial_\beta\hat{\lambda}\}_\star , \\
[W^G_{\hat{A};\hat{\lambda}}, W^D_\theta]\hat{A}_\mu &= W^G_{\hat{A};-W^D_\theta\hat{\lambda}}\hat{A}_\mu + \theta^{\delta\rho}\{\partial_\delta\hat{A}_\mu, \partial_\rho\hat{\lambda}\}_\star .
\end{aligned} \tag{42}$$

5.3 Gauge covariance, covariant representation and Seiberg-Witten differential equation

One may ask (the reason is given below) whether there exists a 'rotation' in $(\hat{A},\theta)$ space so that the 'rotated fields' preserve individually the mixed commutators (40). To be concrete, what we look for is a splitting

$$W^?_{\hat{A}+\theta} \equiv W^?_{\hat{A}} + W^?_\theta = \tilde{W}^?_{\hat{A}} + \tilde{W}^?_\theta , \tag{43}$$

$$[\tilde{W}^?_{\hat{A}}, W^G_{\hat{A};\hat{\lambda}}] = W^G_{\hat{A};\hat{\lambda}^?_{\hat{A}}} , \qquad [\tilde{W}^?_\theta, W^G_{\hat{A};\hat{\lambda}}] = W^G_{\hat{A};\hat{\lambda}^?_\theta} , \tag{44}$$

for appropriate field-dependent gauge parameters $\hat{\lambda}^?_{\hat{A}}$ and $\hat{\lambda}^?_\theta$. Because of (40), each of the two relations in (44) is of course the consequence of the other relation. Furthermore, we impose the condition that the splitting should be universal in the sense $\tilde{W}^?_\theta = W^?_\theta(\theta^{\rho\sigma})\frac{d}{d\theta^{\rho\sigma}}$:

$$\begin{aligned}
\tilde{W}^?_{\hat{A}} &= W^?_{\hat{A}} - W^?_\theta(\theta^{\rho\sigma})\int d^4x\, \mathrm{tr}\Big(\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}\frac{\delta}{\delta\hat{A}_\mu}\Big) , \\
\tilde{W}^?_\theta &= W^?_\theta + W^?_\theta(\theta^{\rho\sigma})\int d^4x\, \mathrm{tr}\Big(\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}\frac{\delta}{\delta\hat{A}_\mu}\Big) \equiv W^?_\theta(\theta^{\rho\sigma})\frac{d}{d\theta^{\rho\sigma}} .
\end{aligned} \tag{45}$$

The notation $\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}$ is for the time being just a symbol for a field-dependent quantity with three Lorentz indices and power-counting dimension 3. Inserted into (44) one gets the *equivalent conditions*

$$-\mathrm{i}\big[\tilde{W}^?_{\hat{A}}\hat{A}_\mu,\hat{\lambda}\big]_\star - W^G_{\hat{A};\hat{\lambda}}\big(\tilde{W}^?_{\hat{A}}(\hat{A}_\mu)\big) = \hat{D}_\mu\big(\hat{\lambda}^?_{\hat{A}} - \tilde{W}^?_{\hat{A}}(\hat{\lambda})\big)\;, \tag{46}$$

$$W^?_\theta(\theta^{\rho\sigma})\Big(-\mathrm{i}\Big[\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}},\hat{\lambda}\Big]_\star + \frac{1}{2}\{\partial_\rho\hat{A}_\mu,\partial_\sigma\hat{\lambda}\}_\star - W^G_{\hat{A};\hat{\lambda}}\Big(\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}\Big)\Big) = \hat{D}_\mu\big(\hat{\lambda}^?_\theta - \tilde{W}^?_\theta(\hat{\lambda})\big)\;. \tag{47}$$

Whereas (47) cannot be solved without prior knowledge of the result[4], we can trivially solve (46) by a covariance ansatz:

$$\tilde{W}^T_{\hat{A};\tau} = W^G_{\hat{A};\hat{\lambda}^T_\tau} + \int d^4x\,\mathrm{tr}\Big(\hat{F}_{\tau\mu}\frac{\delta}{\delta\hat{A}_\mu}\Big)\;, \tag{48}$$

$$\tilde{W}^R_{\hat{A};\alpha\beta} = W^G_{\hat{A};\hat{\lambda}^R_{\alpha\beta}} + \int d^4x\,\mathrm{tr}\Big(\Big(\frac{1}{2}\{\hat{X}_\alpha,\hat{F}_{\beta\mu}\}_\star - \frac{1}{2}\{\hat{X}_\beta,\hat{F}_{\alpha\mu}\}_\star - W^R_{\theta;\alpha\beta}(\theta^{\rho\sigma})\hat{\Omega}_{\rho\sigma\mu}\Big)\frac{\delta}{\delta\hat{A}_\mu}\Big)\;, \tag{49}$$

$$\tilde{W}^D_{\hat{A}} = W^G_{\hat{A};\hat{\lambda}^D} + \int d^4x\,\mathrm{tr}\Big(\Big(\frac{1}{2}\{\hat{X}^\delta,\hat{F}_{\delta\mu}\}_\star - W^D_\theta(\theta^{\rho\sigma})\hat{\Omega}_{\rho\sigma\mu}\Big)\frac{\delta}{\delta\hat{A}_\mu}\Big)\;, \tag{50}$$

where $\hat{X}^\mu = x^\mu + \theta^{\mu\nu}\hat{A}_\nu$ are the covariant coordinates [6, 7] and $\hat{\Omega}_{\rho\sigma\mu}$ is a polynomial in the covariant quantities $\theta, \hat{X}, \hat{F}, \hat{D}\ldots\hat{D}\hat{F}$ which is antisymmetric in ρ,σ and of power-counting dimension 3. For physical reasons (e.g. quantization) an $\hat{X}$-dependence of $\hat{\Omega}_{\rho\sigma\mu}$ should be excluded. We denote (48)–(50) as *covariant transformations* of the noncommutative gauge field $\hat{A}$, because these transformations reduce in the commutative case to the 'gauge-covariant conformal transformations' of Jackiw [8, 9].

It follows from (38) and (43) that $\tilde{W}^?_\theta$ and thus $\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}$ are (up to a gauge transformation) precisely the missing piece to complete (49) and (50) to an invariance of the action,

$$(\tilde{W}^R_{\hat{A};\alpha\beta} + \tilde{W}^R_{\theta;\alpha\beta})\hat{\Sigma} = 0\;, \qquad (\tilde{W}^D_{\hat{A};\alpha\beta} + \tilde{W}^D_{\theta;\alpha\beta})\hat{\Sigma} = 0\;. \tag{51}$$

Applying (48)–(50) to the NCYM action (31) we obtain for $\hat{\Omega}_{\rho\sigma\mu} = 0$

$$\tilde{W}^T_{\hat{A};\tau}\hat{\Sigma} = 0\;, \tag{52}$$

$$\tilde{W}^R_{\hat{A};\alpha\beta}\hat{\Sigma} = \frac{1}{g^2}\int d^4x\,\mathrm{tr}\Big(\theta_{\alpha\rho}\hat{F}^{\rho\sigma}\star\hat{T}_{\beta\sigma} - \theta_{\beta\rho}\hat{F}^{\rho\sigma}\star\hat{T}_{\alpha\sigma}\Big)\;, \tag{53}$$

$$\tilde{W}^D_{\hat{A}}\hat{\Sigma} = \frac{1}{g^2}\int d^4x\,\mathrm{tr}\Big(\theta^\delta{}_\rho\hat{F}^{\rho\sigma}\star\hat{T}_{\delta\sigma}\Big)\;, \tag{54}$$

where the quantity

$$\hat{T}_{\mu\nu} = \frac{1}{2}\hat{F}_{\mu\rho}\star\hat{F}_\nu{}^\rho + \frac{1}{2}\hat{F}_{\nu\rho}\star\hat{F}_\mu{}^\rho - \frac{1}{4}g_{\mu\nu}\hat{F}_{\rho\sigma}\star\hat{F}^{\rho\sigma} \tag{55}$$

resembles (but is not) the energy-momentum tensor. The calculation uses however the symmetry $\hat{T}_{\mu\nu} = \hat{T}_{\nu\mu}$ (a consequence of the symmetrical product in (49)) and tracelessness

[4] One can make of course an ansatz for $\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}$ with free coefficients to be determined by (47).

$g^{\mu\nu}\hat{T}_{\mu\nu} = 0$. We give in Appendix A details of the computation of (53). As we show in Appendix B, the first (rotational) condition in (51) has, reinserting $\hat{\Omega}_{\rho\sigma\mu}$, the solution

$$\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} = -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma \hat{A}_\mu + \hat{F}_{\sigma\mu}\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho \hat{A}_\mu + \hat{F}_{\rho\mu}\big\}_\star + \hat{\Omega}_{\rho\sigma\mu} \,, \tag{56}$$

which is also compatible with the second (dilatational) condition in (51). The solution (56) is for $\hat{\Omega}_{\rho\sigma\mu} = 0$ known as the Seiberg-Witten differential equation [1]. It is now straightforward to check (47) for an arbitrary field-dependent gauge parameter $\hat{\lambda}$. The gauge parameters in (45) are

$$\hat{\lambda}^T_\tau = \hat{A}_\tau \,, \quad \hat{\lambda}^R_{\alpha\beta} = \frac{1}{4}\{2x_\alpha + \theta_\alpha{}^\rho \hat{A}_\rho, \hat{A}_\beta\}_\star - \frac{1}{4}\{2x_\beta + \theta_\beta{}^\rho \hat{A}_\rho, \hat{A}_\alpha\}_\star \,, \quad \hat{\lambda}^D = \frac{1}{2}\{x^\delta, \hat{A}_\delta\}_\star \,. \tag{57}$$

5.4 θ-expansion of noncommutative gauge transformations

The meaning of the condition (44) is easy to understand: $\tilde{W}^?_\theta$ applied to a gauge-invariant functional remains gauge-invariant. Because $\tilde{W}^?_\theta(\theta^{\rho\sigma})$ commutes with $W^G_{\hat{A};\hat{\lambda}}$, we conclude with the notation $\frac{d}{d\theta^{\rho\sigma}} = \frac{\partial}{\partial\theta^{\rho\sigma}} + \int d^4x \,\mathrm{tr}\big(\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} \frac{\delta}{\delta \hat{A}_\mu}\big)$, see (45), that

$$\Big[\frac{d}{d\theta^{\rho\sigma}}, W^G_{\hat{A};\hat{\lambda}}\Big] = W^G_{\hat{A};\hat{\lambda}_{\rho\sigma}(\hat{\lambda})} \,, \tag{58}$$

where $\hat{\lambda}_{\rho\sigma}(\hat{\lambda})$ is determined by $\hat{\lambda}$ and the choice $\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}$. In particular, we conclude from (58) that

$$\frac{d^n\Gamma}{d\theta^{\rho_1\sigma_1}\dots d\theta^{\rho_n\sigma_n}} \quad \text{is gauge-invariant if } \Gamma \text{ is gauge-invariant.} \tag{59}$$

Given any first-order differential equation $\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} = \Phi_{\rho\sigma\mu}[\hat{A},\theta]$ we can express $\hat{A}$ in terms of θ and the initial value A at $\theta = 0$. In the same way, the first-order differential equation expresses any (sufficiently regular) functional $\Gamma[\hat{A},\theta]$ in terms of θ and the initial value A:

$$\Gamma[A,\theta] := \sum_{n=0}^{\infty} \frac{1}{n!}\,\theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}\Big(\frac{d^n\Gamma[\hat{A},\theta]}{d\theta^{\rho_1\sigma_1}\dots d\theta^{\rho_n\sigma_n}}\Big)_{\theta=0} \,. \tag{60}$$

The special choice (56) of the differential equation has due to (59) the distinguished property that

$$W^G_{\hat{A};\hat{\lambda}}\big(\Gamma[\hat{A},\theta]\big) = 0 \quad \Rightarrow \quad W^G_{A;\lambda=\hat{\lambda}|_{\theta=0}}\Big(\sum_{n=0}^{N}\frac{1}{n!}\,\theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}\Big(\frac{d^n\Gamma[\hat{A},\theta]}{d\theta^{\rho_1\sigma_1}\dots d\theta^{\rho_n\sigma_n}}\Big)_{\theta=0}\Big) = 0 \,. \tag{61}$$

In other words, *any approximation up to order N in θ of a noncommutatively gauge-invariant functional $\Gamma[\hat{A},\theta]$ is invariant under commutative gauge transformations if the θ-evolution is given by* (56)*, i.e. the solution of* (44). We stress that the noncommutative conformal transformations (23)–(25) and their commutators (40) with gauge transformations enabled us to compute the gauge-equivalent θ-expansion of Seiberg and Witten directly (without an ansatz) via the equivalent but much simpler solution of (51) for the trivially obtained covariant transformations (45).

Our condition (44) is more general than the original gauge-equivalence requirement [1] by Seiberg and Witten. To see this we consider the θ-expansion of $W^G_{\hat{A};\hat{\lambda}}\hat{A}_\mu$ taking (58) into account, where $\hat{\lambda}$ is allowed to depend on $\hat{A}$. To demonstrate the relation we consider the term to second order in θ:

$$\begin{aligned}\frac{d^2 W^G_{\hat{A};\hat{\lambda}}\hat{A}_\mu}{d\theta^{\rho_1\sigma_1}d\theta^{\rho_2\sigma_2}} &= \frac{d}{d\theta^{\rho_1\sigma_1}}\Big(\Big[\frac{d}{d\theta^{\rho_2\sigma_2}}, W^G_{\hat{A};\hat{\lambda}}\Big] + W^G_{\hat{A};\hat{\lambda}}\frac{d}{d\theta^{\rho_2\sigma_2}}\Big)\hat{A}_\mu \\ &= \Big(W^G_{\hat{A};\hat{\lambda}_{\rho_1\sigma_1}(\hat{\lambda}_{\rho_2\sigma_2}(\hat{\lambda}))} + W^G_{\hat{A};\hat{\lambda}_{\rho_2\sigma_2}(\hat{\lambda})}\frac{d}{d\theta^{\rho_1\sigma_1}} + W^G_{\hat{A};\hat{\lambda}_{\rho_1\sigma_1}(\hat{\lambda})}\frac{d}{d\theta^{\rho_2\sigma_2}} + W^G_{\hat{A};\hat{\lambda}}\frac{d^2}{d\theta^{\rho_1\sigma_1}d\theta^{\rho_2\sigma_2}}\Big)\hat{A}_\mu \;.\end{aligned}$$

Setting $\theta \to 0$, generalizing it to any order n and inserting the result into the Taylor expansion (60) we obtain

$$\begin{aligned}\big(W^G_{\hat{A};\hat{\lambda}}\hat{A}_\mu\big)[A,\theta] &= W^G_{A;\lambda[\hat{\lambda};A,\theta]}\big(\hat{A}_\mu[A,\theta]\big) \;, \\ \lambda[\hat{\lambda};A,\theta] &= \big(\hat{\lambda}\big)_{\theta=0} + \theta^{\rho\sigma}\big(\hat{\lambda}_{\rho\sigma}(\hat{\lambda})\big)_{\theta=0} + \tfrac{1}{2}\theta^{\rho_1\sigma_1}\theta^{\rho_2\sigma_2}\big(\hat{\lambda}_{\rho_1\sigma_1}(\hat{\lambda}_{\rho_2\sigma_2}(\hat{\lambda}))\big)_{\theta=0} + \dots \;.\end{aligned} \tag{62}$$

Eq. (62) is the original Seiberg-Witten gauge-equivalence [1] iff $\big(\hat{\lambda}_{\rho\sigma}(\hat{\lambda})\big)_{\theta=0} = 0$. In other words, our approach via (44)—which leads to the same θ-expansion as the Seiberg-Witten requirement, see (61)—is more general.

5.5 θ-expansion of noncommutative conformal transformations

According to (60) let us compute the θ-expansion of the noncommutative conformal transformation of a functional $\Gamma[\hat{A},\theta]$ approximated up to order N in θ,

$$\sum_{n=0}^{N}\frac{1}{n!}\,\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\,\Big(\frac{d^n(W^?_{\hat{A}+\theta}\Gamma[\hat{A},\theta])}{d\theta^{\rho_1\sigma_1}\dots d\theta^{\rho_n\sigma_n}}\Big)_{\theta=0} \;. \tag{63}$$

As a typical example we regard the $n=2$ term in this series, which we derive by the following procedure. Before putting $\theta=0$ we consider

$$\begin{aligned}T^?_2 &:= \theta^{\rho_1\sigma_1}\theta^{\rho_2\sigma_2}\frac{d^2(W^?_{\hat{A}+\theta}\Gamma[\hat{A};\theta])}{d\theta^{\rho_1\sigma_1}d\theta^{\rho_2\sigma_2}} \\ &= \theta^{\rho_1\sigma_1}\frac{d}{d\theta^{\rho_1\sigma_1}}\Big(\theta^{\rho_2\sigma_2}\frac{d(W^?_{\hat{A}+\theta}\Gamma[\hat{A};\theta])}{d\theta^{\rho_2\sigma_2}}\Big) - \theta^{\rho_2\sigma_2}\frac{d(W^?_{\hat{A}+\theta}\Gamma[\hat{A};\theta])}{d\theta^{\rho_2\sigma_2}} \;.\end{aligned} \tag{64}$$

The crucial property we use is the identity

$$\Big[W^?_{\hat{A}+\theta}, \theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}}\Big] = 0 \;, \tag{65}$$

which is valid for a very general class of differential equations. See Appendix C for details. Thus,

$$
\begin{aligned}
T_2^{?} &= W^{?}_{\hat{A}+\theta}\Big(\theta^{\rho_1\sigma_1}\frac{d}{d\theta^{\rho_1\sigma_1}}\Big(\theta^{\rho_2\sigma_2}\frac{d\Gamma[\hat{A};\theta]}{d\theta^{\rho_2\sigma_2}}\Big) - \theta^{\rho_2\sigma_2}\frac{d\Gamma[\hat{A};\theta]}{d\theta^{\rho_2\sigma_2}}\Big) = W^{?}_{\hat{A}+\theta}\Big(\theta^{\rho_1\sigma_1}\theta^{\rho_2\sigma_2}\frac{d^2\Gamma[\hat{A};\theta]}{d\theta^{\rho_1\sigma_1}d\theta^{\rho_2\sigma_2}}\Big) \\
&= \big(W^{?}_{\hat{A}+\theta}(\theta^{\rho_1\sigma_1})\theta^{\rho_2\sigma_2} + \theta^{\rho_1\sigma_1}W^{?}_{\hat{A}+\theta}(\theta^{\rho_2\sigma_2})\big)\frac{d^2\Gamma[\hat{A};\theta]}{d\theta^{\rho_1\sigma_1}d\theta^{\rho_2\sigma_2}}\Big) + \theta^{\rho_1\sigma_1}\theta^{\rho_2\sigma_2}W^{?}_{\hat{A}+\theta}\Big(\frac{d^2\Gamma[\hat{A};\theta]}{d\theta^{\rho_1\sigma_1}d\theta^{\rho_2\sigma_2}}\Big) \\
&= \theta^{\rho_1\sigma_1}\theta^{\rho_2\sigma_2}\Big(\frac{\partial W^{?}_{\theta}(\theta^{\rho\sigma})}{\partial\theta^{\rho_1\sigma_1}}\frac{d^2\Gamma[\hat{A};\theta]}{d\theta^{\rho\sigma}d\theta^{\rho_2\sigma_2}} + \frac{\partial W^{?}_{\theta}(\theta^{\rho\sigma})}{\partial\theta^{\rho_2\sigma_2}}\frac{d^2\Gamma[\hat{A};\theta]}{d\theta^{\rho_1\sigma_1}d\theta^{\rho\sigma}} + W^{?}_{\hat{A}+\theta}\Big(\frac{d^2\Gamma[\hat{A};\theta]}{d\theta^{\rho_1\sigma_1}d\theta^{\rho_2\sigma_2}}\Big)\Big)\;,
\end{aligned}
\tag{66}
$$

using the linearity of $W^{?}_{\theta}(\theta^{\rho\sigma})$ in θ. We can now omit the leading factors of θ from $T_2^{?}$ in (64) and (66), generalize it to any order n and put $\theta = 0$:

$$
\begin{aligned}
\Big(\frac{d^n(W^{?}_{\hat{A}+\theta}\Gamma[\hat{A},\theta])}{d\theta^{\rho_1\sigma_1}\dots d\theta^{\rho_n\sigma_n}}\Big)_{\theta=0} &= \sum_{i=1}^{n}\frac{\partial W^{?}_{\theta}(\theta^{\rho\sigma})}{\partial\theta^{\rho_i\sigma_i}}\Big(\frac{d^n\Gamma[\hat{A};\theta]}{d\theta^{\rho_2\sigma_1}\dots d\theta^{\rho_{i-1}\sigma_{i-1}}d\theta^{\rho\sigma}d\theta^{\rho_{i+1}\sigma_{i+1}}\dots d\theta^{\rho_n\sigma_n}}\Big)_{\theta=0} \\
&\quad + W^{?}_{A}\Big(\frac{d^n\Gamma[\hat{A};\theta]}{d\theta^{\rho_1\sigma_1}\dots d\theta^{\rho_n\sigma_n}}\Big)_{\theta=0}\;.
\end{aligned}
\tag{67}
$$

Note that from $W^{?}_{\hat{A}+\theta}$ at $\theta = 0$ there survives only the commutative conformal transformation $W^{?}_{A}$ defined in (14)–(16). Inserted into (63) we get the final result

$$
\begin{aligned}
&\sum_{n=0}^{N}\frac{1}{n!}\,\theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}\,\Big(\frac{d^n(W^{?}_{\hat{A}+\theta}\Gamma[\hat{A},\theta])}{d\theta^{\rho_1\sigma_1}\dots d\theta^{\rho_n\sigma_n}}\Big)_{\theta=0} \\
&\qquad = W_{A+\theta}\Big(\sum_{n=0}^{N}\frac{1}{n!}\,\theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}\,\Big(\frac{d^n\Gamma[\hat{A},\theta]}{d\theta^{\rho_1\sigma_1}\dots d\theta^{\rho_n\sigma_n}}\Big)_{\theta=0}\Big)\;.
\end{aligned}
\tag{68}
$$

This result can be formulated as

Theorem *Acting with the noncommutative conformal transformations (translation, rotation, dilatation) on action functionals $\Gamma[\hat{A},\theta]$ and applying the Seiberg-Witten map is* <u>*identical*</u> *to the action of the commutative translation, rotation and dilatation operations, respectively, on $\Gamma[\hat{A}[A,\theta],\theta]$.*

The result means that with the noncommutative conformal symmetries there are—after Seiberg-Witten map—no further symmetries associated than the standard commutative conformal symmetries. Thus, the noncommutative conformal symmetries do not give any hints for the renormalization of noncommutative Yang-Mills theories.

6 Quantization

Passing from a classical action with gauge symmetry to quantum field theory one must introduce gauge-fixing terms to the action in order to define the propagator. Here we repeat this construction for the noncommutative Yang-Mills theory.

The NCYM theory is enlarged by the fields $\hat{c}, \hat{\bar{c}}, \hat{B}$ which transform according to the following representation of (10):

$$W^T_{\hat{A}+\hat{c}+\hat{\bar{c}}+\hat{B}+\theta;\tau} = W^T_{\hat{A}+\theta;\tau} + \int d^4x \,\mathrm{tr}\Big(\partial_\tau \hat{c}\,\frac{\delta}{\delta\hat{c}} + \partial_\tau \hat{\bar{c}}\,\frac{\delta}{\delta\hat{\bar{c}}} + \partial_\tau \hat{B}\,\frac{\delta}{\delta\hat{B}}\Big) , \tag{69}$$

$$\begin{aligned} W^R_{\hat{A}+\hat{c}+\hat{\bar{c}}+\hat{B}+\theta;\alpha\beta} = W^R_{\hat{A}+\theta;\alpha\beta} + \int d^4x \,\mathrm{tr}\Big(&\Big(\frac{1}{2}\{x_\alpha, \partial_\beta \hat{c}\}_\star - \frac{1}{2}\{x_\beta, \partial_\alpha \hat{c}\}_\star\Big)\frac{\delta}{\delta\hat{c}} \\ &+ \Big(\frac{1}{2}\{x_\alpha, \partial_\beta \hat{\bar{c}}\}_\star - \frac{1}{2}\{x_\beta, \partial_\alpha \hat{\bar{c}}\}_\star\Big)\frac{\delta}{\delta\hat{\bar{c}}} \\ &+ \Big(\frac{1}{2}\{x_\alpha, \partial_\beta \hat{B}\}_\star - \frac{1}{2}\{x_\beta, \partial_\alpha \hat{B}\}_\star\Big)\frac{\delta}{\delta\hat{B}}\Big) , \end{aligned} \tag{70}$$

$$\begin{aligned} W^D_{\hat{A}+\hat{c}+\hat{\bar{c}}+\hat{B}+\theta} = W^D_{\hat{A}+\theta} + \int d^4x \,\mathrm{tr}\Big(&\frac{1}{2}\{x^\delta, \partial_\delta \hat{c}\}_\star \frac{\delta}{\delta\hat{c}} + \Big(\frac{1}{2}\{x^\delta, \partial_\delta \hat{\bar{c}}\}_\star + 2\hat{\bar{c}}\Big) \\ &+ \Big(\frac{1}{2}\{x^\delta, \partial_\delta \hat{B}\}_\star + 2B\Big)\frac{\delta}{\delta\hat{B}}\Big) . \end{aligned} \tag{71}$$

The noncommutative BRST transformations are given by

$$\hat{s}\hat{A}_\mu = \hat{D}_\mu \hat{c} , \qquad \hat{s}\hat{c} = -\mathrm{i}c \star c , \qquad \hat{s}\hat{\bar{c}} = \hat{B} , \qquad \hat{s}\hat{B} = 0 . \tag{72}$$

It is then not difficult to verify that the standard gauge-fixing action

$$\hat{\Sigma}_{gf} = \int d^4x \,\mathrm{tr}\Big(\hat{s}\Big[\hat{\bar{c}} \star \Big(\partial^\mu \hat{A}_\mu + \frac{\alpha}{2}\hat{B}\Big)\Big]\Big) \tag{73}$$

is conformally invariant:

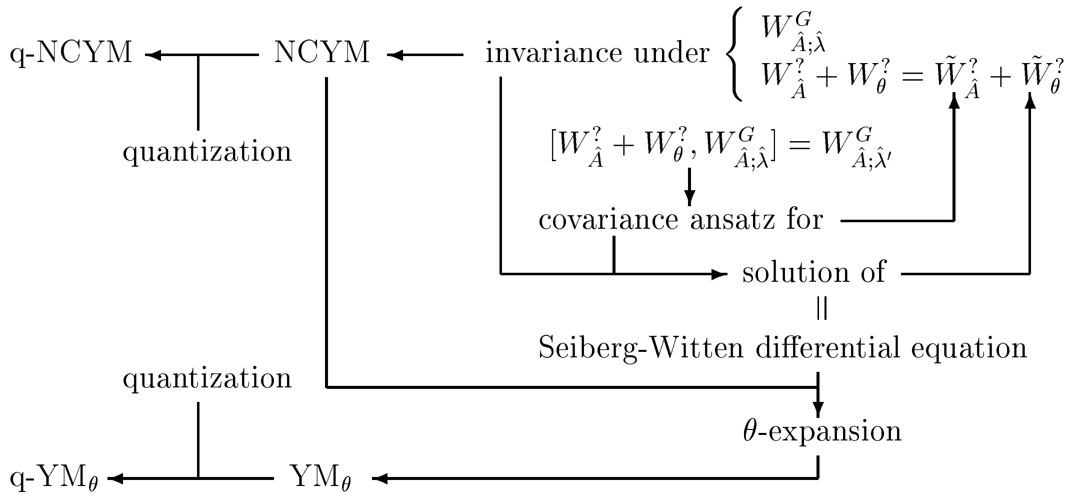
$$W^T_{\hat{A}+\hat{c}+\hat{\bar{c}}+\hat{B}+\theta;\tau}\hat{\Sigma}_{gf} = 0 , \qquad W^R_{\hat{A}+\hat{c}+\hat{\bar{c}}+\hat{B}+\theta;\alpha\beta}\hat{\Sigma}_{gf} = 0 , \qquad W^D_{\hat{A}+\hat{c}+\hat{\bar{c}}+\hat{B}+\theta}\hat{\Sigma}_{gf} = 0 . \tag{74}$$

Loop calculations based on $\hat{\Sigma} + \hat{\Sigma}_{gf}$ in (31) and (73) suffer from infrared divergences [3].

To circumvent the IR-problem one can however use the θ-expansion of the NCYM action leading to a gauge field theory on commutative space-time coupled to an external field θ. This action is quantized according to the analogous formulae as above, omitting everywhere the hat symbolizing noncommutative objects and replacing the $\star$-product by the ordinary product. This approach was used in [10] to compute the one-loop photon selfenergy in θ-expanded Maxwell theory and in [5] to show renormalizability of the photon selfenergy to all orders in $\hbar$ and θ.

7 Summary and outlook

We have established rigid conformal transformations (23)–(25) for the noncommutative Yang-Mills field $\hat{A}$. Our results related to these transformations can be summarized as follows.



The (classical) noncommutative Yang-Mills action (31) is invariant under the Lie algebra $\mathcal{L}$ of gauge transformations $W^G_{\hat{A};\hat{\lambda}}$ and the sum $W^?_{\hat{A}} + W^?_\theta$ of conformal transformations of $\hat{A}$ and θ. The commutation relations $[W^?_{\hat{A}} + W^?_\theta, W^G_{\hat{A};\hat{\lambda}}] = W^G_{\hat{A};\hat{\lambda}'}$ in $\mathcal{L}$ suggest a *covariant* splitting $W^?_{\hat{A}} + W^?_\theta = \tilde{W}^?_{\hat{A}} + \tilde{W}^?_\theta$. The relation $[\tilde{W}^?_{\hat{A}}, W^G_{\hat{A};\hat{\lambda}}] = W^G_{\hat{A};\hat{\lambda}''}$ is trivially solved by a covariance ansatz. Then, the covariant complement $\tilde{W}^?_\theta$ is simply obtained from invariance of the NCYM action under $\tilde{W}^?_{\hat{A}} + \tilde{W}^?_\theta$ transformation. The solution for $\tilde{W}^?_\theta$ is given by the Seiberg-Witten differential equation (56). What we have thus achieved is a more transparent—and less restrictive—derivation of the Seiberg-Witten differential equation which does not require the usual ansatz of gauge equivalence.

Interpreting the Seiberg-Witten differential equation as an evolution equation we can express the noncommutative Yang-Mills field $\hat{A}$ in terms of its initial value A. The resulting θ-expansion of the NCYM action is due to the covariance $[\tilde{W}^?_\theta, W^G_{\hat{A};\hat{\lambda}}] = W^G_{\hat{A};\hat{\lambda}'''}$ invariant under *commutative* gauge transformations. Moreover, noncommutative conformal transformations reduce after θ-expansion to commutative conformal transformations. In this way we associate to the NCYM theory a gauge theory YM$_\theta$ on commutative space-time for a commutative gauge field A coupled to a translation-invariant external field θ. Both gauge theories can be quantized by adding appropriate gauge-fixing terms and yield the two quantum field theories q-NCYM and q-YM$_\theta$, respectively. It is unclear in which sense these two quantum field theories are equivalent. At least on a perturbative level the quantum field theories q-NCYM and q-YM$_\theta$ are completely different.

Loop calculations [3] and power-counting analysis [4] for q-NCYM reveal a new type of infrared singularities which so far could not be treated. Loop calculations [10] for q-YM$_\theta$ are free of infrared problems but lead apparently to an enormous amount of ultraviolet singularities. This is not necessarily a problem. For instance, all UV-singularities in the photon selfenergy are *field redefinitions* [5] which are possible in presence of a field $\theta^{\mu\nu}$ of negative power-counting dimension. For higher N-point Green's functions the situation becomes more and more involved and a renormalization seems to be impossible without a symmetry for the θ-expanded NCYM-action. We had hoped in the beginning of the work on this paper that this symmetry searched for could be the Seiberg-Witten expansion of the noncommutative conformal symmetries. As we have seen in Section 5.5 this is not the case and the complete renormalization of NCYM theory remains an open problem.

We have proved that the noncommutative gauge field is an irreducible representation of the *undeformed* conformal Lie algebra. The noncommutative spin-$\frac{1}{2}$ representations for fermions have been worked out in [16]. This shows that classical concepts of particles and fields extend without modification to a noncommutative space-time. We believe this makes life in a noncommutative world more comfortable.

Of course much work remains to be done. First we have considered a very special noncommutative geometry of a constant $\theta^{\mu\nu}$. This assumption should finally be relaxed; at least the treatment of those non-constant $\theta^{\mu\nu}$ which are Poisson bivectors as in [18] seems to be possible. The influence of the modified concept of locality on causality and unitarity of the S-matrix must be studied. Previous results [19, 20] with different consequences according to whether the electrical components of $\theta^{\mu\nu}$ are zero must be invariantly formulated in terms of the signs of the two invariants $\theta^{\mu\nu}\theta_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}$. Eventually the renormalization puzzle for noncommutative Yang-Mills theory ought to be solved.

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A Covariant $\hat{A}$-rotation of the NCYM action

Let us give here the calculations leading to the result (53). The first input is the $\hat{A}$-variation of the NCYM action (31)

$$\frac{\delta\hat{\Sigma}}{\delta\hat{A}_\mu(x)} = \frac{1}{g^2}\big(\hat{D}_\kappa\hat{F}^{\kappa\mu}\big)(x) \ . \tag{A.1}$$

Inserted into (49), for $\hat{\Omega}_{\rho\sigma\mu}=0$, we obtain

$$\begin{aligned}\tilde{W}^R_{\hat{A};\alpha\beta}\hat{\Sigma} &= \frac{1}{2g^2}\int d^4x\,\mathrm{tr}\Big((\hat{X}_\alpha\star\hat{F}_{\beta\mu}+\hat{F}_{\beta\mu}\star\hat{X}_\alpha-\hat{X}_\beta\star\hat{F}_{\alpha\mu}-\hat{F}_{\alpha\mu}\star\hat{X}_\beta)\star\hat{D}_\kappa\hat{F}^{\kappa\mu}\Big)\\ &= \frac{1}{2g^2}\int d^4x\,\mathrm{tr}\Big(\hat{X}_\alpha\star\big(\hat{D}_\kappa\big\{\hat{F}_{\beta\mu},\hat{F}^{\kappa\mu}\big\}_\star-\big\{\hat{D}_\kappa(\hat{F}_{\beta\mu}),\hat{F}^{\kappa\mu}\big\}_\star\big)\\ &\qquad\qquad -\hat{X}_\beta\star\big(\hat{D}_\kappa\big\{\hat{F}_{\alpha\mu},\hat{F}^{\kappa\mu}\big\}_\star-\big\{\hat{D}_\kappa(\hat{F}_{\alpha\mu}),\hat{F}^{\kappa\mu}\big\}_\star\big)\Big) \ .\end{aligned} \tag{A.2}$$

Now we use the Bianchi identity $\hat{D}_\alpha\hat{F}_{\beta\gamma}+\hat{D}_\beta\hat{F}_{\gamma\alpha}+\hat{D}_\gamma\hat{F}_{\alpha\beta}=0$ and the antisymmetry in κ,μ to rewrite

$$\hat{D}_\kappa(\hat{F}_{\beta\mu})\star\hat{F}^{\kappa\mu} = \frac{1}{2}\hat{D}_\beta(\hat{F}_{\kappa\mu})\star\hat{F}^{\kappa\mu} \tag{A.3}$$

and similarly for the other terms in (A.2). We then obtain

$$\begin{aligned}\tilde{W}^R_{\hat{A};\alpha\beta}\hat{\Sigma} &= \frac{1}{g^2}\int d^4x\,\mathrm{tr}\Big(\hat{X}_\alpha\star\hat{D}_\kappa\Big(\frac{1}{2}\big\{\hat{F}_{\beta\mu},\hat{F}^{\kappa\mu}\big\}_\star-\frac{1}{8}\delta^\kappa_\beta\big\{\hat{F}_{\mu\nu},\hat{F}^{\mu\nu}\big\}_\star\Big)\\ &\qquad\qquad -\hat{X}_\beta\star\hat{D}_\kappa\Big(\frac{1}{2}\big\{\hat{F}_{\alpha\mu},\hat{F}^{\kappa\mu}\big\}_\star-\frac{1}{8}\delta^\kappa_\alpha\big\{\hat{F}_{\mu\nu},\hat{F}^{\mu\nu}\big\}_\star\Big)\Big)\\ &= \frac{1}{g^2}\int d^4x\,\mathrm{tr}\Big(\hat{D}_\kappa\Big(\hat{X}_\alpha\star\hat{T}_\beta{}^\kappa-\hat{X}_\beta\star\hat{T}_\alpha{}^\kappa\Big)-\hat{D}_\kappa(\hat{X}_\alpha)\star\hat{T}_\beta{}^\kappa+\hat{D}_\kappa(\hat{X}_\beta)\star\hat{T}_\alpha{}^\kappa\Big)\ ,\end{aligned}\tag{A.4}$$

where we have used (55) and the derivation property of $\hat{D}_\kappa$. Note that the total derivative $\int d^4x\,\mathrm{tr}(\hat{D}_\kappa\hat{J}^\kappa_{\alpha\beta})$ in (A.4) vanishes. The result (53) follows now from

$$\hat{D}_\kappa\hat{X}_\alpha = g_{\alpha\kappa}+\theta_\alpha{}^\nu\hat{F}_{\kappa\nu}\ ,\tag{A.5}$$

which is easily derived from the formulae in Section 2, and the symmetry $\hat{T}_{\alpha\beta}=\hat{T}_{\beta\alpha}$.

B Derivation of the Seiberg-Witten differential equation

We first compute the explicit θ-dependence of the $\star$-product according to the last term in (27),

$$W^R_{\theta;\alpha\beta}\hat{\Sigma} = -\frac{1}{g^2}\int d^4x\,\mathrm{tr}\Big(\theta_{\alpha\rho}\partial^\rho\hat{A}^\sigma\star\Big\{\frac{1}{2}\partial_\beta\hat{A}_\nu,\hat{F}_\sigma{}^\nu\Big\}_\star-\theta_{\beta\rho}\partial^\rho\hat{A}^\sigma\star\Big\{\frac{1}{2}\partial_\alpha\hat{A}_\nu,\hat{F}_\sigma{}^\nu\Big\}_\star\Big)\ .\tag{B.1}$$

Then, (45) and (A.1) yield

$$\begin{aligned}\tilde{W}^R_{\theta;\alpha\beta}\hat{\Sigma} &= \mathrm{rhs(B.1)}+\frac{1}{g^2}\int d^4x\,\mathrm{tr}\Big(\big(\delta^\rho_\alpha\theta_\beta{}^\sigma-\delta^\rho_\beta\theta_\alpha{}^\sigma+\delta^\sigma_\alpha\theta^\rho{}_\beta-\delta^\sigma_\beta\theta^\rho{}_\alpha\big)\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}\star\hat{D}_\kappa\hat{F}^{\kappa\mu}\Big)\\ &= \mathrm{rhs(B.1)}+\frac{2}{g^2}\int d^4x\,\mathrm{tr}\Big(\theta_\alpha{}^\sigma\hat{D}_\kappa\Big(\frac{d\hat{A}_\mu}{d\theta^{\beta\sigma}}\Big)\star\hat{F}^{\kappa\mu}-\theta_\beta{}^\sigma\hat{D}_\kappa\Big(\frac{d\hat{A}_\mu}{d\theta^{\alpha\sigma}}\Big)\star\hat{F}^{\kappa\mu}\Big)\ ,\end{aligned}\tag{B.2}$$

where rhs(B.1) stands for the right hand side of (B.1). Inserting (53), (B.1) and (B.2) into the first condition (51) and splitting the result into the independent parts with coefficients $\theta_{\alpha\rho}/g^2$ and $\theta_{\beta\rho}/g^2$ we find for the first one

$$\begin{aligned}0 &= \int d^4x\,\mathrm{tr}\Big(\hat{F}^{\rho\sigma}\star\hat{T}_{\beta\sigma}-\frac{1}{2}\partial^\rho\hat{A}^\sigma\star\big\{\partial_\beta\hat{A}_\nu,\hat{F}_\sigma{}^\nu\big\}_\star+2g^{\rho\sigma}\hat{D}_\kappa\Big(\frac{d\hat{A}_\mu}{d\theta^{\beta\sigma}}\Big)\star\hat{F}^{\kappa\mu}\Big)\\ &= \int d^4x\,\mathrm{tr}\Big(-\frac{1}{2}\partial^\rho\hat{A}^\sigma\star\big\{\hat{D}_\nu\hat{A}_\beta,\hat{F}_\sigma{}^\nu\big\}_\star-\frac{1}{8}\partial^\rho\hat{A}_\beta\star\big\{\hat{F}_{\mu\nu},\hat{F}^{\mu\nu}\big\}_\star-\frac{1}{2}\hat{D}^\sigma\hat{A}^\rho\star\big\{\hat{F}_{\beta\nu},\hat{F}_\sigma{}^\nu\big\}_\star\\ &\qquad+\frac{1}{8}\hat{D}_\beta\hat{A}^\rho\star\big\{\hat{F}_{\mu\nu},\hat{F}^{\mu\nu}\big\}_\star+2g^{\rho\sigma}\hat{D}_\kappa\Big(\frac{d\hat{A}_\mu}{d\theta^{\beta\sigma}}\Big)\star\hat{F}^{\kappa\mu}\Big)\\ &= \int d^4x\,\mathrm{tr}\Big(g^{\rho\sigma}\Big(-\frac{1}{2}\big\{\partial_\sigma\hat{A}_\mu,\hat{D}_\nu\hat{A}_\beta\big\}_\star-\frac{1}{2}\big\{\hat{D}_\mu\hat{A}_\sigma,\hat{F}_{\beta\nu}\big\}_\star\\ &\qquad-\frac{1}{8}\big\{\hat{F}_{\sigma\beta},\hat{F}_{\mu\nu}\big\}_\star+2\hat{D}_\mu\Big(\frac{d\hat{A}_\nu}{d\theta^{\beta\sigma}}\Big)\Big)\star\hat{F}^{\mu\nu}\Big)\end{aligned}$$

$$
\begin{aligned}
&= \int d^4x \,\mathrm{tr}\Big(g^{\rho\sigma}\Big(\frac{1}{4}\big\{\hat{D}_\mu\hat{A}_\beta, \partial_\sigma\hat{A}_\nu + \hat{F}_{\sigma\nu}\big\}_\star - \frac{1}{4}\big\{\hat{D}_\mu\hat{A}_\sigma, \partial_\beta\hat{A}_\nu + \hat{F}_{\beta\nu}\big\}_\star \\
&\qquad - \frac{1}{8}\big\{\hat{F}_{\sigma\beta}, \hat{F}_{\mu\nu}\big\}_\star + 2\hat{D}_\mu\Big(\frac{d\hat{A}_\nu}{d\theta^{\beta\sigma}}\Big)\Big) \star \hat{F}^{\mu\nu}\Big) \ ,
\end{aligned} \tag{B.3}
$$

where we have used several times cyclicity of the trace, the identity $\hat{F}_{\beta\nu} = \partial_\beta\hat{A}_\nu - \hat{D}_\nu\hat{A}_\beta$ and the antisymmetry of $\hat{F}_{\mu\nu}$. Now we consider

$$
\begin{aligned}
&\int d^4x \,\mathrm{tr}\Big(\big\{\hat{A}_\beta, \hat{D}_\mu(\partial_\sigma\hat{A}_\nu + \hat{F}_{\sigma\nu})\big\}_\star \star \hat{F}^{\mu\nu}\Big) = \int d^4x \,\mathrm{tr}\Big(\big\{\hat{A}_\beta, \hat{D}_\mu\hat{D}_\nu\hat{A}_\sigma + 2\hat{D}_\mu\hat{F}_{\sigma\nu}\big\}_\star \star \hat{F}^{\mu\nu}\Big) \\
&= \int d^4x \,\mathrm{tr}\Big(\big\{\hat{A}_\beta, -\frac{\mathrm{i}}{2}\big[\hat{F}_{\mu\nu}, \hat{A}_\sigma\big]_\star + \hat{D}_\sigma\hat{F}_{\mu\nu}\big\}_\star \star \hat{F}^{\mu\nu}\Big) \\
&= \int d^4x \,\mathrm{tr}\Big(\frac{\mathrm{i}}{4}\big[\hat{A}_\beta, \hat{A}_\sigma\big]_\star \star \big\{\hat{F}_{\mu\nu}, \hat{F}^{\mu\nu}\big\}_\star - \frac{1}{2}\hat{D}_\sigma\hat{A}_\beta \star \big\{\hat{F}_{\mu\nu}, \hat{F}^{\mu\nu}\big\}_\star\Big) \ ,
\end{aligned} \tag{B.4}
$$

where we have used the Bianchi identity and integrated by parts. Antisymmetrizing in β, σ we obtain

$$
\begin{aligned}
&\int d^4x \,\mathrm{tr}\Big(\big\{\hat{A}_\beta, \hat{D}_\mu(\partial_\sigma\hat{A}_\nu + \hat{F}_{\sigma\nu})\big\}_\star \star \hat{F}^{\mu\nu} - \big\{\hat{A}_\sigma, \hat{D}_\mu(\partial_\beta\hat{A}_\nu + \hat{F}_{\beta\nu})\big\}_\star \star \hat{F}^{\mu\nu}\Big) \\
&= \int d^4x \,\mathrm{tr}\Big(-\frac{1}{2}\big\{\hat{F}_{\sigma\beta}, \hat{F}_{\mu\nu}\big\}_\star \star \hat{F}^{\mu\nu}\Big) \ .
\end{aligned} \tag{B.5}
$$

Combining (B.3) and (B.5) we arrive at

$$
0 = \int d^4x \,\mathrm{tr}\Big(\hat{D}_\mu\Big(\frac{1}{4}\big\{\hat{A}_\beta, \partial_\sigma\hat{A}_\nu + \hat{F}_{\sigma\nu}\big\}_\star - \frac{1}{4}\big\{\hat{A}_\sigma, \partial_\beta\hat{A}_\nu + \hat{F}_{\beta\nu}\big\}_\star + 2\frac{d\hat{A}_\nu}{d\theta^{\beta\sigma}}\Big) \star \hat{F}^{\mu\nu}\Big) \ , \tag{B.6}
$$

which leads after reinsertion of $\hat{\Omega}_{\rho\sigma\mu}$ to the Seiberg-Witten differential equation (56).

C The commutator between rotation and total θ-variation

We will prove here eq. (65) in the case of rotation. As usual it is sufficient to evaluate the commutator on $\hat{A}_\mu$ and on $\theta^{\mu\nu}$. The last one is zero because rotation and dilatation of θ commute, see (10). In fact the commutator will vanish for a very general class of differential equations. Let

$$
\theta^{\rho\sigma}\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} = \theta^{\rho\sigma}\Phi_{\rho\sigma\mu} \ , \tag{C.1}
$$

where $\Phi_{\rho\sigma\mu}$ is a polynomial in[5] $\hat{A}$ and θ with power counting dimension 3. We assume that $\Phi_{\rho\sigma\mu}$ transforms as a tensor under rotation

$$
\begin{aligned}
W^R_{\hat{A}+\theta;\alpha\beta}\Phi_{\rho\sigma\mu} &= \frac{1}{2}\{x_\alpha, \partial_\beta\Phi_{\rho\sigma\mu}\}_\star - \frac{1}{2}\{x_\beta, \partial_\alpha\Phi_{\rho\sigma\mu}\}_\star \\
&\quad + g_{\rho\alpha}\Phi_{\beta\sigma\mu} - g_{\rho\beta}\Phi_{\alpha\sigma\mu} + g_{\sigma\alpha}\Phi_{\rho\beta\mu} - g_{\sigma\beta}\Phi_{\rho\alpha\mu} + g_{\mu\alpha}\Phi_{\rho\sigma\beta} - g_{\mu\beta}\Phi_{\rho\sigma\alpha} \ .
\end{aligned} \tag{C.2}
$$

[5] Φ may also depend on the coordinates. In this case however, (C.2) should also involve rotation of the coordinates.

We find

$$\begin{aligned}\Big[W^R_{\hat A+\theta;\alpha\beta},\theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}}\Big]\hat A_\mu &= W^R_{\hat A+\theta;\alpha\beta}\left(\theta^{\rho\sigma}\Phi_{\rho\sigma\mu}\right)\\
&\quad-\theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}}\big(\tfrac{1}{2}\{x_\alpha,\partial_\beta\hat A_\mu\}_\star-\tfrac{1}{2}\{x_\beta,\partial_\alpha\hat A_\mu\}_\star+g_{\alpha\mu}\hat A_\beta-g_{\beta\mu}\hat A_\alpha\big)\\
&=\theta_\alpha{}^\rho\left(\Phi_{\rho\beta\mu}-\Phi_{\beta\rho\mu}\right)-\theta_\beta{}^\rho\left(\Phi_{\rho\alpha\mu}-\Phi_{\alpha\rho\mu}\right)\\
&\quad+\theta^{\rho\sigma}\big(\tfrac{1}{2}\{x_\alpha,\partial_\beta\Phi_{\rho\sigma\mu}\}_\star-\tfrac{1}{2}\{x_\beta,\partial_\alpha\Phi_{\rho\sigma\mu}\}_\star+g_{\rho\alpha}\Phi_{\beta\sigma\mu}-g_{\rho\beta}\Phi_{\alpha\sigma\mu}\\
&\quad+g_{\sigma\alpha}\Phi_{\rho\beta\mu}-g_{\sigma\beta}\Phi_{\rho\alpha\mu}+g_{\mu\alpha}\Phi_{\rho\sigma\beta}-g_{\mu\beta}\Phi_{\rho\sigma\alpha}\big)\\
&\quad-\theta^{\rho\sigma}\left(\tfrac{1}{2}\{x_\alpha,\partial_\beta\Phi_{\rho\sigma\mu}\}_\star-\tfrac{1}{2}\{x_\beta,\partial_\alpha\Phi_{\rho\sigma\mu}\}_\star+g_{\alpha\mu}\Phi_{\rho\sigma\beta}-g_{\beta\mu}\Phi_{\rho\sigma\alpha}\right)\\
&=0\,. \end{aligned}\tag{C.3}$$

Now, one checks that $\frac{d\hat A_\mu}{d\theta^{\rho\sigma}}$ from (56) fulfills (C.2), whereby we have proven (65) for rotation. The proof of (65) in the case of dilatation is performed in a similar manner. The translational proof is immediate.

We stress, however, that (65) by no means singles out the Seiberg-Witten differential equation.

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