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Quantum field theories on noncommutative $\mathbb{R}^4$ versus θ-expanded quantum field theories

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Abstract. I recall the main motivation to study quantum field theories on noncommutative spaces and comment on the most-studied example, the noncommutative $\mathbb{R}^4$. That algebra is given by the $\star$-product which can be written in (at least) two ways: in an integral form or an exponential form. These two forms of the $\star$-product are adapted to different classes of functions, which, when using them to formulate field theory, lead to two versions of quantum field theories on noncommutative $\mathbb{R}^4$. The integral form requires functions of rapid decay and a (preferably smooth) cut-off in the path integral, which therefore should be evaluated by exact renormalisation group methods. The exponential form is adapted to analytic functions with arbitrary behaviour at infinity, so that Feynman graphs can be used to compute the path integral (without cut-off) perturbatively.

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0 Disclaimer

This is not a review. The organisers of the Hesselberg 2002 workshop on "Theory of renormalisation and regularisation", Ryszard Nest, Florian Scheck and Elmar Vogt, asked me to present something about θ-deformed quantum field theories and to prepare some notes for the proceedings. In the following I will present the logic behind and the results of my own work on this subject. Objectivity and completeness are not the aim of this presentation. I have quoted references where I knew of them, statements without citation do not mean that they are new.

These notes go far beyond my presentation at the Hesselberg workshop. They reflect my current point of view[1], the formulation of which evolved through the typing of papers to be found in the hep-th arXiv and the preparation of invited talks for workshops, conferences and invitations in Wien, Nottingham, Jena, Marseille, Leipzig, München, Hesselberg, Leipzig (again), Oberwolfach, Hamburg and Trieste (Wien will follow). I am grateful to the organisers of these events for invitation and hospitality, as well as to my friends for discussions and cooperation.

1 Farewell to manifolds

Half a century of high energy physics has drawn the following picture of the microscopic world: There are matter fields and carriers of interactions between them. Four different types of interactions exist: electromagnetic, weak and strong interactions as well as gravity. The traditional mathematical language to describe these structures of physics is that of fibre bundles. The base manifold M of these bundles is a four-dimensional metric space with line element $ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$. Matter fields ψ are sections of a vector bundle V over M. The carriers of electromagnetic, weak and strong interactions are described by connection one-forms A of $U(1)$, $SU(2)$ and $SU(3)$ principal fibre bundles, respectively. Gravity is the determination of the metric g by the one-forms A and sections ψ, and vice-versa.

The dynamics of (A, ψ, g) is governed by an action functional $\Gamma[A, \psi, g]$, which yields the equations of motions when varied with respect to A, ψ, g. The complete action functional for the phenomenologically most successful model, the standard model of particle physics, is an ugly patchwork of unrelated pieces when expressed in terms of (A, ψ, g).

Next there is a clever calculus, called *quantum field theory*, which as the input takes the action functional Γ and as the output returns *numbers*. On the other hand, there are experiments which also produce numbers. There is a

[1] Changes in v2 are due to an e-mail exchange with Mohammad Sheikh-Jabbari on the different $\star$-products and discussions with Dorothea Bahns and Klaus Fredenhagen who, in particular, convinced me that Minkowskian noncommutative field theories are different.

Changes in v3 go back to very useful comments by Edwin Langmann who explained to me that the two versions of the $\star$-product which in the previous versions were regarded as different products are actually two extensions of *the same product* to different classes of functions.

remarkable agreement[2] of up to 10^{-11} between corresponding numbers calculated by quantum field theory and those coming from experiment. This tells us two things: The action functional (here: of the standard model) is very well chosen and, in particular, quantum field theory is an extraordinarily successful calculus.

There is however, apart from the description of strong interactions at low energy, a tiny problem: one of the basic assumptions of quantum field theory is not realised in nature. First, the metric g is considered in quantum field theory as an external parameter, and—mostly—the calculus works only if the metric is that of Euclidian or Minkowski space, $g_{\mu\nu} = \delta_{\mu\nu}$ or $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, respectively. But let us ignore this and assume for a moment that quantum field theory works on any (pseudo-) Riemannian manifold. Let us then ask how we measure technically the geometry. The building blocks of a manifold are the *points* labelled by coordinates $\{x^\mu\}$ in a given chart. Points enter quantum field theory via the sections $\psi(x)$ and $A(x)$, i.e. the *values* of the fields at the point labelled by $\{x^\mu\}$. This observation provides a way to "visualize" the points: we have to prepare a distribution of matter which is sharply localised around $\{x^\mu\}$. For a perfect visualisation we need a δ-distribution of the matter field. This is physically not possible, but one would think that a δ-distribution could be arbitrarily well approximated. However, that is not the case, there are limits of localisability long before the δ-distribution is reached [1].

Let us assume there is a matter distribution which is believed to have two separated peaks within a space-time region R of diameter d. How do we test this conjecture? We perform a scattering experiment in the hope to find interferences which tell us about the internal structure in the region R. We clearly need test particles of de Broglie wave length $\lambda = \frac{\hbar c}{E} \lesssim \mathrm{d}$, otherwise we observe a single peak even if there is a double peak. For $\lambda \to 0$ the gravitational field of the test particles becomes important. The gravitational field created by an energy E can be measured in terms of the Schwarzschild radius

$$r_s = \frac{2GE}{c^4} = \frac{2G\hbar}{\lambda c^3} \gtrsim \frac{2G\hbar}{\mathrm{d}c^3}\;, \tag{1}$$

where G is Newton's constant. If the Schwarzschild radius r_s becomes larger than the radius $\frac{\mathrm{d}}{2}$, the inner structure of the region R can no longer be resolved (it is behind the horizon). Thus, $\frac{\mathrm{d}}{2} \geq r_s$ leads to the condition

$$\frac{\mathrm{d}}{2} \gtrsim \ell_P := \sqrt{\frac{G\hbar}{c^3}}\;, \tag{2}$$

which means that the Planck length ℓ_P is the fundamental length scale below of which length measurements become meaningless. Space-time cannot be a manifold.

[2] There are of course experimental data which so far could not be reproduced theoretically, such as the energy spectrum of hadrons.

2 Spectral triples

What does this mean for quantum field theory? It means that we cannot trust traditional quantum field theories like the (quantum) standard model because they rely on *non-existing information* about the short-distance structure of physics which determines the loop calculations.

What else can we take for space-time? A lattice? The disadvantage of the lattice is that *symmetries*, which are guiding principles in quantum field theory, are lost. There are also problems with the spin structure. Lattice calculations are regarded as a mathematically rigorous method, but at the end mostly the continuum limit is desired in which the symmetries are intended to be restored.

The lattice approach points into the right direction. A lattice is a metric space but not a differentiable manifold. What we would like to have as candidates for space-time is a class of metric spaces which are equipped with a differential calculus and, additionally, a spin structure to allow for fermions. Such objects exist in mathematics, they are called *spectral triples* [2, 3]. They are *noncommutative geometries* [4, 5] which are the closest generalisation of differentiable spin manifolds. There are good reasons to believe that spectral triples are the right framework for physics.

1. The language in terms of which spectral triples are formulated comes from field theory: Besides the algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$ (which alone are only good for measure theory), to describe *metric spaces with spin structure* one also needs a Dirac operator $\mathcal{D}$, the chirality γ^5 and the charge conjugation J, see [3].

2. The standard model of particle physics looks much simpler when formulated in the language of spectral triples[3]. This is first of all due to the understanding of the Higgs field as a component of a gauge field living on a spectral triple. The $(\phi^4 - m^2\phi^2)$ Higgs potential comes from the same source as the Maxwell Lagrangian $F_{\mu\nu}F^{\mu\nu}$, and the Yukawa coupling of the Higgs with the fermions has the same origin as the minimal coupling of the gauge fields with the fermions. But the connection is much deeper, for instance, the spectral triple description *enforces* the following (in the language of Yang-Mills-Higgs models unrelated) features [6]:
   (i) weak interactions break parity maximally
   (ii) weak interactions suffer spontaneous breakdown
   (iii) strong interactions do not break parity
   (iv) strong interactions do not suffer spontaneous breakdown

3. The separation of gauge fields and gravity starts to disappear: Yang-Mills fields, Higgs fields and gravitons are all regarded as *fluctuations* of the free

[3] In fact the axioms of spectral triples [3] are tailored such that the (Euclidian) standard model is a spectral triple.

Dirac operator [3]. The spectral action

$$\Gamma = \operatorname{trace} \chi\Big(z\frac{\mathcal{D}^2}{\Lambda^2}\Big) , \qquad\qquad \chi(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1 \\ 0 & \text{for } t > 1 \end{cases} \tag{3}$$

(which is the weighted sum of the eigenvalues of $\mathcal{D}^2$ up to the cut-off Λ^2) of the single fluctuated Dirac operator $\mathcal{D}$ gives the complete bosonic action of the standard model, the Einstein-Hilbert action (with cosmological constant) and an additional Weyl action term in one stroke [7]. The parameter z in (3) is the "noncommutative coupling constant" [6]. Assuming the spectral action (3) to produce the bare action at the (grand unification) energy scale Λ, the renormalisation group equation based on the one-loop β-functions leads to a Higgs mass of $182 \dots 201\,\text{GeV}$ [6].

There are of course technical difficulties with spectral triples, such as the restriction to compact spaces with Euclidian signature, but it is clear that spectral triples are a very promising strategy. For attempts to overcome Euclidian signature see [8, 9] and for an extension to non-compact spaces [10].

The strength of the spectral triple approach is that it leads immediately to classical action functionals with a lot of symmetries, even on spaces other than manifolds. We can feed the spectral action functional into our calculus of quantum field theory in order to produce numbers to be compared with experiments. One of the formulations of that calculus, the path integral approach, is perfectly adapted to spectral triples. All one needs are *labels* Φ for the degrees of freedom of the spectral action $\Gamma_{cl}[\Phi]$ in order to write down (at least formally) the measure $\mathcal{D}[\Phi]$ for the (Euclidian) path integral

$$Z[J] = \int \mathcal{D}[\Phi]\, \exp\Big(-\frac{1}{\hbar}\Gamma_{cl}[\Phi] - \big\langle J, \Phi\big\rangle\Big) . \tag{4}$$

The source J is an appropriate element of the dual space of the Φ's. Everything interesting (in a Euclidian quantum field theory) can be computed out of $Z[J]$. It is not important how one labels the degrees of freedom, because $Z[J]$ is invariant under a change of variables [11].

However, we cannot completely exclude the possibility that quantum field theory is implicitly built upon the assumption that the action functional taken as input lives on a manifold. The best way to test whether the standard calculus of quantum field theory extends to spectral triples is to apply it to examples which are *deformations* of a manifold. Let us assume there is a family of spectral triples which are distinguished by a set of parameters θ such that for $\theta \to 0$ we recover an ordinary manifold. Then we should expect that the family of numbers computed out of (4) for any θ tends for $\theta \to 0$ to the numbers computed for the manifold case. Otherwise something is wrong. It is unlikely that the problem (if any) lies in the formula (4) itself, which is very appealing. However, the evaluation of

(4) often involves formal manipulations which may work in one case but fail in another one. We should be careful.

I should mention that the situation is much more difficult in the case of Minkowskian signature of the metric. Apart from the difficulty to extend the definition of spectral triples to geometries with non-Euclidian signature and the mathematical problems of the non-Euclidian path integral, there is evidence now [12] that formal Wick rotation in the Feynman graph computations based on the path integral (4) does not yield the correct theory in the noncommutative case.

Before going to the example let us remind ourselves what the challenge was. We need a replacement for the space-time manifold which is not based on the notion of points. The replacement is expected to be a spectral triple, but in order to compare the outcome with experiments, we have to be sure that the calculus of quantum field theory can be applied. This is why we are interested in spectral geometries other than manifolds on which quantum field theoretical computations are *possible to perform*. We do not claim that our examples are the correct description of the real world.

3 The noncommutative torus

It is time for an example. The simplest noncommutative spectral triple is the noncommutative d-torus, see e.g. [13]. A basis for the algebra $\mathbb{T}^d_\theta$ of the noncommutative d-torus is given by unitarities U^p labelled by $p = \{p_\mu\} \in \mathbb{Z}^d$, with $U^p(U^p)^* = (U^p)^*U^p = 1$. The multiplication is defined by

$$U^pU^q = \mathrm{e}^{\mathrm{i}\pi\theta^{\mu\nu}p_\mu q_\nu}U^{p+q} , \qquad \mu,\nu = 1,\dots,d , \quad \theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{R} . \tag{5}$$

Elements $a \in \mathbb{T}^d_\theta$ have the following form:

$$a = \sum_{p\in\mathbb{Z}^d} a_pU^p , \qquad a_p \in \mathbb{C} , \quad \|p\|^n|a_p| \to 0 \text{ for } \|p\| \to \infty . \tag{6}$$

If $\theta^{\mu\nu} \notin \mathbb{Q}$ (rational numbers) one can define partial derivatives

$$\partial_\mu U^p := -\mathrm{i}p_\mu U^p , \tag{7}$$

which satisfy the Leibniz rule and Stokes' law with respect to the integral

$$\int a = a_0 , \tag{8}$$

where a is given by (6). The algebra $\mathbb{T}^d_\theta$ gives rise to a Hilbert space by GNS construction with respect to (8), and the partial derivatives (7) yield a Dirac operator. Algebra, Hilbert space and Dirac operator extend to a spectral triple satisfying all axioms. For details (and a discussion of the rational case $\theta^{\mu\nu} \in \mathbb{Q}$) see [14]. The noncommutative torus was the first noncommutative space where field theory has been studied [15].

The spectral action (3) for this spectral triple reads

$$\Gamma = \frac{1}{4g^2} \int F_{\mu\nu} F^{\mu\nu} \;, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}(A_\mu A_\nu - A_\nu A_\mu) \;, \tag{9}$$

where $A_\mu = A_\mu^* \in \mathbb{T}^d_\theta$. Then the path integral (4) is evaluated in terms of Feynman graphs, which involve *sums*, not integrals, over the discrete loop momenta. At the one-loop level it is possible to extract the pole parts of these sums via ζ-function techniques [16]. The result is that the quantum field theory associated to the classical action (9) is divergent but (for $d = 4$ dimensions) one-loop renormalisable (divergences are multiplicatively removable and the Ward identities are satisfied) [16].

Everything is perfect so far. Unfortunately, nobody was able to investigate this model at two and more loops, for the simple reason that sums are more difficult to evaluate than integrals. It is dangerous here to approximate the sums by integrals, because the critical question is the behaviour at small p, see below. The most important property of the torus is that the zero mode $p = 0$ decouples, and the next-to-zero modes are "far away" from zero ($p \in \mathbb{Z}^d$). Taking $p \in \mathbb{R}^d$ to deal with integrals, the zero mode decouples as well, but the next-to-zero modes are infinitesimally close to zero. Due to the ease of the computations, much more work has been performed on the non-compact analogue of the noncommutative 4-torus—the noncommutative $\mathbb{R}^4$.

4 The noncommutative $\mathbb{R}^4$

Therefore, let us pass to the noncommutative $\mathbb{R}^4$. The algebra $\mathbb{R}^4_\theta$ is given by the space $\mathcal{S}(\mathbb{R}^4)$ of (piecewise) Schwartz class functions of rapid decay, equipped with the multiplication rule [17]

$$\begin{gathered}(a \star b)(x) = \int \frac{d^4k}{(2\pi)^4} \int d^4y \; a(x{+}\tfrac{1}{2}\theta{\cdot}k)\, b(x{+}y)\, \mathrm{e}^{\mathrm{i}k\cdot y} \;, \\ (\theta{\cdot}k)^\mu = \theta^{\mu\nu} k_\nu \;, \quad k{\cdot}y = k_\mu y^\mu \;, \quad \theta^{\mu\nu} = -\theta^{\nu\mu} \;. \end{gathered} \tag{10}$$

There is again a θ, which however is completely different from the one in (6). The entries $\theta^{\mu\nu}$ in (10) have the *dimension* of an area whereas in (6) they are numbers (for the torus everything can be measured in terms of the radii). There is also no rational situation for the $\mathbb{R}^4_\theta$. Note that the product (10) is associative but noncommutative, and $(a \star b)(x)\big|_{\theta=0} = a(x)b(x)$.

It is interesting to perform a Taylor expansion of (10) about $\theta = 0$:

$$\begin{aligned}(a \star_\omega b)(x) &:= \sum_{n=0}^\infty \frac{1}{n!} \theta^{\alpha_1\beta_1} \cdots \theta^{\alpha_n\beta_n} \Big(\frac{\partial^n (a \star b)(x)}{\partial\theta^{\alpha_1\beta_1} \dots \partial\theta^{\alpha_n\beta_n}} \Big)_{\theta=0} \\ &= \Big(\mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\alpha\beta} \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial z^\beta}} \; a(x{+}y)\, b(x{+}z) \Big)_{z=y=0} \;. \end{aligned} \tag{11}$$

What is the relation between $\star$ and $\star_\omega$? I first thought that they are completely different products. I am grateful to Edwin Langmann for explaining to me that $\star$ and $\star_\omega$ are actually the same products, the point is that the derivatives in (11) are actually generalised derivatives in the sense of distribution theory. There is a class of functions on which $\star$ and $\star_\omega$ (with the derivatives taken literally) coincide, these are analytic functions of rapid decay. Then, depending on how one extends the class of functions to less regular ones, different forms for displaying the product are preferred.

The $\star$-product (10) has excellent smoothing properties, and the multiplier algebra $\mathcal{M}_\theta$, the set of all distributions which when $\star$-multiplied with elements of $\mathbb{R}^4_\theta$ give again elements of $\mathbb{R}^4_\theta$, is very big [17]. The $\star$-product is clearly *non-local*: to the value of $a \star b$ at x there contribute values of a, b at points far away from x. The form (10) is very convenient for piecewise Schwartz class functions, in particular for functions of compact support, as the computation of a two-dimensional example in Appendix A shows. This calculation shows that the $\star$-product has some surprising (at least to me) behaviour at very short distances. Very similar calculations have already been performed in [18], with similar conclusions. The most impressive behaviour is shown in Figure 3. The $\star$-product completely smoothes away the ("extremely-localised" [18]) modes of support within an area θ. It is apparent that θ acts as a cut-off (or a horizon in terms of gravitational physics), a cut-off which preserves all symmetries! Thus, the $\mathbb{R}^4_\theta$ with $\star$-product (10) is an excellent model for space-time.

The focus of the $\star_\omega$-product is a different one. In order to interpret the partial derivatives literally, one has to stay within the class of analytic functions. However, there is no need of rapid decay at infinity. The $\star_\omega$-product is e.g. defined for polynomials of finite degree or for plane waves:

$$\mathrm{e}^{\mathrm{i}p_\mu x^\mu} \star_\omega \mathrm{e}^{\mathrm{i}q_\nu x^\nu} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{\alpha\beta}p_\alpha q_\beta}\, \mathrm{e}^{\mathrm{i}(p_\mu+q_\mu)x^\mu} \;. \tag{12}$$

There is no need to assemble the plane waves to wave packets of rapid decay at infinity. The non-locality of the $\star_\omega$-product (11) is hidden. To the value of $a \star_\omega b$ at x there contribute values of a, b in an infinitesimal neighbourhood of x only, but taking the derivatives literally requires a, b to be analytic, which means that the infinitesimal neighbourhood of x contains all information about a, b on the entire $\mathbb{R}^4$.

5 The geometry of $\mathbb{R}^4_\theta$

In my opinion a thorough investigation of the spectral geometry of $(\mathbb{R}^4_\theta, \mathcal{H}, \mathcal{D})$ must precede any quantum field theoretical computations of models on $\mathbb{R}^4_\theta$. Unfortunately history went differently, so let me explain what we have failed to do so far.

The geometry of $(\mathbb{R}^4_\theta, \mathcal{H}, \mathcal{D})$ cannot be the geometry of a spectral triple [3], because the spectrum of the Dirac operator $\mathcal{D}$ is continuous. It rather fits into the axioms of "non-compact spectral triples" [10]. In this framework the dimension of $(\mathbb{R}^4_\theta, \mathcal{H}, \mathcal{D})$ equals zero, not four[4], because $f \in \mathcal{S}(\mathbb{R}^4)$ is trace-class so that $f|\mathcal{D}|^{-n}$ has vanishing Dixmier trace. Thus, it is the requirement of rapid decay at infinity which brings the dimension down to zero. Taking the $\star_\omega$ product for a suitable class of analytic functions, we can keep the spectral dimension at four. This is related to the notion of star triples in [10].

The geometry is extracted from a spectral triple via *states*—linear functionals $\chi : \mathbb{R}^4_\theta \to \mathbb{C}$ such that[5] $\chi(1) = 1$ and $\chi(a^* \star a) \geq 0$ for all $a \in \mathbb{R}^4_\theta$. We can view such a state as an element of the multiplier algebra $\mathcal{M}_\theta$ through the formula

$$\chi(a) = \int d^4x\, \chi(x) a(x)\;, \quad \int d^4x\, \chi(x) = 1\;, \quad \int d^4x\, \big(\chi \star a^* \star a\big)(x) \geq 0\;. \qquad (13)$$

The space of states is made to a metric space by means of Connes' distance formula

$$\mathrm{dist}(\chi_1, \chi_2) = \sup_{a \in \mathcal{A}_\theta} \Big\{ \big|\chi_1(a) - \chi_2(a)\big| \;:\quad \big\|[\mathcal{D}, a]\big\|_{\mathcal{B}(\mathcal{H})} \leq 1 \Big\}\;. \qquad (14)$$

In the commutative case, for the states labelled by points according to $\chi_y(x) = \delta^4(x - y)$, this formula returns the geodesic distance of the points. For the standard model one recovers a discrete Kaluza-Klein geometry in five dimensions [21].

The states on commutative space suggest immediately to try whether $\chi_y(x) = \delta^4(x - y)$ are states on $\mathbb{R}^4_\theta$ as well. The answer is no[6]. According to Appendix B (which is copied from [17]) for the two-dimensional case, there are functions $a \in \mathbb{R}^2_\theta$ and points $x \in \mathbb{R}^2$ such that $(a^* \star a)(x) < 0$. Moreover, the algebra $\mathbb{R}^2_\theta$ (functions of rapid decay at infinity) is identified with the algebra of matrices of infinite size. The consequence is that field theory on $\mathbb{R}^4_\theta$ is rather a matrix theory than a traditional field theory on Euclidian space, see [22]. I would like to mention that the matricial basis was crucial for Langmann's class of exactly solvable quantum field theories in odd dimensions [23].

Moreover, Figure 3 in Appendix A tells us that the product of two fields, both of which with support in the interior of the area θ, is, to a high accuracy, zero. All this means that $\mathbb{R}^2_\theta$ is divided into cells of area θ, and fields on $\mathbb{R}^2_\theta$ are characterised by assigning to each cell a value. This picture is easily generalised to $\mathbb{R}^4_\theta$. Now, since the interaction of fields on $\mathbb{R}^4_\theta$ is smeared over the cell of size $|\det\theta|^{\frac{1}{2}}$, one would expect that a quantum field theory on $\mathbb{R}^4_\theta$ is free of divergences [20]. Performing the calculation of Feynman graphs, however, one does encounter divergences, and these are *worse* than on commutative space-time. See sec. 7.

[4]The spectral triple for the noncommutative 4-torus has dimension four, not zero! The Hochschild dimension of $\mathbb{R}^4_\theta$ drops down to zero as well [19].

[5]The 1 in $\chi(1)$ is thought to be the limit of a sequence of appropriate elements of $\mathbb{R}^4_\theta$.

[6]As a consequence, $\mathbb{R}^d_\theta$ is not the algebra of functions on some manifold.

How can we understand this puzzle? The Feynman rules are nothing but the perturbative evaluation of the path integral (4). It seems that this kind of evaluation of (4) somehow brings the dimension of $\mathbb{R}^d_\theta$ back to d. The crucial question here is the definition of the measure $\mathcal{D}[\phi]$ in the path integral. The idea is to integrate over all possible fields on $\mathbb{R}^d_\theta$. This is most conveniently done by taking a basis, for instance the matricial basis f_{mn} of Appendix B in the two-dimensional case:

$$\phi(x) = \sum_{m,n=0}^{\infty} \phi_{mn} f_{mn}(x) \;. \tag{15}$$

Thus, the measure should be $\mathcal{D}[\phi] = \prod_{m,n=0}^{\infty} d\phi_{mn}$. Next we have to specify the domain of integration. The Feynman rules correspond to integrating *all* ϕ_{mn} from $-\infty$ to ∞. Obviously this is not compatible with the requirement that the $\{\phi_{mn}\}$ represent an element of $\mathbb{R}^2_\theta$, which imposes [17]

$$\sum_{m,n=0}^{\infty} \big((2m{+}1)^{2k}(2n{+}1)^{2k}\,|\phi_{mn}|^2\big)^{\frac{1}{2}} < \infty \qquad \text{for all } k \;. \tag{16}$$

Integrating all ϕ_{mn} from $-\infty$ to ∞ we actually include functions with unrestricted behaviour at infinity, and those functions lead to a spectral dimension bigger than zero. In other words, the discussion of the geometry of $\mathbb{R}^4_\theta$ tells us that the usual Feynman rules are not adequate[7] to quantum field theory on $\mathbb{R}^4_\theta$. It is neither surprising nor a problem that the standard Feynman graph approach to quantum field theories on $\mathbb{R}^4_\theta$ fails miserably (see also the next sections).

A possibility to stay within the class of fields of rapid decay at infinity is to introduce a cut-off in the path integral measure, e.g. $\mathcal{D}[\phi] = \prod_{m,n=0}^{L} d\phi_{mn}$. A different view of that cut-off is to regard all modes ϕ_{mn} as included, but with the integral performed over the interval $[0,0]$ instead of $[-\infty,\infty]$ if $m > L$ or $n > L$. This prescription can then be deformed into a smooth cut-off with all ϕ_{mn} included, but with $|\phi_{mn}|$ being of rapid decay at infinity. In this way we integrate indeed over fields on $\mathbb{R}^d_\theta$. The cut-off can be arbitrarily large and the cut-off function arbitrarily chosen; we stay within the allowed class of fields as long as there is a cut-off somewhere when approaching infinity.

Remarkably, this smooth cut-off version of the path integral is exactly the way one proceeds in the exact renormalisation group approach to renormalisation [24], see also [25]. We see thus that the smooth cut-off in the exact renormalisation group is not just a convenient trick to compute the path integral, it is the direct consequence of the zero-dimensional geometry[8] of $\mathbb{R}^d_\theta$. The philosophy

[7]The same criticism applies to canonical quantisation because the amplitudes cannot be promoted to harmonic oscillators at quantum numbers approaching infinity.

[8]In that sense, the renormalisation group approach for commutative field theories is linked to fields of rapid decay in momentum space. It seems to be a degeneracy of commutative geoemtry that this restriction leads to the same results as the Feynman graph appraoch.

is then to redefine the theory in such a way that—once specifying normalisation conditions—everything becomes independent of the cut-off function and the value of the cut-off. There are first results [26] that at least scalar field theories on $\mathbb{R}^4_\theta$ are renormalisable within the exact renormalisation group approach.

6 The Feynman graph approach to quantum field theories on $\mathbb{R}^4_\theta$

We have argued in the last section that the Feynman graph approach to quantum field theories on $\mathbb{R}^4_\theta$ does not correctly reflect the geometry of $\mathbb{R}^4_\theta$. There has been, however, an enormous amount of work along this line, which deserves a few comments. We need a notation to distinguish the Feynman graph approach from the true zero-dimensional $\mathbb{R}^4_\theta$. Let us call the four-dimensional space where Feynman graph computations are performed $\mathbb{R}^4_{nc}$.

The first contribution was the one-loop investigation of U(1) Yang-Mills theory on $\mathbb{R}^4_{nc}$ by Martín and Sánchez-Ruiz [27]. They found that all one-loop pole terms of this model in dimensional regularisation[9] can be removed by multiplicative renormalisation (minimal subtraction) in a way preserving the BRST symmetry. This is completely analogous to the situation on the noncommutative 4-torus [16]. Shortly later there appeared also an investigation of super-Yang-Mills theory on $C^\infty(\mathbb{R}) \times \mathbb{T}^2_\theta$ [28]. In the following two years similar one-loop calculations were performed for the $\mathbb{R}^d_{nc}$-analogue of any existing commutative model.

The reason why these models became so attractive was the completely unexpected discovery of quadratic infrared-like divergences, first in quantum field theories of scalar fields [29], which ruled out a perturbative renormalisation at higher loop order. At that time it was an open question whether this is an artifact of scalar fields or really a general feature. We have shown in [30] using power-law estimations for Bessel functions that the sub-sector of Yang-Mills theory on $\mathbb{R}^4_{nc}$ given by repeated one-loop ghost propagator self-insertions is renormalisable to any loop order. Shortly later it was demonstrated, however, that there are one-loop Green's functions in Yang-Mills theory on $\mathbb{R}^4_{nc}$ which have quadratic and linear infrared-like divergences [31], which prevent any renormalisation beyond one-loop.

In my opinion the most valuable contribution to field theories on $\mathbb{R}^d_{nc}$ are the two articles [32, 33] by Chepelev and Roiban, in which they investigated the convergence behaviour of massive quantum field theories *at any loop order*. The essential technique is the representation of Feynman graphs as ribbon graphs, drawn on an (oriented) Riemann surface with boundary, to which the external legs of the graph are attached. There are two important qualifiers for such a ribbon graph, the index and the cycle number. The index is declared to be one if the external lines attach to boundary components "inside" and "outside" of

[9] There is of course a problem extending θ to complex dimensions, this is however discussed in [27].

the graph, otherwise zero. The cycle number is the number of homologically non-trivial cycles of the Riemann surface of the total graph wrapped by the (sub)graph. Using this language and sophisticated tricks for the manipulation of determinants, Chepelev and Roiban were able to prove that in order to have convergence of the integral, each subgraph must have one of the following properties:

1. The index is one and the external momenta are non-exceptional.

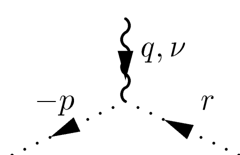
2. If the index is zero or the index is one but the external momenta are non-exceptional, then the power-counting degree of divergence of the graph is smaller than the dimension d times the number of cycles.

Thus, noncommutativity (index one and presence of cycles due to non-planarity) improves the convergence of the integral. Integrals associated to planar sectors are to be renormalised as in commutative quantum field theories, they are not a problem. One has to make sure however, that there are no divergences in non-planar sectors. It turned out that there are two dangerous classes of non-planar divergences, which in [33] are called "Rings" and "Com". Rings consists of a chain of divergent graphs stacked onto the same cycle, they induce the problem first observed in [29]. Com's are index-one graphs with exceptional external momenta due to momentum conservation, they correspond to non-local divergences of the type $(\int \phi \star \phi)^2$. In massless models they are catastrophic. Unfortunately this problem seems to be completely ignored in literature.

7 The non-locality of the divergences

All this is well-known by now and can be looked up in the literature, nevertheless I would like to demonstrate the problem with quantum field theories on $\mathbb{R}^4_{nc}$ by computing the ghost loop contribution to the one-loop gluon two-point function. The necessary Feynman rules adapted to the (Euclidian) BPHZL renormalisation scheme [34] are given by

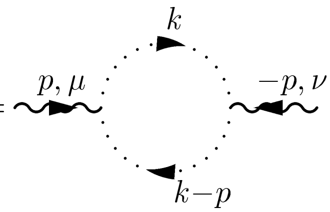
$$= -\frac{1}{p^2 + (s-1)^2 M^2} \;, \tag{17}$$



$$= 2\mathrm{i}\, p^\nu \sin(\tfrac{1}{2}\theta^{\alpha\beta} q_\alpha r_\beta) \;, \tag{18}$$

for ghost propagator and ghost-gluon vertex, respectively. One has $p+q=0$ in (17) and $p+q+r=0$ in (18). We then compute the graph

$$\gamma^{\mu\nu}(p,s) =$$



$$= \hbar \int \frac{d^4k}{(2\pi)^4} \, \frac{k^\mu (k-p)^\nu \Big\{ 2 - \mathrm{e}^{\mathrm{i}\theta^{\alpha\beta} p_\alpha k_\beta} - \mathrm{e}^{-\mathrm{i}\theta^{\alpha\beta} p_\alpha k_\beta} \Big\}}{(k^2 + (s-1)^2 M^2)((k-p)^2 + (s-1)^2 M^2)} \,. \tag{19}$$

The integral as it stands in (19) is meaningless. We have to define a renormalisation scheme which assigns to the graph in (19) a meaningful integral. Here one has to distinguish between the planar part corresponding to the factor 2 in in { } and the non-planar part corresponding to the phase factors in { }. Let us first look at the planar part. The integral is quadratically divergent, and according to the BPHZL scheme we replace the integrand $I^{\mu\nu}(k;p,s)$ by the Taylor subtracted integrand

$$(1-R)[I^{\mu\nu}(k;p,s)]\Big|_{s=1} := (1 - t^1_{p,s-1})(1 - t^2_{p,s})[I^{\mu\nu}(k;p,s)]\Big|_{s=1} \,, \tag{20}$$

where $t^\omega_{p,s'}[I]$ is the Taylor expansion of I about $p=0$ and $s'=0$ up to total degree ω. In the example (19) we have for the integrand without the factor 2 in { }

$$\begin{aligned} R[I^{\mu\nu}(k;p,s)] &= \frac{k^\mu k^\nu}{(k^2)^2} + p_\rho \Big(- \frac{k^\mu g^{\nu\rho}}{(k^2)^2} + \frac{2k^\mu k^\nu k^\rho}{(k^2)^3} \Big) \\ &\quad + p_\rho p_\sigma \Big(- \frac{2k^\mu k^\sigma g^{\nu\rho}}{(k^2+M^2)^3} + \frac{4k^\mu k^\nu k^\rho k^\sigma}{(k^2+M^2)^4} \Big) \\ &\quad + (s-1)^2 \Big(\frac{2M^2 k^\mu k^\nu}{(k^2+M^2)^3} + \frac{12 M^4 k^\mu k^\nu}{(k^2+M^2)^4} \Big) \\ &\quad + p_\rho (s-1) \Big(\frac{4M^2 k^\mu g^{\nu\rho}}{(k^2+M^2)^3} - \frac{12 M^2 k^\mu k^\nu k^\rho}{(k^2+M^2)^4} \Big) \,. \end{aligned} \tag{21}$$

Passing to $s=1$ and omitting the integrand which is odd under $k \to -k$, we now get for the planar part in (19)

$$\begin{aligned} \gamma^{\mu\nu}_{planar,ren}(p,M) = 2\hbar \int \frac{d^4k}{(2\pi)^4} \Big(& \frac{k^\mu (k-p)^\nu}{k^2 (k-p)^2} - \frac{k^\mu k^\nu}{(k^2)^2} \\ & - p_\rho p_\sigma \Big(- \frac{2k^\mu k^\sigma g^{\nu\rho}}{(k^2+M^2)^3} + \frac{4k^\mu k^\nu k^\rho k^\sigma}{(k^2+M^2)^4} \Big) \Big) \,. \end{aligned} \tag{22}$$

The integral (22) is absolutely convergent, see [34].

Let us now compute the difference between $\gamma^{\mu\nu}_{planar}$ and $\gamma^{\mu\nu}_{planar,ren}$ in position space:

$$\begin{aligned}
&\int \frac{d^4p}{(2\pi)^4}\Big(\gamma^{\mu\nu}_{planar}(p) - \gamma^{\mu\nu}_{planar,ren}(p,M)\Big)\,\mathrm{e}^{-\mathrm{i}p_\lambda(x-y)^\lambda} \\
&= 2\hbar\,\delta^4(x-y)\int\frac{d^4k}{(2\pi)^4}\,\frac{k^\mu k^\nu}{(k^2)^2} \\
&+ 2\hbar\,\frac{\partial^2\delta^4(x-y)}{\partial x^\rho\partial x^\sigma}\int\frac{d^4k}{(2\pi)^4}\Big(\frac{2k^\mu k^\sigma g^{\nu\rho}}{(k^2+M^2)^3} - \frac{4k^\mu k^\nu k^\rho k^\sigma}{(k^2+M^2)^4}\Big)\,. \qquad (23)
\end{aligned}$$

The result is zero unless $x = y$. We recall now that the Fourier transformation of (22) is the ghost loop contribution to the gluon two-point correlation function $\big\langle A^\mu(x)A^\nu(y)\big\rangle$. In other words, replacing the meaningless integral (19) by the renormalised one (22), we have only redefined (in fact correctly defined) the product of the distributions $A^\mu(x)$ and $A^\nu(y)$ at coinciding points. This is precisely the freedom which one has in a local quantum field theory [35].

But what about the non-planar part? Although not being absolutely convergent, the oscillating phase (see also Figure 3 in Appendix A) renders the integral actually convergent—provided that $p \neq 0$. Thus the first possibility is to keep the non-planar part untouched in the renormalisation scheme. But now there is a problem for $p \to 0$. Note that the original (ill-defined) integral (19) had no problem at all for $p \to 0$, in fact the integral was zero for $p = 0$. But since we removed from the planar part its first Taylor coefficients about $p = 0$ in order to render the planar part integrable for $k \to \infty$, the singular behaviour of the non-planar part for $p \to 0$ is no longer compensated. For the one-loop graph it is not a terrible problem[10], but inserting this result declared as finite as a subgraph into a bigger divergent graph, the singular behaviour at $p \to 0$ makes the bigger graph non-integrable. We therefore find a fake infrared divergence, which is only due to our (obviously wrong) renormalisation prescription which treated the planar and non-planar parts *differently*. This is the so-called UV/IR-mixing, a name which is not very appropriate.

Since the above treatment of the non-planar part was unsuccessful, let us also remove the first Taylor coefficients about $p = 0$ from the non-planar part. This Taylor expansion must not be applied to the momenta in the phases, because the result would be an even worser divergence in k and not a milder one. The only possibility is to define the renormalised total graph as

$$\begin{aligned}
\gamma^{\mu\nu}_{ren}(p,M) = \hbar\int\frac{d^4k}{(2\pi)^4}\Big\{2 - \mathrm{e}^{\mathrm{i}\theta^{\alpha\beta}p_\alpha k_\beta} - \mathrm{e}^{-\mathrm{i}\theta^{\alpha\beta}p_\alpha k_\beta}\Big\}\Big(\frac{k^\mu(k-p)^\nu}{k^2(k-p)^2} - \frac{k^\mu k^\nu}{(k^2)^2} \\
- p_\rho p_\sigma\Big(-\frac{2g^{\nu\rho}k^\mu k^\sigma}{(k^2+M^2)^3} + \frac{4k^\mu k^\nu k^\rho k^\sigma}{(k^2+M^2)^4}\Big)\Big)\,. \qquad (24)
\end{aligned}$$

[10] As long as one is not interested in producing numbers to be compared with experiments!

Now the integral converges absolutely, in particular there is no problem any more for $p \to 0$. We have to verify, however, that the change from $\gamma^{\mu\nu}(p)$ to $\gamma^{\mu\nu}_{ren}(p, M)$ is compatible with locality. In the planar part this change amounts to a redefinition of the product of distributions at coinciding points. Let us thus evaluate the change in the non-planar part, again in position space:

$$
\begin{aligned}
&\int \frac{d^4p}{(2\pi)^4}\Big(\gamma^{\mu\nu}_{non-planar}(p) - \gamma^{\mu\nu}_{non-planar,ren}(p,M)\Big)\,\mathrm{e}^{-\mathrm{i}p_\lambda(x-y)^\lambda} \\
&= -\hbar\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\Big(\frac{k^\mu k^\nu}{(k^2)^2} - p_\rho p_\sigma\Big(\frac{2g^{\nu\rho}k^\mu k^\sigma}{(k^2+M^2)^3} - \frac{4k^\mu k^\nu k^\rho k^\sigma}{(k^2+M^2)^4}\Big)\Big) \\
&\qquad\qquad\qquad\qquad \times\Big\{\mathrm{e}^{-\mathrm{i}p_\alpha((x-y)^\alpha+\theta^{\alpha\beta}k_\beta)} + \mathrm{e}^{-\mathrm{i}p_\alpha((x-y)^\alpha-\theta^{\alpha\beta}k_\beta)}\Big\} \\
&= -\frac{2\hbar}{(2\pi)^4\,\det\theta}\bigg(\frac{(\theta^{-1}{\cdot}(x{-}y))^\mu(\theta^{-1}{\cdot}(x{-}y))^\nu}{((\theta^{-1}{\cdot}(x{-}y))^2)^2} + 2\frac{(\theta^{-1})^{\mu\sigma}(\theta^{-1})_\sigma^{\ \nu}}{((\theta^{-1}{\cdot}(x{-}y))^2+M^2)^3} \\
&\quad + 4\frac{(\theta^{-2}{\cdot}(x{-}y))^\mu(\theta^{-2}{\cdot}(x{-}y))^\nu}{((\theta^{-1}{\cdot}(x{-}y))^2+M^2)^4} + 4\frac{(\theta^{-1})_{\alpha\beta}(\theta^{-1})^{\alpha\beta}(\theta^{-1}{\cdot}(x{-}y))^\mu(\theta^{-1}{\cdot}(x{-}y))^\nu}{((\theta^{-1}{\cdot}(x{-}y))^2+M^2)^4} \\
&\quad - 8\frac{(\theta^{-3}{\cdot}(x{-}y))^\mu(\theta^{-1}{\cdot}(x{-}y))^\nu}{((\theta^{-1}{\cdot}(x{-}y))^2+M^2)^4} - 8\frac{(\theta^{-1}{\cdot}(x{-}y))^\mu(\theta^{-3}{\cdot}(x{-}y))^\nu}{((\theta^{-1}{\cdot}(x{-}y))^2+M^2)^4} \\
&\quad + 32\frac{(\theta^{-1}{\cdot}(x{-}y))^\mu(\theta^{-1}{\cdot}(x{-}y))^\nu\ (\theta^{-2}{\cdot}(x{-}y))^\rho(\theta^{-2}{\cdot}(x{-}y))_\rho}{((\theta^{-1}{\cdot}(x{-}y))^2+M^2)^5}\bigg).
\end{aligned}
\tag{25}
$$

There are contributions for $x \neq y$ in the non-planar part. In other words, we have changed the non-planar part in a *non-local* way in order to achieve absolute convergence. This is not allowed in a local quantum field theory, which means that our model on $\mathbb{R}^4_{nc}$ is not renormalisable in the framework of local quantum field theories.

On the other hand, the result (25) is exactly what one should expect for a quantum field theory on $\mathbb{R}^4_\theta$: Since the physical information cannot be localised at individual points it must now be allowed to modify the product of distributions not only at coinciding points but for the whole extended region of volume $|\det\theta|^{\frac{1}{2}}$ in which information can be concentrated. Unfortunately, this idea is not very well implemented in the above calculation. The subtraction term is too much localised in the planar part is not enough localised in the non-planar part. In my opinion, the origin of this problem is the wrong choice of the measure in the path integral which is used to derive the Feynman rules, see sec 5.

8 θ-expanded field theories: general remarks

Let us now come to quantum field theories based on the Taylor-expanded $\star_\omega$-product (11) regarded order by order in θ. The philosophy here is to consider the Taylor expansion $\star_\omega$ up to some finite order in θ only. In this way we obtain a *local* field theory on ordinary Euclidian or Minkowski space for which standard

Feynman graph techniques can safely be applied. The only novelty is the presence of external fields $\theta^{\alpha\beta}$ of power-counting dimension -2 which couple to the commutative fields via partial derivatives. When restricting the product

$$
\begin{aligned}
(\phi_1 \star_\omega \phi_2)(x) &= \phi_1(x)\,\phi_2(x) \\
&\qquad + \frac{\mathrm{i}}{2}\theta^{\alpha\beta}\partial_\alpha\phi_1(x)\,\partial_\beta\phi_2(x) - \frac{1}{8}\theta^{\alpha\beta}\theta^{\gamma\delta}\partial_\alpha\partial_\delta\phi_1(x)\,\partial_\beta\partial_\gamma\phi_2(x) + \dots
\end{aligned} \tag{26}
$$

to some finite order, nothing is noncommutative, the second term on the r.h.s. can equally well be written as $\frac{\mathrm{i}}{2}\theta^{\alpha\beta}\partial_\beta\phi_2(x)\,\partial_\alpha\phi_1(x)$.

The most interesting field theories are gauge theories[11]. The prototype is Maxwell theory, the action functional of which, written in terms of the $\star_\omega$-product, reads

$$
\Gamma[A] = \int d^4x\,\Big(- \frac{1}{4g^2}F_{\mu\nu}(x)\star_\omega F^{\mu\nu}(x)\Big)\;, \tag{27}
$$

$$
\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\mu A_\nu - \mathrm{i}A_\mu \star_\omega A_\nu + \mathrm{i}A_\nu \star_\omega A_\mu \\
&= \partial_\mu A_\nu - \partial_\mu A_\nu + \theta^{\alpha\beta}\partial_\alpha A_\mu\,\partial_\beta A_\nu - \frac{1}{24}\theta^{\alpha\beta}\theta^{\gamma\delta}\theta^{\epsilon\zeta}\partial_\alpha\partial_\gamma\partial_\epsilon A_\mu\,\partial_\beta\partial_\delta\partial_\zeta A_\nu + \dots\;.
\end{aligned} \tag{28}
$$

Now, (27) is an action functional for commutative boring photons, which is invariant under the infinitesimal gauge transformation

$$
\begin{aligned}
A_\mu &\mapsto A_\mu + (\partial_\mu\lambda - \mathrm{i}A_\mu \star_\omega \lambda + \mathrm{i}\lambda \star_\omega A_\mu) \\
&= A_\mu + \partial_\mu\lambda + \theta^{\alpha\beta}\partial_\alpha A_\mu\,\partial_\beta\lambda - \frac{1}{24}\theta^{\alpha\beta}\theta^{\gamma\delta}\theta^{\epsilon\zeta}\partial_\alpha\partial_\gamma\partial_\epsilon A_\mu\,\partial_\beta\partial_\delta\partial_\zeta\lambda + \dots\;.
\end{aligned} \tag{29}
$$

But how is this possible, an action functional for photons which transform in a very strange way? The answer was given by Seiberg and Witten [36]: The photon is written in (27) and (29) only in an extremely inconvenient way. There is a change of variables

$$
\begin{aligned}
A_\mu &= A'_\mu - \frac{1}{2}\theta^{\alpha\beta}A'_\alpha(2\partial_\beta A'_\mu - \partial_\mu A'_\beta) + \dots\;, \\
\lambda &= \lambda' - \frac{1}{2}\theta^{\alpha\beta}A'_\alpha\partial_\beta\lambda' + \dots\;,
\end{aligned} \tag{30}
$$

which brings (27) and (29) into the more pleasant form

$$
\begin{aligned}
\Gamma[A'] = \int d^4x\,\Big(&- \frac{1}{4g^2}F'_{\mu\nu}(x)F'^{\mu\nu}(x) \\
&- \frac{1}{2g^2}\theta^{\alpha\beta}F'_{\alpha\mu}(x)F'_{\beta\nu}(x)F'^{\mu\nu}(x) + \frac{1}{8g^2}\theta^{\alpha\beta}F'_{\alpha\beta}(x)F'_{\mu\nu}(x)F'^{\mu\nu}(x) + \dots\Big)\;, \\
F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu\;, &\qquad \Gamma[A'] \text{ invariant under } A'_\mu \mapsto A'_\mu + \partial_\mu\lambda'\;.
\end{aligned} \tag{31}
$$

[11] To the best of our knowledge, there are no fundamental scalar fields in nature—remember that the Higgs field is a noncommutative gauge field, and that supersymmetry is not found so far.

The last line in (31) is exact in θ, it looks much more familiar. Actually Seiberg and Witten formulated their result differently. They interpreted the transformation (30) leading from (27) to (31) as an equivalence between a noncommutative gauge theory and a commutative gauge theory. Now there is a puzzle. Namely, from the noncommutative geometrical background, the noncommutative field theory is given by a spectral triple which can never be expressed in the language of manifolds. How can there be a map to a commutative field theory? The solution is simple, but it took me a long time to understand it: The initial formulation (27) was already in the framework of commutative local geometry, because already there the $\star_\omega$ product was restricted to some finite order in θ. The transformation (30) is merely a *convenient change of variables within the same commutative framework*. The (very difficult) limit where the order of θ goes to infinity is not discussed in this approach.

9 Lorentz invariance and Seiberg-Witten differential equation

One may ask whether the Taylor expansion (11) leading from the non-local $\star$-product to the local $\star_\omega$-product up to finite order in θ, applied to a truly noncommutative action functional $\Gamma[\hat{A}]$, can produce the θ-expanded action functional in the Seiberg-Witten transformed form (31) in a single stroke, i.e. without passing through (27). This is possible indeed, it has something to do with symmetry transformations of the noncommutative theory.

There has been a lot of confusion concerning the question of Lorentz invariance of field theories on $\mathbb{R}^4_\theta$. Once and for all, symmetries in the noncommutative world are automorphisms of the algebra [3]. The algebra $\mathbb{R}^4_\theta$ is determined by θ and the question is how θ is characterised. We follow [1] and agree that θ is characterised by the two Lorentz invariants $\theta^{\mu\nu}\theta_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}$ when discarding dilatation and by the ratio of these two when including dilatation. The individual components $\theta^{\mu\nu}$ (with respect to a given basis) do not have a physical meaning. The algebra is $\mathbb{R}^4_\theta$, not $\mathbb{R}^4_{\theta^{\mu\nu}}$.

Let us be more explicit. Infinitesimal field transformations are implemented by Ward identity operators

$$W = \sum_i \Big\langle \delta\hat{\Phi}_i[\hat{\Phi}_k], \frac{\delta}{\delta\hat{\Phi}_i}\Big\rangle\,, \tag{32}$$

where the index i labels the different sorts of fields, here denoted $\hat{\Phi}_i$. The Ward identity operator (32) acts on (sufficiently regular) functionals $\Gamma[\hat{\Phi}_i]$ in a derivational manner:

$$W\Gamma[\hat{\Phi}_i] = \sum_j \Big\langle \delta\hat{\Phi}_j[\hat{\Phi}_k], \frac{\delta\Gamma[\hat{\Phi}_i]}{\delta\hat{\Phi}_j}\Big\rangle = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\Gamma[\hat{\Phi}_i + \epsilon\delta\hat{\Phi}_i[\hat{\Phi}_k]] - \Gamma[\hat{\Phi}_i]\Big)\,. \tag{33}$$

We are interested in a set $\mathcal{S}$ of symmetry transformations of the action functional, $W^I\Gamma = 0$, $I \in \mathcal{S}$. This set is required to be complete, $[W^I, W^{I'}] =$

$\sum_n W^{I_n}$, $I_n \in \mathcal{S}$. In particular, we are interested in gauge transformation G and Lorentz transformation L which satisfy

$$[W^L, W^L] \subset W^L\;, \qquad [W^G, W^G] \subset W^G\;, \qquad [W^G, W^L] \subset W^G\;. \tag{34}$$

The Lorentz transformation has for the field $\hat{A}$ of Yang-Mills theory on $\mathbb{R}^4_\theta$ the symbolic form

$$W^L = \Big\langle \delta^L \hat{A}, \frac{\delta}{\delta \hat{A}} \Big\rangle + \Big\langle \delta^L \theta, \frac{\delta}{\delta \theta} \Big\rangle\;, \tag{35}$$

it is a symmetry of the Yang-Mills action functional, and (34) is satisfied [37]. It is essential that in (35) the sum of the $\hat{A}$ and the θ-transformation appears, the individual transformations do not have any meaning. Neither they are symmetries of the action functional, nor they fulfil (34). But if one really insists on transforming $\hat{A}$ only, then at least this transformation $\tilde{W}^L_{\hat{A}}$, which cannot be a symmetry of the action functional, must satisfy

$$[\tilde{W}^L_{\hat{A}}, W^G] \subset W^G\;. \tag{36}$$

The condition (36) guarantees that $\tilde{W}^L_{\hat{A}} \Gamma[\hat{A}] \neq 0$, which can be regarded as the particle Lorentz symmetry breaking, is a gauge-invariant quantity [37]. Otherwise $\tilde{W}^L_{\hat{A}}$ is completely unphysical. It is then somehow natural to make the ansatz

$$\begin{aligned} &W^L = \tilde{W}^L_{\hat{A}} + \tilde{W}^L_\theta\;, \\ &\tilde{W}^L_{\hat{A}} = \Big\langle \delta^L \hat{A} - \delta^L \theta\, \frac{d\hat{A}}{d\theta}, \frac{\delta}{\delta \hat{A}} \Big\rangle\;, \qquad \tilde{W}^L_\theta = \Big\langle \delta^L \theta, \frac{\delta}{\delta \theta} \Big\rangle + \Big\langle \delta^L \theta\, \frac{d\hat{A}}{d\theta}, \frac{\delta}{\delta \hat{A}} \Big\rangle\;, \end{aligned} \tag{37}$$

where $\frac{d\hat{A}}{d\theta}$ is, for the time being, just a symbol. The condition (36) determines $\frac{d\hat{A}}{d\theta}[\hat{A}]$, which thus becomes a concrete (but not unique) function of $\hat{A}$. The equation $\frac{d\hat{A}}{d\theta} = \frac{d\hat{A}}{d\theta}[\hat{A}]$ looks formally like a differential equation—the Seiberg-Witten differential equation. Now we can define the following Taylor expansion of the action functional $\Gamma[\hat{A}]$:

$$\Gamma^{(n)}[A] := \sum_{j=0}^{n} \frac{1}{j!} (\theta)^j \Big(\big(\tilde{W}^1_\theta\big)^j \Gamma[\hat{A}] \Big)_{\theta=0}\;, \qquad \delta^1 \theta := 1\;, \qquad A := \big(\hat{A}\big)_{\theta=0}\;. \tag{38}$$

By construction, the action functional $\Gamma^{(n)}[A]$ describes a commutative Yang-Mills theory (coupled to the external field θ) which is invariant under commutative gauge and Lorentz transformations at any cut-off order n in θ, see [37]. We have thus obtained (31) up to any desired order in a single stroke.

10 Quantisation of θ-expanded field theories

From a physical point of view, θ-expanded quantum field theories are not so interesting, because they are local and therefore show all the the problems discussed in sec. 1. They have a very interesting structure, though, because the appearance of a field θ of power-counting dimension -2 makes them power-counting *non-renormalisable*. It could seem, therefore, that it is not very useful to study such a model as a quantum field theory. However, at the same time where θ leads to an explosion of the number of divergences, it also provides the means to absorb a considerable fraction of these divergences through field redefinitions. A field redefinition is a non-linear generalisation of the usual wave function renormalisation, a generalisation which is possible precisely because there is a field of negative power-counting dimension. And there could be symmetries in the θ-expanded action which would prevent the appearance of other divergences. There is thus a race between the number of divergences created by θ and the number of divergences absorbable by (unphysical) field redefinitions or avoided by symmetries.

The winner is probably the creator of divergences, but this is a conjecture only. In this case, although there is at any given order n in θ a finite number of new interaction terms only, the theory looses all predictability in the limit $n \to \infty$. There are however signs for hope. First, all superficial divergences in the photon self-energy in θ-expanded Maxwell theory are field redefinitions, to all order n in θ and any loop order [38]. For the photon self-energy the field redefinitions win the race.

A direct search for symmetries was not successful so far so that the only chance to detect them is to perform some loop calculations. Due to the extremely rich tensorial structure in presence of θ, these calculations are extremely difficult to perform, even for the one-loop photon self-energy in θ-deformed Maxwell theory to second order in θ [39]. The photon three-point function which is of at least third order in θ is already beyond the means.

The simplest model to study other Green's functions than the self-energy is θ-deformed QED. I have computed in [40] all divergent one-loop Green's functions up to first order in θ. The result was astonishing. Although not renormalisable at the considered order, there was in the massless case only a single divergence more than those absorbable by field redefinitions, where four exceeding divergent terms were to expect. In the massive case (where the mass term is inserted directly into the Dirac action) things become really bad so that this work suggests that fermion masses should be introduced via a Higgs mechanism.

The results of [40] provide a very strong signal that new symmetries in θ-expanded field theories exist indeed. Since the initial action functional comes via (38) from an action functional on $\mathbb{R}^4_\theta$, it seems plausible that these symmetries are already present in the truly noncommutative field theory. For me this is the justification to study θ-expanded quantum field theories: Although being completely different from quantum field theories on $\mathbb{R}^4_\theta$, the otherwise unphysical θ-expanded models may provide valuable information about the symmetries of the really interesting noncommutative models. My feeling is that these symmetries come through the spectral action. The spectral action is invariant under all unitarities of the Hilbert space, not only those coming from the algebra. The problem is to make this idea explicit.

The loop calculations of [39, 40] were performed for the θ-expanded action which comes out of (38), with the standard commutative gauge invariance (31). As we have shown in [41], very similar computations are possible when starting directly from the action functional for the $\star_\omega$-product, see (27). The only difference is that now the gauge symmetry is non-linearly realised so that the whole machinery of external fields and Slavnov-Taylor identities must be used. It is not sufficient to write down the BRST transformations only. We looked as in [40] at θ-expanded QED up to first order in θ, and to our great surprise we found—up to field redefinitions—exactly the same result as in [40]. This seems to indicate that the Seiberg-Witten map (30) is an unphysical change of variables also on quantum level.

This is true to some extent, but there is a subtlety. One can perform the change of variables before or after quantisation. Changing the variables $\Phi' = \Phi'[\Phi]$ after quantization, i.e. performing a change of the dummy integration variables in the path integral (4), one obtains exactly the same Green's functions. This was to expect from the general equivalence theorem [11]. The changes in the Feynman rules from Φ' to Φ are compensated by graphs involving the modified source term $\langle J, \Phi'[\Phi]\rangle$. In principle one would also expect contributions from field redefinition ghosts, but here the propagator equals 1 so that there is no contribution at least for certain regularisation schemes. On the other hand, changing the variables in the action functional before inserting it into the path integral, the outcome is expected to be different. However, at first order in θ only, the difference to the other method is a field redefinition.

11 Outlook

Trial-and-error is the best method to start exploring a new world. We have collected a big amount of empirical data on Feynman graph computations of quantum field theories on noncommutative $\mathbb{R}^4$. These theories are one-loop renormalisable and show at higher loop order a new type of infrared-like non-local divergences. Any model one can possibly think of has been studied. Everything is covered by the power-counting theorem [33] (when extended to the massless case à la Lowenstein). This is the most rigorous result so far. On the Taylor expanded side, θ-expanded field theories suggest that there are new symmetries. Further going loop calculations are not possible in future due to the enormous complexity of the outcome. Thus, the trial-and-error epoch has finished.

Now it is time for a more systematic approach. As argued in sec. 5, the

Feyman graph approach does not correctly reflect the geometry of $\mathbb{R}^4_\theta$. Instead, one has to introduce a smooth cut-off in the path integral and to compute it directly with methods of the exact renormalisation group approach [24, 25].

A An example of the $\star$-product in two dimensions

We consider the following function on $\mathbb{R}^2$

$$
\begin{aligned}
f_{\vec{a},\vec{L}}(\vec{x}) &= \prod_{i=1}^{2} f_{a_i,L_i}(x_i)\;, \\
f^N_{a_i,L_i}(x_i) &= \begin{cases} \cos(\frac{x_i-a_i}{L_i}) & \text{for } a_i - \frac{(2N+1)\pi L_i}{2} \leq x_i \leq a_i + \frac{(2N+1)\pi L_i}{2} \\ \qquad 0 & \text{for } |x_i-a_i| > \frac{(2N+1)\pi L_i}{2} \end{cases}
\end{aligned}
\tag{A.1}
$$

Clearly $f^N_{\vec{a},\vec{L}}(\vec{x}) \in \mathcal{S}(\mathbb{R}^2)$ (piecewise) for finite N because for multi-indices $\alpha = \{\alpha_i\}$ and $\beta = \{\beta_i\}$ one has $|(x)^\alpha(\partial_x)^\beta f^N_{\vec{a},\vec{L}}(\vec{x})| \leq \prod_{i=1}^2 L_i^{-\beta_i}\big(|a_i| + \frac{(2N+1)\pi L_i}{2}\big)^{\alpha_i}$. It is now an elementary calculation to compute the $\star$-product (10) of two functions (A.1):

$$
\begin{aligned}
&\big(f^N_{\vec{a},\vec{L}} \star f^N_{\vec{b},\vec{L}'}\big)(\vec{x}) \qquad\qquad (\text{A.2})\\
&= \Big(\frac{1}{4}\frac{(-\mathrm{i})}{2\pi} \sum_{\epsilon,\epsilon'=\pm 1} \mathrm{e}^{-\mathrm{i}(\epsilon\frac{x_1-a_1}{L_1}+\epsilon'\frac{x_2-b_2}{L_2'}+\epsilon\epsilon'\frac{\theta}{2L_1L_2'})} \\
&\quad\times \sum_{\epsilon'',\epsilon'''=\pm 1} \epsilon''\epsilon'''\,\mathcal{G}\big[\tfrac{2L_1L_2'}{\theta}\big(\tfrac{x_1-a_1}{L_1}+\epsilon'''\tfrac{(2N+1)\pi}{2}+\epsilon'\tfrac{\theta}{2L_1L_2'}\big)\big(\tfrac{x_2-b_2}{L_2'}+\epsilon''\tfrac{(2N+1)\pi}{2}+\epsilon\tfrac{\theta}{2L_1L_2'}\big)\big]\Big) \\
&\times \Big(\frac{1}{4}\frac{\mathrm{i}}{2\pi} \sum_{\epsilon,\epsilon'=\pm 1} \mathrm{e}^{\mathrm{i}(\epsilon\frac{x_2-a_2}{L_2}+\epsilon'\frac{x_1-b_1}{L_1'}+\epsilon\epsilon'\frac{\theta}{2L_2L_1'})} \\
&\quad\times \sum_{\epsilon'',\epsilon'''=\pm 1} \epsilon''\epsilon'''\,\mathcal{G}\big[-\tfrac{2L_2L_1'}{\theta}\big(\tfrac{x_2-a_2}{L_2}+\epsilon'''\tfrac{(2N+1)\pi}{2}+\epsilon'\tfrac{\theta}{2L_2L_1'}\big)\big(\tfrac{x_1-b_1}{L_1'}+\epsilon''\tfrac{(2N+1)\pi}{2}+\epsilon\tfrac{\theta}{2L_2L_1'}\big)\big]\Big)\;,
\end{aligned}
$$

where $\theta \equiv \theta^{12} = -\theta^{21}$,

$$
\mathcal{G}[u] := \sum_{n=1}^{\infty} \frac{(\mathrm{i}u)^n}{n\,n!} = \mathrm{ci}(u) - \gamma_E - \ln(u) + \mathrm{i}\,\mathrm{si}(u)\;, \tag{A.3}
$$

$$
\mathrm{ci}(u) = -\int_u^\infty \frac{dt\,\cos t}{t} = \gamma_E + \ln(u) + \int_0^u \frac{dt\,(\cos t - 1)}{t}\;, \quad \mathrm{si}(u) = \int_0^u \frac{dt\,\sin t}{t}\;,
$$

and $\gamma_E = 0.577216\ldots$. In the limit $N \to \infty$ one recovers the $\star_\omega$ product of the cosine functions:

$$
\begin{aligned}
&\big(f^\infty_{\vec{a},\vec{L}} \star_\omega f^\infty_{\vec{b},\vec{L}'}\big)(\vec{x}) \\
&= \Big(\frac{1}{4}\sum_{\epsilon,\epsilon'=\pm 1} \mathrm{e}^{-\mathrm{i}(\epsilon\frac{x_1-a_1}{L_1}+\epsilon'\frac{x_2-b_2}{L_2'}+\epsilon\epsilon'\frac{\theta}{2L_1L_2'})}\Big)\Big(\frac{1}{4}\sum_{\epsilon,\epsilon'=\pm 1} \mathrm{e}^{\mathrm{i}(\epsilon\frac{x_2-a_2}{L_2}+\epsilon'\frac{x_1-b_1}{L_1'}+\epsilon\epsilon'\frac{\theta}{2L_2L_1'})}\Big) \\
&= \Big(\cos\big(\tfrac{x_1-a_1}{L_1}\big)\cos\big(\tfrac{x_2-a_2}{L_2}\big)\Big) \star_\omega \Big(\cos\big(\tfrac{x_1-b_1}{L_1'}\big)\cos\big(\tfrac{x_2-b_2}{L_2'}\big)\Big)\;.
\end{aligned}
\tag{A.4}
$$

It is illuminating to plot (A.2) for various values of θ and N. For simplicity we choose $a_i = b_i = 0$ and $L_i = L'_i = L$. The result for $N \in \{0, 1\}$ is shown in Figure 1 for $\theta = L^2$ and the cut with the plane $x_1 = x_2$ for $\theta \in \{0.1L^2, L^2, 3L^2\}$ in Figure 2. Actually the way one should read Figure 2 is the following. One should

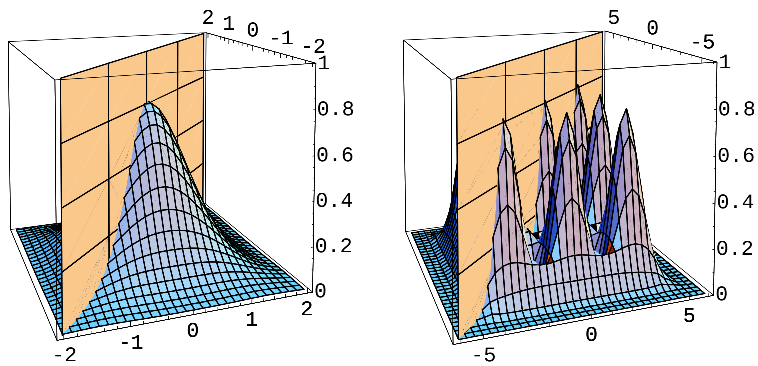


Figure 1: The functions $f^0 \star f^0$ (left) and $f^1 \star f^1$ (right) at $\theta = L^2$, where $f^N \equiv f^N_{(0,0),(L,L)}$. The x_1, x_2 axes are in units of L. The cut with the plane $x_1 = x_2$ is shown in Figure 2 for various values of θ.

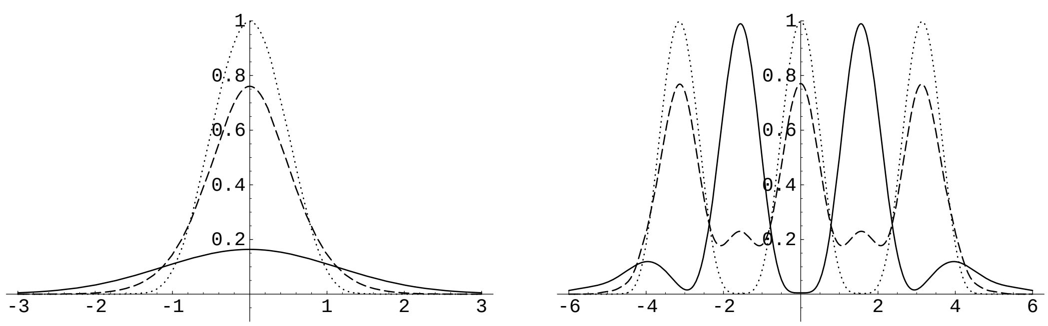


Figure 2: The cut $x_1 = x_2$ through the functions $f^0 \star f^0$ (left) and $f^1 \star f^1$ (right) at $\theta = 0.1L^2$ (dots), $\theta = L^2$ (dashes) and $\theta = 3L^2$ (solid), where $f^N \equiv f^N_{(0,0),(L,L)}$. The $\star$-product is *smooth and non-local*. For $\theta \to 0$ the commutative case is well approximated.

regard θ as fixed and what varies is the characteristic length L. For $L^2 \gg \theta$ the influence of θ can be neglected, and the $\star$-product agrees to high precision with the usual commutative product of functions. For $L \ll \theta$ the situation is drastically different. The $\star$-product is distributed over a region of size $\sqrt{\theta}$, whatever L is, at the same time the amplitudes are damped. This is impressively shown in Figure 3, where the value of the product at 0 is plotted over $\log_{10}(L^2/\theta)$. If the functions are extremely localised, i.e. if $\left(\frac{(2N+1)\pi L}{2}\right)^2 \ll \theta$, their product is zero to a high precision. Thus, θ acts as a horizon: Oscillations contained in an area smaller than θ are smoothed away. They do not carry any physical information. See also [18].

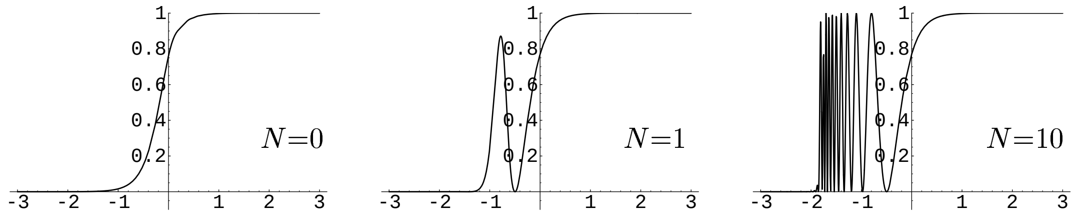


Figure 3: The value $(f^N \star f^N)(0)$ over $\log_{10}(\frac{L^2}{\theta})$ for $N \in \{0,1,10\}$, where $f^N \equiv f^N_{(0,0),(L,L)}$. This shows in a striking manner that the $\star$-product θ acts as a horizon. Oscillations of characteristic area smaller than θ are filtered out.

B The matricial basis of $\mathbb{R}^2_\theta$

The following is copied from [17], adapted to our notation. It proves that evaluation at $x \in \mathbb{R}^2$ is not a state on $\mathbb{R}^2_\theta$.

The Gaussian

$$f_0(x) = 2\mathrm{e}^{-\frac{1}{\theta}(x_1^2+x_2^2)} , \tag{B.1}$$

with $\theta \equiv \theta^{12} = -\theta^{21} > 0$, is an idempotent,

$$\begin{aligned}(f_0 \star f_0)(x) &= 4\int d^2y \int \frac{d^2k}{(2\pi)^2}\, \mathrm{e}^{-\frac{1}{\theta}(2x^2+y^2+2x\cdot y+x\cdot\theta\cdot k+\frac{1}{4}\theta^2k^2)+\mathrm{i}k\cdot y} \\ &= \frac{\theta}{\pi}\int d^2k\, \mathrm{e}^{-\frac{1}{\theta}(x^2+x\cdot\theta\cdot k+\mathrm{i}(k\cdot x)\theta+\frac{1}{2}\theta^2k^2)} = f_0(x) .\end{aligned} \tag{B.2}$$

We consider creation and annihilation operators

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(x_1+\mathrm{i}x_2) , & \bar{a} &= \frac{1}{\sqrt{2}}(x_1-\mathrm{i}x_2) , \\ \frac{\partial}{\partial a} &= \frac{1}{\sqrt{2}}(\partial_1-\mathrm{i}\partial_2) , & \frac{\partial}{\partial \bar{a}} &= \frac{1}{\sqrt{2}}(\partial_1+\mathrm{i}\partial_2) .\end{aligned} \tag{B.3}$$

For any $f \in \mathbb{R}^2_\theta$ we have

$$\begin{aligned} (a\star f)(x) &= a(x)f(x) + \frac{\theta}{2}\frac{\partial f}{\partial \bar{a}}(x) , & (f\star a)(x) &= a(x)f(x) - \frac{\theta}{2}\frac{\partial f}{\partial \bar{a}}(x) , \\ (\bar{a}\star f)(x) &= \bar{a}(x)f(x) - \frac{\theta}{2}\frac{\partial f}{\partial a}(x) , & (f\star \bar{a})(x) &= \bar{a}(x)f(x) + \frac{\theta}{2}\frac{\partial f}{\partial a}(x) .\end{aligned} \tag{B.4}$$

This implies $\bar{a}^{\star m} \star f_0 = 2^m \bar{a}^m f_0$, $f_0 \star a^{\star n} = 2^n a^n f_0$ and

$$a \star \bar{a}^{\star m} \star f_0 = \begin{cases} m\theta(\bar{a}^{\star(m-1)} \star f_0) & \text{for } m \geq 1 \\ 0 & \text{for } m = 0 \end{cases}$$

$$f_0 \star a^{\star n} \star \bar{a} = \begin{cases} n\theta(f_0 \star a^{\star(n-1)}) & \text{for } n \geq 1 \\ 0 & \text{for } n = 0 \end{cases} \tag{B.5}$$

where $a^{\star n} = a \star a \star \cdots \star a$ (n factors) and similarly for $\bar{a}^{\star m}$. Now, defining

$$\begin{aligned} f_{mn} &:= \frac{1}{\sqrt{n!m!\,\theta^{m+n}}}\,\bar{a}^{\star m} \star f_0 \star a^{\star n} \\ &= \frac{1}{\sqrt{n!m!\,\theta^{m+n}}} \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!\, 2^{m+n-2k}\, \theta^k\, \bar{a}^{m-k}\, a^{n-k} f_0 \;, \end{aligned} \tag{B.6}$$

(the second line is proved by induction) it follows from (B.5) and (B.2) that

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x) \;. \tag{B.7}$$

The multiplication rule (B.7) identifies the $\star$-product with the ordinary matrix product:

$$\begin{aligned} a(x) &= \sum_{m,n=0}^{\infty} a_{mn} f_{mn}(x) \;, & b(x) &= \sum_{m,n=0}^{\infty} b_{mn} f_{mn}(x) \\ \Rightarrow \quad (a \star b)(x) &= \sum_{m,n=0}^{\infty} (ab)_{mn} f_{mn}(x) \;, & (ab)_{mn} &= \sum_{k=0}^{\infty} a_{mk} b_{kn} \;. \end{aligned} \tag{B.8}$$

In order to describe elements of $\mathbb{R}^2_\theta$ the sequences $\{a_{mn}\}$ must be of rapid decay [17]:

$$\sum_{m,n=0}^{\infty} a_{mn} f_{mn} \in \mathbb{R}^2_\theta \qquad \text{iff} \qquad \sum_{m,n=0}^{\infty} \big((2m{+}1)^{2k}(2n{+}1)^{2k}|a_{mn}|^2\big)^{\frac{1}{2}} < \infty \quad \text{for all } k \;. \tag{B.9}$$

Finally, using (B.2) we compute

$$\begin{aligned} \int d^2x\, f_{mn}(x) &= \frac{1}{\sqrt{m!n!\,\theta^{m+n}}} \int d^2x\, \big(\bar{a}^{\star m} \star f_0 \star f_0 \star a^{\star n}\big)(x) \\ &= \frac{1}{\sqrt{m!n!\,\theta^{m+n}}} \int d^2x\, \big(f_0 \star a^{\star n} \star \bar{a}^{\star m} \star f_0\big)(x) \\ &= \delta_{mn} \int d^2x f_0(x) = 2\pi\theta\delta_{mn} \;. \end{aligned} \tag{B.10}$$

Now we return to the question of states. We clearly have

$$(f^*_{mn} \star f_{mn})(x) = (f_{nm} \star f_{mn})(x) = f_{nn}(x) \;, \tag{B.11}$$

and $f_{11}(x) = \frac{2}{\theta}\big(4x_1^2 + 4x_2^2 - \theta\big)\mathrm{e}^{-\frac{1}{\theta}(x_1^2+x_2^2)} < 0$ for $4x_1^2 + 4x_2^2 < \theta$. Thus, δ-distributions cannot be states on $\mathbb{R}^d_\theta$. On the other hand, (B.11) and (B.10) imply that $\chi_n(x) = \frac{1}{2\pi\theta} f_{nn}(x)$ are states on $\mathbb{R}^2_\theta$. The basis f_{mn} was used in [23] to construct a new class of exactly solvable quantum field theories.

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