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Non-Compact Spectral Triples with Finite Volume

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Dedicated to Alain Connes on the occasion of his 60th birthday

Abstract. In order to extend the spectral action principle to non-compact spaces, we propose a framework for spectral triples where the algebra may be non-unital but the resolvent of the Dirac operator remains compact. We show that an example is given by the supersymmetric harmonic oscillator which, interestingly, provides two different Dirac operators. This leads to two different representations of the volume form on the Hilbert space, and only their product is the grading operator. The index of the even-to-odd part of each of these Dirac operators is 1.

We also compute the spectral action for the corresponding Connes-Lott two-point model. There is an additional harmonic oscillator potential for the Higgs field, whereas the Yang-Mills action is unchanged. The total Higgs potential shows a two-phase structure with smooth transition between them: In the spontaneously broken phase below a critical radius, all fields are massive, with the Higgs field mass slightly smaller than the NCG prediction. In the unbroken phase above the critical radius, gauge fields and fermions are massless, whereas the Higgs field remains massive.

1. Introduction

One of the greatest achievements of noncommutative geometry [**1**] is the conceptual understanding of the Standard Model of particle physics. This was not reached in one step. It took more than 15 years

- from the first appearance of the Higgs potential in noncommutative models [**2, 3**]
- via the two-sheeted universe of Connes-Lott [**4**] with its bimodule structure [**1**],
- the discovery of the real structure [**5**] (which eliminated one redundant $U(1)$ group),
- the understanding of gauge fields as inner fluctuations in an axiomatic setting [**6**] and the move from the Dixmier trace based action functional to the spectral action principle [**7**], which unifies the Standard Model with gravity,

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- the superseding of the unimodularity condition [**8**] (which eliminated the second redundant $U(1)$ group),
- to the spectacular rebirth [**9**] with the explanation [**10**] of the $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ Standard Model matrix algebra as the distinguished maximal subalgebra of $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ compatible with a non-trivial first order condition (i.e. Majorana masses) and a six-dimensional real structure (i.e. charge conjugation).

There is one important message of this evolution: One should never be completely satisfied with one's achievements! The description given in Alain Connes' book [**1**] definitely has its beauty. The little annoyance with the redundant $U(1)$ found its solution in the real structure [**5**] which soon was realised as a key to unlocking the secrets of spin manifolds [**6**] in noncommutative geometry. This axiomatic setting initiated many examples of noncommutative manifolds and culminated in the recent spectral characterisation of manifolds [**11**].

Let me give a wish list for further improvements—not as a criticism of the model, but rather as a possible source of insight.

(1) Quantisation. The outcome of the spectral action principle is a classical action functional valid at a distinguished (grand unification) scale. It is connected to the scale realised in a particle accelerator by the renormalisation group flow. This flow can be computed by rules from perturbative quantum field theory. The input is not directly the spectral action, but a gauge-fixed version of it which involves Faddeev-Popov ghosts. It is highly desirable to include these ghosts in the spectral action, because in this way unitary invariance is realised as cohomology of the BRS complex. We may speculate that the BRS cohomology of the spectral action is deeply connected to the wealth of noncommutative cohomology theories. As a starting point one might use results of Perrot [**12**], who identifies the BRS coboundary as the de Rham differential in the loop space $C^\infty(S^1,\mathcal{U}(\mathcal{A}))$ and connects the chiral anomaly with the local index formula [**13**].

(2) Big desert. The present form of the spectral action is based on the big desert hypothesis which asserts that, apart from the Higgs boson, all particles relevant at the grand unification scale are already discovered. The minor mismatch between observed and predicted $U(1)$ coupling constant (see Figure 1 in [**9**]) might suggest some new physics in the desert. Candidates include supersymmetry and dark matter, but also noncommutativity of space itself could alter the slope of the running $U(1)$ coupling.

The latter question concerning the renormalisation group flow of field theories on noncommutative geometries was intensely studied in the last decade. After unexpected difficulties with UV/IR-mixing, we established perturbative renormalisability of scalar field theories on Moyal-deformed Euclidean space [**14, 15**]. The key is a deformation also of the differential calculus, namely from the Laplace operator to the harmonic oscillator Schrödinger operator. It turned out indeed that the combined Moyal-harmonic oscillator deformation removes the Landau ghost of the commutative scalar model [**16**] by altering the slope of the running coupling constant [**17**]. Since the $U(1)$-part of the Standard Model has the same Landau ghost problem, we might expect that, once the Standard Model has been grounded in an appropriate noncommutative geometry, the three running couplings of Figure 1 in [**9**] will eventually intersect in a single point.

The first step in this programme is to construct a spectral triple with its canonically associated spectral action for the combined Moyal-harmonic oscillator deformation. The present paper achieves an intermediate goal: We construct and investigate a *commutative* harmonic oscillator spectral triple. Its Moyal isospectral deformation will be treated in [**18**], building on ideas developed in [**19**]. The main obstacle was to identify a Dirac operator whose square is the harmonic oscillator Hamiltonian of [**14**]. The solution which we give in this paper is deeply connected to supersymmetric quantum mechanics [**20**], in particular to Witten's approach to Morse theory [**21**]. It would be interesting to reformulate Witten's results in noncommutative index theory using the spectral triple we suggest.

(3) Time. The spectral action relies on compact Euclidean geometry. For the Standard Model one typically chooses the manifold $S^3 \times S^1$, where S^3 is for "space" and S^1 for "temperature", not "time". Although the universe is filled with thermal background radiation, it is desirable to allow for a genuine time evolution of the spectral geometry. In fact, noncommutative von Neumann algebras carry their own time evolution through the modular automorphism group, and it has been argued [**22**] that this is the source of the physical time flow. So far the modular automorphisms seem disconnected from the spectral action. The most ambitious project to reconcile time development and spectral geometry within generally covariant quantum field theory was initiated by Paschke and Verch [**23**].

(4) Compactness. As mentioned above, the spectral action presumes compactness, namely, compactness of the resolvent of the Dirac operator. The example we study in this paper shows that compactness of the resolvent does not imply spatial compactness. It is eventually a matter of experiment to determine the type of compactness of the universe.

The paper is organised as follows: We propose in Section 2 a definition of non-unital spectral triples, but with compactness of the resolvent of the Dirac operator. We show in Section 3 that the supersymmetric harmonic oscillator is an example of such a spectral triple: In Section 3.1 we introduce the supercharges in a slightly generalised framework and briefly discuss their cohomology. The supercharges give rise to two distinct Dirac operators. In Section 3.2 we identify for the harmonic oscillator the algebra and the smooth part of the Hilbert space. In Section 3.3 and Appendix A we compute the dimension spectrum. The novel orientability structure is studied in Section 3.4, and Section 3.5 discusses the index formula for the Dirac operators. The spectral action is computed in Section 4 and Appendix B. In the final Section 5 we study the solution of the equations of motion.

2. Non-compact spectral triples

Motivated by the spectral characterisation of manifolds [**11**], we propose here a definition of spectral triples which does not require the algebra to be unital. There are several proposals in the literature for a non-compact generalisation of spectral triples; see [**24**] and references therein. To include $\mathbb{R}^d$ with its standard Dirac operator, these proposals relax the compactness of the resolvent of $\mathcal{D}$ to the requirement that $\pi(a)(\mathcal{D}+\mathrm{i})^{-1}$ is compact for all $a \in \mathcal{A}$. However, compactness of the resolvent (or similar regularisation [**25**]) is essential for a well-defined spectral action. Moreover, the usual Dirac operator on $\mathbb{R}^d$ is not suited for an index formula [**26**]. We therefore keep compactness of the resolvent (and thus exclude standard $\mathbb{R}^d$), but to achieve this in the non-compact situation we are forced to give up (at least in our example)

(1) the universality of dimensions,
(2) the connection between volume form and $\mathbb{Z}_2$-grading.

We give some comments after the definition. To simplify the presentation we require the algebra to be commutative; the noncommutative generalisation involves the real structure J.

Definition 1. *A (possibly non-compact) commutative spectral triple with finite volume $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a (possibly non-unital) commutative and involutive algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$ and a selfadjoint unbounded operator $\mathcal{D}$ in $\mathcal{H}$ with compact resolvent fulfilling the conditions 1-5 below.*

(1) `Regularity and dimension spectrum.` *For any $a \in \mathcal{A}$, both a and $[\mathcal{D}, a]$ belong to $\bigcap_{n=1}^{\infty} \mathrm{dom}(\delta^n)$, where $\delta T := [\langle\mathcal{D}\rangle, T]$ and $\langle\mathcal{D}\rangle := (\mathcal{D}^2+1)^{\frac{1}{2}}$.*
For any element ϕ of the algebra $\Psi_0(\mathcal{A})$ generated by $\delta^m a$ and $\delta^m[\mathcal{D}, a]$, with $a \in \mathcal{A}$, the function $\zeta_\phi(z) := \mathrm{Tr}(\phi\langle\mathcal{D}\rangle^{-z})$ extends holomorphically to $\mathbb{C} \setminus \mathrm{Sd}$ for some discrete set $\mathrm{Sd} \subset \mathbb{C}$ (the dimension spectrum), and all poles of ζ_ϕ at $z \in \mathrm{Sd}$ are simple.

(2) `Metric dimension.` *The maximum $d := \max\{r \in \mathbb{R} \cap \mathrm{Sd}\}$ belongs to $\mathbb{N}$. The noncommutative integral $⨍ a\langle\mathcal{D}\rangle^{-d}$ is finite for any $a \in \mathcal{A}$ and positive for positive elements of $\mathcal{A}$.*

(3) `Orientability.` *For the preferred unitisation*
$$\mathcal{B} := \{b \in \mathcal{A}'' \ : \quad b, [\mathcal{D}, b] \in \bigcap_{n \in \mathbb{N}} \mathrm{dom}(\delta^m)\}\ ,$$
there is a Hochschild d-cycle $\boldsymbol{c} \in Z_d(\mathcal{B}, \mathcal{B})$, i.e. a finite sum of terms $b_0 \otimes b_1 \otimes \cdots \otimes b_d$. Its representation $\boldsymbol{\gamma} := \pi_{\mathcal{D}}(\boldsymbol{c})$, with $\pi_{\mathcal{D}}(b_0 \otimes b_1 \otimes \cdots \otimes b_d) := b_0[\mathcal{D}, b_1] \cdots [\mathcal{D}, b_d]$, satisfies $\boldsymbol{\gamma}^2 = 1$ and $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}$. Additionally, $\boldsymbol{\gamma}$ defines the volume form on $\mathcal{A}$, i.e.
$$\phi_{\boldsymbol{\gamma}}(a_0, \ldots, a_d) := ⨍ \big(\boldsymbol{\gamma} a_0[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_d]\langle\mathcal{D}\rangle^{-d}\big)$$
provides a non-vanishing Hochschild d-cocycle $\phi_{\boldsymbol{\gamma}}$ on $\mathcal{A}$.

(4) `First order.` *$[[\mathcal{D}, b], b'] = 0$ for all $b, b' \in \mathcal{B}$.*

(5) `Finiteness.` *The subspace $\mathcal{H}_\infty := \bigcap_{k=0}^{\infty} \mathrm{dom}(\mathcal{D}^k) \subset \mathcal{H}$ is a finitely generated projective $\mathcal{A}$-module $e\mathcal{A}^n$, for some $n \in \mathbb{N}$ and some projector $e = e^2 = e^* \in M_n(\mathcal{B})$. The composition of the noncommutative integral with the induced Hermitian structure $(\ |\ ) : \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{A}$ coincides with the scalar product $\langle\ ,\ \rangle$ on $\mathcal{H}_\infty$,*
$$\langle\xi, \eta\rangle = ⨍ \Big((\xi|\eta)\,\langle\mathcal{D}\rangle^{-d}\Big)\ , \qquad \xi, \eta \in \mathcal{H}_\infty\ .$$

The dimension spectrum was introduced by Connes and Moscovici [**13**] precisely to describe by a local formula the lower-dimensional pieces in the Chern character that are ignored by the top-dimensional Hochschild cohomology class. The local index formula was generalised in [**27**] to a larger class of examples. We are interested in a similar situation. For non-unital algebras we may have the characteristic values of the resolvent of $\mathcal{D}$ run as $\mathcal{O}(n^{-\frac{1}{p}})$ for p greater than the metric dimension d. The dimension spectrum is the right tool to deal with this case.

It would be interesting to know whether Definition 1, despite its differences from Connes' original definition [**11**], allows reconstruction of a manifold structure on the spectrum $X = \mathrm{Spec}(A)$ of the norm closure A of $\mathcal{A}$. At first sight, the construction of candidates for local charts only uses the measure λ on X defined by the noncommutative integral $\lambda(f) = \fint f\langle\mathcal{D}\rangle^{-d}$ for $f \in A = C(X)$ and the fact that the Hilbert space $\mathcal{H}$ is precisely the L^2-closure of $\mathcal{H}_\infty$ with respect to λ. The details of how $\fint f\langle\mathcal{D}\rangle^{-d}$ is constructed, whether as a state-independent Dixmier trace or as a residue in the dimension spectrum, do not seem to enter. In particular, Lemma 2.1 of [**11**] holds: if $1 \in \mathcal{A}$, then $\mathcal{B} = \mathcal{A}$ (in the notation of Definition 1), so that conditions (3),(4),(5) are the same as in [**11**], with the sole exception that $\boldsymbol{\gamma}$ is not necessarily the $\mathbb{Z}_2$-grading for even d or $\boldsymbol{\gamma} = 1$ for odd d. However, this was only used for uniqueness of the noncommutative integral, which we achieve alternatively from the dimension spectrum. But [**11**, §9] makes heavy use of the asymptotics of the eigenvalues of $\langle\mathcal{D}\rangle^{-1}$ to prove injectivity of the local charts; we do not know how to achieve this from the dimension spectrum.

3. A spectral triple for the harmonic oscillator

3.1. Supersymmetric quantum mechanics. Supersymmetric quantum mechanics provides an elegant approach to exactly solvable quantum-mechanical models [**20**] and is also a powerful tool in mathematics [**21**]. Our notation is a compromise between [**20**] and [**21**].

Let X be a d-dimensional smooth manifold with trivial cotangent bundle and ∂_μ, for $\mu = 1, \dots, d$, be the basis of the tangent space T_xX induced by the coordinate functions. On the Hilbert space $L^2(X)$ we consider the unbounded operators

$$a_\mu = e^{-\omega h}\partial_\mu e^{\omega h} = \partial_\mu + W_\mu\,, \qquad a_\mu^\dagger = -e^{\omega h}\partial_\mu e^{-\omega h} = -\partial_\mu + W_\mu\,, \tag{1}$$

where h is some real-valued function on X, the Morse function [**21**], and $W_\mu(x) = \omega(\partial_\mu h)(x)$. It is convenient to keep the frequency ω separate from h. The resulting commutation relations are

$$[a_\mu, a_\nu] = [a_\mu^\dagger, a_\nu^\dagger] = 0\,, \qquad [a_\mu, a_\nu^\dagger] = 2\omega\partial_\mu\partial_\nu h\,. \tag{2}$$

We define fermionic ladder operators $b^\mu, b^{\dagger\mu}$ which satisfy the anticommutation relations

$$\{b^\mu, b^\nu\} = 0\,, \qquad \{b^{\dagger\mu}, b^{\dagger\nu}\} = 0\,, \qquad \{b^\mu, b^{\dagger\nu}\} = \delta^{\mu\nu}\,. \tag{3}$$

We also let all mixed commutators vanish, $[a_\mu^{(\dagger)}, b^{(\dagger)\nu}] = 0$. We introduce the supercharges $\mathfrak{Q}, \mathfrak{Q}^\dagger$ by

$$\mathfrak{Q} := a_\mu \otimes b^{\dagger\mu}\,, \qquad \mathfrak{Q}^\dagger := a_\mu^\dagger \otimes b^\mu\,. \tag{4}$$

Unless otherwise stated, we use Einstein's summation convention, i.e. summation over a pair of upper/lower Greek indices from 1 to d is understood. The supercharges satisfy

$$\{\mathfrak{Q},\mathfrak{Q}\} = \{\mathfrak{Q}^\dagger,\mathfrak{Q}^\dagger\} = 0\;, \quad \{\mathfrak{Q},\mathfrak{Q}^\dagger\} =: \mathfrak{H}\;, \qquad [\mathfrak{Q},\mathfrak{H}] = [\mathfrak{Q}^\dagger,\mathfrak{H}] = 0\;. \tag{5}$$

The Hamiltonian $\mathfrak{H}$ introduced by the anticommutator reads explicitly (index raising by $\delta^{\mu\nu}$)

$$\begin{aligned}\mathfrak{H} &= \frac{1}{2}\delta^{\mu\nu}\{a_\mu,a_\nu^\dagger\}\otimes 1 + \frac{1}{2}[a_\mu,a_\nu^\dagger]\otimes[b^{\dagger\mu},b^\nu] \\ &= \big(-\partial_\mu\partial^\mu + \omega^2(\partial_\mu h)(\partial^\mu h)\big)\otimes 1 + \omega(\partial_\mu\partial_\nu h)\otimes[b^{\dagger\mu},b^\nu]\;.\end{aligned} \tag{6}$$

The supercharges give rise to *two* anticommuting Dirac operators

$$\mathcal{D}_1 = \mathfrak{Q}+\mathfrak{Q}^\dagger\;, \qquad\qquad \mathcal{D}_2 = \mathrm{i}\mathfrak{Q}-\mathrm{i}\mathfrak{Q}^\dagger\;, \tag{7}$$

$$\mathcal{D}_i^2 = \mathfrak{H} \quad \text{for } i=1,2\;, \qquad \mathcal{D}_1\mathcal{D}_2+\mathcal{D}_2\mathcal{D}_1 = 0\;. \tag{8}$$

We let $|0\rangle_f$ be the fermionic vacuum with $b^\mu|0\rangle_f = 0$. By repeated application of $b^{\dagger\mu}$ one constructs out of $|0\rangle_f$ the 2^d-dimensional fermionic Hilbert space $\bigwedge(\mathbb{C}^d)$ in which we label the standard orthonormal basis as follows:

$$|s_1,\dots,s_d\rangle_f = (b^{\dagger 1})^{s_1}\dots(b^{\dagger d})^{s_d}|0\rangle_f\;, \qquad s_\mu\in\{0,1\}\;. \tag{9}$$

The fermionic number operator is $N_f = b^\dagger_\mu b^\mu$, with

$$N_f|s_1,\dots,s_d\rangle_f = (s_1+\dots+s_d)|s_1,\dots,s_d\rangle_f\;.$$

The fermionic Hilbert space is $\mathbb{N}$-graded by $\bigwedge(\mathbb{C}^d) = \bigoplus_{p=0}^d \Lambda^p(\mathbb{C}^d)$ with $\dim(\Lambda^p(\mathbb{C}^d)) = \binom{d}{p}$. Accordingly, the total Hilbert space $\mathcal{H} = L^2(X)\otimes\bigwedge(\mathbb{C}^d)$ is graded by the fermion number, $\mathcal{H} = \bigoplus_{p=0}^d \mathcal{H}_p$. Note that $\mathfrak{Q}:\mathcal{H}_p\to\mathcal{H}_{p+1}$ and $\mathfrak{Q}^\dagger:\mathcal{H}_p\to\mathcal{H}_{p-1}$. The induced $\mathbb{Z}_2$-grading operator is

$$\Gamma = (-1)^{N_f}\;, \qquad \Gamma^2 = 1\;,\; \Gamma=\Gamma^*\;, \qquad \Gamma\mathcal{D}_i+\mathcal{D}_i\Gamma = 0\;. \tag{10}$$

Let $B_p(\omega)$ be the dimension of the p-th cohomology group of $\mathfrak{Q}$, i.e. the number of linearly independent $\psi_p\in\ker\mathfrak{Q}\cap\mathcal{H}_p$ that cannot be written as $\psi_p = \mathfrak{Q}\eta_{p-1}$ for some $\eta\in\mathcal{H}_{p-1}$. According to Witten [**21**], $B_p(\omega)$ coincides with the Betti number B_p and is deeply connected with the Morse index M_p for the function h: Let x_α be a critical point of h, i.e. $(\partial_\mu h)(x)=0$. If $\partial_\mu\partial_\nu h$ is regular at each of these critical points, then M_p is the number of critical points at which $\partial_\mu\partial_\nu h$ has p negative eigenvalues. The weak Morse inequalities $M_p\geq B_p$ follow from the eigenvalue problem for $\mathfrak{H}$ in the limit of large ω.

By Hodge theory, which relies on the Hilbert space structure, every generator of the p-th cohomology group of $\mathfrak{Q}$ has a unique representative ψ which is also $\mathfrak{Q}^\dagger$-exact (and thus belongs to $\ker\mathfrak{H}$). Since the $b^\mu, b^{\dagger\mu}$ generate linearly independent subspaces, this means (no summation over $\bar\mu,\bar\nu$)

$$(a_{\bar\mu}\otimes b^{\dagger\bar\mu})\psi = 0 \quad\text{and}\quad (a^\dagger_{\bar\nu}\otimes b^{\bar\nu})\psi = 0 \quad \text{for all } \bar\mu,\bar\nu = 1,\dots,d\;. \tag{11}$$

The only candidates are (up to a multiplicative constant)

$$\psi_0 = e^{-\omega h}|0\rangle_f \quad\text{and}\quad \psi_d = e^{\omega h}b^{\dagger 1}\cdots b^{\dagger d}|0\rangle_f\;. \tag{12}$$

For compact manifolds, where both $e^{\pm\omega h}$ are integrable, this yields $B_0 = 1$ and $B_d = 1$ as the only non-vanishing Betti numbers. In the non-compact case one should choose $e^{-\omega h}$ integrable, so that $e^{\omega h}$ is not integrable, and hence $B_p = \delta_{p0}$.

Of course, this behaviour is due to the assumption of a trivial cotangent bundle. For more interesting topology one should define the smooth subspace of the Hilbert space as a finitely generated projective module.

3.2. The harmonic oscillator. In the following we propose a spectral triple in the sense of Definition 1 with objects related to the harmonic oscillator. We will check the axioms, but no attempt will be made to reconstruct a manifold.

The harmonic oscillator is obtained from the Morse function $h = \frac{\|x\|^2}{2} = \frac{1}{2}\delta^{\mu\nu}x_\mu x_\nu$ on the manifold $\mathbb{R}^d$. This leads to the relation

$$[a_\mu, a_\nu^\dagger] = 2\omega\delta_{\mu\nu} \; , \tag{13}$$

which in turn permits a complete reconstruction of the eigenfunctions by repeated application of $a_\mu^\dagger, b^{\dagger\nu}$ to the ground state $\psi_0 = |0\rangle_b \otimes |0\rangle_f \in \ker \mathfrak{H}$, with $|0\rangle_b = (\frac{\omega}{\pi})^{\frac{d}{4}} e^{-\frac{\omega}{2}\|x\|^2}$. Defining

$$|n_1, \dots, n_d\rangle_b = \frac{1}{\sqrt{n_1! \dots n_d! (2\omega)^{n_1+\dots+n_d}}} (a_1^\dagger)^{n_1} \cdots (a_d^\dagger)^{n_d} |0\rangle_b \; , \qquad n_\mu \in \mathbb{N} \; , \tag{14}$$

the tensor products $|n_1, \dots, n_d\rangle_b \otimes |s_1, \dots, s_d\rangle_f$ of (14) with (9) form an orthonormal basis of the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^d) \otimes \mathbb{C}^{2^d} \simeq L^2(\mathbb{R}^d) \otimes \bigwedge(\mathbb{C}^d)$.

There are two ways of viewing the Hamiltonian (6). In the $L^2(\mathbb{R}^d)$-representation, we have

$$\mathfrak{H} = H \otimes 1 + \omega \otimes \Sigma \; , \qquad H = -\partial_\mu \partial^\mu + \omega^2 x_\mu x^\mu \; , \qquad \Sigma = [b_\mu^\dagger, b_\mu] \; , \tag{15}$$

i.e. the total Hamiltonian is the sum of the harmonic oscillator Hamiltonian and ω times the spin matrix Σ. This representation will be useful when considering the algebra $\mathcal{A}$ later on which is also realised in the $L^2(\mathbb{R}^d)$-representation. In the $\ell^2(\mathbb{N}^d)$-representation, we have

$$\mathcal{D}_1^2 = \mathcal{D}_2^2 = \mathfrak{H} = a_\mu^\dagger a^\mu \otimes 1 + 2\omega \otimes b_\mu^\dagger b^\mu = 2\omega(N_b + N_f) \; , \tag{16}$$

which is up to a factor of 2ω the supersymmetric number operator:

$$\begin{aligned} &\mathcal{D}_i^2(|n_1, \dots, n_d\rangle_b \otimes |s_1, \dots, s_d\rangle_f) \\ &\quad = \Big(2\omega \sum_{\mu=1}^{d} (n_\mu + s_\mu)\Big)(|n_1, \dots, n_d\rangle_b \otimes |s_1, \dots, s_d\rangle_f) \; . \end{aligned} \tag{17}$$

In particular, the kernel of $\mathcal{D}_i$ is one-dimensional, and the resolvent of $\mathcal{D}_i$ is compact. To deal with the kernel, we introduce

$$\langle \mathcal{D} \rangle := (\mathcal{D}_1^2 + 1)^{\frac{1}{2}} = (\mathcal{D}_2^2 + 1)^{\frac{1}{2}} \; , \qquad \delta T := [\langle \mathcal{D} \rangle, T] \quad \text{for } T \in \mathcal{B}(\mathcal{H}) \; . \tag{18}$$

Counting the number of eigenvalues $\leq N$ one finds that $\langle \mathcal{D} \rangle^{-1}$ is a noncommutative infinitesimal of order $2d$, and $\langle \mathcal{D} \rangle^{-p}$ is trace-class for $p > 2d$. Formula (17) also shows that

$$\mathcal{H}_\infty := \bigcap_{m \geq 0} \mathrm{dom}(\mathcal{D}^n) = \mathcal{S}(\mathbb{N}^d) \otimes \bigwedge(\mathbb{C}^d) \simeq \mathcal{S}(\mathbb{R}^d) \otimes \bigwedge(\mathbb{C}^d) \simeq \big(\mathcal{S}(\mathbb{R}^d)\big)^{2^d} \; , \tag{19}$$

which is required to be a finitely generated projective module over the algebra of the spectral triple. We are interested here in the commutative case, so that we are led to consider the algebra

$$\mathcal{A} = \mathcal{S}(\mathbb{R}^d) \tag{20}$$

of Schwartz class functions with standard commutative product. The Hermitian structure is pointwise the scalar product in $\bigwedge(\mathbb{C}^d)$, i.e. $(\xi|\eta) = \sum_{i=1}^{2^d} \xi_i^* \eta_i$ for $\xi = (\xi_1, \dots, \xi_{2^d}), \eta = (\eta_1, \dots, \eta_{2^d}) \in \mathcal{H}_\infty = \big(\mathcal{S}(\mathbb{R}^d)\big)^{2^d}$.

As usual, we represent the algebra $\mathcal{A}$ on $\mathcal{H}$ by pointwise multiplication in $L^2(\mathbb{R}^d)$:

$$f(\psi \otimes \rho) := (f\psi) \otimes \rho \qquad \text{for } f \in \mathcal{A}\,,\ \psi \in L^2(\mathbb{R}^d)\,,\ \rho \in \textstyle\bigwedge(\mathbb{C}^d)\,. \tag{21}$$

The action of $\mathcal{A}$ commutes with $b^\mu, b^{\dagger\mu}$ so that we obtain

$$[\mathcal{D}_1, f] = \partial_\mu f \otimes (b^{\dagger\mu} - b^\mu)\,, \qquad [\mathcal{D}_2, f] = \partial_\mu f \otimes (\mathrm{i}b^{\dagger\mu} + \mathrm{i}b^\mu)\,. \tag{22}$$

In particular, the first-order condition is satisfied. For $f \in \mathcal{A}$, the expansion coefficients $\langle n_1, \dots, n_d | f | n_1', \dots, n_d' \rangle$ are Schwartz sequences in n_μ, n_μ'. Therefore, f and $[\mathcal{D}_i, f]$ belong for any $m \in \mathbb{N}$ to the domain of δ^m.

We show in joint work with V. Gayral [**18**] (which supersedes [**19**]), that the Moyal-deformation of $\mathcal{S}(\mathbb{R}^d)$ together with the same Dirac operator and Hilbert space forms a noncommutative spectral triple in the sense of Definition 1, i.e. an isospectral deformation.

3.3. Dimension spectrum. In this subsection we take for $\mathcal{D}$ either of $\mathcal{D}_1$ or $\mathcal{D}_2$. We consider the algebra $\Psi_0(\mathcal{A})$ generated by $\delta^m f$ and $\delta^m[\mathcal{D}, f]$. As $\langle \mathcal{D} \rangle^{-z}$ is trace-class for $\mathrm{Re}(z) > 2d$, the ζ-function $\zeta_\phi(z) := \mathrm{Tr}(\phi \langle \mathcal{D} \rangle^{-z})$ exists for such $z \in \mathbb{C}$ and $\phi \in \Psi_0(\mathcal{A})$ and can possibly be extended to a meromorphic function on $\mathbb{C}$. The following theorem identifies the poles and the structure of the residues:

THEOREM 2. *The spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *has dimension spectrum* $\mathrm{Sd} = d - \mathbb{N}$ *and hence metric dimension* d. *All poles of* ζ_ϕ *at* $z \in \mathrm{Sd}$ *are simple with local residues, i.e. for* $\phi = \delta^{n_1} f_1 \cdots \delta^{n_v} f_v$, *any residue* $\mathrm{res}_{z \in \mathrm{Sd}} \zeta_\phi(z)$ *is a finite sum of* $\displaystyle\int_{\mathbb{R}^d} dx\ x^{\alpha_0} (\partial^{\alpha_1} f_1) \cdots (\partial^{\alpha_v} f_v)$, *where* α_i *are multi-indices. The analogous result holds when* f_i *in* ϕ *is replaced by* $[\mathcal{D}, f_i]$.

This theorem is the central result of this paper. We give the rather long proof in Appendix A.

A special case of the proof of Theorem 2 is the computation of the Dixmier trace:

PROPOSITION 3. $\displaystyle ⨍ f \langle \mathcal{D} \rangle^{-d} = \frac{1}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d+2}{2})} \int_{\mathbb{R}^d} dx\ f(x)$ *for any* $f \in \mathcal{A}$.

Proof. As the dimension spectrum is simple, the Dixmier trace can be computed as a residue [**28**], is independent of the state ω, and defines unambiguously the noncommutative integral:

$$⨍ f \langle \mathcal{D} \rangle^{-d} = \mathrm{res}_{s=1} \mathrm{Tr}(f \langle \mathcal{D} \rangle^{-sd})\,. \tag{23}$$

Taking $v = 1$ and $n_1 = 0$ in (82) and inserting $\det Q$ and Q^{-1} from (85) and (86) as well as (80), we have

$$⨍ f \langle \mathcal{D} \rangle^{-d} = \mathrm{res}_{s=1} \bigg(\frac{1}{\Gamma(\frac{sd}{2})} \int_0^\infty dt_0\ t_0^{\frac{sd}{2}-1} e^{-t_0} \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d}\ \hat{f}(p) \frac{e^{-\frac{p^2}{\omega \tanh(\omega t_0)}}}{\tanh^d(\omega t_0)} \bigg)\,. \tag{24}$$

We write $\hat{f}(p) = \hat{f}(0) + p_\mu \frac{\partial \hat{f}}{\partial p_\mu}(0) + p_\mu p_\nu \int_0^1 d\lambda\, (1-\lambda) \frac{\partial^2 \hat{f}}{\partial p_\mu \partial p_\nu}(\lambda p_\mu)$ and get

$$
\begin{aligned}
(25)\qquad & \frac{1}{\Gamma(\frac{sd}{2})} \int_0^\infty dt_0\, t_0^{\frac{sd}{2}-1} e^{-t_0} \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d}\, \hat{f}(0) \frac{e^{-\frac{p^2}{\omega \tanh(\omega t_0)}}}{\tanh^d(\omega t_0)} \\
&= \frac{\hat{f}(0)}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{sd}{2})} \int_0^\infty dt_0\, t_0^{\frac{(s-1)d}{2}-1} e^{-t_0} \underbrace{\Big(\frac{\omega t_0}{\tanh(\omega t_0)}\Big)^{\frac{d}{2}}}_{g(t_0)} \\
&= \frac{\hat{f}(0)}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\frac{(s-1)d}{2})}{\Gamma(\frac{sd}{2})} + \frac{\hat{f}(0)}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{sd}{2})} \int_0^\infty dt_0\, t_0^{\frac{(s-1)d}{2}} e^{-t_0} \int_0^1 d\lambda\, g'(\lambda t_0)\,.
\end{aligned}
$$

As $|g'(y)| \leq \frac{d}{2} y^{\frac{d}{2}-1}$ for all $y \in \mathbb{R}_+$, we have

$$
(26)\qquad \Big| \int_0^\infty dt_0\, t_0^{\frac{(s-1)d}{2}} e^{-t_0} \int_0^1 d\lambda\, g'(\lambda t_0) \Big| \leq \int_0^\infty dt_0\, t_0^{\frac{s}{2}-1} e^{-t_0} = \Gamma(\tfrac{s}{2})\,,
$$

which is regular for $s=1$. The first-order term $p_\mu \frac{\partial \hat{f}}{\partial p_\mu}(0)$ does not contribute as an odd function in p. In the remainder, $\int_0^1 d\lambda (1-\lambda) \frac{\partial^2 \hat{f}}{\partial p_\mu \partial_\nu}(\lambda p_\mu)$ is bounded, and

$$
(27)\qquad \int \frac{dp}{(2\pi)^d}\, p_\mu p_\nu \frac{e^{-\frac{p^2}{\omega \tanh(\omega t_0)}}}{\tanh^d(\omega t_0)} = \frac{\omega^2}{2} \frac{\delta_{\mu\nu}}{(4\pi)^{\frac{d}{2}}} \Big(\frac{\omega}{\tanh(\omega t_0)}\Big)^{\frac{d}{2}-1}
$$

provides another factor of t_0 so that the remainder does not contribute to the residue at $s=1$. The assertion follows from $\hat{f}(0) = \int_{\mathbb{R}^d} dx\, f(x)$. □

Therefore, with the normalisation $\langle \xi, \eta \rangle = \frac{1}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d+2}{2})} \int_{\mathbb{R}^d} dx\, (\xi|\eta)$ of the scalar product in $\mathcal{H}$, the finiteness condition is satisfied.

It remains to discuss the orientability, for which we need the algebra

$$
(28)\qquad \mathcal{B} := \{ b \in \mathcal{A}'' \;:\; b, [\mathcal{D}, b] \in \bigcap_{m \in \mathbb{N}} \mathrm{dom}(\delta^m) \}\,.
$$

Clearly, $\mathcal{B}$ is unital and commutative; we now show that it contains the plane waves $u_\mu = e^{ix_\mu}$.

Lemma 4. $u_\mu = e^{ix_\mu} \in \mathcal{B}$.

Proof. From (73), which applies without change to $T = u_\mu$, we get (no summation over μ)

$$
(29)\qquad \delta^n u_\mu = \frac{(-\mathrm{i})^n}{\pi^n} \int_0^\infty \prod_{i=1}^n \frac{d\lambda_i \sqrt{\lambda_i}}{\langle \mathcal{D} \rangle^2 + \lambda_i} \{\underbrace{\partial_\mu, \dots, \{\partial_\mu}_{n \text{ derivatives}}, e^{\mathrm{i}x^\mu}\} \dots \} \prod_{j=1}^n \frac{1}{\langle \mathcal{D} \rangle^2 + \lambda_j}\,.
$$

We have

$$
\begin{aligned}
(30)\qquad & \Big(\prod_{i=1}^n \frac{1}{A + \lambda_i} \Big) B \\
&= \Big(\sum_{S \in \{1,2,\dots,n\}} (-1)^{|S|} \Big(\prod_{i \in S} \frac{1}{A + \lambda_i} \Big) (\mathrm{ad}(A))^{|S|}(B) \Big) \Big(\prod_{j=1}^n \frac{1}{A + \lambda_j} \Big)\,,
\end{aligned}
$$

where the sum runs over all subsets $S \subset \{1, 2, \dots, n\}$ including the empty set. After relabelling of the $|S|$ elements of S, which gives a factor $\binom{n}{|S|}$, we have

$$\delta^n(u_\mu) = \frac{(-\mathrm{i})^n}{\pi^n} \sum_{k=0}^{n} \binom{n}{k} \mathrm{i}^k \times \int_0^\infty \prod_{i=1}^{k} \frac{1}{\langle\mathcal{D}\rangle^2 + \lambda_i} \{ \underbrace{\partial_\mu, \dots, \{\partial_\mu}_{n+k \text{ derivatives}}, e^{\mathrm{i}x^\mu}\} \dots \} \prod_{j=1}^{n} \frac{d\lambda_j \sqrt{\lambda_j}}{(\langle\mathcal{D}\rangle^2 + \lambda_j)^2} . \tag{31}$$

The anticommutators can be arranged as a finite sum with $r \le n$ derivatives on the right and $l \le k$ derivatives on the left of $e^{\mathrm{i}x^\mu}$. Each such term is estimated by

$$\begin{aligned} &\Big\| \int_0^\infty \prod_{i=1}^{k} \frac{1}{\langle\mathcal{D}\rangle^2 + \lambda_i} (\partial_\mu)^l e^{\mathrm{i}x^\mu} (\partial_\mu)^r \langle\mathcal{D}\rangle^{-n} \prod_{j=1}^{n} \frac{d\lambda_j \sqrt{\lambda_j} \langle\mathcal{D}\rangle}{(\langle\mathcal{D}\rangle^2 + \lambda_j)^2} \Big\| \\ &\le \big\| \langle\mathcal{D}\rangle^{-2k} (\partial_\mu)^l \big\| \, \big\| (\partial_\mu)^r \langle\mathcal{D}\rangle^{-n} \big\| \Big\| \int_0^\infty \frac{d\lambda \sqrt{\lambda} \langle\mathcal{D}\rangle}{(\langle\mathcal{D}\rangle^2 + \lambda)^2} \Big\|^n , \end{aligned} \tag{32}$$

which is bounded because the integral in the second line evaluates to $\frac{\pi}{2}$. □

By the same arguments one shows that the algebra $C_b^\infty(\mathbb{R}^d)$ of smooth bounded functions with all derivatives bounded is contained in $\mathcal{B}$, and it is plausible that actually $\mathcal{B} = C_b^\infty(\mathbb{R}^d)$.

3.4. Orientability. Here the distinction between $\mathcal{D}_1$ and $\mathcal{D}_2$ is crucial again. It follows from the standard example of the compact case that

$$\boldsymbol{c} = \sum_{\sigma \in S_d} \epsilon(\sigma) \frac{\mathrm{i}^{\frac{d(d-1)}{2}}}{d!} (u_1 \cdots u_d)^{-1} \otimes u_{\sigma(1)} \otimes \cdots u_{\sigma(d)} \in Z_d(\mathcal{B}, \mathcal{B}) \tag{33}$$

is a Hochschild d-cycle, $b\boldsymbol{c} = 0$. From (22) and (3) we obtain

$$\begin{aligned} \boldsymbol{\gamma}_1 &:= \pi_{\mathcal{D}_1}(\boldsymbol{c}) = \mathrm{i}^{\frac{d(d+1)}{2}} (b^{\dagger 1} - b^1) \cdots (b^{\dagger d} - b^d) , \\ \boldsymbol{\gamma}_2 &:= \pi_{\mathcal{D}_2}(\boldsymbol{c}) = \mathrm{i}^{\frac{d(d+3)}{2}} (b^{\dagger 1} + b^1) \cdots (b^{\dagger d} + b^d) . \end{aligned} \tag{34}$$

Both $\boldsymbol{\gamma}_i$ commute with every element of $\mathcal{A}$ or $\mathcal{B}$. Using the anticommutation relations (3) and $(b^\mu)^* \equiv b^{\dagger\mu}$, we have

$$\boldsymbol{\gamma}_1^2 = 1 = \boldsymbol{\gamma}_2^2 , \qquad \boldsymbol{\gamma}_1^* = \boldsymbol{\gamma}_1 , \quad \boldsymbol{\gamma}_2^* = \boldsymbol{\gamma}_2 . \tag{35}$$

Decomposing the fermionic part of the Dirac operators $\mathcal{D}_i$ in $b^{\dagger\mu} \pm b^\mu$, we have

$$(b^{\dagger\mu} \pm b^\mu)\boldsymbol{\gamma}_1 = \pm(-1)^d \boldsymbol{\gamma}_1 (b^{\dagger\mu} \pm b^\mu) , \quad (b^{\dagger\mu} \pm b^\mu)\boldsymbol{\gamma}_2 = \mp(-1)^d \boldsymbol{\gamma}_2 (b^{\dagger\mu} \pm b^\mu) . \tag{36}$$

Therefore, $b^{\dagger\mu} \pm b^\mu$ and hence the $\mathcal{D}_i$ *always* (d even or odd) anticommute with the product $\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2$, which turns out to be (up to a factor) the $\mathbb{Z}_2$-grading $(-1)^{N_f}$ of the Hilbert space:

$$(-\mathrm{i})^d \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2 = \mathrm{i}^d \boldsymbol{\gamma}_2\boldsymbol{\gamma}_1 = (b^1 b^{\dagger 1} - b^{\dagger 1} b^1) \cdots (b^d b^{\dagger d} - b^{\dagger d} b^d) = (-1)^{N_f} . \tag{37}$$

This is quite different from conventional spectral triples [**11**] with a single operator $\mathcal{D}$.

3.5. The index formula. We let $\mathcal{H} = \mathcal{H}_{ev} \oplus \mathcal{H}_{odd}$ be the decomposition into even and odd subspaces with respect to the grading $(-1)^{N_f}$ induced by the fermion number operator N_f. The $\mathcal{D}_i$ are off-diagonal in this decomposition, $\mathcal{D}_i = \mathcal{D}_i^+ + \mathcal{D}_i^-$, with $\mathcal{D}_i^+ = \mathcal{D}_i\big|_{\mathcal{H}_{ev}} : \mathcal{H}_{ev} \to \mathcal{H}_{odd}$ and $\mathcal{D}_i^- = (\mathcal{D}_i^+)^* = \mathcal{D}_i\big|_{\mathcal{H}_{odd}} : \mathcal{H}_{odd} \to \mathcal{H}_{ev}$.

There is a well-defined index problem for $\mathcal{D}_i^+$ due to Elliott, Natsume and Nest [**26**]. The $\mathcal{D}_i^+$ are elliptic pseudodifferential operators in the sense of Shubin [**29**] with symbol $\mathfrak{a}_i$. Then, the analytic index

$$\text{index}\,(\mathcal{D}_i^+) = \dim \ker \mathcal{D}_i^+ - \dim \ker \mathcal{D}_i^- \tag{38}$$

can be computed by an index formula for the symbol $\mathfrak{a}_i$ as described below.

Following [**26**], we associate to (appropriate) operators $\mathcal{P}_\mathfrak{a} : \mathcal{S}(\mathbb{R}^n;\mathbb{C}^k) \to \mathcal{S}(\mathbb{R}^n;\mathbb{C}^k)$ the symbol $\mathfrak{a} \in M_k(C^\infty(T^*\mathbb{R}^n))$ by

$$(\mathcal{P}_\mathfrak{a}\eta)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n\times\mathbb{R}^n} d\xi\, dy\; e^{\mathrm{i}\langle x-y,\xi\rangle}\,\mathfrak{a}_i(x,\xi)\,\eta(y)\;, \qquad \eta \in \mathcal{S}(\mathbb{R}^n;\mathbb{C}^k)\;. \tag{39}$$

The symbol $\mathfrak{a}$ is said to be *elliptic of order* m if there exist $C, R > 0$ such that $\mathfrak{a}(x,\xi)^*\mathfrak{a}(x,\xi) \geq C(\|x\|^2 + \|\xi\|^2)^m 1_k$ for $\|x\|^2 + \|\xi\|^2 \geq R$.

For $m > 0$ one defines the graph projector

$$e_\mathfrak{a} = \begin{pmatrix} (1+\mathfrak{a}^*\mathfrak{a})^{-1} & (1+\mathfrak{a}^*\mathfrak{a})^{-1}\mathfrak{a} \\ \mathfrak{a}^*(1+\mathfrak{a}^*\mathfrak{a})^{-1} & \mathfrak{a}^*(1+\mathfrak{a}^*\mathfrak{a})^{-1}\mathfrak{a} \end{pmatrix} \in M_{2k}(C(T^*\mathbb{R}^n)) \tag{40}$$

and the matrix $\hat{e}_\mathfrak{a} = e_\mathfrak{a} - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in M_{2k}(C_0(T^*\mathbb{R}^n))$, i.e. $\hat{e}_\mathfrak{a}$ vanishes at infinity for $m > 0$ (the entries of $\hat{e}_\mathfrak{a}$ are of order $-m$). Using continuous fields of C^*-algebras, the following index theorem is proven in [**26**]:

Theorem 5. *If $\mathcal{P}_\mathfrak{a}$ is an elliptic pseudodifferential operator of positive order, then*

$$\text{index}\,(\mathcal{P}_\mathfrak{a}) = \frac{1}{(2\pi\mathrm{i})^n n!}\int_{T^*\mathbb{R}^n} \mathrm{tr}\big(\hat{e}_\mathfrak{a}(d\hat{e}_\mathfrak{a})^{2n}\big)\;, \tag{41}$$

where $T^*\mathbb{R}^n$ is oriented by $dx_1 \wedge d\xi_1 \wedge \cdots \wedge dx_n \wedge d\xi_n > 0$.

Let us return to our example. Restricting $\mathcal{D}_i^+$ to the even part of $\mathcal{H}_\infty$, we regard $\mathcal{D}_i^+$ as an operator $\mathcal{D}_i^+ : \mathcal{S}(\mathbb{R}^d;\mathbb{C}^{2^{d-1}}) \to \mathcal{S}(\mathbb{R}^d;\mathbb{C}^{2^{d-1}})$. The symbol $\mathfrak{a}_i \in M_{2^{d-1}}(C^\infty(T^*\mathbb{R}^d))$ of $\mathcal{D}_i^+$ is obtained from the action of $\mathfrak{Q}, \mathfrak{Q}^\dagger$ on the basis $e^{\mathrm{i}\langle\xi,x\rangle}|s_1,\dots,s_d\rangle_f$. For example, we have for $d = 2$ in the matrix bases $\left(\begin{smallmatrix} |0,0\rangle_f \\ |1,1\rangle_f \end{smallmatrix}\right)$ of $\big(\bigwedge(\mathbb{C}^d)\big)_{ev}$ and $\left(\begin{smallmatrix} |1,0\rangle_f \\ |0,1\rangle_f \end{smallmatrix}\right)$ of $\big(\bigwedge(\mathbb{C}^d)\big)_{odd}$ the representation

$$\mathfrak{a}_1(x_1,x_2,\xi_1,\xi_2) = \begin{pmatrix} \mathrm{i}\xi_1 + \omega x_1 & -(-\mathrm{i}\xi_2 + \omega x_2) \\ \mathrm{i}\xi_2 + \omega x_2 & -\mathrm{i}\xi_1 + \omega x_1 \end{pmatrix}\;. \tag{42}$$

The product $\mathfrak{a}_i(x,\xi)^*\mathfrak{a}_i(x,\xi)$ is the restriction of the symbol of H to the even subspace. This implies

$$\mathfrak{a}_i(x,\xi)^*\mathfrak{a}_i(x,\xi) = (\omega^2\|x\|^2 + \|\xi\|^2)1_{2^{d-1}}\;, \tag{43}$$

i.e. ellipticity of order 1 if $\omega > 0$. Note that the usual Dirac operator $\mathrm{i}\gamma^\mu\partial_\mu$ on $\mathbb{R}^d$ is not elliptic in this sense.

For $d = 2$ an already lengthy computation shows

$$\mathrm{tr}\big(\hat{e}_{\mathfrak{a}_1}\, d\hat{e}_{\mathfrak{a}_1} \wedge d\hat{e}_{\mathfrak{a}_1} \wedge d\hat{e}_{\mathfrak{a}_1} \wedge d\hat{e}_{\mathfrak{a}_1}\big) = -\frac{96\omega^2\, dx_1 \wedge d\xi_1 \wedge dx_2 \wedge d\xi_2}{(1+\omega^2x_1^2+\omega^2x_2^2+\xi_1^2+\xi_2^2)^5}\;, \tag{44}$$

which yields

$$\text{index}\,(\mathcal{D}_1^+) = \frac{1}{(2\pi \mathrm{i})^2 \cdot 2} \int_0^\infty 2\pi x\, dx \int_0^\infty 2\pi\xi\, d\xi\, \frac{(-96\omega^2)}{(1+\omega^2x^2+\xi^2)^5} = 1\;. \tag{45}$$

This is of course expected in any dimension d: the (one-dimensional) kernel of $\mathcal{D}_i^+$ is spanned by the Gaußian $e^{-\frac{\omega}{2}\|x\|^2}|0,\dots,0\rangle_f$, and the cokernel is trivial.

4. The spectral action for the $U(1)$-Higgs model

In the Connes-Lott spirit [**4**] we take the tensor product of the $(d=4)$-dimensional spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D}_1)$ with the finite Higgs spectral triple $(\mathbb{C}\oplus\mathbb{C},\mathbb{C}^2,M\sigma_1,\sigma_3)$, which is even with $\mathbb{Z}_2$-grading σ_3. Here, M is a real number, and σ_k are the Pauli matrices. For the bosonic sector considered here only the spectrum of $\mathcal{D}_i$ matters, so that $\mathcal{D}_1$ and $\mathcal{D}_2$ give identical results. The total Dirac operator $\mathcal{D}=\mathcal{D}_1\otimes\sigma_3+1\otimes M\sigma_1$ of the product triple becomes

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}_1 & M \\ M & -\mathcal{D}_1 \end{pmatrix}\;. \tag{46}$$

In this representation, the algebra is $\mathcal{A}\oplus\mathcal{A}\ni(f,g)$ with diagonal action by pointwise multiplication on $\mathcal{H}_{tot}=\mathcal{H}\oplus\mathcal{H}$. The commutator of $\mathcal{D}$ with (f,g) is

$$[\mathcal{D},(f,g)] = \begin{pmatrix} \partial_\mu f\otimes(b^{\dagger\mu}-b^\mu) & M(g-f) \\ M(f-g) & -\partial_\mu g\otimes(b^{\dagger\mu}-b^\mu) \end{pmatrix}\;. \tag{47}$$

This shows that selfadjoint fluctuated Dirac operators $\mathcal{D}_A=\mathcal{D}+\sum_i a_i[\mathcal{D},b_i]$ are of the form

$$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_1+\mathrm{i}A_\mu\otimes(b^{\dagger\mu}-b^\mu) & \phi\otimes 1 \\ \bar{\phi}\otimes 1 & -\mathcal{D}_1-\mathrm{i}B_\mu\otimes(b^{\dagger\mu}-b^\mu) \end{pmatrix}\;, \tag{48}$$

for real fields $A_\mu=\overline{A_\mu}$, $B_\mu=\overline{B_\mu}\in\mathcal{A}$ and a complex field $\phi\in\mathcal{A}$. The square of $\mathcal{D}_A$ is

$$\mathcal{D}_A^2 = \begin{pmatrix} H\otimes 1+\omega\otimes\Sigma+\mathrm{i}F_A+|\phi|^2\otimes 1 & D_\mu\phi\otimes(b^{\dagger\mu}-b^\mu) \\ -\overline{D_\mu\phi}\otimes(b^{\dagger\mu}-b^\mu) & H\otimes 1+\omega\otimes\Sigma+\mathrm{i}F_B+|\phi|^2\otimes 1 \end{pmatrix}\;, \tag{49}$$

where

$$D_\mu\phi := \partial_\mu\phi+\mathrm{i}(A_\mu-B_\mu)\phi\;, \tag{50}$$

$$\begin{aligned} F_A &:= \{\mathcal{D}_1, A_\mu\otimes(b^{\dagger\mu}-b^\mu)\}+\mathrm{i}A_\mu A_\nu\otimes(b^{\dagger\mu}-b^\mu)(b^{\dagger\nu}-b^\nu) \\ &= (-\{\partial_\mu,A^\mu\}-\mathrm{i}A_\mu A^\mu)\otimes 1+\frac{1}{4}F^A_{\mu\nu}\otimes[b^{\dagger\mu}-b^\mu, b^{\dagger\nu}-b^\nu] \end{aligned} \tag{51}$$

and similarly for F_B. Here, $F^A_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the $U(1)$-curvature (field strength), and the explicit appearance of x has dropped out in F_A because of $\{b^{\dagger\mu}+b^\mu, b^{\dagger\nu}-b^\nu\}=0$.

According to the spectral action principle [**6, 7**], the bosonic action depends only on the spectrum of the Dirac operator. Thus, by functional calculus, the most general form of the bosonic action is

$$S(\mathcal{D}_A) = \mathrm{Tr}\big(\chi(\mathcal{D}_A^2)\big) = \int_0^\infty dt\;\mathrm{Tr}(e^{-t\mathcal{D}_A^2})\hat{\chi}(t)\;, \tag{52}$$

for some function $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ for which the operator trace exists. The second equality is obtained by Laplace transformation, which produces the inverse Laplace transform $\hat{\chi}$ of $\chi(s) = \int_0^\infty dt\, e^{-st}\hat{\chi}(t)$. One has

$$\chi_z := \int_0^\infty dt\, t^z \hat{\chi}(t) = \begin{cases} \dfrac{1}{\Gamma(-z)} \displaystyle\int_0^\infty ds\, s^{-z-1}\chi(s) & \text{for } z \notin \mathbb{N}\,, \\ (-1)^k \chi^{(k)}(0) & \text{for } z = k \in \mathbb{N}\,. \end{cases} \tag{53}$$

To compute the traces $\mathrm{Tr}(e^{-t\mathcal{D}_A^2})$ we write $\mathcal{D}_A^2 = \mathrm{H}_0 - V$, with $\mathrm{H}_0 := H + \omega\Sigma$, and consider the Duhamel expansion

(54)

$$\begin{aligned} &e^{-t_0(\mathrm{H}_0-V)} \\ &= e^{-t_0\mathrm{H}_0} - \int_0^{t_0} dt_1\, \frac{d}{dt_1}\big(e^{-(t_0-t_1)(\mathrm{H}_0-V)}e^{-t_1\mathrm{H}_0}\big) \\ &= e^{-t_0\mathrm{H}_0} + \int_0^{t_0} dt_1\, \big(e^{-(t_0-t_1)(\mathrm{H}_0-V)}Ve^{-t_1\mathrm{H}_0}\big) \\ &= e^{-t_0\mathrm{H}_0} + \int_0^{t_0} dt_1\, \big(e^{-(t_0-t_1)\mathrm{H}_0}Ve^{-t_1\mathrm{H}_0}\big) \\ &+ \int_0^{t_0} dt_1 \int_0^{t_0-t_1} dt_2\, \big(e^{-(t_0-t_1-t_2)\mathrm{H}_0}Ve^{-t_2\mathrm{H}_0}Ve^{-t_1\mathrm{H}_0}\big) + \dots \\ &+ \int_0^{t_0} dt_1 \dots \int_0^{t_0-t_1-\dots-t_{n-1}} dt_n\, \big(e^{-(t_0-t_1-\dots-t_n)\mathrm{H}_0}(Ve^{-t_n\mathrm{H}_0})\cdots(Ve^{-t_1\mathrm{H}_0})\big) + \dots \\ &= e^{-t_0\mathrm{H}_0} + \sum_{n=1}^\infty t_0^n \int_{\Delta^n} d^n\alpha \Big(e^{-t_0(1-|\alpha|)\mathrm{H}_0}\prod_{j=1}^n (Ve^{-t_0\alpha_j\mathrm{H}_0})\Big)\,, \end{aligned}$$

where the integration is performed over the standard n-simplex $\Delta^n := \{\alpha := (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n\,,\ \alpha_i \geq 0\,,\ |\alpha| := \alpha_1 + \dots + \alpha_n \leq 1\}$.

Using $\mathrm{tr}(e^{\omega\Sigma t}) = (2\cosh(\omega t))^4$ and the Mehler kernel (76), the vacuum contribution without V is

$$\begin{aligned} \mathrm{Tr}(e^{-t(H+\omega\Sigma)\otimes 1_2}) &= \big(2\,\mathrm{tr}(e^{\omega\Sigma t})\big)\int_{\mathbb{R}^4} dx\, e^{-tH}(x,x) \\ &= 2(2\cosh(\omega t))^4 \cdot \Big(\frac{\omega}{2\pi\sinh(2\omega t)}\Big)^2 \int_{\mathbb{R}^4} dx\, e^{-\omega\tanh(\omega t)\|x\|^2} \\ &= \frac{2}{\tanh^4(\omega t)}\,. \end{aligned} \tag{55}$$

With $\coth^4(\omega t) = \frac{1}{(\omega t)^4} + \frac{4}{3(\omega t)^2} + \frac{26}{45} + \mathcal{O}(t^2)$ we get under the usual assumption $\chi^{(k)}(0) = 0$ for $k = 1, 2, 3, \dots$ the asymptotic expansion[1]

$$S_0(\mathcal{D}_A) = \frac{2\chi_{-4}}{\omega^4} + \frac{8\chi_{-2}}{3\omega^2} + \frac{52\chi_0}{45}\,. \tag{56}$$

[1]The Laplace transformation for the vacuum contribution can be performed exactly. For powers of $\coth x = \frac{1+e^{-2x}}{1-e^{-2x}}$ we have

$$\Big(\frac{1+y}{1-y}\Big)^n = 1 + \sum_{k=1}^\infty \underbrace{\frac{(k+n-1)!}{k!}\, {}_2F_1\Big(\begin{matrix} -k\,,\ -n \\ 1-k-n \end{matrix}\Big| -1\Big)}_{=F_n(k)} y^k\,.$$

For the further computation we distinguish the vertices (see (49), (50) and (51))

(57) $$\begin{aligned} V_1 &:= \mathrm{diag}\big(\mathrm{i}\{\partial^\mu, A_\mu\}\otimes 1, \mathrm{i}\{\partial^\mu, B_\mu\}\otimes 1\big)\;, \\ V_2 &:= \mathrm{diag}\big(-A_\mu A^\mu\otimes 1 - |\phi^2|\otimes 1, -B_\mu B^\mu \otimes 1 - |\phi^2|\otimes 1\big)\;, \\ V_3 &:= \mathrm{diag}\big(-\mathrm{i}F^A_{\mu\nu}\otimes \tfrac{1}{4}[b^{\dagger\mu}-b^\mu, b^{\dagger\nu}-b^\nu],\; -\mathrm{i}F^B_{\mu\nu}\otimes \tfrac{1}{4}[b^{\dagger\mu}-b^\mu, b^{\dagger\nu}-b^\nu]\big)\;, \\ V_4 &= \begin{pmatrix} 0 & -D_\mu\phi\otimes(b^{\dagger\mu}-b^\mu) \\ \overline{D_\mu\phi}\otimes(b^{\dagger\mu}-b^\mu) & 0 \end{pmatrix}\;. \end{aligned}$$

We compute the traces of the spectral action in the same way as the residues of the ζ-function in Appendix A. The main step consists in computing the following trace:

(58) $$S_{t_1,\dots,t_v}(\tilde{V}_1,\dots,\tilde{V}_v) := \mathrm{Tr}\Big(\tilde{V}_1 e^{-t_1 H}\tilde{V}_2 e^{-t_2 H}\cdots \tilde{V}_v e^{-t_v H}\Big)\;,$$

either with $\tilde{V}_i = f_i$ or $\tilde{V}_i = -\mathrm{i}\{\partial_\mu, f_i^\mu\} = -\mathrm{i}(\partial_\mu f_i^\mu) - 2\mathrm{i} f_i^\mu \partial_\mu$. We realise this alternative as $\tilde{V}_i = f_i^{1-n_i}\{-\mathrm{i}\partial_\mu, f^\mu\}^{n_i}$ with $n_i\in\{0,1\}$:

(59)

$$\begin{aligned} &S^{n_1\dots n_v}_{t_1,\dots t_v}(f_1,\dots,f_v) \\ &= \sum_{k_1=0}^{n_1}\cdots\sum_{k_v=0}^{n_v}\omega^{k_1+\dots+k_v}\int_{(\mathbb{R}^4\times\mathbb{R}^4)^v}\Big(\prod_{i=1}^v \frac{dx_i dp_i}{(2\pi)^4}\Big) \\ &\times\Big(\prod_{i=1}^v \hat{f}_i^{1-n_i}(p_i)\Big(\hat{f}_i^{\mu_1}(p_i) p_{i,\mu_i}^{1-k_i} P^{k_i}_{\mu_i}\Big(2\omega t_i, \frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_{i+1}}\Big)\Big)^{n_i}\Big)\Big(\prod_{i=1}^v e^{-t_i H}(x_i, x_{i+1}) e^{\mathrm{i}p_i x_i}\Big) \\ &= \sum_{k_1=0}^{n_1}\cdots\sum_{k_v=0}^{n_v}\int_{(\mathbb{R}^4)^v}\Big(\prod_{i=1}^v\frac{dp_i}{(2\pi)^4}\Big)\frac{\omega^{k_1+\dots+k_v}}{(2\sinh(\omega(t_1+\dots+t_v)))^4} \\ &\times\Big(\prod_{i=1}^v \hat{f}_i^{1-n_i}(p_i)\Big(\hat{f}_i^{\mu_1}(p_i) p_{i,\mu_i}^{1-k_i} P^{k_i}_{\mu_i}\Big(2\omega t_i, \frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_{i+1}}\Big)\Big)^{n_i}\Big)e^{-\frac{1}{4}pQ^{-1}p}\;, \end{aligned}$$

where P_μ and Q^{-1} are given in (83) and (86). From the formulae analogous to (88) and (90) we thus obtain

(60)

$$\begin{aligned} &S^{n_1\dots n_v}_{t_1,\dots t_v}(f_1,\dots,f_v) \\ &= \sum_{\substack{k_1+r_{11}+\dots+r_{1v}=n_1,\dots,\\ k_1+r_{v1}+\dots+r_{vv}=n_v,\\ r_{ii}=0\;,\; r_{ij}=r_{ji}}}\int_{(\mathbb{R}^4)^v}\Big(\prod_{i=1}^v\frac{dp_i}{(2\pi)^4}\Big)\frac{1}{(2\sinh(\omega t))^4}\Big(\prod_{i=1}^v \hat{f}_i^{1-n_i}(p_i)\big(\hat{f}_i^{\mu_i}(p_i)\big)^{n_i}\Big) \\ &\times\Big(\prod_{i=1}^v\Big(\sum_{j\neq i}\frac{\sinh(\omega t_{ji})}{\sinh(\omega t)}p_{j,\mu_i}\Big)^{k_i}\Big)\Big(\prod_{i\le j}\Big(2\omega\delta_{\mu_i\mu_j}\frac{\cosh(\omega t_{ji})}{\sinh(\omega t)}\Big)^{r_{ij}}\Big)e^{-\frac{1}{4}pQ^{-1}p}\;, \end{aligned}$$

Particular values are $F_1(k)=2$, $F_2(k)=4k$, $F_3(k)=8k^2+4$, $F_4(k)=16k^3+32k$ and $F_5(k)=32k^4+160k^2+48$. Inserted into (52) we obtain after Laplace transformation

$$S_0(\mathcal{D}_A) = 2\chi(0) + \sum_{k=1}^\infty (32k^3+64k)\chi(2\omega k)\;.$$

where $t_{ji} := t_j + \cdots + t_{i-1} - t_i - \cdots - t_{j-1}$ and $t := t_1 + \cdots + t_v$

For the spectral action we are interested in the small-t behaviour. From (86) we know that the singularity in $\sinh^{-4-\sum_{i<j} r_{ij}}(\omega t)$ is protected by $\exp(-\frac{(p_1+\cdots+p_v)^2}{4\omega\tanh(\omega t)})$ unless the total momentum is conserved. Thus, Taylor-expanding the prefactor about $p_v = -(p_1 + \cdots + p_{v-1})$ up to order ρ and Gaußian integration in p_v yields

$$S^{n_1,\ldots n_v}_{t_1,\ldots,t_v} = \mathcal{O}(t^{-2-\lfloor\frac{n_1+\cdots+n_v}{2}\rfloor+\lceil\frac{\rho}{2}\rceil}) .$$

To obtain the spectral action, there are apart from the (at most) t-neutral matrix trace the v integrations over $t_1, \ldots, t_v$ which contribute another power of t^v. If there are v_i vertices of type V_i present, with $v_1 + \cdots + v_4 = v$, then $n_1 + \cdots + n_v = v_1$, and we have for such a contribution

$$S_t(V_1^{v_1}, \ldots, V_4^{v_4}) = \mathcal{O}(t^{-2+v_2+v_3+v_4+\lceil\frac{v_1}{2}\rceil+\lceil\frac{\rho}{2}\rceil}) .$$

Only the non-positive exponents contribute to the asymptotic expansion so that it suffices to compute the following traces of vertex combinations:

(1) V_2 with Taylor expansion up to order $\rho = 2$ (V_3 and V_4 are traceless, and in V_1 alone there is necessarily $k_1 = n_1 = 1$ and then no sum over $i \neq j$),
(2) V_1V_1 with Taylor expansion up to order $\rho = 2$,
(3) V_1V_2, V_2V_1 and $V_1V_1V_2$, $V_1V_2V_1$, $V_2V_1V_1$ with Taylor expansion up to order $\rho = 0$,
(4) V_2V_2, V_3V_3 and V_4V_4 with Taylor expansion up to order $\rho = 0$ (mixed products are traceless),
(5) $V_1V_1V_1$ and $V_1V_1V_1V_1$ with Taylor expansion up to order $\rho = 0$.

We compute these vertex combinations in Appendix B. The spectral action is the sum of (100), (104), (107), (109), (111), (113) and (115). Altogether, the spectral action of the Abelian Higgs model reads

$$\begin{aligned}S(\mathcal{D}_A) = &\frac{2\chi_{-4}}{\omega^4} + \frac{8\chi_{-2}}{3\omega^2} + \frac{52\chi_0}{45} \\ &+ \frac{\chi_0}{\pi^2}\int_{\mathbb{R}^4} dx\,\Big\{\frac{5}{12}(F_A^{\mu\nu}F_{A\mu\nu} + F_B^{\mu\nu}F_{B\mu\nu}) + \overline{D_\mu\phi}(D^\mu\phi) \\ &\qquad - \frac{2\chi_{-1}}{\chi_0}|\phi|^2 + |\phi|^4 + 2\omega^2\|x\|^2|\phi|^2\Big\}(x) .\end{aligned} \tag{61}$$

The scalar sector (putting $A = B = 0$ and ignoring the constant) is almost identical to the commutative version of the renormalisable ϕ^4-action [**14**],

$$S(\mathcal{D}_A)|_{A=b=0} = \frac{\chi_0}{\pi^2}\int_{\mathbb{R}^4} dx\,\Big\{\partial_\mu\bar\phi(\partial^\mu\phi) + 2\omega^2\|x\|^2|\phi|^2 - \frac{2\chi_{-1}}{\chi_0}|\phi|^2 + |\phi|^4\Big\}(x) . \tag{62}$$

The crucial difference is the negative mass squared term, which leads to a drastically different vacuum structure, as shown in the next section.

5. Field equations

We can assume the solution of the corresponding equation of motion to be given by $A = B = 0$ and ϕ a real function. Then, the Euler-Lagrange equation reads

$$-\Delta\phi + 2\omega^2\|x\|^2\phi + 2\phi^3 - 2\tfrac{\chi_{-1}}{\chi_0}\phi = 0 . \tag{63}$$

In terms of the rescaled radius $r = 2^{\frac{1}{4}}\sqrt{\omega}\|x\|$ and the rescaled field $\phi = \frac{\pi}{\sqrt{2}\chi_0}\varphi$ we have the rotationally invariant equation

$$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)\,, \qquad \mu^2 = \frac{\chi_{-1}}{\sqrt{8}\omega\chi_0}\,, \quad \lambda = \frac{\pi^2}{\sqrt{2}\omega\chi_0}\,. \tag{64}$$

We expand φ in terms of eigenfunctions of the four-dimensional harmonic oscillator,

$$\varphi = \frac{2}{\sqrt{\lambda}}\sum_{n=0}^{\infty} c_n\varphi_n\,, \qquad \varphi_n := e^{-\frac{r^2}{2}}L_n^1(r^2)\,, \qquad \Big(-\frac{d^2}{dr^2} - \frac{3}{r}\frac{d}{dr} + r^2\Big)\varphi_n = 4(n+1)\varphi_n\,. \tag{65}$$

We are thus left with the equation

$$\sum_{n=0}^{\infty} c_n(\mu^2 - n - 1)\varphi_n = \sum_{k,l,m=0}^{\infty} c_k c_l c_m \varphi_k\varphi_l\varphi_m \tag{66}$$

or, using the orthogonality relation,

$$c_n(\mu^2-n-1) = \frac{1}{(n+1)}\sum_{k,l,m=0}^{\infty} c_k c_l c_m \int_0^{\infty} dt\, e^{-2t}\, t\, L_k^1(t)\, L_l^1(t)\, L_m^1(t)\, L_n^1(t)\,. \tag{67}$$

The generating function $(1-z)^{-\alpha-1}\exp(-\frac{xz}{1-z}) = \sum_{k=0}^{\infty} L_k^{\alpha}(t)z^k$ is used to obtain

$$c_n(\mu^2 - n - 1) = \sum_{k,l,m=0}^{\infty} \frac{c_k c_l c_m}{k!l!m!}\Big(\frac{d^k}{dw^k}\frac{d^l}{dy^l}\frac{d^m}{dz^m}\frac{(1 - yz - yw - wz + 2wyz)^n}{(2 - y - z - w + yzw)^{n+2}}\Big)_{w=y=z=0}\,. \tag{68}$$

With a cut-off N for the matrix indices, this equation can be solved numerically. It turns out that except for a region about $r = 4\mu^2$ the convergence is quite good. Figure 1 contains plots of the vacuum solution $\varphi_{vac}(r)$ for $4\mu^2 = 9$ and $4\mu^2 = 13$ compared with the ellipse $\varphi^2 + \frac{1}{4}r^2 = \mu^2$. We learn that $\varphi_{vac}(r) < \sqrt{\mu^2 - \frac{1}{4}r^2}$ due to the negative curvature $\frac{1}{\varphi}(\varphi'' + \frac{3}{r}\varphi') < 0$ which effectively reduces μ^2. For $r > 2\mu$ we should have $\varphi_{vac}(r) = 0$ as the only solution[2]. We also expect that for $\mu \to \infty$, where the ellipse becomes flat, the vacuum solution approaches its limiting ellipse. This limit is connected to the limit $\omega \to 0$, i.e $r = 2^{\frac{1}{4}}\sqrt{\omega}\|x\| \to 0$. In this limit the usual constant Higgs vacuum is recovered:

$$\lim_{\omega\to 0}\phi^2 = \frac{\pi^2}{2\chi_0^2}\frac{4\mu^2}{\lambda} = \frac{\chi_{-1}}{\chi_0^2} = \text{const}\,. \tag{69}$$

For finite ω the cut-off for φ_{vac} at $r = 2\mu$ implies that φ_{vac} is an integrable function.

The vacuum solution

$$\frac{2}{\sqrt{\lambda}}\varphi_{vac} = \sqrt{\frac{4\mu^2}{\lambda}}\frac{\varphi_{vac}}{\mu} = \sqrt{\frac{2\chi_{-1}}{\pi^2}}\frac{\varphi_{vac}}{\mu} \tag{70}$$

[2]The numerical convergence in the figure is bad for $r \approx 2\mu$.

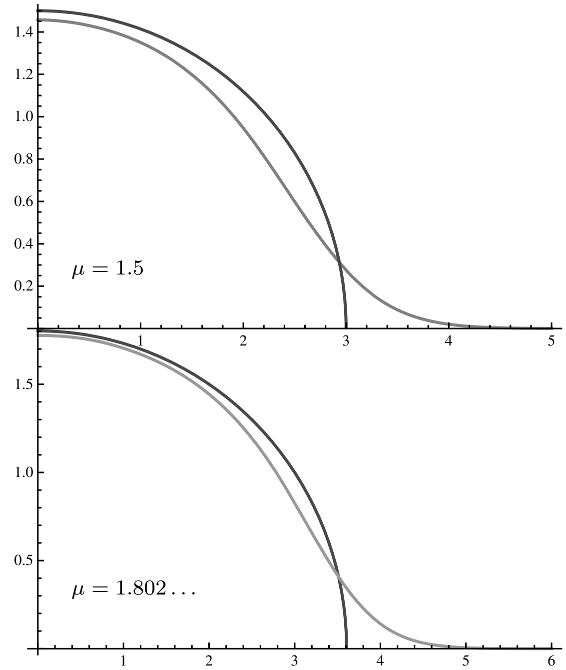

FIGURE 1. The lower curve at $r = 0$ shows $\varphi_{vac}(r)$ in units of $\frac{2}{\sqrt{\lambda}}$, with cut-off at $N = 10$. The upper curve at $r = 0$ is the ellipse $\varphi^2 + \frac{1}{4}r^2 = \mu^2$. The error is below 1% for $r < 1.8\mu$. The true curve $\varphi_{vac}(r)$ is expected to stay always below the ellipse and to connect smoothly (at least C^2) to $\varphi_{vac} = 0$ for $r > 2\mu$.

sets the scale for the bare masses of gauge fields and fermions. On the other hand, the bare mass of the Higgs field is obtained from the shift of the Higgs potential into its minimum and therefore reads

$$\sqrt{\sqrt{2}\omega((r^2 - 4\mu^2) + 12\mu^2 \frac{\varphi_{vac}^2}{\mu^2})} = \sqrt{\frac{4\chi_{-1}}{\chi_0}} \frac{\sqrt{\frac{3}{2}\varphi_{vac}^2 - \frac{1}{2}\mu^2 + \frac{1}{8}r^2}}{\mu} . \tag{71}$$

We compare in Figure 2 the scale $\frac{\varphi_{vac}}{\mu}$ of gauge field mass with the scale $\frac{1}{\mu}\sqrt{\frac{3}{2}\varphi_{vac}^2 - \frac{1}{2}\mu^2 + \frac{1}{8}r^2}$ of the bare Higgs mass. Reinserting ω we obtain the following *two-phase structure*:

- *A spontaneously broken phase for* $\omega^2\|x\|^2 < \frac{\chi_{-1}}{\chi_0}$.
  Fermions, gauge fields and Higgs field are all massive, with the Higgs mass slightly smaller than the prediction from noncommutative geometry [**9**]. In particular, this phase is the only existing one in the limit $\omega \to 0$, and in this limit the NCG prediction is recovered.
- *An unbroken phase for* $\omega^2\|x\|^2 > \frac{\chi_{-1}}{\chi_0}$.
  Fermions and gauge fields are massless, whereas the Higgs field remains massive.

The model we have studied is a toy model. But, as it is a noncommutative geometry like that of the NCG-formulation of the Standard Model [**9**], it is ultimately an experimental question to set limits on the frequency parameter ω. To

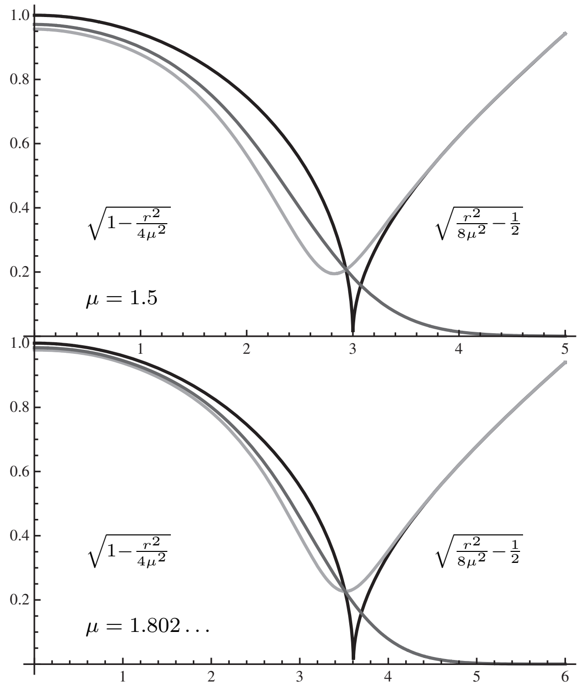

FIGURE 2. The scale $\frac{\varphi_{vac}}{\mu}(r)$ (middle curve at $r=0$) of the gauge field mass compared with the scale $\frac{1}{\mu}\sqrt{\frac{3}{2}\varphi_{vac}^2(r) - \frac{1}{2}\mu^2 + \frac{1}{8}r^2}$ of the Higgs field mass (lowest curve at $r = 0$) and the limiting ellipse $s^2 + \frac{r^2}{4\mu^2} = 1$ and hyperbola $\frac{r^2}{8\mu^2} - s^2 = \frac{1}{2}$. Cut-off again at $N = 10$. The true curve $\frac{\varphi_{vac}}{\mu}(r)$ should always stay below the ellipse and connect smoothly to $\frac{\varphi_{vac}}{\mu} = 0$ for $r > 2\mu$. The true curve $\frac{1}{\mu}\sqrt{\frac{3}{2}\varphi_{vac}^2(r) - \frac{1}{2}\mu^2 + \frac{1}{8}r^2}$ should stay below $\frac{\varphi_{vac}}{\mu}$ for $r < 2\mu$, whereas for $r > 2\mu$ one should exactly have $\frac{1}{\mu}\sqrt{\frac{3}{2}\varphi_{vac}^2 - \frac{1}{2}\mu^2 + \frac{1}{8}r^2}(r) = \sqrt{\frac{r^2}{8\mu^2} - \frac{1}{2}}$.

be compatible with both high energy and cosmological data, ω has to be extremely small. We definitely live in the spontaneously broken phase $\omega^2\|x\|^2 < \frac{\chi_{-1}}{\chi_0}$, and the observable universe is very close to $\omega^2\|x\|^2 = 0$. Nevertheless, a regulating $\omega \neq 0$ has some nice consequences such as integrability of the Higgs vacuum and integrability of the cosmological constant.

One may speculate how an $\omega \neq 0$ can be detected. We mentioned the reduction of the ratio between Higgs mass and Z mass compared with the NCG prediction. However, in the presence of $\omega \neq 0$ the β-functions must be recomputed so that at the moment no prediction is possible. In cosmology, limits for ω could be obtained from precision measurements of the ratio between the proton mass and the electron mass at far distance. The electron mass which governs the atomic spectra via the Rydberg frequency should vary in the same way as the Higgs scale $\frac{\varphi_{vac}}{\mu}$. On the other hand, the proton mass arises mainly from broken scale invariance in QCD and therefore can be regarded as constant. This means that the gravitational energy of a standard star is constant whereas its transition into radiation energy might vary with the position of the star in the universe. Observational limits on such a variation would limit the value of ω.

Another observable consequence could be a variation of the cosmological constant. The Higgs potential at the vacuum solution is negative and hence reduces the volume term of the cosmological constant. Thus, the effective cosmological constant would increase with the radius (the masses of gauge fields and fermions dissolve into the cosmological constant).

6. Conclusion and perspectives

We have proposed a definition for non-compact spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ where the algebra is allowed to be non-unital but the resolvent of the operator $\mathcal{D}$ remains compact. The metric dimension is defined via the dimension spectrum; it is (in general) different from the noncommutative dimension given by the decay rate of the characteristic values of the resolvent.

Our definition excludes non-compact manifolds with the standard Dirac operator, but this is necessary for a well-defined index problem and a well-defined spectral action in the non-compact case. An example for our definition is given by operators $\mathcal{D}$ which are square roots of the d-dimensional harmonic oscillator Hamiltonian $-\Delta + \omega^2 x^2$. These square roots are constructed by conjugation of the partial derivatives with $e^{\pm\omega h}$, where h is the Morse function. This relates to supersymmetric quantum mechanics, in particular to a special case of Witten's work [**21**] on Morse theory.

The most involved piece of work was the computation of the dimension spectrum which showed that the metric dimension is the oscillator dimension and that all residues of the operator zeta function are local. Due to its relation to supersymmetry, there are in fact two Dirac operators $\mathcal{D}_1$ and $\mathcal{D}_2$, which define two distinct images $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ of the d-dimensional volume form, and only the product $\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2$ defines the $\mathbb{Z}_2$-grading.

We have computed the spectral action for the corresponding Connes-Lott two-point model. In contrast to standard $\mathbb{R}^d$, the spectral action is finite also in the cosmological constant part. The result is an Abelian Higgs model with additional harmonic oscillator potential for the Higgs field. The resulting field equations show a phase transition phenomenon: There is a spontaneously broken phase below a critical radius determined by the oscillator frequency ω, which for small enough ω is qualitatively identical to standard Higgs models. Possible observable consequences are discussed at the end of the previous section. Above the critical radius we have an unbroken phase with massless gauge fields. This phase is necessary to have an integrable vacuum solution for the Higgs field.

The class of spectral triples we proposed deserves further investigation. We show with V. Gayral [**18**] that there is an isospectral Moyal deformation of the harmonic oscillator spectral triple. Some ideas appeared already in our preprint [**19**] with H. Grosse, but the mathematical structure was unclear at that point. The field equations of the preprint [**19**] are correct, but their "solution" is wrong. It misses the phase transitions which we first observed for the commutative model in the present paper. We expect that the phase structure is much richer in the Moyal-deformed model. A hint can already be found in the pure gauge field sector, which leads in terms of "covariant coordinates" to the field equation $[X^\mu, [X_\mu, X_\nu]] = 0$.

This equation has the Moyal deformation $[X_\mu, X_\nu] = \mathrm{i}\Theta_{\mu\nu} = \text{const}$ as a solution, but also commutative coordinates $[X_\mu, X_\nu] = 0$; the preferred solution arises from a subtle interplay with the boundary conditions. One may speculate that these boundary conditions change with the temperature of the universe, so that the (non)commutative geometry could emerge through a cascade of phase transitions when the universe cools down. The Moyal-deformed harmonic oscillator spectral triple could serve as an excellent toy model to study these transitions.

On the mathematical side, the relation to supersymmetric quantum mechanics needs further study. In particular, a real structure (or better several real structures) must be identified to reduce the multiplicity of the action of the algebra from its present value 2^d to $2^{\frac{d}{2}}$ in order to support a Spin^c structure. One should also allow for a non-trivial projection e to define the smooth subspace $\mathcal{H}_\infty = e\mathcal{A}^n$ of the Hilbert space. The corresponding action of $\mathcal{D}_i$ or its components $\mathfrak{Q}, \mathfrak{Q}^\dagger$ would then permit a complete reformulation of Witten's approach [**21**] to Morse theory in the framework of spectral triples and noncommutative index theory.

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Appendix A. Proof of Theorem 2

Let $\mathcal{D}$ denote $\mathcal{D}_1$ or $\mathcal{D}_2$. The spectral identity $A = \dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{d\lambda}{\sqrt{\lambda}} \frac{A^2}{A^2+\lambda}$ for a positive selfadjoint operator A leads to

(72) $$\delta T = \frac{1}{\pi}\int_0^\infty d\lambda\,\sqrt{\lambda}\frac{1}{\langle\mathcal{D}\rangle^2+\lambda}[\mathcal{D}^2, T]\frac{1}{\langle\mathcal{D}\rangle^2+\lambda}\;.$$

From (15) we recall that $\mathcal{D}^2 = H + \omega\Sigma$, where $H = -\partial^\mu\partial_\mu + \omega^2 x_\mu x^\mu$ and $\Sigma = [b_\mu^\dagger, b^\mu]$ satisfy $[H, \Sigma] = 0$. This implies

(73)
$$\delta^n T = \sum_{k=0}^m \binom{n}{k}\big(\omega\,\mathrm{ad}(\Sigma)\big)^{n-k}\bigg(\frac{1}{\pi^n}\int_0^\infty \prod_{i=1}^n \frac{d\lambda_i\,\sqrt{\lambda_i}}{\langle\mathcal{D}\rangle^2+\lambda_i}(\mathrm{ad}(H))^k(T)\prod_{j=1}^n \frac{1}{\langle\mathcal{D}\rangle^2+\lambda_j}\bigg).$$

The case $T = [\mathcal{D}_1, f] = \partial_\mu f \otimes (b^{\dagger\mu} - b^\mu)$ or $T = [\mathcal{D}_2, f] = \partial_\mu f \otimes (\mathrm{i}b^{\dagger\mu} + \mathrm{i}b^\mu)$ is also reduced to $T = f$; only $\mathrm{ad}(\Sigma)$ distinguishes them, and each application of $\mathrm{ad}(\Sigma)$ makes δT more regular. It is therefore sufficient to study $T = f$ and $k = n$. Using

$[H,f] = -(\Delta f) - 2(\partial^\mu f)\partial_\mu = -\{\partial_\mu, \partial^\mu f\}$, we have

$$\delta^n f = \sum_{k=0}^{n} \binom{n}{k} 2^k \frac{(-1)^n}{\pi^n} \int_0^\infty \prod_{i=1}^{n} \frac{d\lambda_i \sqrt{\lambda_i}}{\langle\mathcal{D}\rangle^2 + \lambda_i} \times (\Delta^{n-k}\partial^{\mu_1}\cdots\partial^{\mu_k} f)\partial_{\mu_1}\cdots\partial_{\mu_k} \prod_{j=1}^{n} \frac{1}{\langle\mathcal{D}\rangle^2 + \lambda_j} . \tag{74}$$

By linearity, it suffices to consider $\phi = (\delta^{n_1} f_1)\cdots(\delta^{n_v} f_v)$. The most convenient way is to compute $\zeta_\phi(z)$ as a trace over position space kernels,

$$\begin{aligned}\zeta_\phi(z) &:= \mathrm{Tr}\big((\delta^{n_1} f_1)\cdots(\delta^{n_v} f_v)\langle\mathcal{D}\rangle^{-z}\big) \\ &= \mathrm{tr}\bigg(\int_0^\infty dt_0\, \frac{t_0^{\frac{z}{2}-1}}{\Gamma(\frac{z}{2})} \int_{(\mathbb{R}^d)^v} \Big(\prod_{i=1}^{v} dy_i\Big)(\delta^{n_1} f_1)(y_1,y_2)\cdots(\delta^{n_{v-1}} f_{v-1})(y_{v-1},y_v) \\ &\qquad \times (\delta^{n_v} f_v)(y_v,y_0)(e^{-t_0\langle\mathcal{D}\rangle^2})(y_0,y_1)\bigg) .\end{aligned} \tag{75}$$

The remaining trace tr is taken in $\bigwedge(\mathbb{C}^d)$. Further evaluation is possible thanks to the d-dimensional Mehler kernel

$$e^{-tH}(x,y) = \Big(\frac{\omega}{2\pi\sinh(2\omega t)}\Big)^{\frac{d}{2}} e^{-\frac{\omega}{4}\coth(\omega t)\|x-y\|^2 - \frac{\omega}{4}\tanh(\omega t)\|x+y\|^2} , \tag{76}$$

for $x, y \in \mathbb{R}^d$, which solves the differential equation $(\frac{d}{dt} + H_x)e^{-tH}(x,y) = 0$ with initial condition $\lim_{t\to 0} e^{-tH}(x,y) = \delta(x-y)$. Uniqueness of the solution implies

$$\int_{\mathbb{R}^d} dy\; e^{-t_1 H}(x,y) e^{-t_2 H}(y,z) = e^{-(t_1+t_2)H}(x,z) . \tag{77}$$

We can therefore recombine left and right Mehler kernels

$$\frac{1}{\langle\mathcal{D}\rangle^2 + \lambda_{i,j_i}} = \int_0^\infty dt_{i,j_i}\; e^{-t_{i,j_i}(H+\omega\Sigma+1+\lambda_{i,j_i})} \tag{78}$$

in (74) and integrate over λ_{i,j_i}:

$$\begin{aligned}&(\delta^{n_i} f_i)(y_i,y_{i+1}) \\ &= \sum_{k_i=0}^{n_i} \binom{n_i}{k_i} 2^{k_i} \frac{(-1)^{n_i}}{(2\sqrt{\pi})^{n_i}} \int_0^\infty \prod_{j_i=1}^{n_i} \frac{dt_{i,j_i} ds_{i,j_i}}{(t_{i,j_i}+s_{i,j_i})^{\frac{3}{2}}} e^{-(1+\omega\Sigma)(S_i+T_i)} \\ &\times \int_{\mathbb{R}^d} dx_i\, e^{-S_i H}(y_i,x_i)(\Delta^{n_i-k_i}\partial^{\mu_1^i}\cdots\partial^{\mu_{k_i}^i} f_i)(x_i) \frac{\partial^{k_1}}{\partial x_i^{\mu_1^i}\cdots\partial x_i^{\mu_{k_i}^i}} e^{-T_i H}(x_i,y_{i+1}) ,\end{aligned} \tag{79}$$

where $S_i := \sum_{j_i=1}^{n_i} s_{j_i}$ and $T_i := \sum_{j_i=1}^{n_i} t_{j_i}$. We insert this into (75), move $e^{-S_1 H}$ under the trace to the end, and perform the y_i-integrations which combine the Mehler kernels into $e^{-\frac{\tau_i}{2\omega}H}(x_i,x_{i+1})$, with $\tau_i = 2\omega(T_i + S_{i+1} + \delta_{iv}t_0)$ and the convention $v+1 \equiv 1$. The remaining trace in $\bigwedge(\mathbb{C}^d)$ is

$$\mathrm{tr}(e^{-\Sigma y}) = \mathrm{tr}(e^{-y[b^{\dagger 1},b^1]}\cdots e^{-y[b^{\dagger d},b^d]}) = (2\cosh y)^d . \tag{80}$$

Now the k_i partial derivatives of the Mehler kernel read

(81)
$$\begin{aligned}
&\sum_{k_i=0}^{n_i}\binom{n_i}{k_i}2^{k_i}(-1)^{n_i}(\Delta^{n_i-k_i}\partial^{\mu^i_1}\cdots\partial^{\mu^i_{k_i}}f_i)(x_i)\frac{\partial^{k_1}}{\partial x_i^{\mu^i_1}\cdots\partial x_i^{\mu^i_{k_i}}}e^{-\frac{\tau_i H}{2\omega}}(x_i,x_{i+1})\\
&=\sum_{k_i+2l_i+r_i=n_i}\frac{n_i!}{l_i!k_i!r_i!}\omega^{n_i-k_i-l_i}(-1)^{k_i+l_i}2^{l_i}\coth^{l_i}(\tau_i)\big(\Delta^{k_i+l_i}\partial^{\mu^i_1}\cdots\partial^{\mu^i_{r_i}}f_i\big)(x_i)\\
&\qquad\times\Big(\prod_{j=1}^{r_i}\big((x_i-x_{i+1})\coth\tfrac{\tau_i}{2}+(x_i+x_{i+1})\tanh\tfrac{\tau_i}{2}\big)_{\mu^i_j}\Big)e^{-\frac{\tau_i H}{2\omega}}(x_i,x_{i+1})\;.
\end{aligned}$$

We represent the f_i by their Fourier transforms $f_i(x)=\int_{\mathbb{R}^d}\frac{dp_i}{(2\pi)^d}\;\hat{f}_i(p_i)\,e^{\mathrm{i}p_ix_i}$, write the x_i,x_{i+1} in (81) as derivatives with respect to p_i,p_{i+1}, respectively, and obtain after Gaußian integration of the x_i

(82)
$$\begin{aligned}
\zeta_\phi(z)=&\sum_{\substack{k_1+2l_1+r_1\,=\,n_1,\,\dots,\\ k_v+2l_v+r_v\,=\,n_v}}\Big(\prod_{i=1}^{v}\frac{n_i!}{l_i!k_i!r_i!}\omega^{n_i-k_i}\Big)\frac{1}{\Gamma(\frac{z}{2})(2\sqrt{\pi})^{n_1+\dots+n_v}}\\
&\times\int_0^\infty dt_0\;t_0^{\frac{z}{2}-1}\int_0^\infty\prod_{i=1}^{v}\prod_{j_i=1}^{n_i}\frac{dt_{i,j_i}ds_{i,j_i}}{(t_{i,j_i}+s_{i,j_i})^{\frac{3}{2}}}e^{-(t_0+\sum_{i=1}^n(S_i+T_i))}\\
&\times\big(2\cosh\tfrac{\tau_1+\dots+\tau_v}{2}\big)^d\Big(\prod_{i=1}^{v}\Big(\frac{2}{\omega}\coth\tau_i\Big)^{l_i}\Big(\frac{\omega}{2\pi\sinh\tau_i}\Big)^{\frac{d}{2}}\Big)\\
&\times\int_{(\mathbb{R}^d)^v}\Big(\prod_{i=1}^{v}\frac{dp_i}{(2\pi)^d}\Big)\Big(\prod_{i=1}^{v}(p_i^2)^{k_i+l_i}p_i^{\mu^i_1}\cdots p_i^{\mu^i_{r_i}}\hat{f}_i(p_i)\Big)\\
&\qquad\times\Big(\prod_{i=1}^{v}\prod_{j=1}^{r_i}P_{\mu^i_j}\Big(\tau_i;\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_{i+1}}\Big)\Big)\Big(\frac{\sqrt{\pi}^{dv}\,e^{-\frac{1}{4}pQ^{-1}p}}{(\det Q)^{\frac{d}{2}}}\Big)\;,
\end{aligned}$$

where

(83)
$$P_{\mu^i_j}\Big(\tau_i;\frac{\partial}{\partial p_i},\frac{\partial}{\partial p_{i+1}}\Big):=\coth\frac{\tau_i}{2}\Big(\frac{\partial}{\partial p_i^{\mu^i_j}}-\frac{\partial}{\partial p_{i+1}^{\mu^i_j}}\Big)+\tanh\frac{\tau_i}{2}\Big(\frac{\partial}{\partial p_i^{\mu^i_j}}+\frac{\partial}{\partial p_{i+1}^{\mu^i_j}}\Big)$$

and

(84)
$$Q=\frac{\omega}{2}\begin{pmatrix}
\frac{\sinh(\tau_v+\tau_1)}{\sinh\tau_v\sinh\tau_1} & \frac{-1}{\sinh\tau_1} & 0 & \cdots & 0 & \frac{-1}{\sinh\tau_v}\\
\frac{-1}{\sinh\tau_1} & \frac{\sinh(\tau_1+\tau_2)}{\sinh\tau_1\sinh\tau_2} & \frac{-1}{\sinh\tau_2} & \ddots & \ddots & 0\\
0 & \frac{-1}{\sinh\tau_2} & \frac{\sinh(\tau_2+\tau_3)}{\sinh\tau_2\sinh\tau_3} & \ddots & \ddots & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & \frac{\sinh(\tau_{v-2}+\tau_{v-1})}{\sinh\tau_{v-2}\sinh\tau_{v-1}} & \frac{-1}{\sinh\tau_{v-1}}\\
\frac{-1}{\sinh\tau_v} & 0 & 0 & \cdots & \frac{-1}{\sinh\tau_{v-1}} & \frac{\sinh(\tau_{v-1}+\tau_v)}{\sinh\tau_{v-1}\sinh\tau_v}
\end{pmatrix}.$$

By Gauß-Jordan elimination and multiple use of the addition theorems for sinh it is straightforward to compute the determinant and the inverse of the symmetric matrix Q (the result also holds for $v = 1$):

$$\det Q = \Big(\frac{\omega}{2}\Big)^v \frac{4\sinh^2(\frac{1}{2}(\tau_1+\dots+\tau_v))}{\prod_{i=1}^v \sinh\tau_i}\;, \tag{85}$$

$$(Q^{-1})_{ij} = \frac{1}{\omega\tanh(\frac{1}{2}(\tau_1+\dots+\tau_v))} + \tilde{Q}_{ij}\;, \tag{86}$$

$$\tilde{Q}_{ij} = -\frac{2\sinh(\frac{1}{2}(\tau_i+\dots+\tau_{j-1}))\sinh(\frac{1}{2}(\tau_j+\dots+\tau_{i-1}))}{\omega\sinh(\frac{1}{2}(\tau_1+\dots+\tau_v))}\;, \tag{87}$$

where in $\tilde{Q}_{ij}$ one of the chains $\tau_i+\dots+\tau_{j-1}$ or $\tau_j+\dots+\tau_{i-1}$ passes through the index $v\equiv 0$. The determinant can also be obtained from the fact that for $p=0$ we just have the trace over the concatenation of Mehler kernels (77).

The action of $(P_{\mu^i_j})$ on $e^{-\frac{1}{4}pQ^{-1}p}$ is partitioned into k'_i out of r_i single contractions, l'_i double contractions and r_{ij} halves of mixed contraction with another index $j\neq i$ such that $k'_i+l'_i+\sum_{j\neq i} r_{ij} = r_i$. Their number is $\frac{r_i!}{2^{l'_i} l'_i! k'_i! r_{i1}!\cdots r_{iv}!}$ if we put $r_{ii}=0$ and $r_{ij}=r_{ji}$. Together with the multiplying factor $p_i^{\mu^i_j}$, a single contraction gives a factor

$$p_i^{\mu^i} P_{\mu^i}(-\tfrac{1}{4}pQ^{-1}p) = -\frac{p_i^2}{\omega} - \sum_{j\neq i}\frac{\sinh(\frac{\tau_i+\dots+\tau_{j-1}-\tau_j-\dots-\tau_{i-1}}{2})}{\omega\sinh(\frac{\tau_1+\dots+\tau_v}{2})} p_i p_j\;. \tag{88}$$

A double contraction with respect to the same index i gives a factor

$$p_i^{\mu^i} p_i^{\nu^i} P_{\mu^i} P_{\nu^i}(-\tfrac{1}{4}pQ^{-1}p) = \Big(-\frac{4\coth\tau_i}{\omega} + \frac{2}{\omega}\coth(\tfrac{\tau_1+\dots+\tau_v}{2})\Big)p_i^2\;. \tag{89}$$

A mixed contraction with respect to different indices $i\neq j$ gives a factor

$$p_i^{\mu^i} p_j^{\mu^j} P_{\mu^i} P_{\mu^j}(-\tfrac{1}{4}pQ^{-1}p) = 2\frac{\cosh(\frac{\tau_j+\dots+\tau_{i-1}-\tau_i-\dots-\tau_{j-1}}{2})}{\omega\sinh(\frac{\tau_1+\dots+\tau_v}{2})} p_i p_j\;. \tag{90}$$

We insert these formulae into (82) and notice that the sum over l_i, l'_i combines to a joint sum (with new index l_i) involving only the factor $\frac{p_i^2}{\omega}\coth(\frac{1}{2}(\tau_1+\dots+\tau_v))$ from (89), whereas $\coth\tau_i$ cancels. In the same way, the sum over k_i, k'_i cancels the term $-p_i^2$ from (88) so that only the sum over $j\neq i$ remains:

(91)

$$\begin{aligned}
\zeta_\phi(z) = &\sum_{\substack{k_1+2l_1+r_1=n_1,\dots,\\ k_v+2l_v+r_v=n_v\\ r_1+\dots+r_v=2m}} \;\sum_{\substack{r_{11}+\dots+r_{1v}=r_1,\dots,\\ r_{v1}+\dots+r_{vv}=r_v}} \Big(\prod_{i=1}^v \frac{n_i!}{l_i!k_i!}\Big)\frac{2^m\omega^{l_1+\dots+l_v+m}}{\Gamma(\frac{z}{2})(2\sqrt{\pi})^{n_1+\dots+n_v}}\\
&\times\int_0^\infty dt_0\, t_0^{\frac{z}{2}-1}\int_0^\infty \prod_{i=1}^v\prod_{j_i=1}^{n_i}\frac{dt_{i,j_i}ds_{i,j_i}}{(t_{i,j_i}+s_{i,j_i})^{\frac{3}{2}}}\frac{e^{-(t_0+\sum_{i=1}^n(S_i+T_i))}}{\big(\tanh\frac{\tau_1+\dots+\tau_v}{2}\big)^{d+l_1+\dots+l_v}}\\
&\times\int_{(\mathbb{R}^d)^v}\Big(\prod_{i=1}^v\frac{dp_i}{(2\pi)^d}\Big)\Big(\prod_{i<j}\frac{1}{r_{ij}!}\Big(\frac{\cosh\big(\frac{\tau_j+\dots+\tau_{i-1}}{2}-\frac{\tau_i+\dots+\tau_{j-1}}{2}\big)}{\sinh(\frac{\tau_1+\dots+\tau_v}{2})}p_ip_j\Big)^{r_{ij}}\Big)\\
&\times\Big(\prod_{i=1}^v\Big(\sum_{j\neq i}\frac{\sinh\big(\frac{\tau_j+\dots+\tau_{i-1}}{2}-\frac{\tau_i+\dots+\tau_{j-1}}{2}\big)}{\sinh(\frac{\tau_1+\dots+\tau_v}{2})}p_ip_j\Big)^{k_i}(p_i^2)^{l_i}\hat{f}_i(p_i)\Big)e^{-\frac{1}{4}pQ^{-1}p}.
\end{aligned}$$

The zeta-function potentially has a singularity for $\tau = \tau_1 + \cdots + \tau_v \to 0$ of order $\tau^{\frac{z+k_1+\cdots+k_v}{2}-d}$. The contribution $\frac{z}{2}$ is from $dt_0\ t_0^{\frac{z}{2}-1}$, the measure $\prod \frac{dtds}{(t+s)^{\frac{3}{2}}}$ contributes $\frac{n_1+\cdots+n_v}{2}$, and $(\tanh\frac{\tau}{2})^{-d-l_1-\cdots-l_v}(\sinh\frac{\tau}{2})^{-r_{12}+\cdots+r_{v-1,v}}$ contribute $-(d+l_1+\cdots+l_v+\frac{r_1+\cdots+r_v}{2})$. However, the independence of the leading term in Q^{-1}_{ij} from i,j shows that this singularity is protected by $e^{-\frac{(p_1+\cdots+p_v)^2}{\omega\tanh\frac{\tau}{2}}}$ unless the total momentum is conserved, $p_1+\cdots+p_v=0$. The remaining singularity is identified by a Taylor expansion in p_v about $\bar{p}_v := -(p_1+\cdots+p_{v-1})$ up to order ρ to be determined later:

(92)
$$\begin{aligned} F(p_1,\dots,p_v) &= \sum_{|\alpha|\leq\rho}\frac{(p_v-\bar{p}_v)^\alpha}{\alpha!}\frac{\partial^{|\alpha|}F}{\partial p_v^\alpha}(p_1,\dots,p_{v-1},\bar{p}_v)\\ &+\sum_{|\alpha|=\rho+1}\frac{(p_v-\bar{p}_v)^\alpha}{\rho!}\int_0^1 d\lambda\,(1-\lambda)^\rho\frac{\partial^{|\alpha|}F}{\partial p_v^\alpha}(p_1,\dots,p_{v-1},\bar{p}_v+\lambda(p_v-\bar{p}_v))\;, \end{aligned}$$

where α is a multi-index. Together with the measure dp_v, the last line combines with $\tanh^{-d}(\frac{\tau}{2})$ to a factor $dP\ P^{\rho+1}\,e^{-P^2}\tanh^{\frac{\rho+1-d}{2}}(\frac{\tau}{2})$, where $P = \frac{p_v-\bar{p}_v}{\sqrt{\tanh\frac{\tau}{2}}}$. For sufficiently large but finite ρ we shall see in (96) that the potential singularity in $t_0^{\frac{z}{2}}$ is cancelled so that the last line of (92) is regular. The bilinear form in the exponent has the form

(93)
$$e^{-\frac{1}{4}pQ^{-1}p} = e^{-\frac{(p_v-\bar{p}_v)^2}{4\omega\tanh\frac{\tau}{2}}-\frac{1}{2}(p_v-\bar{p}_v)q-\frac{1}{2}\bar{p}_v\sum_{j=1}^{v-1}\tilde{Q}_{vj}p_j-\frac{1}{4}\sum_{i,j=1}^{v-1}\tilde{Q}_{ij}p_ip_j}\;,\qquad q:=\sum_{j=1}^{v-1}\tilde{Q}_{vj}p_j\;.$$

We can thus perform the Gaußian integration over p_v and obtain for the restricted zeta function ζ^r, where the second line of (92) is removed:

(94)
$$\begin{aligned} &\zeta^r_\phi(z)\\ &=\sum_{\substack{k_1+2l_1+r_{11}+\dots+r_{1v}=n_1,\dots,\\ k_1+2l_1+r_{v1}+\dots+r_{vv}=n_v,\\ r_{ii}=0\;,\;r_{ij}=r_{ji}}}\frac{n_1!\cdots n_v!}{\Gamma(\frac{z}{2})\pi^{\frac{d}{2}}(2\sqrt{\pi})^{n_1+\cdots+n_v}}\\ &\times\int_0^\infty dt_0\,t_0^{\frac{z}{2}-1}\int_0^\infty\prod_{i=1}^v\prod_{j_i=1}^{n_i}\frac{dt_{i,j_i}ds_{i,j_i}}{(t_{i,j_i}+s_{i,j_i})^{\frac{3}{2}}}\,e^{-t}\Big(\frac{\omega}{\tanh(\omega t)}\Big)^{\frac{d}{2}+\sum_{i=1}^v l_i+\sum_{i<j}r_{ij}}\\ &\times\int_{(\mathbb{R}^d)^{v-1}}\Big(\prod_{i=1}^{v-1}\frac{dp_i}{(2\pi)^d}\Big)\sum_{|\alpha|\leq\rho}\frac{(-2)^{|\alpha|}}{\alpha!}\frac{\partial^{|\alpha|}}{\partial q^\alpha}\Big(e^{\frac{\omega}{4}q^2\tanh(\omega t)-\frac{1}{2}\bar{p}_v\sum_{j=1}^{v-1}\tilde{Q}_{vj}p_j-\frac{1}{4}\sum_{i,j=1}^{v-1}\tilde{Q}_{ij}p_ip_j}\Big)\\ &\times\frac{\partial^{|\alpha|}}{\partial p_v^\alpha}\Bigg(\Bigg(\prod_{i<j}\frac{\Big(\frac{2\cosh(\omega t_{ij})}{\cosh(\omega t)}p_ip_j\Big)^{r_{ij}}}{r_{ij}!}\Bigg)\Bigg(\prod_{i=1}^v\frac{\Big(\sum_{j\neq i}\frac{\sinh(\omega t_{ij})}{\sinh(\omega t)}p_ip_j\Big)^{k_i}}{k_i!}\frac{(p_i^2)^{l_i}}{l_i!}\hat{f}_i(p_i)\Bigg)\Bigg)_{p_v\mapsto\bar{p}_v}\;, \end{aligned}$$

where $t = \frac{1}{2\omega}\tau = t_0 + \sum_{i=1}^{v}(T_i + S_i)$ and $t_{ij} = \frac{1}{2\omega}(\tau_j + \cdots + \tau_{i-1}) - \frac{1}{2\omega}(\tau_i + \cdots + \tau_{j-1})$. The q-derivatives and the quadratic form in the exponent become with $\tilde{Q}_{ij} = \frac{\cosh(\omega t_{ij}) - \cosh(\omega t)}{\omega \sinh(\omega t)}$

$$
\begin{aligned}
(95)\qquad & \sum_{|\alpha|\leq\rho} \frac{(-2)^{|\alpha|}}{\alpha!}\frac{\partial^{|\alpha|}}{\partial q^\alpha}\Big(e^{\frac{\omega}{4}q^2\tanh(\omega t) - \frac{1}{2}\bar{p}_v\sum_{j=1}^{v-1}\tilde{Q}_{vj}p_j - \frac{1}{4}\sum_{i,j=1}^{v-1}\tilde{Q}_{ij}p_ip_j}\Big)\frac{\partial^{|\alpha|}}{\partial p_v^\alpha} \\
&= \sum_{|\alpha|+2a\leq\rho} (\omega\tanh(\omega t))^{a+|\alpha|} e^{-\big(\sum_{i,j=1}^{v-1}\frac{\sinh(\omega t_{ij}^-)\sinh(\omega t_{ij}^+)}{2\omega\sinh(2\omega t)}p_ip_j\big)} \\
&\qquad \times \frac{1}{a!}\Big(\frac{\partial^2}{\partial p_v^\mu \partial p_{v\mu}}\Big)^a \frac{1}{\alpha!}\Big(\sum_{j=1}^{v-1}\frac{2\sinh(\omega\frac{t+t_{vj}}{2})\sinh(\omega\frac{t+t_{jv}}{2})}{\omega\sinh(\omega t)}p_j\Big)^\alpha \frac{\partial^{|\alpha|}}{\partial p_v^\alpha}\;,
\end{aligned}
$$

where $t_{ij}^- = t + t_{kv}\big|_{k=\min(i,j)}$ and $t_{ij}^+ = t + t_{vk}\big|_{k=\max(i,j)}$. Note that (95) is bounded for all t.

We insert (95) into (94). We change the integration variables to $t_0 = (1-u)t$, $\sum_{i=1}^{v}(S_i + T_i) = ut$ with integration over t from 0 to ∞, over u from 0 to 1 and over the surface Δ given by $\sum_{i=1}^{v}(S_i + T_i) = 1$. We write the denominators $\frac{1}{\sinh(\omega t)} = \frac{1}{\omega t}\cdot\frac{\omega t}{\sinh(\omega t)}$ and $\frac{1}{\tanh(\omega t)} = \frac{1}{\omega t}\cdot\frac{\omega t}{\tanh(\omega t)}$ and expand the bounded (at 0) fractions $\frac{\omega t}{\sinh(\omega t)}$ and $\frac{\omega t}{\tanh(\omega t)}$ into a Taylor series in (ωt). The numerators in hyperbolic functions of (ωt) and (ωt_{ij}) and $\frac{1}{\cosh(\omega t)}$ are expanded into a Taylor series in their arguments. Then, for each term in the sum, the u,t-integral is of the form

$$
\begin{aligned}
(96)\quad & \frac{1}{\Gamma(\frac{z}{2})}\int_0^\infty dt\, t^{(\frac{z}{2}-\frac{d}{2}+\frac{k_1+\cdots+k_v}{2}+a+2|\alpha|+b-1)}e^{-t}\int_0^1 du(1-u)^{\frac{z}{2}-1}u^{\frac{n_1+\cdots+n_v}{2}+c-1} \\
&= \frac{\Gamma(\frac{z}{2}-\frac{d}{2}+\frac{k_1+\cdots+k_v}{2}+a+2|\alpha|+b)\,\Gamma(\frac{n_1+\cdots+n_v}{2}+c)}{\Gamma(\frac{z}{2}+\frac{n_1+\cdots+n_v}{2}+c)}\;,
\end{aligned}
$$

where the integers $b \geq c \geq 0$ arise from the Taylor expansion. The remaining integration over the simplex Δ is regular because from the Taylor expansion only positive powers of the integration variables appear. From (96) we deduce the following information about the pole structure:

- For $z \notin \mathbb{Z}$ or for $z > d$ there is no pole.
- For $z = d - N$ with $N \in \mathbb{N}$, and $n_1, \ldots, n_v$ such that $z + n_1 + \cdots + n_v$ is even, there is a pole for a finite (and non-vanishing) number of index combinations and finite Taylor order $\rho = d + n_1 + \cdots + n_v - k_1 - \cdots - k_v$.

This concludes the proof that $\mathrm{Sd} = d - \mathbb{N}$.

It remains to characterise the nature of the residues. From (94) we conclude that the residues are given by the integral over $p_1, \ldots, p_{v-1}$ of an integrand which is a polynomial in $p_1, \ldots, p_{v-1}$ times $\prod_{i=1}^{v-1}\hat{f}_i(p_i)$ times possible derivatives of $\hat{f}_v(\bar{p}_v)$. Reconstructing the p_v-variable by a δ-function and integrating by parts the derivatives of $\hat{f}_v(\bar{p}_v)$, the residue becomes a finite sum of the form

$$(97)\qquad \mathrm{res}_{z=d-N}(\zeta(z))$$

$$= \sum_{\alpha_0,\dots,\alpha_v} c_{\alpha_0\dots\alpha_v} \int_{(\mathbb{R}^d)^v} \Big(\prod_{i=1}^v \frac{dp_i}{(2\pi)^d}\Big) \int_{\mathbb{R}^d} dx\; e^{\mathrm{i}(p_1+\dots+p_v)x} x^{\alpha_0} \prod_{i=1}^v p_i^{\alpha_i} \hat{f}_i(p_i)$$
$$= \sum_{\alpha_0,\dots,\alpha_v} \int_{\mathbb{R}^d} dx\; c_{\alpha_0\dots\alpha_v} (-\mathrm{i})^{|\alpha_1|+\dots+|\alpha_v|} x^{\alpha_0} \prod_{i=1}^v (\partial^{\alpha_i} f_i)(x_i)\;,$$

where the α_j are multi-indices which contract to a Lorentz scalar. The prefactor $c_{\alpha_0\dots\alpha_v}$ results from the integration over the t-variables. Thus, the residues are local. □

We would like to stress that it was important to keep track of the combinatorial factors which led to the cancellation of denominators $\frac{1}{\sinh\tau_i}$. Such denominators in the final formula (94) would be fatal because in that case the u-integral of (96) would produce a hypergeometric function instead of the beta function and therefore an infinite sum for the residue, which could be non-local.

Appendix B. Vertices contributing to the spectral action

We compute here the individual vertex contributions (54) to the spectral action. This is done by inserting the vertices (57) into (60) and then computing the t_i-integrals.

B.1. V_2**.** The contribution of a single V_2-vertex is

(98) $$S_t(V_2) = \int_0^t dt_1\; \mathrm{tr}(e^{-\omega\Sigma t}) S_t^0(f)\;, \quad f = -2|\phi|^2 - A_\mu A^\mu - B_\mu B^\mu\;.$$

With $\mathrm{tr}(e^{-\omega\Sigma t}) = (2\cosh(\omega t))^4$ we have after second order Taylor expansion, ignoring the remainder and the odd first-order term,

(99) $$S_t(V_2) = \int_{\mathbb{R}^4} \frac{dp}{(2\pi)^4} \frac{t}{(\tanh(\omega t))^4} \Big(\hat{f}(0) + \frac{1}{2} p_\mu p_\nu \frac{\partial^2 \hat{f}}{\partial p_\mu \partial p_\nu}(0)\Big) e^{-\frac{p^2}{4\omega\tanh(\omega t)}}$$
$$= \frac{\omega^2 t}{\pi^2 \tanh^2(\omega t)} \Big(\hat{f}(0) + \omega\tanh(\omega t)\delta_{\mu\nu} \frac{\partial^2 \hat{f}}{\partial p_\mu \partial p_\nu}(0)\Big)$$
$$= \frac{\omega^2 t}{\pi^2 \tanh^2(\omega t)} \int_{\mathbb{R}^4} dx \Big(f(x) - \omega \|x\|^2 \tanh(\omega t)\, f(x)\Big)\;,$$

after Fourier transformation $\hat{f}(p) = \int_{\mathbb{R}^4} dx\; e^{-ipx} f(x)$. Inserting f we obtain after Laplace transformation the leading terms of the asymptotic expansion to

(100) $$S_2(\mathcal{D}_A) = \frac{\chi_{-1}}{\pi^2} \int_{\mathbb{R}^4} dx\; \big(-2|\phi|^2 - A_\mu A^\mu - B_\mu B^\mu\big)(x)$$
$$+ \frac{\chi_0}{\pi^2} \int_{\mathbb{R}^4} dx\; (\omega^2 |x|^2 \big(2|\phi|^2 + A_\mu A^\mu + B_\mu B^\mu\big)(x)\;.$$

B.2. V_1V_1**.** The contribution of two V_1-vertices is

(101) $$S_t(V_1, V_1) = \int_0^t dt_1 \int_0^{t-t_1} dt_2\; \mathrm{tr}(e^{-\omega\Sigma t}) S^{1,1}_{t_2, t-t_2}(-A, -A) + (A \mapsto B)\;.$$

This is the most involved computation. To (60) there are the two contributions $k_1 = k_2 = 1$ up to order 0 and $r_{12} = r_{21} = 1$ with Taylor expansion about $p_2 = -p_1$ up to order 2:

(102)

$$
\begin{aligned}
&S_t(V_1,V_1)\\
&=\int_0^t dt_1\int_0^{t-t_1}dt_2\int_{(\mathbb{R}^4)^2}\frac{dp_1dp_2}{(2\pi)^8}\frac{1}{\tanh^4(\omega t)}\;e^{-\frac{(p_1+p_2)^2}{4\omega\tanh(\omega t)}+p_1p_2\frac{\sinh(\omega t_2)\sinh(\omega(t-t_2))}{\omega\sinh(\omega t)}}\\
&\times\Big\{\hat{A}^\mu(p_1)\hat{A}^\nu(-p_1)\Big(\frac{\sinh^2(\omega(t-2t_2))}{\sinh^2(\omega t)}p_{1\mu}p_{1\nu}+2\omega\delta_{\mu\nu}\frac{\cos(\omega(t-2t_2))}{\sinh(\omega t)}\Big)\\
&+(p_1+p_2)^\rho\hat{A}^\mu(p_1)\frac{\partial\hat{A}^\nu}{\partial p_2^\rho}(-p_1)\cdot 2\omega\delta_{\mu\nu}\frac{\cos(\omega(t-2t_2))}{\sinh(\omega t)}\\
&+\frac{1}{2}(p_1+p_2)^\rho(p_1+p_2)^\sigma\hat{A}^\mu(p_1)\frac{\partial^2\hat{A}^\nu}{\partial p_2^\rho\partial p_2^\sigma}(-p_1)\cdot 2\omega\delta_{\mu\nu}\frac{\cos(\omega(t-2t_2))}{\sinh(\omega t)}\Big\}+(A\mapsto B)\\
&=\int_0^t dt_1\int_0^{t-t_1}dt_2\int_{\mathbb{R}^4}\frac{dp_1}{(2\pi)^4}\frac{\omega^2}{\pi^2\tanh^2(\omega t)}\;e^{-\frac{\sinh(2\omega t_2)\sinh(2\omega(t-t_2))}{2\omega\sinh(2\omega t)}p_1^2}\\
&\times\Big\{\hat{A}^\mu(p_1)\hat{A}^\nu(-p_1)\Big(\frac{\sinh^2(\omega(t-2t_2))}{\sinh^2(\omega t)}p_{1\mu}p_{1\nu}+2\omega\delta_{\mu\nu}\frac{\cos(\omega(t-2t_2))}{\sinh(\omega t)}\Big)\\
&+4\omega p_1^\rho\hat{A}^\mu(p_1)\frac{\partial\hat{A}_\mu}{\partial p_2^\rho}(-p_1)\cdot\frac{\sinh(\omega t_2)\sinh(\omega(t-t_2))}{\cos(\omega t)}\frac{\cos(\omega(t-2t_2))}{\sinh(\omega t)}\\
&+\omega\Big(2\delta^{\rho\sigma}\omega\tanh(\omega t)+4p_1^\rho p_1^\sigma\frac{\sinh^2(\omega t_2)\sinh^2(\omega(t-t_2))}{\cos^2(\omega t)}\Big)\\
&\qquad\times\hat{A}^\mu(p_1)\frac{\partial^2\hat{A}_\mu}{\partial p_2^\rho\partial p_2^\sigma}(-p_1)\cdot\frac{\cos(\omega(t-2t_2))}{\sinh(\omega t)}\Big\}+(A\mapsto B)\;.
\end{aligned}
$$

Up to $\mathcal{O}(t)$ this reduces to

(103)

$$
\begin{aligned}
&S_t(V_1,V_1)\\
&=\int_0^t dt_1\int_0^{t-t_1}dt_2\int_{\mathbb{R}^4}\frac{dp_1}{(2\pi)^4}\frac{\omega^2}{\pi^2\tanh^2(\omega t)}\\
&\times\Big\{\hat{A}^\mu(p_1)\hat{A}^\nu(-p_1)\Big(\frac{\sinh^2(\omega(t-2t_2))}{\sinh^2(\omega t)}p_{1\mu}p_{1\nu}\\
&\qquad-\delta_{\mu\nu}\frac{\cosh(\omega(t-2t_2))}{\sinh(\omega t)}\frac{\sinh(2\omega t_2)\sinh(2\omega(t-t_2))}{\sinh(2\omega t)}p_1^2\Big)\\
&+2\omega\hat{A}^\mu(p_1)\hat{A}_\mu(-p_1)\frac{\cosh(\omega(t-2t_2))}{\sinh(\omega t)}+2\omega^2\hat{A}^\mu(p_1)\frac{\partial^2\hat{A}_\mu}{\partial p_2^\rho\partial p_{2\rho}}(-p_1)\frac{\cosh(\omega(t-2t_2))}{\cosh(\omega t)}\Big\}\\
&+(A\mapsto B)\\
&=\int_{\mathbb{R}^4}\frac{dp_1}{(2\pi)^4}\frac{\omega^2}{\pi^2\tanh^2(\omega t)}\\
&\times\Big\{\hat{A}^\mu(p_1)\hat{A}^\nu(-p_1)\Big(\Big(\frac{t}{4\omega\tanh(\omega t)}-\frac{t^2}{4\sinh^2(\omega t)}\Big)p_{1\mu}p_{1\nu}-\delta_{\mu\nu}\frac{t\tanh(\omega t)}{6\omega}p_1^2\Big)
\end{aligned}
$$

$$+ t\,\hat{A}^\mu(p_1)\hat{A}_\mu(-p_1) + (\omega t)\tanh(\omega t)\,\hat{A}^\mu(p_1)\frac{\partial^2\hat{A}_\mu}{\partial p_2^\rho \partial p_{2\rho}}(-p_1)\Big\} + (A \mapsto B)\;.$$

After Fourier and Laplace transformation, the leading contribution to the spectral action becomes

$$\begin{aligned}(104)\qquad S_{11}(\mathcal{D}_A) &= \frac{\chi_{-1}}{\pi^2}\int_{\mathbb{R}^4} dx\, \big(A_\mu A^\mu + B_\mu B^\mu\big)(x)\\ &\quad - \frac{\chi_0}{\pi^2}\int_{\mathbb{R}^4} dx\, (\omega^2\|x\|^2)\big(A_\mu A^\mu + B_\mu B^\mu\big)(x)\\ &\quad - \frac{\chi_0}{12\pi^2}\int_{\mathbb{R}^4} dx\, \big(F^A_{\mu\nu}F^{A\mu\nu} + F^B_{\mu\nu}F^{B\mu\nu}\big)(x)\;.\end{aligned}$$

B.3. V_2V_2, V_3V_3, V_4V_4. We have

$$\begin{aligned}(105)\qquad &\sum_{i=2}^4 S_t(V_i,V_i)\\ &= \int_0^t dt_1 \int_0^{t-t_1} dt_2 \Big\{\mathrm{tr}(e^{-\omega\Sigma t})\Big(S^{0,0}_{t_2,t-t_2}(-|\phi|^2 - A_\mu A^\mu, -|\phi|^2 - A_\mu A^\mu)\\ &\qquad\qquad + S^{0,0}_{t_2,t-t_2}(-|\phi|^2 - B_\mu B^\mu, -|\phi|^2 - B_\mu B^\mu)\Big)\\ &\quad + \mathrm{tr}\Big(\frac{\mathrm{i}}{4}[b^{\dagger\mu} - b^\mu, b^{\dagger\nu} - b^\nu]e^{-\omega\Sigma t_2}\frac{\mathrm{i}}{4}[b^{\dagger\rho} - b^\rho, b^{\dagger\sigma} - b^\sigma]e^{-\omega\Sigma(t-t_2)}\Big)\\ &\qquad \times \Big(S^{0,0}_{t_2,t-t_2}(F^A_{\mu\nu}, F^A_{\rho\sigma}) + S^{0,0}_{t_2,t-t_2}(F^B_{\mu\nu}, F^B_{\rho\sigma})\Big)\\ &\quad + \mathrm{tr}\Big((b^{\dagger\mu} - b^\mu)e^{-\omega\Sigma t_2}(b^{\dagger\nu} - b^\nu)e^{-\omega\Sigma(t-t_2)}\Big)\\ &\qquad \times \Big(S^{0,0}_{t_2,t-t_2}(-D_\mu\phi, \overline{D_\nu\phi}) + S^{0,0}_{t_2,t-t_2}(\overline{D_\mu\phi}, -D_\nu\phi)\Big)\Big\}\;.\end{aligned}$$

Since the $S^{0,0}_{t_2,t-t_2}$ are at least $\mathcal{O}(t^{-2})$, only the $\mathcal{O}(t^0)$-parts of $e^{-\omega\Sigma t_2}$ and $e^{-\omega\Sigma(t-t_2)}$ will contribute to the spectral action. Now the traces in $\bigwedge(\mathbb{C}^4)$ are easy to compute:

$$\begin{aligned}(106)\qquad \mathrm{tr}\big(e^{\omega\Sigma t}\big) &= (2\cosh(\omega t))^4\;,\\ \mathrm{tr}\big((b^{\dagger\mu} - b^\mu)e^{-\omega\Sigma t_2}(b^{\dagger\nu} - b^\nu)e^{-\omega\Sigma(t-t_2)}\big) &= -16\delta^{\mu\nu} + \mathcal{O}(t)\;,\\ \mathrm{tr}\big(\tfrac{\mathrm{i}}{4}[b^{\dagger\mu}-b^\mu, b^{\dagger\nu}-b^\nu]e^{-\omega\Sigma t_2}\tfrac{\mathrm{i}}{4}[b^{\dagger\rho}-b^\rho, b^{\dagger\sigma}-b^\sigma]e^{-\omega\Sigma(t-t_2)}\big) &= 8(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho}) + \mathcal{O}(t).\end{aligned}$$

After Taylor expansion about $p_2 = -p_1$ up to order 0, integration over p_2, t_1, t_2 and Laplace transformation, we obtain

$$\begin{aligned}(107)\qquad (S_{22}+S_{33}+S_{44})(\mathcal{D}_A) &= \frac{\chi_0}{2\pi^2}\int_{\mathbb{R}^4} dx \Big\{2\overline{D_\mu\phi}(D^\mu\phi) + (|\phi|^2 + A_\mu A^\mu)^2\\ &\qquad + F^A_{\mu\nu}F^{A\mu\nu} + (|\phi|^2 + B_\mu B^\mu)^2 + F^B_{\mu\nu}F^{B\mu\nu}\Big\}(x)\;.\end{aligned}$$

B.4. V_1V_2, V_2V_1. With the abbreviation $f_{\phi A} := |\phi|^2 + A_\mu A^\mu$, we have

$$\begin{aligned}(108)\qquad &S_t(V_1,V_2) + S_t(V_2,V_1)\\ &= \int_0^t dt_1 \int_0^{t-t_1} dt_2\; \mathrm{tr}(e^{-\omega\Sigma t})\Big(S^{1,0}_{t_2,t-t_2}(-A, -f_{\phi A}) + S^{0,1}_{t_2,t-t_2}(-f_{\phi A}, -A)\Big)\end{aligned}$$

$$
\begin{aligned}
&+ (A \mapsto B) \\
&= \int_0^t dt_1 \int_0^{t-t_1} dt_2 \int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{dp_1 dp_2}{(2\pi)^8} \frac{1}{\tanh^4(\omega t)} \\
&\times \Big(p_{2,\mu} \hat{A}^\mu(p_1)\hat{f}_{\phi A}(p_2) - p_{1,\mu}\hat{A}^\mu(p_2)\hat{f}_{\phi A}(p_1)\Big) \frac{\sinh(\omega(t-2t_2))}{\sinh(\omega t)} e^{-\frac{1}{4}pQ^{-1}p} + (A \mapsto B) \\
&= \int_0^t dt_1 \int_0^{t-t_1} dt_2 \int_{\mathbb{R}^4} \frac{dp_1}{(2\pi)^4} \frac{\omega^2}{\pi^2 \tanh^2(\omega t)} \frac{\sinh(\omega(t-2t_2))}{\sinh(\omega t)} \\
&\times \Big(- p_{1,\mu}\hat{A}^\mu(p_1)\hat{f}_{\phi A}(-p_1) - p_{1,\mu}\hat{A}^\mu(-p_1)\hat{f}_{\phi A}(p_1)\Big) + (A \mapsto B) + \mathcal{O}(t) \\
&= \mathcal{O}(t) \,.
\end{aligned}
$$

We thus have

$$
S_{12}(\mathcal{D}_A) = 0 \,. \tag{109}
$$

B.5. $V_1V_1V_2$**,** $V_1V_2V_1$**,** $V_2V_1V_1$**.** Only the $k_i = 0$ terms in (60) contribute to the leading order. With the abbreviation $f_{\phi A} := |\phi|^2 + A_\mu A^\mu$, these give

(110)

$$
\begin{aligned}
&S_t(V_1, V_1, V_2) + S_t(V_1, V_2, V_1) + S_t(V_2, V_1, V_1) \\
&= \int_0^t dt_1 \int_0^{t-t_1} dt_2 \int_0^{t-t_1-t_2} dt_3 \,\mathrm{tr}(e^{-\omega\Sigma t}) \Big(S^{1,1,0}_{t_3,t_2,t-t_2-t_3}(-A,-A,-f_{\phi A}) \\
&\qquad + S^{1,0,1}_{t_3,t_2,t-t_2-t_3}(-A,-f_{\phi A},-A) + S^{0,1,1}_{t_3,t_2,t-t_2-t_3}(-f_{\phi A},-A,-A)\Big) + (A \mapsto B) \\
&= \int_0^t dt_1 \int_0^{t-t_1} dt_2 \int_0^{t-t_1-t_2} dt_3 \int_{(\mathbb{R}^4)^3} \frac{dp_1\, dp_2\, dp_3}{(2\pi)^{12}} \frac{-2\omega}{\tanh^4(\omega t)\sinh(\omega t)} \\
&\times \Big(\hat{A}_\mu(p_1)\hat{A}^\mu(p_2)\hat{f}_{\phi A}(p_3)\cosh(\omega(t-2t_3)) \\
&\qquad + \hat{A}_\mu(p_1)\hat{f}_{\phi A}(p_2)\hat{A}^\mu(p_3)\cosh(\omega(t-2t_2-2t_3)) \\
&\qquad + \hat{f}_{\phi A}(p_1)\hat{A}_\mu(p_2)\hat{A}^\mu(p_3)\cosh(\omega(t-2t_2))\Big) e^{-\frac{1}{4}pQ^{-1}p} + (A \mapsto B) + \mathcal{O}(t) \\
&= \int_0^t dt_1 \int_0^{t-t_1} dt_2 \int_0^{t-t_1-t_2} dt_3 \int_{(\mathbb{R}^4)^2} \frac{dp_1\, dp_2}{(2\pi)^8} \frac{(-2\omega^3)}{\pi^2\tanh^2(\omega t)\sinh(\omega t)} \\
&\times \hat{A}_\mu(p_1)\hat{A}^\mu(p_2)\hat{f}_{\phi A}(-p_1-p_2)\Big(\cosh(\omega(t-2t_3)) + \cosh(\omega(t-2t_2-2t_3)) \\
&\qquad\qquad + \cosh(\omega(t-2t_2))\Big) + (A \mapsto B) + \mathcal{O}(t) \\
&= \int_{(\mathbb{R}^4)^2} \frac{dp_1\, dp_2}{(2\pi)^8} \frac{(-\omega^2 t^2)}{\pi^2 \tanh^2(\omega t)} \hat{A}_\mu(p_1)\hat{A}^\mu(p_2)\hat{f}_{\phi A}(-p_1-p_2) + (A \mapsto B) + \mathcal{O}(t) \,.
\end{aligned}
$$

After Fourier and Laplace transformation we obtain

$$
S_{112}(\mathcal{D}_A) = -\frac{\chi_0}{\pi^2} \int_{\mathbb{R}^4} dx \Big\{ A_\mu A^\mu(|\phi|^2 + A_\nu A^\nu) + B_\mu B^\mu(|\phi|^2 + B_\nu B^\nu)\Big\}(x) \,. \tag{111}
$$

B.6. $V_1V_1V_1$**.** The leading order in (60) is given by the $(k_1 = 1, r_{23} = 1)$ and the other two cyclic permutations:

(112)

$$
\begin{aligned}
&S_t(V_1, V_1, V_1)\\
&= \int_0^t dt_1 \int_0^{t-t_1} dt_2 \int_0^{t-t_1-t_2} dt_3 \;\mathrm{tr}(e^{-\omega\Sigma t}) S^{1,1,1}_{t_3,t_2,t-t_2-t_3}(-A,-A,-A) + (A \mapsto B)\\
&= \int_0^t dt_1 \int_0^{t-t_1} dt_2 \int_0^{t-t_1-t_2} dt_3 \int_{(\mathbb{R}^4)^3} \frac{dp_1\,dp_2\,dp_3}{(2\pi)^{12}} \frac{-2\omega}{\tanh^4(\omega t)\sinh^2(\omega t)} \hat{A}_\mu(p_1)\hat{A}_\nu(p_2)\hat{A}_\rho(p_3)\\
&\times \Big((p_2^\mu \sinh(\omega(t-2t_3)) + p_3^\mu \sinh(\omega(t-2t_2-2t_3)))\delta^{\nu\rho}\cosh(\omega(t-2t_2))\\
&\quad + (p_3^\nu \sinh(\omega(t-2t_2)) + p_1^\nu \sinh(\omega(2t_3-t)))\delta^{\rho\mu}\cosh(\omega(t-2t_2-2t_3))\\
&\quad + (p_1^\rho \sinh(\omega(2t_2+2t_3-t)) + p_2^\rho \sinh(\omega(2t_2-t)))\delta^{\mu\nu}\cosh(\omega(t-2t_3))\Big)\\
&\times e^{-\frac{1}{4}pQ^{-1}p} + (A \mapsto B)\\
&= \int_0^t dt_1 \int_0^{t-t_1} dt_2 \int_0^{t-t_1-t_2} dt_3 \int_{(\mathbb{R}^4)^3} \frac{dp_1\,dp_2\,dp_3}{(2\pi)^{12}} \frac{-2\omega}{\tanh^4(\omega t)\sinh^2(\omega t)} \hat{A}_\mu(p_1)\hat{A}_\nu(p_2)\hat{A}_\rho(p_3)\\
&\times p_2^\mu \Big(\sinh(\omega(2t_2-2t_3)) + \sinh(\omega(4t_3+2t_2-2t) + \sinh(\omega(2t-4t_2-2t_3))\Big)\\
&+ (A \mapsto B)\\
&= \mathcal{O}(t)\;.
\end{aligned}
$$

(The integral without $e^{-\frac{1}{4}pQ^{-1}p}$ cancels exactly.) We thus have

$$S_{111}(\mathcal{D}_A) = 0\;. \tag{113}$$

B.7. $V_1V_1V_1V_1$**.** The leading order in (60) is given by the three possibilities with $k_i = 0$:

(114)

$$
\begin{aligned}
&S_t(V_1, V_1, V_1, V_1)\\
&= \int_0^t dt_1 \int_0^{t-t_1} dt_2 \int_0^{t-t_1-t_2} dt_3 \int_0^{t-t_1-t_2-t_3} dt_4 \;\mathrm{tr}(e^{-\omega\Sigma t})\\
&\qquad \times S^{1,1,1,1}_{t_4,t_3,t_2,t-t_2-t_3-t_4}(-A,-A,-A,-A) + (A \mapsto B)\\
&= \int_{(\mathbb{R}^4)^3} \frac{dp_1\,dp_2\,dp_3}{(2\pi)^{12}} \frac{(2\omega)^2}{\tanh^4(\omega t)\sinh^2(\omega t)} \hat{A}_\mu(p_1)\hat{A}_\nu(p_2)\hat{A}_\rho(p_3)\hat{A}_\sigma(p_4)\\
&\times \int_0^t dt_1 \int_0^{t-t_1} dt_2 \int_0^{t-t_1-t_2} dt_3 \int_0^{t-t_1-t_2} dt_3 \int_0^{t-t_1-t_2-t_3} dt_4 \Big(\cosh(\omega t_{21})\cosh(\omega t_{43})\delta^{\mu\nu}\delta^{\rho\sigma}\\
&\qquad + \cosh(\omega t_{31})\cosh(\omega t_{42})\delta^{\mu\rho}\delta^{\nu\sigma} + \cosh(\omega t_{41})\cosh(\omega t_{32})\delta^{\mu\sigma}\delta^{\nu\rho}\Big) + (A \mapsto B)\;,
\end{aligned}
$$

with $t_{21} = t - 2t_4$, $t_{43} = t - 2t_2$, $t_{31} = t - 2t_3 - 2t_4$, $t_{42} = t - 2t_2 - 2t_3$, $t_{41} = t - 2t_2 - 2t_3 - 2t_4$ and $t_{32} = t - 2t_3$. Taylor expansion in p_4 and Gaußian integration over $\frac{dp_4}{(2\pi)^4}$ yield, as usual, a factor $\frac{\omega^2}{\pi^2}\tanh(\omega t)$ and an exponential function that can be ignored in leading order. The $t_1, \dots, t_4$ integrals evaluate to $\frac{t^2\sinh^2(\omega t)}{8\omega^2}$, so that we conclude

$$S_{1111}(\mathcal{D}_A) = \frac{\chi_0}{2\pi^2} \int_{\mathrm{R}^4} dx \, \Big\{ A_\mu A^\mu A_\nu A^\nu + B_\mu B^\mu B_\nu B^\nu \Big\}(x) \,. \tag{115}$$

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