Renormalisation of Noncommutative Quantum Field Theory

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Abstract

We recall some models for noncommutative space-time and discuss quantum field theories on these deformed spaces. We discribe the IR/UV mixing disease, which implies nonrenormalizability for a large class of models. We review the power-counting analysis of field theories on the Moyal plane in momentum space and our recent renormalisation proof of noncommutative ϕ^4-theory based on renormalisation group techniques for dynamical matrix models. Some further developments of dynamical matrix models are mentioned.

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1 Introduction

Four-dimensional quantum field theory suffers from infrared and ultraviolet divergences as well as from the divergence of the renormalized perturbation expansion. Despite the impressive agreement between theory and experiments and despite many attempts, these problems are not settled and remain a big challenge for theoretical physics. Furthermore, attempts to formulate a quantum theory of gravity have not yet been fully successful. It is astonishing that the two pillars of modern physics, quantum field theory and general relativity, seem incompatible. This convinced physicists to look for more general descriptions: After the formulation of supersymmetry and supergravity string theory was developed, and anomaly cancellation forced the introduction of six additional dimensions. On the other hand, loop gravity was formulated, and led to spin networks and space-time foams. Both approaches are not fully satisfactory. A third impulse came from noncommutative geometry developed by Alain Connes, providing a natural interpretation of the Higgs effect at the classical level. This finally led to noncommutative quantum field theory, which is the subject of this contribution. It allows one to incorporate fluctuations of space into quantum field theory. There are of course relations among these three developments. In particular, the field theory limit of string theory leads to certain noncommutative field theory models, and some models defined over fuzzy spaces are related to spin networks.

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We know from quantum mechanics that any measurement uncertainty (enforced by principles of Nature and not due to lack of experimental skills) goes hand in hand with noncommutativity. To the best of our knowledge, the possibility that geometry loses its meaning in quantum physics was first[1] considered by Schrödinger [2]. On the other hand Heisenberg suggested to use coordinate uncertainty relations to ameliorate the short-distance singularities in the quantum theory of fields. His idea (which appeared later [3]) inspired Peierls in the treatment of electrons in a strong external magnetic field [4]. Via Pauli and Oppenheimer the idea came to Snyder, who was the first to write down uncertainty relations between coordinates [5].

The uncertainty relations for coordinates were revived by Doplicher, Fredenhagen and Roberts [6] as a means to avoid gravitational collapse when localising events with extreme precision.

1.1 Noncommutative geometry

Space-time differs from the quantum mechanical phase space. Field theory has to be defined over it. Thus, apart from only describing the algebra of space-time operators, we have to realize the geometry of gauge fields, fermions, differential calculi, Dirac operators and action functionals associated with this algebra. Fortunately for us, the relevant mathematical framework—*noncommutative geometry*—has been developed, foremost by Alain Connes [7, 8]. Related monographs are [9, 10, 11, 12].

Noncommutative geometry is the reformulation of geometry in an algebraic and functional-analytic language, in this way permitting an enormous generalization. In physics, the most important achievement of noncommutative geometry is to overcome the distinction between *continuous* and *discrete* spaces, in the same way as quantum mechanics unified the concepts of waves and particles.

Eventually, noncommutative geometry achieved via the spectral action principle [13] a true unification of the Standard Model with general relativity on the level of classical field theories. Kinematically, Yang-Mills fields, Higgs fields and gravitons are all regarded as *fluctuations* of the free Dirac operator [14]. The spectral action

$$S = \mathrm{trace}\, \chi\Big(z \frac{\mathcal{D}^2}{\Lambda^2}\Big) , \tag{1}$$

(which is the weighted sum of the eigenvalues of $\mathcal{D}^2$ up to the cut-off Λ^2) of the single fluctuated Dirac operator $\mathcal{D}$ gives the complete bosonic action of the Standard Model, the Einstein-Hilbert action (with cosmological constant) and an additional Weyl action term in one stroke [13].

Of course, the unification of the standard model with general relativity via the spectral action is of limited value so long as it is not achieved at the level of quantum field theory. On the other hand, the arguments of [6] make clear that this will not be possible with almost commutative geometries (products of commutative geometries with matrices). Space-time has to be noncommutative itself. The complete problem of a gravitational

[1] Actually, Riemann himself speculated in his famous Habilitationsvortrag [1] about the possibility that the hypotheses of geometry lose their validity in the infinitesimally small regime.

dynamics of the noncommutative space-time being too difficult to treat, the first step is to consider field theory on noncommutative background spaces.

2 Some models for noncommutative space (-time)

2.1 The Moyal plane

The best-studied candidate for noncommutative space-time is the Moyal plane [15, 16], which was identified as a solution of the uncertainty conditions for coordinate operators [6]. The (D-dimensional) Moyal plane $\mathbb{R}^D_\theta$ is characterised by the *non-local* $\star$-product

$$(a \star b)(x) := \int d^D y \frac{d^D k}{(2\pi)^D} a\big(x+\tfrac{1}{2}\theta\cdot k\big) b(x+y)\, \mathrm{e}^{\mathrm{i}ky} , \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R} . \tag{2}$$

Here, $a, b \in \mathcal{S}(\mathbb{R}^D)$ are (complex-valued) Schwartz class functions of rapid decay. The entries $\theta^{\mu\nu}$ in (2) have the dimension of an area. Generalisations of (2) to deformations of C^*-algebras are considered in [17].

Using the identity $\int \frac{d^D k}{(2\pi)^D}\, \mathrm{e}^{\mathrm{i}k\cdot(x-y)} = \delta(x-y)$ it is not difficult to prove that the $\star$-product (2) is associative $((a\star b)\star c)(x) = (a\star(b\star c))(x)$ and non-commutative, $a\star b \neq b\star a$. Moreover, complex conjugation is an involution, $\overline{a \star b} = \overline{b} \star \overline{a}$. One has the important property

$$\int d^D x\ (a \star b)(x) = \int d^D x\ a(x) b(x) . \tag{3}$$

Partial derivatives are derivations, $\partial_\mu(a \star b) = (\partial_\mu a) \star b + a \star (\partial_\mu b)$. For various proofs (such as in [18]) one needs the fact that for each $f \in \mathbb{R}^D_\theta$ there exist $f_1, f_2 \in \mathbb{R}^D_\theta$ with $f = f_1 \star f_2$, see [19].

The Moyal product (2) has its origin in quantum mechanics, in particular in Weyl's operator calculus. Wigner introduced the useful concept of the phase space distribution function [20]. Then, Groenewold [15] and Moyal [16] showed that quantum mechanics can be formulated on classical phase space using the *twisted product* concept. In particular, Moyal proposed the "sine-Poisson bracket" (nowadays called Moyal bracket), which is the analogue of the quantum mechanical commutation relations. The twisted product was extended from Schwartz class functions to (appropriate) tempered distributions by Gracia-Bondía and Várilly. The programme of Groenewold and Moyal culminated in the axiomatic approach of *deformation quantisation* [21, 22]. The problem of lifting a given Poisson structure to an associative $\star$-product was solved by Kontsevich [23]. Cattaneo and Felder [24] found a physical derivation of Kontsevich's formula in terms of a path integral quantisation of a Poisson sigma model [25]. The Moyal plane is a spectral triple [18] and the spectral action has been computed [26, 27].

There is a (unfortunately more popular) different version of the $\star$-product,

$$(a \star b)(x) = \exp\Big(\mathrm{i}\theta^{\mu\nu} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial z^\nu}\Big) a(y) b(z)\Big|_{y=z=x} , \tag{4}$$

which is obtained by the following steps from (2):

- Taylor expansion of $a(x+\frac{1}{2}\theta\cdot k)$ about $k=0$,
- repeated representation of $k_\mu \mathrm{e}^{\mathrm{i}k\cdot y} = -\mathrm{i}\frac{\partial}{\partial y^\mu}\mathrm{e}^{\mathrm{i}k\cdot y}$,
- integration by parts in y,
- k-integration yielding $\int \frac{d^Dk}{(2\pi)^D}\mathrm{e}^{\mathrm{i}k\cdot y} = \delta(y)$,
- y-integration.

Of course, as the Taylor expansion is involved, at least one of the functions a, b has to be analytic. Actually, the formula (4) is an *asymptotic expansion* of the $\star$-product (2) which becomes exact under the conditions given in [28]. We would like to stress that the most important property concerning physics is the *non-locality* of the $\star$-product (2), not its non-commutativity. To the value of $a\star b$ at the point x there are contributions of individual values of the functions a, b far away from x. This non-locality is hidden in (4): At first sight it seems to be local, as only the derivatives of a, b at x contribute to $(a\star b)(x)$. However, the point is that analyticity is required, where the information about a function is not localised at all.

A third version of the $\star$-product which is particularly useful for field theory in momentum space is obtained by expressing on the rhs of (2) the functions by their Fourier transforms[2]. This yields

$$(a\star b)(x) = \int \frac{d^Dp}{(2\pi)^D}\,\mathrm{e}^{-\mathrm{i}px}\int \frac{d^Dq}{(2\pi)^D}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu}p_\mu q_\nu}\,\hat{a}(p-q)\hat{b}(q)\;. \tag{5}$$

Being a non-compact space, the algebra $\mathbb{R}^D_\theta$ cannot have a unit. For various reasons, the restriction of the $\star$-product to Schwartz class functions should be relaxed. That extension to tempered distributions was performed in [19]. A good summary is the appendix of [29]. Since (2) is smooth, for T being a tempered distribution and $f, g\in\mathcal{S}(\mathbb{R}^D)$ one defines the product $T\star f$ via

$$\langle T\star f, g\rangle := \langle T, f\star g\rangle\;, \tag{6}$$

and similarly for $f\star T$. Both $T\star f$ and $f\star T$ are smooth functions, but not necessarily of Schwartz class. The set of those T for which $T\star f$ is of Schwartz class is the left multiplier algebra $M_L(\mathbb{R}^D_\theta)$, and similarly for $M_R(\mathbb{R}^D_\theta)$ (which is different). Then, the *Moyal algebra* is defined as $M(\mathbb{R}^D_\theta) := M_L(\mathbb{R}^D_\theta)\cap M_R(\mathbb{R}^D_\theta)$. It is a unital algebra (in fact the largest compactification of $\mathbb{R}^D_\theta$) and contains also the coordinate functions x^μ and the "plane waves" $\mathrm{e}^{\mathrm{i}p_\mu x^\mu}$. In fact, the famous commutation relation $[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}$ holds in $M(\mathbb{R}^D_\theta)$ and not in $\mathbb{R}^D_\theta$. The Moyal algebra is huge so that for practical purposes appropriate subalgebras must be considered [19, 18]. There are several surprises on $M(\mathbb{R}^D_\theta)$: For instance, the Dirac δ-distribution belongs to $M(\mathbb{R}^D_\theta)$, with $\delta\star\delta = \frac{2^D}{\det\theta}1$. On the other hand, $\mathrm{e}^{\frac{2\mathrm{i}}{a}x^1x^2}\in M(\mathbb{R}^2_\theta)$ iff $|a|\neq\theta_1$, $\theta_1 := \theta^{12} = -\theta^{21}$. This proves, by the way, that for different θ the Moyal algebras $M(\mathbb{R}^D_\theta)$ are different.

[2]We use the convention that $f(x) = \int \frac{d^Dp}{(2\pi)^D}\,\mathrm{e}^{-\mathrm{i}px}\hat{f}(p)$ and $\hat{f}(p) = \int d^Dx\,\mathrm{e}^{\mathrm{i}px}f(x)$.

Traditionally, physicists expand the algebra $\mathbb{R}^D_\theta$ into the Weyl basis (plane waves) $\mathrm{e}^{\mathrm{i}p_\mu x^\mu}$, which has the advantage that the resulting computations are similar to the usual treatment of commutative field theories in momentum space. For both mathematical investigations (see e.g. [19, 18]) and our recent renormalisability proof [30] it is, however, much more convenient to use the harmonic oscillator basis given by the eigenfunctions of the Hamiltonian $H = \frac{1}{2}x_\mu x^\mu$. In $D = 2$ dimensions one has [31, 19]

$$H \star f_{mn} = \theta_1(m + \tfrac{1}{2})f_{mn} \;, \qquad f_{mn} \star H = \theta_1(n + \tfrac{1}{2})f_{mn} \;. \tag{7}$$

$$f_{mn}(x) = \frac{2}{\sqrt{n!m!\,\theta_1^{m+n}}}\,\bar{a}^{\star m} \star \mathrm{e}^{-\frac{2H}{\theta_1}} \star a^{\star n} \;, \tag{8}$$

where $a = \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}x_2)$ and $\bar{a} = \frac{1}{\sqrt{2}}(x_1 - \mathrm{i}x_2)$.

The eigenfunctions f_{mn} have the remarkable property that

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x) \;, \qquad \int d^2x\, f_{mn} = (2\pi)\sqrt{\det\theta}\,\delta_{mn} \;. \tag{9}$$

Thus, the f_{mn} behave like infinite standard matrices with entry 1 at the intersection of the $(m+1)^{\mathrm{st}}$ row with the $(n+1)^{\mathrm{st}}$ column, and with entry 0 everywhere else. In fact, the decomposition

$$\mathbb{R}^2_\theta \ni a(x) = \sum_{m,n=0}^{\infty} a_{mn} f_{mn}(x) \tag{10}$$

defines a Fréchet algebra isomorphism between $\mathbb{R}^2_\theta$ and the matrix algebra of rapidly decreasing double sequences $\{a_{mn}\}$ for which

$$r_k(a) := \Big(\sum_{m,n=0}^{\infty} \theta_1^{2k}(m + \tfrac{1}{2})^k (n + \tfrac{1}{2})^k |a_{mn}|^2\Big)^{\frac{1}{2}} \tag{11}$$

is finite for all $k \in \mathbb{N}$, see [19].

Both the $f_{mn}(x)$ and their Fourier transforms are given by Laguerre polynomials in the radial direction and Fourier modes in the angular direction. On one hand, this makes clear that the f_{mn} form a basis of the two-dimensional Moyal plane. On the other hand, restricting the matrix base to finite matrices f_{mn}, $n, m \leq N$, corresponds to a cut-off both in position space and momentum space.

Further, we note that the f_{mn} are also the common eigenfunctions of the Landau Hamiltonian

$$H_L^{\pm} = \frac{1}{2}(\mathrm{i}\partial_\mu \pm A_\mu)(\mathrm{i}\partial^\mu \pm A^\mu) \;, \qquad A_\mu = \frac{1}{2}B_{\mu\nu}x^\nu \;. \tag{12}$$

If $B_{\mu\nu} = 4(\theta^{-1})_{\mu\nu}$, and thus $B := \frac{4}{\theta_1}$, one has

$$H_L^{+} f_{mn} = B(m + \tfrac{1}{2})f_{mn} \;, \qquad H_L^{-} f_{mn} = B(n + \tfrac{1}{2})f_{mn} \;. \tag{13}$$

Thus, the harmonic oscillator basis has the additional merit of diagonalising the Landau Hamiltonian. This observation was the starting point of various exact solutions of quantum field theories on noncommutative phase space [32, 33, 34].

For more information about the noncommutative $\mathbb{R}^D$ we refer to [19, 35, 18].

2.2 *The noncommutative torus*

The Moyal plane is closely related to the noncommutative torus, which is the best-studied noncommutative space [36, 37]. A basis for the algebra $\mathbb{T}^D_\theta$ of the noncommutative D-torus is given by unitaries U^p labelled by $p = \{p_\mu\} \in \mathbb{Z}^D$, with $U^p(U^p)^* = (U^p)^* U^p = 1$. The multiplication is defined by

$$U^p U^q = \mathrm{e}^{\mathrm{i}\pi\theta^{\mu\nu} p_\mu q_\nu} U^{p+q} , \qquad \mu, \nu = 1, \ldots, D , \quad \theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{R} . \tag{14}$$

Elements $a \in \mathbb{T}^d_\theta$ have the following form:

$$a = \sum_{p \in \mathbb{Z}^d} a_p U^p , \qquad a_p \in \mathbb{C} , \quad \|p\|^n |a_p| \to 0 \text{ for } \|p\| \to \infty . \tag{15}$$

If $\theta^{\mu\nu} \notin \mathbb{Q}$ (irrational case) one can define partial derivatives

$$\partial_\mu U^p := -\mathrm{i} p_\mu U^p , \tag{16}$$

which satisfy the Leibniz rule and Stokes' law with respect to the integral

$$\int a = a_0 , \tag{17}$$

where a is given by (15).

An excellent presentation of the noncommutative torus was given by Rieffel [38].

Other interesting noncommutative spectral triples are the Connes-Landi spheres [39] and the (mostly spherical) examples found by Connes and Dubois-Violette [40].

2.3 *Fuzzy spaces*

The fuzzy sphere [41] is one of the simplest noncommutative spaces. It is obtained by truncating the representations of $su(2)$. The algebra S^2_N is identified with the mappings from the representation space $\frac{N}{2}$ of $su(2)$ to itself, thus with the algebra $M_{N+1}(\mathbb{C})$. The fuzzy sphere S^2_N is generated by $\hat{X}_i$, $i = 1, 2, 3$, which form an $su(2)$-Lie algebra with suitable rescaling, identified by the requirement that the Casimir operator still fulfils the defining relation of the two-sphere as an operator:

$$[\hat{X}_i, \hat{X}_j] = \sum_{k=1}^{3} \mathrm{i}\lambda_N \epsilon_{ijk} \hat{X}_k , \qquad \sum_{i=1}^{3} \hat{X}_i \hat{X}_i = R^2 , \qquad \frac{R}{\lambda_N} = \sqrt{\tfrac{N}{2}\big(\tfrac{N}{2} + 1\big)} . \tag{18}$$

One has to give a precise description of the embeddings of these algebras for different N. Then, for fixed radius R, one recovers the commutative algebra of the ordinary sphere, $\lambda = 0$, in the limit $N \to \infty$ [42].

The Lie algebra $su(2)$ generated by J_i, $i = 1, 2, 3$, acts on $a \in S^2_N$ by the adjoint action

$$J_i a = \frac{1}{\lambda} [\hat{X}_i, a] . \tag{19}$$

Thus, an element $a \in S^2_N$ can be represented by $a = \sum_{l=0}^{N} \sum_{m=-l}^{l} a_{lm}\Psi_{lm}$, where

$$\sum_{i=1}^{3} J_i^2 \Psi_{lm} = l(l+1)\Psi_{lm} \ , \qquad J_3\Psi_{lm} = m\Psi_{lm} \ , \qquad \frac{4\pi}{N+1}\mathrm{tr}\big(\overline{\Psi_{lm}}\Psi_{l'm'}\big) = \delta_{ll'}\delta_{mm'} \ . \tag{20}$$

For comments on field theoretical models, see Section 4. Other fuzzy spaces include the fuzzy $\mathbb{C}P^2$ [43, 44] and the q-deformed fuzzy sphere [45, 46].

3 Classical field theory on noncommutative spaces

Since classical field theories can be geometrically described, it is not difficult to write down classical action functionals on noncommutative spaces. The first example of this type was Yang-Mills theory on the noncommutative torus. Another example is the noncommutative geometrical description of the Standard Model recalled briefly in Section 1.1.

3.1 Field theory on the noncommutative torus

The noncommutative torus became popular to field theorists when Connes, Douglas and Schwarz [39] proposed to compactify M-theory on such a space. M-theory lives in higher dimensions so that some of them must be compactified to give a realistic model. Compactifying on a noncommutative instead of a commutative torus amounts to turning on a constant background 3-form C. An alternative interpretation based on D-branes on tori in presence of a Neveu-Schwarz B-field was given by Douglas and Hull. Similar effects are obtained in boundary conformal field theory [47]. There are also other noncommutative spaces which arise as limiting cases of string theory [48].

Later, the appearance of noncommutative field theory in the zero-slope limit of type II string theory was thoroughly investigated by Seiberg and Witten [49]. Moreover, using the results of [50] about instantons on noncommutative $\mathbb{R}^4$, Seiberg and Witten argued that there is an equivalence between the Yang-Mills theories on standard $\mathbb{R}^4$ and noncommutative $\mathbb{R}^4$, which we comment on in Section 5.4.

It should be mentioned that matrix theories were studied long before M-theory was proposed, and that these matrix theories did contain certain noncommutative features. In the large-N limit of two-dimensional $SU(N)$ lattice gauge theory, the number of degrees of freedom is reduced and corresponds to a zero-dimensional model, under the condition that no spontaneous breakdown of the $[U(1)]^4$-symmetry appears. As shown in [51], a spontaneous symmetry breakdown does not appear when twisted boundary conditions [52] are used. In [53], the construction of the twisted Eguchi-Kawai model was extended to any even dimension. Here, the action can be rewritten in terms of noncommuting matrix derivatives $[\Gamma^{(j)},\ .\ ]$, with $[\Gamma^{(2j)},\Gamma^{(2j+1)}] = -2\pi\mathrm{i}/N$.

3.2 Classical action functionals on the Moyal plane

Here, we list for the example of the Moyal plane a few important action functionals for noncommutative field theories. In principle, these action functionals are related to connections on projective modules. To simplify the presentation, we restrict ourselves to trivial modules given by the algebra R^D_θ itself.

The most natural action from the point of view of noncommutative geometry is $U(N)$ Yang-Mills theory in four dimensions:

$$S_{\mathrm{YM}}[A] = \int d^4x \,\mathrm{tr}_{M_N(\mathbb{C})}\Big(\frac{1}{4g^2}F_{\mu\nu}\star F^{\mu\nu}\Big)\;, \tag{21}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}(A_\mu\star A_\nu - A_\nu \star A_\mu)\;, \tag{22}$$

where $A_\mu = A_\mu^* \in \mathbb{R}^4_\theta \otimes M_N(\mathbb{C})$. This action arises from the Connes-Lott action functional [54] and the spectral action principle [26, 27] as well as in the zero-slope limit of string theory [49]. For quantum field theory it has to be extended—as usual—by the ghost sector:

$$S_{\mathrm{gf}} = \int d^4x \,\mathrm{tr}_{M_N(\mathbb{C})}\Big(s\Big\{\bar{c}\star\partial_\mu A^\mu + \frac{\alpha}{2}\bar{c}\star B + \rho^\mu \star A_\mu + \sigma\star c\Big\}\Big)\;, \tag{23}$$

where α is the gauge parameter. The components of $\bar{c}, c, \rho^\mu$ are anticommuting fields and the graded BRST differential s [55] (which commutes with ∂_μ) is defined by

$$\begin{aligned} sA_\mu &= \partial_\mu c - \mathrm{i}(A_\mu \star c - c\star A_\mu)\;, & sc &= \mathrm{i}c\star c\;, \\ s\bar{c} &= B\;, & sB &= s\rho^\mu = s\sigma = 0\;. \end{aligned} \tag{24}$$

The external fields ρ^μ and σ are the Batalin-Vilkovisky antifields [56] relative to A_μ and c, respectively.

There is always a reference frame in which the noncommutativity matrix θ takes the standard form where the only non-vanishing components are $\theta_{2i,2i+1} = -\theta_{2i+1,2i} \equiv \theta_i$. Each of the two-dimensional blocks is invariant under two-dimensional rotations. This means that action functionals which involve the $\star$-product like (21) are invariant under the subgroup $\big(SO(2)\big)^{\frac{D}{2}}$ of the D-dimensional rotation group $SO(D)$.

The Yang-Mills action (21) suggests that action functionals for field theories on $\mathbb{R}^D_\theta$ are simply obtained by replacing the ordinary (commutative) product of functions on Euclidean space by the $\star$-product (2). This procedure leads to the following action for noncommutative ϕ^4-theory:

$$S[\phi] := \int d^Dx \Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{1}{2}\mu^2\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x)\;. \tag{25}$$

It must be stressed, however, that this is a formal procedure and that—in contrast to the Yang-Mills action (21)—the scalar field action (25) does not directly follow from noncommutative geometry or the scaling limit of string theory [49]. In fact, we have proven in [30] that it has to be extended by a harmonic oscillator term.

It was pointed out by Langmann and Szabo [57] that the $\star$-product interaction is (up to rescaling) invariant under a duality transformation between positions and momenta. Indeed, using a modified Fourier transform $\hat{\phi}(p_a) = \int d^4x\; \mathrm{e}^{(-1)^a \mathrm{i} p_{a,\mu} x^\mu_a}\phi(x_a)$, where the subscript a refers to the cyclic order in the $\star$-product, one obtains from the definitions

(2) and (5) and the reality condition $\phi(x) = \overline{\phi(x)}$ the representation

$$
\begin{aligned}
S_{\text{int}}[\phi;\lambda] &= \int d^4x\, \frac{\lambda}{4!}(\phi\star\phi\star\phi\star\phi)(x) \\
&= \int \Big(\prod_{a=1}^4 d^4x_a\Big)\, \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\, V(x_1,x_2,x_3,x_4) && (26a)\\
&= \int \Big(\prod_{a=1}^4 \frac{d^4p_a}{(2\pi)^4}\Big)\, \hat\phi(p_1)\hat\phi(p_2)\hat\phi(p_3)\hat\phi(p_4)\, \hat V(p_1,p_2,p_3,p_4)\;, && (26b)
\end{aligned}
$$

with

$$
\begin{aligned}
\hat V(p_1,p_2,p_3,p_4) &= \frac{\lambda}{4!}(2\pi)^4\delta^4(p_1{-}p_2{+}p_3{-}p_4)\,\cos\Big(\tfrac{1}{2}\theta^{\mu\nu}(p_{1,\mu}p_{2,\nu}{+}p_{3,\mu}p_{4,\nu})\Big)\;, && (27a)\\
V(x_1,x_2,x_3,x_4) &= \frac{\lambda}{4!}\frac{1}{\pi^4\det\theta}\,\delta^4(x_1{-}x_2{+}x_3{-}x_4)\,\cos\Big(2(\theta^{-1})_{\mu\nu}(x_1^\mu x_2^\nu{+}x_3^\mu x_4^\nu)\Big)\;. && (27b)
\end{aligned}
$$

Thus, the replacements

$$
\hat\phi(p) \leftrightarrow \pi^2\sqrt{|\det\theta|}\,\phi(x)\;, \qquad\qquad p_\mu \leftrightarrow \tilde x_\mu := 2(\theta^{-1})_{\mu\nu}x^\nu\;, \tag{28}
$$

exchange the a,b-versions of (26) and (27).

On the other hand, the usual free scalar field action given by $\lambda = 0$ in (25) is not invariant under this duality transformation. In order to achieve this we have to extend the free scalar field action by a harmonic oscillator potential:

$$
S_{\text{free}}[\phi;\mu,\Omega_0] = \int d^4x\,\Big(\frac{1}{2}(\partial_\mu\phi)\star(\partial^\mu\phi) + \frac{\Omega^2}{2}(\tilde x_\mu\phi)\star(\tilde x^\mu\phi) + \frac{\mu^2}{2}\phi\star\phi\Big)(x)\;. \tag{29}
$$

Of course, the oscillator potential breaks translation invariance. For complex scalar fields φ of electric charge Ω, another possibility is given by a constant external magnetic field $B_{\mu\nu} = 4(\theta^{-1})_{\mu\nu}$ via the covariant derivative $D_\mu\varphi := \partial_\mu\varphi + \mathrm{i}\Omega A_\mu\varphi$, with $A_\mu = \frac{1}{2}B_{\mu\nu}x^\nu$:

$$
S^B_{\text{free}}[\varphi;\mu,\Omega] = \int d^4x\,\Big(\frac{1}{2}(D_\mu\varphi)^*\star(D^\mu\varphi) + \frac{\mu^2}{2}\varphi^*\star\varphi\Big)(x)\;. \tag{30}
$$

Adding the interaction term $S_{\text{int}}[\varphi;\lambda] = \frac{\lambda}{4!}\int d^4x\,\varphi\star\varphi^*\star\varphi\star\varphi^*$, the quantum field theory associated with the magnetic field action (30) was analysed and for $\Omega = 1$ exactly solved in [33, 34]. Note that

$$
\begin{aligned}
&S_{\text{free}}[\phi_1;\mu,\Omega] + S_{\text{free}}[\phi_2;\mu,\Omega]\\
&\qquad = \frac{1}{2}S^B_{\text{free}}[\phi_1{+}\mathrm{i}\phi_2;\mu,\Omega] + \frac{1}{2}S^B_{\text{free}}[\phi_1{+}\mathrm{i}\phi_2;\mu,-\Omega]\;. && (31)
\end{aligned}
$$

The interaction mixes ϕ_1,ϕ_2, though.

Now, under the transformation (28) one has for the total action $S = S_{\text{free}} + S_{\text{int}}$

$$
S\big[\phi;\mu,\lambda,\Omega\big] \mapsto \Omega^2 S\Big[\phi;\frac{\mu}{\Omega},\frac{\lambda}{\Omega^2},\frac{1}{\Omega}\Big]\;, \tag{32}
$$

and accordingly for $S^B_{\text{free}}[\varphi,\Omega] + S_{\text{int}}[\varphi,\lambda]$. In the special case $\Omega = 1$ the action $S[\phi;\mu,\lambda,1]$ is invariant under the duality (28) and can be written as a standard matrix model.

4 Regularisation

The philosophy that space-time noncommutativity regularises quantum field theories was made explicit in fuzzy noncommutative spaces [41, 9]. The partition function relative to the scalar field action on the fuzzy sphere,

$$Z[j] = \int \mathcal{D}[\phi]\, \mathrm{e}^{-S[\phi] - \frac{4\pi}{N+1}\mathrm{tr}(j\phi)} ,$$
$$S[\phi] = \frac{4\pi}{N+1}\mathrm{tr}\Big(\frac{1}{2}\sum_{i=1}^{3} \phi J_i^2 \phi + V[\phi]\Big) , \qquad \phi \in S^2_N , \tag{33}$$

where $V[\phi]$ is some polynomial in ϕ, leads to an automatic UV-regularisation [58, 59, 43] of the resulting Feynman graphs. See also [60].

Of course, the standard divergences of the ϕ^4-model on the commutative sphere S^2 will reappear in the limit $N \to \infty$. This limit was investigated in [61]. For the one-loop self-energy in the ϕ^4-model, a finite but non-local difference between the $N \to \infty$ limit of the fuzzy sphere and the ordinary sphere was found. See [62] for similar calculations.

The construction of gauge models on the fuzzy sphere is less obvious. See e.g. [63, 64].

We would also like to mention another construction of finite quantum field theories on noncommutative spaces which is based on point-splitting via tensor products [65, 66].

5 Renormalisation

It is not difficult to write down classical action functionals on noncommutative spaces (see Section 3), but it is not clear that quantum field theories [67, 68, 69] can be defined consistently[3]. As locality is so important in quantum field theory, it is perfectly possible that quantum field theories are implicitly built upon the assumption that the action functional has to live on a (commutative) manifold.

The first results on noncommutative quantum field theories (with an infinite number of degrees of freedom) are due to Filk [70] who showed that the planar graphs of a field theory on the Moyal plane[4] are identical to the commutative theory (and thus have the same divergences). An achievement in [70] which turned out to be important for later work was the definition of the *intersection matrix* of a graph which is read off from its reduction to a rosette. In [71] the persistence of divergences was rephrased in the framework of noncommutative geometry, based on the general definition of a dimension and the noncommutative formulation of external field quantisation. See also [72].

Knowing that divergences persist in quantum field theories on the Moyal plane, the question arises whether these models are renormalisable. It was, therefore, an important step to prove that Yang-Mills theory is one-loop renormalisable on the Moyal plane and on the noncommutative torus [73, 74, 75]. This means that these models are divergent [70], but the one-loop divergences are absorbable in a multiplicative renormalisation of the initial action such that the Ward identities are fulfilled.

[3] This refers to infinite-dimensional quantum field theories. There is no problem with finite-dimensional examples [58, 59].

[4] Filk's model refers to [6] but is formulated in the $\star$-product formalism. It is certainly inspired by the twisted Eguchi-Kawai model [51, 53] discussed in Section 3.1.

In this line of success, it was somewhat surprising when Minwalla, Van Raamsdonk and Seiberg [76] pointed out that there is a new type of infrared-like divergences which makes the renormalisation of scalar field theories on the Moyal plane very unlikely. Non-planar graphs are regulated by the phase factors in the $\star$-product (5), but only if the external momenta of the graph are non-exceptional. Inserting non-planar graphs (declared as regular) as subgraphs into bigger graphs, external momenta of the subgraph are internal momenta for the total graph. As such, exceptional external momenta for the subgraph are realised in the loop integration, resulting in a divergent integral for the total graph. This is the so-called *UV/IR-mixing* problem [76].

5.1 Quantum field theory on the noncommutative torus

The paper [77] inspired many activities on the interface between string/M-theory and noncommutative geometry (we come back to that in Section 5.2). Among others the question was raised whether Yang-Mills theory on the noncommutative torus is renormalisable. See also [78]. We have confirmed one-loop renormalisability in [75]: Using ζ-function techniques and cocycle identities we have extracted the pole parts related to the Feynman graphs and proved that they can be removed by multiplicative renormalisation of the initial action. In particular, the Ward identities are satisfied. See also [79].

Based on ideas developed in [80] on type IIB matrix models, it was shown in [81] that, imposing a natural constraint for the (finite) matrices, the twisted Eguchi-Kawai construction [51] can be generalised to noncommutative Yang-Mills theory on a toroidal lattice. The appearing gauge-invariant operators are the analogues of Wilson loops. This formulation enabled numerical simulations [82, 83] of the various limiting procedures which confirmed conjectures [84] about striped and disordered patterns in the phase diagram of spontaneously broken noncommutative ϕ^4-theory. On the other hand, the limit $N \to \infty$ of the matrix size is mathematically delicate [85]. To deal with that problem, a new formulation [86, 87] of matrix models approximating field theories on the noncommutative torus has been proposed which is based on noncommutative solitons [88].

An important development is the exact (non-perturbative) solution of Yang-Mills theory on the two-dimensional noncommutative torus [89, 90]. This solution is in the same spirit as the original Connes-Rieffel analysis [36, 37], but expands it to completely solve the quantum theory.

5.2 Quantum field theories on the Moyal plane

With the motivation of the Moyal plane in [6], the proof that UV-divergences in quantum field theories persist [70], and the relationship of the noncommutative torus to M-theory [77] and the noncommutative $\mathbb{R}^D$ to type II string theory [91, 50]. The time was ready in 1998 to investigate the renormalisation of quantum field theories on the noncommutative torus and the noncommutative $\mathbb{R}^D$. It is, therefore, not surprising that this question was addressed by different groups at about the same time [73, 74, 75].

Martín and Sánchez-Ruiz [73] investigated U(1) Yang-Mills theory on the noncommutative $\mathbb{R}^4$ at the one-loop level. They found that all one-loop pole terms of this model in dimensional regularisation[5] can be removed by multiplicative renormalisation (minimal subtraction) in a way preserving the BRST symmetry. This is completely analogous to the situation on the noncommutative 4-torus [75]. Shortly after there also appeared an investigation of $(2+1)$-dimensional super-Yang-Mills theory with the two-dimensional space being the noncommutative torus [74].

The paper [49] of Seiberg and Witten from August 1999 made the interface between string theory and noncommutative geometry extremely popular. Thousands of papers on this subject appeared, making it impossible to give an adequate overview. We restrict ourselves to the renormalisation question and refer to the following reviews for further information:

- by Konechny and Schwarz with focus on compactifications of M-theory on noncommutative tori [92] as well as on instantons and solitons on noncommutative $\mathbb{R}^D$ [93],
- by Douglas and Nekrasov [94] and by Szabo [95], both with focus on field theory on noncommutative spaces in relation to string theory,
- by Aref'eva, Belov, Giryavets, Koshelev and Medvedev [96] with focus on string field theory.

A systematic analysis of field theories on noncommutative $\mathbb{R}^D$, to any loop order, was first performed by Chepelev and Roiban [97]. The essential technique is the representation of Feynman graphs as ribbon graphs, drawn on an (oriented) Riemann surface with boundary, to which the external legs of the graph are attached. Using sophisticated mathematical tools (which we review in Section 5.3), Chepelev and Roiban were able to relate the power-counting behaviour to the topology of the graph. Their first conclusion was that a noncommutative field theory is renormalisable iff its commutative counterpart is renormalisable. Then, by computing the non-planar one-loop graphs explicitly, Minwalla, Van Raamsdonk and Seiberg pointed out a serious problem in the renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ and ϕ^3-theory on noncommutative $\mathbb{R}^6$ [76]. It turned out that this problem was simply overlooked in the first version of [97], with the power-counting analysis being correct. A refined proof of the power-counting theorem was given in [98].

Anyway, the problem discovered in [76] made the subject of noncommutative field theories extremely popular. In the following months, an enormous number of articles doing (mostly) one-loop computations of all kinds of models appeared. We do not want to give an overview about these activities and mention only a few papers: the two-loop calculation of ϕ^4-theory [99]; the renormalisation of complex $\phi \star \phi^* \star \phi \star \phi^*$ theory [100], later explained by a topological analysis [98]; computations in noncommutative QED [101]; the calculation of noncommutative $U(1)$ Yang-Mills theory [102], with an outlook to super-Yang-Mills theory; the one-loop analysis of noncommutative $U(N)$ Yang-Mills theory [103].

[5] There is of course a problem extending θ to complex dimensions; this is however discussed in [73].

5.3 The power-counting analysis of Chepelev and Roiban

The previously mentioned one-loop calculations are superseded by the power-counting theorem of Chepelev and Roiban [98] which decides the renormalisability question of (massive, Euclidean) quantum field theories on the Moyal plane to all orders. Roughly speaking, quantum field theories with only logarithmic divergences are renormalisable[6] on the Moyal plane. Still, the 1PI Green's functions do not exist pointwise (at exceptional momenta) so that multiplication with IR-smoothing test functions is necessary. Apart from some exceptional cases such as the $\phi\star\phi^*\star\phi\star\phi^*$ interaction, models with quadratic divergences are not perturbatively renormalisable.

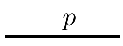
As we have the impression that the work of Chepelev and Roiban is not sufficiently known, we would like to review the main steps for the example of the noncommutative ϕ^4-model arising from the action (25). As usual, the Euclidean quantum field theory is (formally) defined via the partition function,

$$Z[J] := \int \mathcal{D}[\phi]\, \mathrm{e}^{-S[\phi]-\int d^Dx\, J(x)\phi(x)}\;. \tag{34}$$

We suppose here that the fields are expanded in the Weyl basis $\phi(x) = \int \frac{d^Dp}{(2\pi)^D}\,\phi(p)\mathrm{e}^{\mathrm{i}px}$, where $\phi(p)$ are commuting amplitudes of rapid decay in $\|p\|$ and $\mathrm{e}^{\mathrm{i}px}$ is the base of an appropriate subalgebra of the Moyal algebra $M(\mathbb{R}^D_\theta)$. Then, the "measure" of the functional integration is formally defined as $\mathcal{D}[\phi] = \prod_{p\in\mathbb{R}^D} d\phi(p)$.

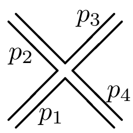
As usual, the integral (34) is solved perturbatively about the solution of the free theory given by $\lambda = 0$. The solution is conveniently organised by *Feynman graphs* built according to Feynman rules out of propagators and vertices. For the noncommutative scalar field action (25), the representation (5) leads to the following rules:

- Due to (3), the propagator is unchanged compared with commutative ϕ^4-theory, but for later purposes is written in double line notation:



$$= \frac{1}{p^2+m^2}\;. \tag{35}$$

- The vertices receive phase factors [70] which depend on the cyclic order of the legs:



$$= \frac{\lambda}{4!}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\sum_{i<j} p_i^\mu p_j^\nu \theta_{\mu\nu}}\;. \tag{36}$$

  There is momentum conservation $p_1+p_2+p_3+p_4 = 0$ at each vertex (due to translation invariance of (25)).

The double line notation reflects the fact that the vertex (36) is invariant only under cyclic permutations of the legs (using momentum conservation). The resulting Feynman graphs are *ribbon graphs* [62, 97] which depend crucially on how the valences of the vertices are connected. For *planar graphs* the total phase factor of the integrand is

[6] The reason is that logarithms are integrable, see [104] for an explicit construction of the estimations.

independent of internal momenta, whereas *non-planar graphs* have a total phase factor which involves internal momenta. Planar graphs are integrated as usual and give (up to symmetry factors) the same divergences as commutative ϕ^4-theory [70]. One would remove these divergences as usual by appropriate normalisation conditions for physical correlation functions. Non-planar graphs require a separate treatment.

There is a closed formula for the integral associated to a noncommutative Feynman graph in terms of the intersection matrices I, J, K (which encode the phase factors) and the incidence matrix $\mathcal{E}$. We give an orientation to each inner line l and let k_l be the momentum flowing through the line l. For each vertex v we define[7]

$$\mathcal{E}_{vl} = \begin{cases} 1 & \text{if } l \text{ emerges from } v\,, \\ -1 & \text{if } l \text{ arrives at } v\,, \\ 0 & \text{if } l \text{ is not attached to } v\,. \end{cases} \tag{37}$$

We let P_v be the total external momentum flowing into the vertex v. Restricting ourselves to 4 dimensions, a 1PI (one-particle irreducible) Feynman graph $\mathcal{G}$ with I internal lines and V vertices gives rise to the integral

$$\begin{aligned}\mathcal{I}_{\mathcal{G}}(P) = \int \prod_{l=1}^{I} \frac{d^4k_l}{(k_l^2+m^2)} \prod_{v=1}^{V} (2\pi)^4 \delta\Big(P_v - \sum_{l=1}^{l} \mathcal{E}_{vl} k_l\Big) \\ \times \exp \mathrm{i}\theta_{\mu\nu}\Big(\sum_{m,n=1}^{I} I^{mn} k_m^\mu k_n^\nu + \sum_{m=1}^{I}\sum_{v=1}^{V} J^{mv} k_m^\mu P_v^\nu + \sum_{v,w=1}^{V} K^{vw} P_v^\mu P_w^\nu\Big)\,.\end{aligned} \tag{38}$$

One can show that $I^{mn}, J^{mv}, K^{vw} \in \{1, -1, 0\}$ after use of momentum conservation [70].

Next, one introduces Schwinger parameters $\frac{1}{k^2+m^2} = \int d\alpha\, \mathrm{e}^{-\alpha(k^2+m^2)}$ and the identity $(2\pi)^4\delta(q_v) = \int d^4y_v\, \mathrm{e}^{\mathrm{i}y_v q_v}$ for each vertex in (38), then completes the squares in k and performs the Gaußian k-integrations[8]. Writing $y_{\bar{v}} = y_V + z_{\bar{v}}$ for $\bar{v} = 1, \dots, V{-}1$ one has $\sum_{v=1}^{V} y_v \mathcal{E}_{vl} = \sum_{\bar{v}=1}^{V-1} z_{\bar{v}} \bar{\mathcal{E}}^{\bar{v}l}$. The y_V-integration yields the overall momentum conservation. It remains to complete the squares for $z_{\bar{v}}$ and finally to evaluate the Gaußian z_v-integrations. The result is [97]

$$\begin{aligned}\mathcal{I}_{\mathcal{G}}(P) = (2\pi)^4\delta\Big(\sum_{v=1}^{V} P_v\Big) \frac{1}{16^I \pi^{2L}} \exp\Big(\mathrm{i}\theta_{\mu\nu} \sum_{v,w=1}^{V} K^{vw} P_v^\mu P_w^\nu\Big) \\ \times \int_0^\infty \prod_{l=1}^{I} d\alpha_l\, \frac{\mathrm{e}^{-\sum_{l=1}^{I}\alpha_l m^2}}{\sqrt{\det\mathcal{A}\det\mathcal{B}}} \exp\Big(-\frac{1}{4}(J\tilde{P})^T \mathcal{A}^{-1}(J\tilde{P}) \\ + \frac{1}{4}\big(\bar{\mathcal{E}}\mathcal{A}^{-1}(J\tilde{P}) + 2\mathrm{i}P'\big)^T \mathcal{B}^{-1}\big(\bar{\mathcal{E}}\mathcal{A}^{-1}(J\tilde{P}) + 2\mathrm{i}P'\big)\Big)\,,\end{aligned} \tag{39}$$

[7]We assume that tadpoles (a line starting and ending at the same vertex) are absent. In the final formula they can be taken into account [98].

[8]This means that the order of integrations is exchanged in an integral which is in general not absolutely convergent. Thus, the result (39) is based on a certain limiting procedure, which is not necessarily unique. That leaves the possibility of circumventing the UV/IR-problems arising from (39) by different limiting procedures.

where

$$\begin{aligned}
\mathcal{A}^{mn}_{\mu\nu} &:= \alpha_m \delta^{mn}\delta_{\mu\nu} - \mathrm{i} I^{mn}\theta_{\mu\nu} \;, & (J\tilde{P})^m_\mu &:= \sum_{v=1}^{V} J^{mv}\theta_{\mu\nu}P^\nu_v \;, \\
\bar{\mathcal{E}}^{\bar{v}l} &:= \mathcal{E}_{\bar{v}l} \quad \text{for } \bar{v} = 1,\dots,V{-}1 \;, & P'^{\bar{v}}_\mu &:= P^\mu_{\bar{v}} \quad \text{for } \bar{v} = 1,\dots,V{-}1 \;, \\
\mathcal{B}^{\bar{v}\bar{w}}_{\mu\nu} &:= \sum_{m,n=1}^{I} \bar{\mathcal{E}}^{\bar{v}m}(\mathcal{A}^{-1})^{\mu\nu}_{mn}\bar{\mathcal{E}}^{\bar{w}n} \;. &&
\end{aligned} \tag{40}$$

The formula (39) is referred to as the parametric integral representation of a noncommutative Feynman graph. See also [76]. Actually, [98] treats a more general case where also derivative couplings are admitted.

Possible divergences of (39) show up in the $\alpha_i \to 0$ behaviour[9]. In order to analyse them one reparametrises the integration domain in (39), similar to the usual procedure described in [69]. For each Hepp sector [106]

$$\alpha_{\pi_1} \leq \alpha_{\pi_2} \leq \cdots \leq \alpha_{\pi_I} \qquad \text{related to a permutation } \pi \text{ of } 1,\dots,I \tag{41}$$

one defines $\alpha_{\pi_i} = \prod_{j=i}^{I} \beta_j^2$, with $0 \leq \beta_I < \infty$ and $0 \leq \beta_j \leq 1$ for $j \neq I$. The leading contribution for small β_j has a topological interpretation.

A ribbon graph can be drawn on a genus-g Riemann surface with possibly several holes to which the external legs are attached [97, 98]. We will say more on ribbon graphs on Riemann surfaces in Section 6.2. We will explain, in particular, how a ribbon graph $\mathcal{G}$ defines a Riemann surface. On such a Riemann surface one considers *cycles*, i.e. equivalence classes of closed paths which cannot be contracted to a point. According to homological algebra, one actually factorises with respect to commutants, i.e. one considers the path $aba^{-1}b^{-1}$ involving two cycles a, b as trivial. We let $c_{\mathcal{G}}(\mathcal{G}_i)$ be the number of non-trivial cycles of the ribbon graph $\mathcal{G}$ wrapped by the subgraph $\mathcal{G}_i$. Next, there may exist external lines m, n such that the graph obtained by connecting m, n has to be drawn on a Riemann surface of genus $g_{mn} > g$. If this happens one defines an index $j(\mathcal{G}) = 1$, otherwise $j(\mathcal{G}) = 0$. The index extends to subgraphs by defining $j_{\mathcal{G}}(\mathcal{G}_i) = 1$ if there are external lines m, n of $\mathcal{G}$ which are already attached to $\mathcal{G}_i$ so that the line connecting m, n wraps a cycle of the additional genus $g \to g_{mn}$ of $\mathcal{G}$.

Now we can formulate the relation between the parametric integral representation and the topology of the ribbon graph. Each sector (41) of the α-parameters defines a sequence of (possibly disconnected) subgraphs $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}_I = \mathcal{G}$, where $\mathcal{G}_i$ is made up of the i doublelines $\pi_1, \dots, \pi_i$ and the vertices to which these lines are attached. If $\mathcal{G}_i$ forms L_i loops it has a power-counting degree of divergence $\omega_i = 4L_i - 2i$. Using sophisticated mathematical techniques on determinants (e.g. the Cauchy-Binet theorem and Jacobi ratio theorem), Chepelev and Roiban have derived in [98] the following leading

[9]The mass term regularises the $\alpha \to \infty$ behaviour of (39). It should be possible to proceed accordingly for massless models using Lowenstein's trick of auxiliary masses [105].

contribution to the integral:

$$\mathcal{I}_{\mathcal{G}}(P) = (2\pi)^4 \delta\Big(\sum_{v=1}^{V} P_v\Big) \frac{1}{8^I \pi^{2L} (\det\theta)^g} \exp\Big(\mathrm{i}\theta_{\mu\nu} \sum_{v,w=1}^{V} K^{vw} P_v^\mu P_w^\nu\Big)$$
$$\times \sum_{\text{Hepp sectors}} \int_0^\infty \frac{d\beta_I \, \mathrm{e}^{-\beta_I^2 m^2}}{\beta_I^{1+\omega_I - 4c_{\mathcal{G}}(\mathcal{G})}} \int_0^1 \Big(\prod_{i=1}^{I-1} \frac{d\beta_i}{\beta_i^{1+\omega_i - 4c_{\mathcal{G}}(\mathcal{G}_i)}}\Big)$$
$$\times \exp\Big(-f_\pi(P) \prod_{i=1}^{I} \frac{1}{\beta^{2j_{\mathcal{G}}(\mathcal{G}_i)}}\Big)\Big(1 + \mathcal{O}(\beta^2)\Big) , \tag{42}$$

where $f_\pi(P) \geq 0$, with equality for *exceptional momenta*. In order to obtain a finite integral $\mathcal{I}_{\mathcal{G}}$, one obviously needs

1. $\omega_i - 4c_{\mathcal{G}}(\mathcal{G}_i) < 0$ for all i if $j(\mathcal{G}) = 0$ or $j(\mathcal{G}) = 1$ but the external momenta are exceptional, or

2. $\omega_i - 4c_{\mathcal{G}}(\mathcal{G}_i) < 0$ or $j_{\mathcal{G}}(\mathcal{G}_i) = 1$ for all i if $j(\mathcal{G}) = 1$ and the external momenta are non-exceptional.

There are two types of divergences where these conditions are violated.

First let the non-planarity be due to internal lines only, $j(\mathcal{G}) = 0$. Since the total graph $\mathcal{G}$ is non-planar, one has $c_{\mathcal{G}}(\mathcal{G}) > 0$ and therefore no superficial divergence. However, there might exist subgraphs $\mathcal{G}_i$ related to a Hepp sector of integration (41) where $\omega_i - 4c_{\mathcal{G}}(\mathcal{G}_i) \geq 0$. Such a situation requires *disconnected*[10] loops wrapping the same handle of the Riemann surface. In this case the integral (39) does not exist unless one introduces a regulator. The problem is that such a subdivergence may appear in graphs with an arbitrary number of external lines. In the commutative theory this also happens, but there one renormalises already the subdivergence. This procedure is based on normalisation conditions, which can only be imposed for *local* divergences. Since a non-planar graph wrapping a handle of a Riemann surface is clearly a non-local object (it cannot be reduced to a point, i.e. a counterterm vertex), it is not possible in the noncommutative case to remove that subdivergence. We are thus forced to use normalisation conditions for the total graph, but as the problem is independent of the number of external legs of the total graph, we finally need an infinite number of normalisation conditions. Hence, the model is not renormalisable in the standard way. This is the UV/IR-mixing problem.

The proposal to treat the UV/IR-mixing problem is a reordering of the perturbation series [76]. More details of this idea are given in [98]. The procedure is promising, but a renormalisation proof based on the resummation of non-planar graphs is still missing[11]. Clearly, the problem is absent in theories with only logarithmic divergences.

[10] I have the impression that the problem with disconnected graphs as discovered by Chepelev and Roiban is completely ignored in the recent literature. Therefore, we have to underline that, in renormalisation schemes for noncommutative quantum field theories which are based on the forest formula, it is not possible to restrict oneself to connected graphs. The reason is that, in contrast to the commutative situation, disconnected subgraphs can be coupled in the noncommutative case via the topology of the Riemann surface defined by the total graph.

[11] we conjecture that the result of such a reordering and resummation procedure would be equivalent to the duality-covariant ϕ^4-action (29)+(26), but we cannot prove this idea.

The second class of problems is found in graphs where the non-planarity is at least partly due to the external legs, $j(\mathcal{G}) = 1$. This means that there is no way to remove possible divergences in these graphs by normalisation conditions. Fortunately, these graphs are superficially finite as long as the external momenta are non-exceptional. Subdivergences are supposed to be treated by a resummation. However, since the non-exceptional external momenta can become arbitrarily close to exceptional ones, these graphs are unbounded: For every $\delta > 0$ one finds non-exceptional momenta $\{p_n\}$ such that $\big|\langle\phi(p_1)\dots\phi(p_n)\rangle\big| > \frac{1}{\delta}$. This problem also arises in models with only logarithmic divergences.

5.4 θ-expanded field theories

The only way to circumvent the power-counting theorem of [98] is a different limiting procedure of the loop calculations. Namely, in intermediate steps one changes the order of integrations of integrals which are not absolutely convergent. One possibility is the use [107] of the Seiberg-Witten map [49] which, however, does not help [108].

In their famous paper on type II string theory in the presence of a Neveu-Schwarz B-field [49], Seiberg and Witten noticed that passing to the zero-slope limit in two different regularisation schemes (point-splitting and Pauli-Villars) gives rise to a Yang-Mills theory either on noncommutative or on commutative $\mathbb{R}^D$. Since the regularisation scheme cannot matter, Seiberg and Witten argued that the two theories must be gauge-equivalent. More generally, under an infinitesimal transformation of θ, which can be related to deformation quantisation as in [49] or simply to a coordinate rotation [109], one has to require that gauge-invariant quantities remain gauge-invariant. This requirement leads to the Seiberg-Witten differential equation

$$\frac{dA_\mu}{d\theta_{\rho\sigma}} = -\frac{1}{8}\big\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\big\}_\star + \frac{1}{8}\big\{A_\sigma, \partial_\rho A_\mu + F_{\rho\mu}\big\}_\star\,, \tag{43}$$

where $\{a,b\}_\star = a\star b + b\star a$.

The differential equation (43) is usually solved by integrating it from an initial condition $A^{(0)}$ at $\theta = 0$ in the spirit of deformation quantisation [21, 22]. Then, A becomes a formal power series in θ and the initial condition $A^{(0)}$. The solution depends on the path of integration, but the difference between paths is a field redefinition [110]. The solution to all orders in θ and lowest order in $A^{(0)}$ was given in [111]. A generating functional for the complete solution of (43) was derived in [112]. The Seiberg-Witten approach was made popular in [113] where it was argued that this is the only way to obtain a finite number of degrees of freedom in non-Abelian noncommutative Yang-Mills theory.

Inserting the solution of the Seiberg-Witten differential equation (43) into the noncommutative Yang-Mills action $\int d^Dx F_{\mu\nu}F^{\mu\nu}$ leads to the so-called θ-expanded field theories. It must be stressed, however, that unless a complete solution to all orders in θ and $A^{(0)}$ is known (which is not the case), the θ-expansion of the noncommutative Yang-Mills action describes a *local* field theory. As such, θ-expanded field theories lose the interesting features of the original field theory on the Moyal plane.

The quantum field theoretical treatment of θ-expanded field theories was initiated in [107]. We have shown that the one-loop divergences to the θ-expanded Maxwell action in second order in θ are gauge-invariant, independent of a linear or a non-linear gauge fixing and independent of the gauge parameter. There is no UV/IR-problem in that approach. We have shown in [114] that these one-loop divergences can be removed by a field redefinition related to the freedom in the Seiberg-Witten map. In fact, the superficial divergences in the photon self-energy are field redefinitions to all orders in θ and any loop order [114]. However, we have shown in [108] that θ-expanded field theories are not renormalisable as regards to more complicated graphs than the self-energy. On the other hand, one of us has found in [108] striking evidence for new symmetries in the θ-expanded action which eliminates several divergences expected from the counting of allowed divergences modulo field redefinitions. Finally, we have shown in [115] that the use of the θ-expanded $\star$-product (4) without application of the Seiberg-Witten map leads (up to field redefinitions) to exactly the same result. Thus, the Seiberg-Witten map is merely an unphysical (but convenient) change of variables [116].

Recently, phenomenological investigations of θ-expanded field theories became popular [117, 118]. However, quantitative statements are delicate because in the presence of a new field $\theta^{\mu\nu}$, many new terms in the action are not only possible but in fact required by renormalisability [115] or the desire to cure the UV/IR-problem [119]. Moreover, deformed spaces are too rigid to be a realistic model [120].

5.5 Noncommutative space-time

We have to stress that all mentioned contributions refer to a Euclidean space and a definition of the quantum field theory via the partition function (the Euclidean analogue of the path integral). It was pointed out in [121] that a simple Wick rotation does *not* give a meaningful theory on Minkowskian space-time, first of all because unitarity is lost [122, 123, 124]. The original proposal [6] of a quantum field theory on noncommutative space-time stayed within the Minkowskian framework, but later work started from Feynman graphs, the admissibility of a Wick rotation taken (erroneously) for granted. To obtain a consistent Minkowskian quantum field theory, it was proposed in [121] to iteratively solve the field equations à la Yang-Feldman [125]. See also [126]. Other possibilities are a functional formalism for the S-matrix [127] and time-ordered perturbation theory [128, 129]. See also [130, 131]. Unfortunately, the resulting Feynman rules become so complicated that apart from tadpole-like diagrams [130] it seems impossible to perform perturbative calculations in time-ordered perturbation theory. Moreover, it seems impossible to preserve Ward identities [132].

On the other hand, the rôle of time in noncommutative geometry is not completely clear. Time should be established around the ideas presented in [133]. For general approaches to Minkowskian noncommutative spaces we refer to [134, 135, 136]. There is a recent proposal [137] to combine spectral geometry with local covariant quantum field theory.

6 Renormalisation of noncommutative ϕ^4-theory to all orders

After the previous unsuccessful attempts to renormalise noncommutative quantum field theories, the last resort is a more careful way of performing the limits in the spirit of Wilson [138] and Polchinski [139]. Early attempts [140, 141] did not notice the new effects in higher-genus graphs of noncommutative field theories, which are not visible in one-loop calculations. A rigorous treatment exists for the large-θ limit [142, 143]. Eventually, the Wilson-Polchinski programme for noncommutative ϕ^4-theory was realised in the series of papers [144, 145, 30]. The main ideas are summarised in [146]. We achieved the remarkable balance of proving renormalisability of the ϕ^4-model to all orders and reconfirming the UV/IR-duality of [76]. Our proof rests on two concepts:

- the use of the harmonic oscillator base of the Moyal plane, which avoids the phase factors appearing in momentum space,
- the renormalisation by flow equations.

The renormalised ϕ^4-model corresponds to the classical action

$$S = \int d^4x \Big(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big)(x) , \qquad (44)$$

with $\tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu}\, x^\nu$. The appearance of the harmonic oscillator term $\frac{\Omega^2}{2}(\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi)$ in the action (44) is a result of the renormalisation proof.

6.1 The ϕ^4-action in the matrix base

We assume for simplicity that $\theta_{12} = -\theta_{21} = \theta_{34} = -\theta_{43}$ are the only non-vanishing components. Expanding the fields in the harmonic oscillator base (8) of the Moyal plane, $\phi(x) = \sum_{m^1,n^1,m^2,n^2 \in \mathbb{N}} \phi_{\substack{m^1 \, n^1 \\ m^2 \, n^2}} f_{m^1 n^1}(x_1, x_2) f_{m^2 n^2}(x_3, x_4)$, the action (44) takes the form

$$S[\phi] = (2\pi\theta)^2 \sum_{m,n,k,l \in \mathbb{N}^2} \Big(\frac{1}{2} \phi_{mn} G_{mn;kl} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \Big) , \qquad (45)$$

$$\begin{aligned} G_{\substack{m^1 \, n^1 \\ m^2 \, n^2};\substack{k^1 \, l^1 \\ k^2 \, l^2}} &= \big(\mu^2 + \tfrac{2+2\Omega^2}{\theta}(m^1+n^1+m^2+n^2+2)\big) \delta_{n^1 k^1} \delta_{m^1 l^1} \delta_{n^2 k^2} \delta_{m^2 l^2} \\ &\quad - \tfrac{2-2\Omega^2}{\theta} \big(\sqrt{k^1 l^1}\, \delta_{n^1+1,k^1} \delta_{m^1+1,l^1} + \sqrt{m^1 n^1}\, \delta_{n^1-1,k^1} \delta_{m^1-1,l^1} \big) \delta_{n^2 k^2} \delta_{m^2 l^2} \\ &\quad - \tfrac{2-2\Omega^2}{\theta} \big(\sqrt{k^2 l^2}\, \delta_{n^2+1,k^2} \delta_{m^2+1,l^2} + \sqrt{m^2 n^2}\, \delta_{n^2-1,k^2} \delta_{m^2-1,l^2} \big) \delta_{n^1 k^1} \delta_{m^1 l^1} . \end{aligned} \qquad (46)$$

The quantum field theory is constructed as a perturbative expansion about the free theory, which is solved by the propagator $\Delta_{mn;kl}$, the inverse of $G_{mn;kl}$. After diagonalisation of $G_{mn;kl}$ (which leads to orthogonal Meixner polynomials, see [147]) and the use of identities for hypergeometric functions one arrives at

$$\begin{aligned} \Delta_{\substack{m^1 \, n^1 \\ m^2 \, n^2};\substack{k^1 \, l^1 \\ k^2 \, l^2}} &= \frac{\theta}{2(1+\Omega)^2} \delta_{m^1+k^1,n^1+l^1} \delta_{m^2+k^2,n^2+l^2} \\ \times & \sum_{v^1 = \frac{|m^1-l^1|}{2}}^{\frac{m^1+l^1}{2}} \sum_{v^2 = \frac{|m^2-l^2|}{2}}^{\frac{m^2+l^2}{2}} B\big(1 + \tfrac{\mu^2\theta}{8\Omega} + \tfrac{1}{2}(m^1+k^1+m^2+k^2) - v^1 - v^2, 1 + 2v^1 + 2v^2\big) \\ \times & {}_2F_1\Bigg(\begin{matrix} 1+2v^1+2v^2 \,,\; \frac{\mu^2\theta}{8\Omega} - \frac{1}{2}(m^1+k^1+m^2+k^2) + v^1 + v^2 \\ 2 + \frac{\mu^2\theta}{8\Omega} + \frac{1}{2}(m^1+k^1+m^2+k^2) + v^1 + v^2 \end{matrix} \Bigg| \frac{(1-\Omega)^2}{(1+\Omega)^2} \Bigg) \Big(\frac{1-\Omega}{1+\Omega} \Big)^{2v^1+2v^2} \end{aligned}$$

$$\times \prod_{i=1}^{2} \sqrt{\binom{n^i}{v^i + \frac{n^i - k^i}{2}} \binom{k^i}{v^i + \frac{k^i - n^i}{2}} \binom{m^i}{v^i + \frac{m^i - l^i}{2}} \binom{l^i}{v^i + \frac{l^i - m^i}{2}}} \,. \tag{47}$$

It is important that the sums in (47) are finite.

6.2 Renormalisation group approach to dynamical matrix models

The (Euclidean) quantum field theory is defined by the partition function

$$Z[J] = \int \mathcal{D}[\phi] \, \exp\Big(- S[\phi] - (2\pi\theta)^2 \sum_{m,n} \phi_{mn} J_{nm} \Big) \,. \tag{48}$$

The idea inspired by Polchinski's renormalisation proof [139] of commutative ϕ^4-theory is to change the weights of the matrix indices in the kinetic part of $S[\phi]$ as a smooth function of an energy scale Λ and to compensate this by a careful adaptation of the effective action $L[\phi,\Lambda]$ such that $Z[J]$ becomes independent of the scale Λ. If the modification of the weights of a matrix index $m \in \mathbb{N}$ is described by a function $K\big(\frac{m}{\theta\Lambda^2}\big)$, then the required Λ-dependence of the effective action is given by the matrix Polchinski equation

$$\Lambda \frac{\partial L[\phi,\Lambda]}{\partial\Lambda} = \sum_{m,n,k,l} \frac{1}{2} \big(2\pi\theta Q_{nm;lk}(\Lambda)\big) \Big(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}} \frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}} - \frac{1}{(2\pi\theta)^2} \frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{mn}\,\partial\phi_{kl}} \Big) \,, \tag{49}$$

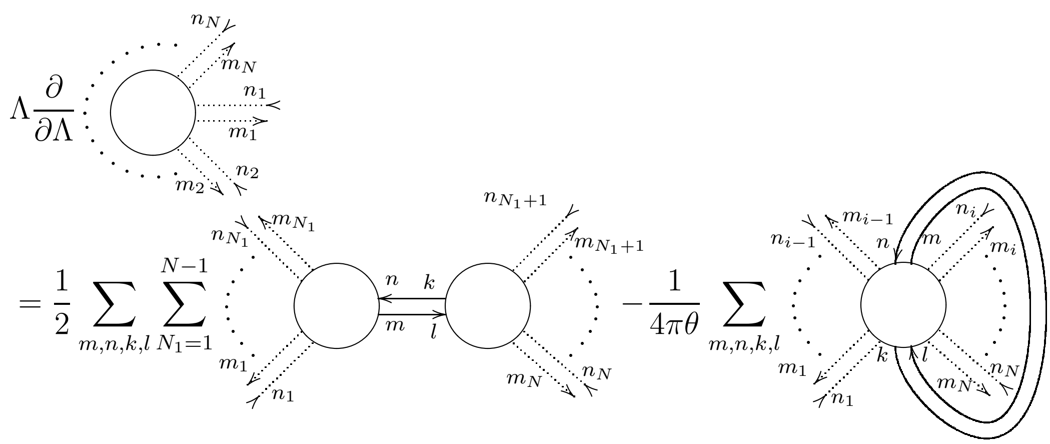
where

$$2\pi\theta Q_{mn;kl}(\Lambda) := \Lambda \frac{\partial}{\partial\Lambda} \Big(\prod_{i \in m^1, m^2, \dots, l^1, l^2} K\Big(\frac{i}{\theta\Lambda^2}\Big) \Delta_{mn;kl}(\Lambda) \Big) \,. \tag{50}$$

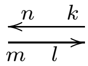
We look for a perturbative solution of the matrix Polchinski equation (49). In terms of the expansion coefficients

$$L[\phi,\Lambda] = \sum_{V=1}^{\infty} \lambda^V \sum_{N=2}^{2V+2} \frac{(2\pi\theta)^{\frac{N}{2}-2}}{N!} \sum_{m_1, n_i \in \mathbb{N}^2} A^{(V)}_{m_1 n_1; \dots; m_N n_N}[\Lambda] \phi_{m_1 n_1} \cdots \phi_{m_N n_N} \tag{51}$$

of the effective action, the matrix Polchinski equation (49) is represented by *ribbon graphs*:



(52)

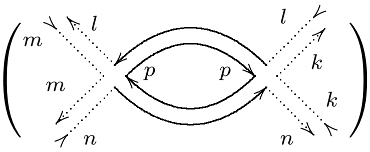
An internal double line  symbolises the propagator $Q_{mn;kl}(\Lambda)$. In this way, very complicated ribbon graphs can be produced which cannot be drawn any longer in a plane. A ribbon graph represents a simplicial complex for a Riemann surface and thus defines the topology of the Riemann surface on which it can be drawn. The Riemann surface is characterised by its genus g computable via the Euler characteristic of the graph, $g = 1 - \frac{1}{2}(L - I + V)$, and the number B of holes. Here, L is the number of single-line loops if we close the external lines of the graph, I is the number of double-line propagators and V the number of vertices. The number B of holes coincides with the number of single-line cycles which carry external legs. Accordingly, we also label the expansion coefficients in (51) by the topology, $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}$.

We have proven in [144] a power-counting estimation for these coefficients which relates the Λ-scaling of a ribbon graph to the topology of the graph and to two asymptotic scaling dimensions of the differentiated cut-off propagator $Q_{mn;kl}(\Lambda)$. As a result, if these scaling dimensions coincide with the classical momentum space dimensions, then all non-planar graphs are suppressed by the renormalisation flow. This is a necessary requirement for the renormalisability of a model. On the other hand, as the expansion coefficients $A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ carry an infinite number of matrix indices, the general power-counting estimation proven in [144] leaves, a priori, an infinite number of divergent planar graphs. These planar graphs require a separate analysis.

6.3 Power-counting behaviour of the noncommutative ϕ^4-model

The key is the integration procedure of the Polchinski equation (52), which involves the entire magic of renormalisation. We consider the example of the planar one-particle irreducible four-point function with two vertices, $A^{(2,1,0)\text{1PI}}_{m_1n_1;\dots;m_Nn_N}$. The Polchinski equation (52) provides the Λ-derivative of that function:

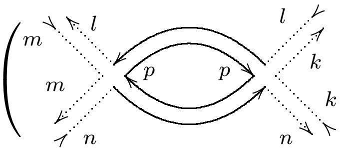
$$\Lambda\frac{\partial}{\partial\Lambda}A^{(2,1,0)\text{1PI}}_{mn;nk;kl;lm}[\Lambda] = \sum_{p\in\mathbb{N}^2} \Bigg(\quad \Bigg)(\Lambda) + \text{ permutations}\,. \qquad (53)$$



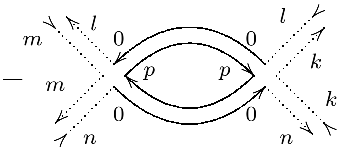
Performing the Λ-integration of (53) from some initial scale Λ_0 (sent to ∞ at the end) down to Λ, we obtain $A^{(2,1,0)\text{1PI}}_{mn;nk;kl;lm}[\Lambda] \sim \ln\frac{\Lambda_0}{\Lambda}$, which diverges for $\Lambda_0 \to \infty$. Renormalisation can be understood as the change of the boundary condition for the integration. Thus, integrating (53) from a renormalisation scale Λ_R up to Λ, we have $A^{(2,1,0)\text{1PI}}_{mn;nk;kl;lm}[\Lambda] \sim \ln\frac{\Lambda}{\Lambda_R}$, and there would be no problem for $\Lambda_0 \to \infty$. However, since there is an *infinite number* of matrix indices and there is no symmetry which could relate the amplitudes, that integration procedure entails an infinite number of initial conditions $A^{(2,1,0)\text{1PI}}_{mn;nk;kl;lm}[\Lambda_R]$. To have a renormalisable model, we can only afford a finite number of integrations from Λ_R

up to Λ. Thus, the correct choice is

$$A^{(2,1,0)1\text{PI}}_{mn;nk;kl;lm}[\Lambda] = -\int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$

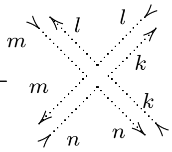


$$-$$

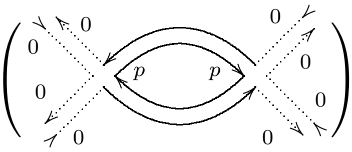


$$\Bigg)[\Lambda']$$

$$+$$



$$\Bigg[\int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$

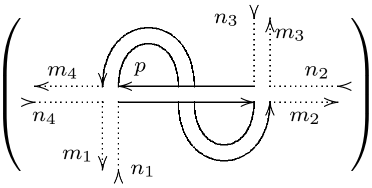


$$\Bigg)[\Lambda'] + A^{(2,1,0)1\text{PI}}_{00;00;00;00}[\Lambda_R]\Bigg] . \tag{54}$$

The second graph in the first line on the rhs and the graph in brackets in the last line are identical, because only the indices on the propagators determine the value of the graph. Moreover, the vertex in the last line in front of the bracket equals 1. Thus, differentiating (54) with respect to Λ we indeed obtain (53). As a further check one can consider (54) for $m = n = k = l = 0$. Finally, the independence of $A^{(2,1,0)1\text{PI}}_{mn;nk;kl;lm}[\Lambda_0]$ on the indices m, n, k, l is built-in. This property is, for $\Lambda_0 \to \infty$, dynamically generated by the model.

There is a similar Λ_0-Λ_R-mixed integration procedure for the planar 1PI two-point functions $A^{(V,1,0)1\text{PI}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$, $A^{(V,1,0)1\text{PI}}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$, $A^{(V,1,0)1\text{PI}}_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}}$ and all other $A^{(V,1,0)1\text{PI}}_{mn;nk;kl;lm}$. These involve in total four different sub-integrations from Λ_R up to Λ. We refer to [30] for details. All other graphs are integrated from Λ_0 down to Λ, e.g.

$$A^{(2,2,0)1\text{PI}}_{m_1n_1;\dots;m_4n_4}[\Lambda] = -\int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$



$$\Bigg)[\Lambda'] . \tag{55}$$

Theorem 1 *The previous integration procedure yields*

$$\begin{aligned} &\big|A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\big| \\ &\leq \big(\sqrt{\theta}\Lambda\big)^{(4-N)+4(1-B-2g)} P^{4V-N}\Big[\frac{\max(\|m_1\|,\|n_1\|,\dots\|n_N\|)}{\theta\Lambda^2}\Big] P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big], \end{aligned} \tag{56}$$

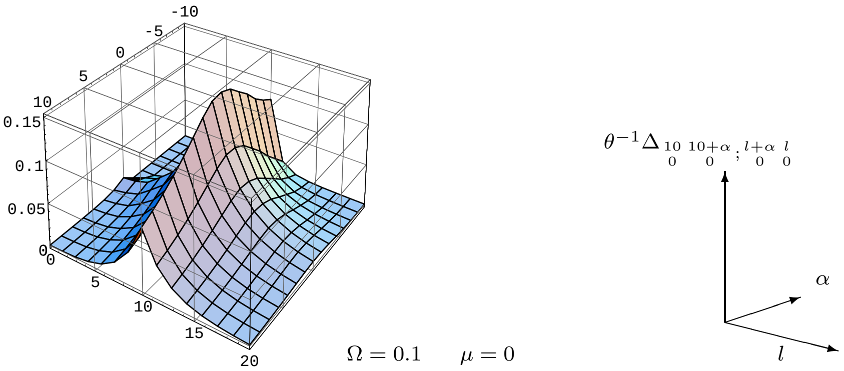
where $P^q[X]$ stands for a polynomial of degree q in X.

Idea of the proof. For the choice $K(x) = 1$ for $0 \leq x \leq 1$ and $K(x) = 0$ for $x \geq 2$ of the cut-off function in (50) one has

$$|Q_{mn;kl}(\Lambda)| < \frac{C_0}{\Omega\theta\Lambda^2}\delta_{m+k,n+l} . \tag{57}$$

Thus, the propagator and the volume of a loop summation have the same power-counting dimensions as a commutative ϕ^4-model in momentum space, giving the total power-counting degree $4 - N$ for an N-point function.

This is (more or less) correct for planar graphs. The scaling behaviour of non-planar graphs is considerably improved by the *quasi-locality* of the propagator:



(58)

As a consequence, for given index m of the propagator $Q_{mn;kl}(\Lambda) = \underset{m\quad l}{\overset{n\quad k}{\leftrightarrows}}$, the contribution to a graph is strongly suppressed unless the other index l on the trajectory through m is close to m. Thus, the sum over l for given m converges and does not alter (apart from a factor Ω^{-1}) the power-counting behaviour of (57):

$$\sum_{l\in\mathbb{N}^2}\Big(\max_{n,k}|Q_{mn;kl}(\Lambda)|\Big) < \frac{C_1}{\theta\Omega^2\Lambda^2}\,. \tag{59}$$

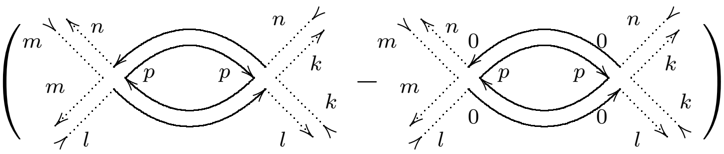
In a non-planar graph like the one in (55), the index n_3—fixed as an external index—localises the summation index $p \approx n_3$. Thus, we save one volume factor $\theta^2\Lambda^4$ compared with a true loop summation as in (54). In general, each hole in the Riemann surface saves one volume factor, and each handle even saves two.

A more careful analysis of (47) shows that also planar graphs get suppressed with $\big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{k^1\\k^2}\substack{l^1\\l^2}}(\Lambda)\big| < \frac{C_2}{\Omega\theta\Lambda^2}\prod_{i=1}^2\big(\frac{\max(m^i,l^i)+1}{\theta\Lambda^2}\big)^{\frac{|m^i-l^i|}{2}}$, for $m^i \leq n^i$, if the index along a trajectory jumps. This leaves the functions $A^{(V,1,0)\mathrm{1PI}}_{mn;nk;kl;lm}$, $A^{(V,1,0)\mathrm{1PI}}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$, $A^{(V,1,0)\mathrm{1PI}}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}$ and $A^{(V,1,0)\mathrm{1PI}}_{\substack{m^1\\m^2+1}\substack{n^1\\n^2+1};\substack{n^1\\n^2}\substack{m^1\\m^2}}$ as the only relevant or marginal ones. In these functions one has to use a discrete version of the Taylor expansion such as

$$\Big|Q_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}}(\Lambda) - Q_{\substack{0\\0}\substack{n^1\\n^2};\substack{n^1\\n^2}\substack{0\\0}}(\Lambda)\Big| < \frac{C_3}{\Omega\theta\Lambda^2}\Big(\frac{\max(m^1,m^2)}{\theta\Lambda^2}\Big)\,, \tag{60}$$

which can be traced back to the Meixner polynomials. The discrete Taylor subtractions are used in the integration from Λ_0 down to Λ in prescriptions like (54):

$$-\int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\sum_{p\in\mathbb{N}^2}\Big(\ \cdots\ \Big)[\Lambda']$$



$$= \int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\int_{\Lambda'}^{\Lambda_0}\frac{d\Lambda''}{\Lambda''}\sum_{p\in\mathbb{N}^2}\Big((Q_{np;pn} - Q_{0p;p0})(\Lambda')Q_{lp;pl}(\Lambda'') + Q_{0p;p0}(\Lambda')(Q_{lp;pl} - Q_{0p;p0})(\Lambda'')\Big) \sim \frac{C(\|n\|+\|l\|)}{\theta\Omega^2\Lambda^2}\,. \tag{61}$$

This explains the polynomial in fractions like $\frac{\|m\|}{\theta\Lambda^2}$ in (56). $\square$

As the estimation (56) is achieved by a finite number of initial conditions at Λ_R (see (54)), the noncommutative ϕ^4-model with oscillator term is renormalisable to all orders in perturbation theory. These initial conditions correspond to normalisation experiments for the mass, the field amplitude, the coupling constant and the oscillator frequency in the bare action related to (44). The resulting one-loop β-functions are computed in [148].

We have also proven renormalisability of the two-dimensional case in [145], where the oscillator frequency required in intermediate steps can be switched off at the end.

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References

[1] B. Riemann, "Ueber die Hypothesen, welche der Geometrie zugrunde liegen," in *Bernhard Riemann's gesammelte mathematische Werke und wissenschaftlicher Nachlaß*, H. Weber, ed. B. G. Teubner, Leipzig, 1892.

[2] E. Schrödinger, "Über die Unanwendbarkeit der Geometrie im Kleinen," *Naturwiss.* **31** (1934) 342–344.

[3] W. Heisenberg, "Die Grenzen der Anwendbarkeit der bisherigen Quantentheorie," *Z. Phys.* **110** (1938) 251–266.

[4] R. Peierls, "Zur Theorie des Diamagnetismus von Leitungselektronen," *Z. Phys.* **80** (1933) 763.

[5] H. S. Snyder, "Quantized space-time," *Phys. Rev.* **71** (1947) 38–41.

[6] S. Doplicher, K. Fredenhagen, and J. E. Roberts, "The quantum structure of space-time at the Planck scale and quantum fields," *Commun. Math. Phys.* **172** (1995) 187–220, `hep-th/0303037`.

[7] A. Connes, *Noncommutative geometry.* Academic Press, San Diego, 1994.

[8] A. Connes, "Noncommutative geometry: Year 2000," `math.qa/0011193`.

[9] J. Madore, *An introduction to noncommutative differential geometry and its physical applications*, vol. 257 of *London Mathematical Society lecture note series.* Cambridge University Press, Cambridge, 2nd ed., 2000.

[10] G. Landi, *An introduction to noncommutative spaces and their geometries*, vol. m51 of *Lecture notes in physics.* Springer-Verlag, Berlin, 1997.

[11] J. C. Várilly, "An introduction to noncommutative geometry," `physics/9709045`.

[12] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa, *Elements of noncommutative geometry.* Birkhäuser advanced texts. Birkhäuser, Boston, 2001.

[13] A. H. Chamseddine and A. Connes, "The spectral action principle," *Commun. Math. Phys.* **186** (1997) 731–750, `hep-th/9606001`.

[14] A. Connes, "Gravity coupled with matter and the foundation of non- commutative geometry," *Commun. Math. Phys.* **182** (1996) 155–176, `hep-th/9603053`.

[15] H. J. Groenewold, "On the principles of elementary quantum mechanics," *Physica* **12** (1946) 405–460.

[16] J. E. Moyal, "Quantum mechanics as a statistical theory," *Proc. Cambridge Phil. Soc.* **45** (1949) 99–124.

[17] M. A. Rieffel, *Deformation quantization for actions of* $\mathbb{R}^d$, vol. 106. Mem. Amer. Math. Soc., Providence, 1993.

[18] V. Gayral, J. M. Gracia-Bondía, B. Iochum, T. Schücker, and J. C. Várilly, "Moyal planes are spectral triples," *Commun. Math. Phys.* **246** (2004) 569–623, `hep-th/0307241`.

[19] J. M. Gracia-Bondía and J. C. Várilly, "Algebras of distributions suitable for phase space quantum mechanics. I," *J. Math. Phys.* **29** (1988) 869–879.

[20] E. P. Wigner, "On the quantum correction for thermodynamic equilibrium," *Phys. Rev.* **40** (1932) 749–760.

[21] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, "Deformation theory and quantization. I. Deformations of symplectic structures," *Ann. Phys.* **111** (1978) 61–110.

[22] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, "Deformation theory and quantization. II. Physical applications," *Ann. Phys.* **111** (1978) 111–151.

[23] M. Kontsevich, "Deformation quantization of Poisson manifolds, I," `q-alg/9709040`.

[24] A. S. Cattaneo and G. Felder, "A path integral approach to the Kontsevich quantization formula," *Commun. Math. Phys.* **212** (2000) 591–611, `math.qa/9902090`.

[25] P. Schaller and T. Strobl, "Poisson structure induced (topological) field theories," *Mod. Phys. Lett.* **A9** (1994) 3129–3136, `hep-th/9405110`.

[26] D. V. Vassilevich, "Non-commutative heat kernel," *Lett. Math. Phys.* **67** (2004) 185–194, `hep-th/0310144`.

[27] V. Gayral and B. Iochum, "The spectral action for Moyal planes," `hep-th/0402147`.

[28] R. Estrada, J. M. Gracia-Bondía, and J. C. Várilly, "On asymptotic expansions of twisted products," *J. Math. Phys.* **30** (1989) 2789–2796.

[29] J. M. Gracia-Bondía, F. Lizzi, G. Marmo, and P. Vitale, "Infinitely many star products to play with," *JHEP* **04** (2002) 026, `hep-th/0112092`.

[30] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base," `hep-th/0401128`. to appear in Commun. Math. Phys.

[31] M. S. Bartlett and J. E. Moyal, "The exact transition probabilities of quantum-mechanical oscillators calculated by the phase-space method," *Proc. Cambridge Phil. Soc.* **45** (1949) 545–553.

[32] E. Langmann, "Interacting fermions on noncommutative spaces: Exactly solvable quantum field theories in $2n + 1$ dimensions," *Nucl. Phys.* **B654** (2003) 404–426, `hep-th/0205287`.

[33] E. Langmann, R. J. Szabo, and K. Zarembo, "Exact solution of noncommutative field theory in background magnetic fields," *Phys. Lett.* **B569** (2003) 95–101, `hep-th/0303082`.

[34] E. Langmann, R. J. Szabo, and K. Zarembo, "Exact solution of quantum field theory on noncommutative phase spaces," *JHEP* **01** (2004) 017, `hep-th/0308043`.

[35] J. C. Várilly and J. M. Gracia-Bondía, "Algebras of distributions suitable for phase space quantum mechanics. II. Topologies on the Moyal algebra," *J. Math. Phys.* **29** (1988) 880–887.

[36] A. Connes, "C^* algèbres et géométrie différentielle," *C. R. Acad. Sci. Paris Sér. A-B* **290** (1980) 599–604, `hep-th/0101093`.

[37] M. A. Rieffel, "C^*-algebras associated with irrational rotations," *Pacific J. Math.* **93** (1981) 415–429.

[38] M. A. Rieffel, "Noncommutative tori—a case study of noncommutative differentiable manifolds," *Contemp. Math.* **105** (1990) 191–211.

[39] A. Connes and G. Landi, "Noncommutative manifolds: The instanton algebra and isospectral deformations," *Commun. Math. Phys.* **221** (2001) 141–159, `math.qa/0011194`.

[40] A. Connes and M. Dubois-Violette, "Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples," *Commun. Math. Phys.* **230** (2002) 539–579, `math.qa/0107070`.

[41] J. Madore, "The fuzzy sphere," *Class. Quant. Grav.* **9** (1992) 69–88.

[42] M. A. Rieffel, "Matrix algebras converge to the sphere for quantum Gromov-Hausdorff distance," *Mem. Amer. Math. Soc.* **168** (2004) 67–91, `math.OA/0108005`.

[43] H. Grosse and A. Strohmaier, "Noncommutative geometry and the regularization problem of 4D quantum field theory," *Lett. Math. Phys.* **48** (1999) 163–179, `hep-th/9902138`.

[44] G. Alexanian, A. P. Balachandran, G. Immirzi, and B. Ydri, "Fuzzy $\mathbb{C}P^2$," *J. Geom. Phys.* **42** (2002) 28–53, `hep-th/0103023`.

[45] H. Grosse, J. Madore, and H. Steinacker, "Field theory on the q-deformed fuzzy sphere. I," *J. Geom. Phys.* **38** (2001) 308–342, `hep-th/0005273`.

[46] H. Grosse, J. Madore, and H. Steinacker, "Field theory on the q-deformed fuzzy sphere. II: Quantization," *J. Geom. Phys.* **43** (2002) 205–240, `hep-th/0103164`.

[47] V. Schomerus, "D-branes and deformation quantization," *JHEP* **06** (1999) 030, `hep-th/9903205`.

[48] A. Yu. Alekseev, A. Recknagel, and V. Schomerus, "Non-commutative world-volume geometries: Branes on SU(2) and fuzzy spheres," *JHEP* **09** (1999) 023, `hep-th/9908040`.

[49] N. Seiberg and E. Witten, "String theory and noncommutative geometry," *JHEP* **09** (1999) 032, `hep-th/9908142`.

[50] N. Nekrasov and A. Schwarz, "Instantons on noncommutative $\mathbb{R}^4$ and $(2,0)$ superconformal six dimensional theory," *Commun. Math. Phys.* **198** (1998) 689–703, `hep-th/9802068`.

[51] A. Gonzalez-Arroyo and M. Okawa, "The twisted Eguchi-Kawai model: A reduced model for large N lattice gauge theory," *Phys. Rev.* **D27** (1983) 2397.

[52] G. 't Hooft, "Some twisted selfdual solutions for the Yang-Mills equations on a hypertorus," *Commun. Math. Phys.* **81** (1981) 267–275.

[53] A. Gonzalez-Arroyo and C. P. Korthals Altes, "Reduced model for large N continuum field theories," *Phys. Lett.* **B131** (1983) 396.

[54] V. Gayral, "The action functional for Moyal planes," *Lett. Math. Phys.* **65** (2003) 147–157, `hep-th/0307220`.

[55] C. Becchi, A. Rouet, and R. Stora, "Renormalization of gauge theories," *Annals Phys.* **98** (1976) 287–321.

[56] I. A. Batalin and G. A. Vilkovisky, "Gauge algebra and quantization," *Phys. Lett.* **B102** (1981) 27–31.

[57] E. Langmann and R. J. Szabo, "Duality in scalar field theory on noncommutative phase spaces," *Phys. Lett.* **B533** (2002) 168–177, `hep-th/0202039`.

[58] H. Grosse, C. Klimčík, and P. Prešnajder, "Towards finite quantum field theory in noncommutative geometry," *Int. J. Theor. Phys.* **35** (1996) 231–244, `hep-th/9505175`.

[59] H. Grosse, C. Klimčík, and P. Prešnajder, "On finite 4D quantum field theory in noncommutative geometry," *Commun. Math. Phys.* **180** (1996) 429–438, hep-th/9602115.

[60] B. Ydri, *Fuzzy physics*. PhD thesis, Syracuse University, Syracuse, 2001. hep-th/0110006.

[61] C.-S. Chu, J. Madore, and H. Steinacker, "Scaling limits of the fuzzy sphere at one loop," *JHEP* **08** (2001) 038, hep-th/0106205.

[62] E. Hawkins, "Noncommutative regularization for the practical man," hep-th/9908052.

[63] S. Iso, Y. Kimura, K. Tanaka, and K. Wakatsuki, "Noncommutative gauge theory on fuzzy sphere from matrix model," *Nucl. Phys.* **B604** (2001) 121–147, hep-th/0101102.

[64] H. Steinacker, "Quantized gauge theory on the fuzzy sphere as random matrix model," *Nucl. Phys.* **B679** (2004) 66–98, hep-th/0307075.

[65] S. Cho, R. Hinterding, J. Madore, and H. Steinacker, "Finite field theory on noncommutative geometries," *Int. J. Mod. Phys.* **D9** (2000) 161–199, hep-th/9903239.

[66] D. Bahns, S. Doplicher, K. Fredenhagen, and G. Piacitelli, "Ultraviolet finite quantum field theory on quantum spacetime," *Commun. Math. Phys.* **237** (2003) 221–241, hep-th/0301100.

[67] N. N. Bogolyubov and D. V. Shirkov, *Introduction to the theory of quantized fields*, vol. 3 of *Interscience monographs in physics and astronomy.* Interscience, New York, 1959.

[68] G. Velo and A. S. Wightman, eds., *Renormalization theory*, Proc. Erice 1975. D. Reidel Publishing, Dordrecht, 1976.

[69] C. Itzykson and J.-B. Zuber, *Quantum field theory.* International series in pure and applied physics. McGraw-Hill, New York, 1980.

[70] T. Filk, "Divergencies in a field theory on quantum space," *Phys. Lett.* **B376** (1996) 53–58.

[71] J. C. Várilly and J. M. Gracia-Bondía, "On the ultraviolet behaviour of quantum fields over noncommutative manifolds," *Int. J. Mod. Phys.* **A14** (1999) 1305, hep-th/9804001.

[72] M. Chaichian, A. Demichev, and P. Prešnajder, "Quantum field theory on noncommutative space-times and the persistence of ultraviolet divergences," *Nucl. Phys.* **B567** (2000) 360–390, hep-th/9812180.

[73] C. P. Martín and D. Sánchez-Ruiz, "The one-loop UV divergent structure of U(1) Yang-Mills theory on noncommutative $\mathbb{R}^4$," *Phys. Rev. Lett.* **83** (1999) 476–479, hep-th/9903077.

[74] M. M. Sheikh-Jabbari, "Renormalizability of the supersymmetric Yang-Mills theories on the noncommutative torus," *JHEP* **06** (1999) 015, hep-th/9903107.

[75] T. Krajewski and R. Wulkenhaar, "Perturbative quantum gauge fields on the noncommutative torus," *Int. J. Mod. Phys.* **A15** (2000) 1011–1030, hep-th/9903187.

[76] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, "Noncommutative perturbative dynamics," *JHEP* **02** (2000) 020, hep-th/9912072.

[77] A. Connes, M. R. Douglas, and A. Schwarz, "Noncommutative geometry and matrix theory: Compactification on tori," *JHEP* **02** (1998) 003, hep-th/9711162.

[78] M. R. Douglas, "Two lectures on D-geometry and noncommutative geometry," in *Nonperturbative aspects of strings, branes and supersymmetry*, M. Duff *et al.*, eds., pp. 131–156. World Scientific, Singapore, 1999. hep-th/9901146.

[79] T. Krajewski, *Géométrie non commutative et interactions fondamentales.* PhD thesis, Université de Provence, Marseille, 1998. math-ph/9903047.

[80] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa, and T. Tada, "Noncommutative Yang-Mills in IIB matrix model," *Nucl. Phys.* **B565** (2000) 176–192, hep-th/9908141.

[81] J. Ambjørn, Y. M. Makeenko, J. Nishimura, and R. J. Szabo, "Finite N matrix models of noncommutative gauge theory," *JHEP* **11** (1999) 029, hep-th/9911041.

[82] W. Bietenholz, F. Hofheinz, and J. Nishimura, "A non-perturbative study of gauge theory on a non-commutative plane," *JHEP* **09** (2002) 009, hep-th/0203151.

[83] W. Bietenholz, F. Hofheinz, and J. Nishimura, "Non-commutative field theories beyond perturbation theory," *Fortsch. Phys.* **51** (2003) 745–752, hep-th/0212258.

[84] S. S. Gubser and S. L. Sondhi, "Phase structure of non-commutative scalar field theories," *Nucl. Phys.* **B605** (2001) 395–424, hep-th/0006119.

[85] G. Landi, F. Lizzi, and R. J. Szabo, "From large N matrices to the noncommutative torus," *Commun. Math. Phys.* **217** (2001) 181–201, hep-th/9912130.

[86] G. Landi, F. Lizzi, and R. J. Szabo, "A new matrix model for noncommutative field theory," *Phys. Lett.* **B578** (2004) 449–458, hep-th/0309031.

[87] G. Landi, F. Lizzi, and R. J. Szabo, "Matrix quantum mechanics and soliton regularization of noncommutative field theory," hep-th/0401072.

[88] R. Gopakumar, S. Minwalla, and A. Strominger, "Noncommutative solitons," *JHEP* **05** (2000) 020, hep-th/0003160.

[89] L. D. Paniak and R. J. Szabo, "Instanton expansion of noncommutative gauge theory in two dimensions," *Commun. Math. Phys.* **243** (2003) 343–387, hep-th/0203166.

[90] L. D. Paniak and R. J. Szabo, "Lectures on two-dimensional noncommutative gauge theory. 2: Quantization," hep-th/0304268.

[91] M. R. Douglas and C. M. Hull, "D-branes and the noncommutative torus," *JHEP* **02** (1998) 008, hep-th/9711165.

[92] A. Konechny and A. Schwarz, "Introduction to M(atrix) theory and noncommutative geometry," *Phys. Rept.* **360** (2002) 353–465 (parts I and II), hep-th/0012145.

[93] A. Konechny and A. Schwarz, "Introduction to M(atrix) theory and noncommutative geometry. II," *Phys. Rept.* **360** (2002) 353–465 (parts I and II), hep-th/0107251.

[94] M. R. Douglas and N. A. Nekrasov, "Noncommutative field theory," *Rev. Mod. Phys.* **73** (2001) 977–1029, hep-th/0106048.

[95] R. J. Szabo, "Quantum field theory on noncommutative spaces," *Phys. Rept.* **378** (2003) 207–299, hep-th/0109162.

[96] I. Ya. Aref'eva, D. M. Belov, A. A. Giryavets, A. S. Koshelev, and P. B. Medvedev, "Noncommutative field theories and (super)string field theories," in *Particles and Fields*, G. A. Alves *et al.*, eds., pp. 1–163. World Scientific, Singapore, 2001. hep-th/0111208.

[97] I. Chepelev and R. Roiban, "Renormalization of quantum field theories on noncommutative $\mathbb{R}^d$. I: Scalars," *JHEP* **05** (2000) 037, hep-th/9911098.

[98] I. Chepelev and R. Roiban, "Convergence theorem for non-commutative Feynman graphs and renormalization," *JHEP* **03** (2001) 001, hep-th/0008090.

[99] I. Ya. Aref'eva, D. M. Belov, and A. S. Koshelev, "Two-loop diagrams in noncommutative ϕ^4_4 theory," *Phys. Lett.* **B476** (2000) 431–436, hep-th/9912075.

[100] I. Ya. Aref'eva, D. M. Belov, and A. S. Koshelev, "A note on UV/IR for noncommutative complex scalar field," hep-th/0001215.

[101] M. Hayakawa, "Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on $\mathbb{R}^4$," hep-th/9912167.

[102] A. Matusis, L. Susskind, and N. Toumbas, "The IR/UV connection in the non-commutative gauge theories," *JHEP* **12** (2000) 002, hep-th/0002075.

[103] L. Bonora and M. Salizzoni, "Renormalization of noncommutative U(N) gauge theories," *Phys. Lett.* **B504** (2001) 80–88, hep-th/0011088.

[104] H. Grosse, T. Krajewski, and R. Wulkenhaar, "Renormalization of noncommutative Yang-Mills theories: A simple example," hep-th/0001182.

[105] J. H. Lowenstein, "Convergence theorems for renormalized Feynman integrals with zero-mass propagators," *Commun. Math. Phys.* **47** (1976) 53–68.

[106] K. Hepp, "Proof of the Bogolyubov-Parasiuk theorem on renormalization," *Commun. Math. Phys.* **2** (1966) 301–326.

[107] A. A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda, and R. Wulkenhaar, "Perturbative analysis of the Seiberg-Witten map," *Int. J. Mod. Phys.* **A17** (2002) 2219–2232, hep-th/0102044.

[108] R. Wulkenhaar, "Non-renormalizability of θ-expanded noncommutative QED," *JHEP* **03** (2002) 024, hep-th/0112248.

[109] A. A. Bichl, J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda, and R. Wulkenhaar, "Noncommutative Lorentz symmetry and the origin of the Seiberg-Witten map," *Eur. Phys. J.* **C24** (2002) 165–176, hep-th/0108045.

[110] T. Asakawa and I. Kishimoto, "Comments on gauge equivalence in noncommutative geometry," *JHEP* **11** (1999) 024, hep-th/9909139.

[111] M. R. Garousi, "Non-commutative world-volume interactions on D-branes and Dirac-Born-Infeld action," *Nucl. Phys.* **B579** (2000) 209–228, hep-th/9909214.

[112] B. Jurčo, P. Schupp, and J. Wess, "Nonabelian noncommutative gauge theory via noncommutative extra dimensions," *Nucl. Phys.* **B604** (2001) 148–180, hep-th/0102129.

[113] B. Jurčo, S. Schraml, P. Schupp, and J. Wess, "Enveloping algebra valued gauge transformations for non-Abelian gauge groups on non-commutative spaces," *Eur. Phys. J.* **C17** (2000) 521–526, hep-th/0006246.

[114] A. A. Bichl, J. M. Grimstrup, H. Grosse, L. Popp, M. Schweda, and R. Wulkenhaar, "Renormalization of the noncommutative photon self-energy to all orders via Seiberg-Witten map," *JHEP* **06** (2001) 013, hep-th/0104097.

[115] J. M. Grimstrup and R. Wulkenhaar, "Quantisation of θ-expanded non-commutative QED," *Eur. Phys. J.* **C26** (2002) 139–151, hep-th/0205153.

[116] S. Kamefuchi, L. O'Raifeartaigh, and A. Salam, "Change of variables and equivalence theorems in quantum field theories," *Nucl. Phys.* **28** (1961) 529–549.

[117] X. Calmet, B. Jurčo, P. Schupp, J. Wess, and M. Wohlgenannt, "The standard model on non-commutative space-time," *Eur. Phys. J.* **C23** (2002) 363–376, hep-ph/0111115.

[118] W. Behr, N. G. Deshpande, G. Duplančić, P. Schupp, J. Trampetić, and J. Wess, "The $Z \to \gamma\gamma$, gg decays in the noncommutative standard model," *Eur. Phys. J.* **C29** (2003) 441–446, hep-ph/0202121.

[119] A. A. Slavnov, "Consistent noncommutative quantum gauge theories?," *Phys. Lett.* **B565** (2003) 246–252, hep-th/0304141.

[120] E. Hawkins, "Noncommutative rigidity," math.qa/0211203.

[121] D. Bahns, S. Doplicher, K. Fredenhagen, and G. Piacitelli, "On the unitarity problem in space/time noncommutative theories," *Phys. Lett.* **B533** (2002) 178–181, hep-th/0201222.

[122] J. Gomis and T. Mehen, "Space-time noncommutative field theories and unitarity," *Nucl. Phys.* **B591** (2000) 265–276, hep-th/0005129.

[123] L. Alvarez-Gaumé, J. L. F. Barbón, and R. Zwicky, "Remarks on time-space noncommutative field theories," *JHEP* **05** (2001) 057, hep-th/0103069.

[124] C.-S. Chu, J. Lukierski, and W. J. Zakrzewski, "Hermitian analyticity, IR/UV mixing and unitarity of noncommutative field theories," *Nucl. Phys.* **B632** (2002) 219–239, hep-th/0201144.

[125] C.-N. Yang and D. Feldman, "The S matrix in the Heisenberg representation," *Phys. Rev.* **79** (1950) 972–978.

[126] D. Bahns, "Unitary quantum field theory on the noncommutative Minkowski space," *Fortsch. Phys.* **51** (2003) 658–663, hep-th/0212266.

[127] C.-h. Rim and J. H. Yee, "Unitarity in space-time noncommutative field theories," *Phys. Lett.* **B574** (2003) 111–120, hep-th/0205193.

[128] Y. Liao and K. Sibold, "Time-ordered perturbation theory on noncommutative spacetime: Basic rules," *Eur. Phys. J.* **C25** (2002) 469–477, hep-th/0205269.

[129] Y. Liao and K. Sibold, "Time-ordered perturbation theory on noncommutative spacetime. II. Unitarity," *Eur. Phys. J.* **C25** (2002) 479–486, hep-th/0206011.

[130] H. Bozkaya, P. Fischer, H. Grosse, M. Pitschmann, V. Putz, M. Schweda, and R. Wulkenhaar, "Space/time noncommutative field theories and causality," *Eur. Phys. J.* **C29** (2003) 133–141, hep-th/0209253.

[131] S. Denk and M. Schweda, "Time ordered perturbation theory for non-local interactions: Applications to NCQFT," *JHEP* **09** (2003) 032, hep-th/0306101.

[132] T. Ohl, R. Rückl, and J. Zeiner, "Unitarity of time-like noncommutative gauge theories: The violation of Ward identities in time-ordered perturbation theory," *Nucl. Phys.* **B676** (2004) 229–242, hep-th/0309021.

[133] A. Connes and C. Rovelli, "Von Neumann algebra automorphisms and time thermodynamics relation in general covariant quantum theories," *Class. Quant. Grav.* **11** (1994) 2899–2918, gr-qc/9406019.

[134] E. Hawkins, "Hamiltonian gravity and noncommutative geometry," *Commun. Math. Phys.* **187** (1997) 471–489, gr-qc/9605068.

[135] A. Strohmaier, "On noncommutative and semi-Riemannian geometry," math-ph/0110001.

[136] T. Kopf and M. Paschke, "A spectral quadruple for de Sitter space," *J. Math. Phys.* **43** (2002) 818–846, math-ph/0012012.

[137] M. Paschke and R. Verch, "Local covariant quantum field theory over spectral geometries," gr-qc/0405057.

[138] K. G. Wilson and J. B. Kogut, "The renormalization group and the epsilon expansion," *Phys. Rept.* **12** (1974) 75–200.

[139] J. Polchinski, "Renormalization and effective Lagrangians," *Nucl. Phys.* **B231** (1984) 269–295.

[140] L. Griguolo and M. Pietroni, "Wilsonian renormalization group and the non-commutative IR/UV connection," *JHEP* **05** (2001) 032, hep-th/0104217.

[141] S. Sarkar, "On the UV renormalizability of noncommutative field theories," *JHEP* **06** (2002) 003, hep-th/0202171.

[142] C. Becchi, S. Giusto, and C. Imbimbo, "The Wilson-Polchinski renormalization group equation in the planar limit," *Nucl. Phys.* **B633** (2002) 250–270, hep-th/0202155.

[143] C. Becchi, S. Giusto, and C. Imbimbo, "The renormalization of non-commutative field theories in the limit of large non-commutativity," *Nucl. Phys.* **B664** (2003) 371–399,

hep-th/0304159.

[144] H. Grosse and R. Wulkenhaar, "Power-counting theorem for non-local matrix models and renormalisation," hep-th/0305066. to appear in Commun. Math. Phys.

[145] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^2$ in the matrix base," *JHEP* **12** (2003) 019, hep-th/0307017.

[146] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ to all orders," hep-th/0403232.

[147] R. Koekoek and R. F. Swarttouw, "The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue," math.CA/9602214.

[148] H. Grosse and R. Wulkenhaar, "The β-function in duality-covariant noncommutative ϕ^4-theory," *Eur. Phys. J.* **C35** (2004) 277–282, hep-th/0402093.