Progress in solving a noncommutative QFT in four dimensions

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(joint work with Harald Grosse)

This report is based on [1].

In previous work [2] we have proven that the following action functional for a ϕ^4-model on four-dimensional Moyal space gives rise to a renormalisable quantum field theory:

$$S = \int d^4x \Big(\frac{1}{2}\phi(-\Delta + \Omega^2\tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x)\;. \tag{1}$$

Here, $\star$ refers to the Moyal product parametrised by the antisymmetric $4{\times}4$-matrix Θ, and $\tilde{x} = 2\Theta^{-1}x$. The model is covariant under the Langmann-Szabo duality transformation and becomes self-dual at $\Omega = 1$. Evaluation of the β-functions for the coupling constants Ω, λ in first order of perturbation theory leads to a coupled dynamical system which indicates a fixed-point at $\Omega = 1$, while λ remains bounded [3, 4]. The vanishing of the β-function at $\Omega = 1$ was next proven in [5] at three-loop order and finally by Disertori, Gurau, Magnen and Rivasseau [6] to all orders of perturbation theory. It implies that there is no infinite renormalisation of λ, and a non-perturbative construction seems possible. The Landau ghost problem is solved.

The action functional (1) is most conveniently expressed in the matrix base of the Moyal algebra [2]. For $\Omega = 1$ it simplifies to

$$S = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi)\;, \tag{2}$$

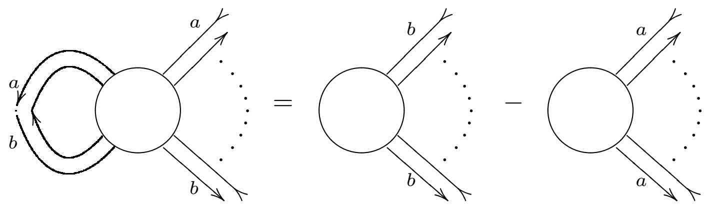
$$H_{mn} = Z\big(\mu^2_{bare} + |m| + |n|\big)\;, \qquad V(\phi) = \frac{Z^2\lambda}{4}\sum_{m,n,k,l\in\mathbb{N}^2_\Lambda}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\;, \tag{3}$$

The model only needs wavefunction renormalisation $\phi \mapsto \sqrt{Z}\phi$ and mass renormalisation $\mu_{bare} \to \mu$, but no renormalisation of the coupling constant [6] or of $\Omega = 1$. All summation indices $m, n, \dots$ belong to $\mathbb{N}^2$, with $|m| := m_1 + m_2$, and $\mathbb{N}^2_\Lambda$ refers to a cut-off in the matrix size.

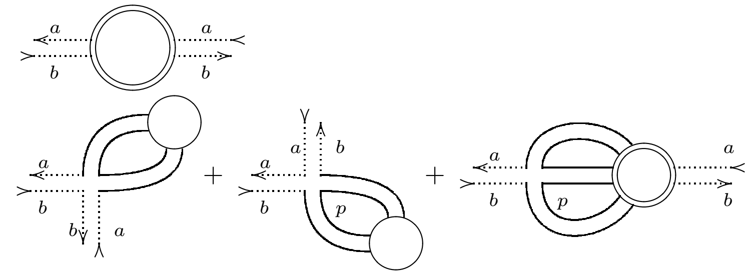
The key step in the proof [6] that the β-function vanishes is the discovery of a Ward identity induced by inner automorphisms $\phi \mapsto U\phi U^\dagger$. Inserting into the connected graphs one special insertion vertex

$$V^{ins}_{ab} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na} \tag{4}$$

is the same as the difference of graphs with external indices b and a, respectively, $Z(|a| - |b|)G^{ins}_{[ab]\dots} = G_{b\dots} - G_{a\dots}$:

(5) $Z(|a|-|b|)$ 

The Schwinger-Dyson equation for the one-particle irreducible two-point function Γ^{ab} reads

(6) $\Gamma_{ab} =$ 

The sum of the last two graphs can be reexpressed in terms of the two-point function with insertion vertex,

$$\Gamma_{ab} = Z^2\lambda \sum_p \Big(G_{ap} + G^{-1}_{ab} G^{ins}_{[ap]b}\Big) = Z^2\lambda \sum_p \Big(G_{ap} - G^{-1}_{ab} \frac{G_{bp}-G_{ba}}{Z(|p|-|a|)}\Big) \tag{7}$$
$$= Z^2\lambda \sum_p \Big(\frac{1}{H_{ap}-\Gamma_{ap}} + \frac{1}{H_{bp}-\Gamma_{bp}} - \frac{1}{H_{bp}-\Gamma_{bp}} \frac{(\Gamma_{bp}-\Gamma_{ab})}{Z(|p|-|a|)}\Big)\;.$$

This is a closed equation for the two-point function alone. It involves the divergent quantities Γ_{bp} and Z, μ_{bare} in H (3). Introducing the renormalised planar two-point function Γ^{ren}_{ab} by Taylor expansion $\Gamma_{ab} = Z\mu^2_{bare} - \mu^2 + (Z-1)(|a|+|b|) + \Gamma^{ren}_{ab}$, with $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$, we obtain a coupled system of equations for Γ^{ren}_{ab}, Z and μ_{bare}. It leads to a closed equation for the renormalised function Γ^{ren}_{ab} alone, which is further analysed in the integral representation.

We replace the indices in $a, b, \dots \mathbb{N}$ by continuous variables in $\mathbb{R}_+$. Equation (7) depends only on the length $|a| = a_1 + a_2$ of indices. The Taylor expansion respects this feature, so that we replace $\sum_{p\in\mathbb{N}^2_\Lambda}$ by $\int_0^\Lambda |p|\,dp$. After a convenient change of variables $|a| =: \mu^2\frac{\alpha}{1-\alpha}$, $|p| =: \mu^2\frac{\rho}{1-\rho}$ and

$$\Gamma^{ren}_{ab} =: \mu^2 \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}\Big(1 - \frac{1}{G_{\alpha\beta}}\Big)\;, \tag{8}$$

and using an identity resulting from the symmetry $G_{0\alpha} = G_{\alpha 0}$, we arrive at [1]:

Theorem 1. *The renormalised planar connected two-point function $G_{\alpha\beta}$ of self-dual noncommutative ϕ^4_4-theory satisfies the integral equation*

$$
\begin{aligned}
(9) \qquad G_{\alpha\beta} = 1 + \lambda\Big(& \frac{1-\alpha}{1-\alpha\beta}\big(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}\big) + \frac{1-\beta}{1-\alpha\beta}\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}\big) \\
& + \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}\big) \\
& - \frac{\alpha(1-\beta)}{1-\alpha\beta}\big(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}\big) + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta}-1)\mathcal{Y}\Big) ,
\end{aligned}
$$

where $\alpha, \beta \in [0,1)$,

$$
\mathcal{L}_\alpha := \int_0^1 d\rho \, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} , \quad \mathcal{M}_\alpha := \int_0^1 d\rho \, \frac{\alpha \, G_{\alpha\rho}}{1-\alpha\rho} , \quad \mathcal{N}_{\alpha\beta} := \int_0^1 d\rho \, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho - \alpha} ,
$$

and $\mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$.

The non-trivial renormalised four-point function fulfils a linear integral equation with the inhomogeneity determined by the two-point function [1].

These integral equations are the starting point for a perturbative solution. In this way, the renormalised correlation functions are directly obtained, without Feynman graph computation and further renormalisation steps. We obtain

$$
\begin{aligned}
(10) \qquad G_{\alpha\beta} = 1 & + \lambda\Big\{A(I_\beta - \beta) + B(I_\alpha - \alpha)\Big\} \\
& + \lambda^2\Big\{AB\big((I_{\overset{\alpha}{\bullet}} - \alpha) + (I_{\overset{\beta}{\bullet}} - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2)+1)\big) \\
& \quad + A\big(\beta I_{\overset{\beta}{\bullet}} - \beta I_\beta\big) - \alpha AB\big((I_\beta)^2 - 2\beta I_\beta + I_\beta\big) \\
& \quad + B\big(\alpha I_{\overset{\alpha}{\bullet}} - \alpha I_\alpha\big) - \beta BA\big((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha\big)\Big\} + \mathcal{O}(\lambda^3) ,
\end{aligned}
$$

where $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$ and the following iterated integrals appear:

$$
\begin{aligned}
(11) \qquad I_\alpha &:= \int_0^1 dx \, \frac{\alpha}{1-\alpha x} = -\ln(1-\alpha) , \\
I_{\overset{\alpha}{\bullet}} &:= \int_0^1 dx \, \frac{\alpha \, I_x}{1-\alpha x} = \mathrm{Li}_2(\alpha) + \frac{1}{2}\big(\ln(1-\alpha)\big)^2 .
\end{aligned}
$$

We conjecture that $G_{\alpha\beta}$ is at any order a polynomial with rational coefficients in α, β, A, B and iterated integrals labelled by rooted trees.

References

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