The harmonic oscillator, its noncommutative dimension and the vacuum of noncommutative gauge theory

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(joint work with Harald Grosse)

Renormalisable scalar quantum field theories on noncommutative Moyal space are characterised by the appearance of a harmonic oscillator potential in the action functional [1]. To make contact with noncommutative geometry, and also to extend the model to gauge theory, one must understand the Moyal-oscillator space as a spectral triple. This amounts to construct a Dirac operator such that its square is the Schrödinger operator of the harmonic oscillator. All attempts to find such a spectral triple in agreement with the original set of axioms [2] failed so far. Recently, it was realised that many interesting noncommutative geometries, such as the standard model [3], have different KO and spectral dimensions.

With this flexibility in mind, it it now not difficult to understand the harmonic oscillator as a spectral triple. We describe here the four-dimensional case. For details we refer to [4]. As the algebra we take the Schwartz class functions $\mathcal{A}_4 = \mathcal{S}(\mathbb{R}^4)$ either with the commutative product or with the Moyal product

$$(1) \qquad (f \star g)(x) = \int d^4y \, \frac{d^4k}{(2\pi)^4} \, f(x+\tfrac{1}{2}\Theta \cdot k) \, g(x+y) \, \mathrm{e}^{i\langle k,y\rangle} \;, \qquad f, g \in \mathcal{A}_4 \;.$$

The Dirac operator $\mathcal{D}_4$ is constructed in the *eight-dimensional Clifford algebra* with generators $\Gamma_1, \dots, \Gamma_8$. In the Moyal case we take

$$(2) \qquad \mathcal{D}_4 = i\Gamma^\mu \partial_\mu + \Omega \Gamma^{\mu+4} \tilde{x}_\mu \;,$$

where $\tilde{x}_\mu := 2(\Theta^{-1}_{\mu\nu} x^\nu$. As usual greek indices run from 1 to 4 and Einstein's sum convention is used. In the commutative case, we replace $\Omega\Gamma^{\mu+4}\tilde{x}_\mu$ by $\omega\Gamma^{\mu+4}x_\mu$. Accordingly, the Hilbert space is $\mathcal{H}_4 = L^2(\mathbb{R}^4) \otimes \mathbb{C}^{16}$. Equivalently, one can regard $\mathbb{C}^{16} \simeq \mathrm{Cliff}(\mathbb{C}^4)$ and realise the first set $\Gamma^1, \dots, \Gamma^4$ as usual Clifford multiplication with standard $4D$ gamma matrices, whereas the action of $\Gamma^{\mu+4}$ is constructed from γ^μ and a graded sign. In the Moyal case, the algebra acts on $\mathcal{H}_4$ by componentwise left Moyal multiplication $L_\star : \mathcal{A}_4 \times \mathcal{H}_4 \to \mathcal{H}_4$.

The only possibility for the grading operator is $\chi_4 = \Gamma_9 := \Gamma_1 \cdots \Gamma_8$. Then, choosing the Clifford generators such that $\Gamma_1, \dots, \Gamma_4$ are real and $\Gamma_5, \dots, \Gamma_8$ purely imaginary, the only possible real structure is $J_4\psi := \Gamma_9\bar{\psi}$ for $\psi \in \mathcal{H}_4$. This means that the geometry has KO-dimension $0 \mod 8$. At first sight, this seems to be related with the dimension of the Clifford algebra and the phase space dimension. However, the KO-dimension is always zero for any harmonic oscillator dimension.

On the other hand, the metric dimension of the triple $(\mathcal{A}_4, \mathcal{D}_4, \mathcal{H}_4)$ is four—if defined in non-compact sense. First, if Θ is given by θ times the standard symplectic form, one finds $\mathcal{D}_4^2 = H_4 \otimes 1_{16} + \tilde{\Omega} \otimes \Sigma_4$, where $H_4 = -\partial^\mu\partial_\mu + \tilde{\Omega}^2 x^\mu x_\mu$, $\tilde{\Omega} := \frac{2\Omega}{\theta}$, and $\Sigma \in M_{16}(\mathbb{C})$ is the traceless spin matrix. Using the Mehler kernel

$$(3) \qquad e^{-tH_4}(x,y) = \Big(\frac{\omega}{2\pi\sinh(2\omega t)}\Big)^2 e^{-\frac{\omega}{4}\coth(\omega t)\|x-y\|^2 - \frac{\omega}{4}\tanh(\omega t)\|x+y\|^2} \;,$$

for $x, y \in \mathbb{R}^4$, and the kernel representation of the Moyal product (1), one can compute the Dixmier trace of $L_\star(f)(\mathcal{D}_4^2+1)^{-\frac{\delta}{2}}$ as a residue:

$$\operatorname{Tr}_\omega\big(L_\star(f)(\mathcal{D}_4^2+1)^{-\frac{\delta}{2}}\big) = \lim_{s\to 1}\frac{(s-1)}{\Gamma(\frac{\delta s}{2})}\int_0^\infty dt\; t^{\frac{\delta s}{2}-3}e^{-t}\,\frac{(\tilde{\Omega}t)^2}{\pi^2(1+\Omega^2)^2\tanh^2(\tilde{\Omega}t)}$$
$$\times\int d^4x\; f(x)\,e^{-\frac{\tilde{\Omega}\tanh(\tilde{\Omega}t)}{1+\Omega^2}\|x\|^2}\;. \tag{4}$$

For Schwartz-class functions f, the dimension spectrum is given by the even integers ≤ 4, so that we regard the maximal value $d=4$ as the spectral dimension. The corresponding trace theorem allows one to express the scalar product for smooth spinors by the Dixmier trace and the standard hermitean structure. However, if we adjoin a unit to the algebra and take $f=1$, then the compact dimension would be 8. The non-compact version is the correct one, because the non-compact L^2-Hilbert space is essential in the quantisation of the spectrum of H_4.

Now we can derive the spectral action [2]. To make the model more interesting, we tensor $(\mathcal{A}_4, \mathcal{D}_4, \mathcal{H}_4, \chi_4)$ with the Connes-Lott two-point spectral triple $(\mathbb{C}\oplus\mathbb{C}, \mathbb{C}^2, M\sigma_1)$. Then, using $[\mathcal{D}_4, L_\star(f)] = i(\Gamma^\mu+\Omega\Gamma^{\mu+4})(\partial_\mu f)$ due to $[x^\nu, f]_\star = i\Theta^{\nu\rho}\partial_\rho f$, the fluctuated Dirac operators $\mathcal{D}_A = \mathcal{D}+\sum_i a_i[\mathcal{D}, b_i]$ are of the form

$$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_4+(\Gamma^\mu+\Omega\Gamma^{\mu+4})L_\star(A_\mu) & \Gamma_9 L_\star(\phi) \\ \Gamma_9 L_\star(\bar{\phi}) & \mathcal{D}_4+(\Gamma^\mu+\Omega\Gamma^{\mu+4})L_\star(B_\mu)\end{pmatrix}\;, \tag{5}$$

for real Yang-Mills fields $A_\mu, B_\mu \in \mathcal{A}_4$ and a complex Higgs field $\phi\in\mathcal{A}_4$.

According to the spectral action principle [2], the most general form of the bosonic action is

$$S(\mathcal{D}_A) = \operatorname{Tr}\big(\chi(\mathcal{D}_A^2)\big) = \int_0^\infty dt\;\operatorname{Tr}(e^{-t\mathcal{D}_A^2})\hat{\chi}(t)\;, \tag{6}$$

where $\hat{\chi}$ is the (inverse) Laplace transform of the weight function χ. We write $\mathcal{D}_A^2 =: H_4 1_{32}+\tilde{\Omega}\Sigma_4 1_2 - V$ and iterate the Duhamel expansion

$$e^{-t_0(\mathrm{H}_0-V)} = e^{-t_0\mathrm{H}_0}+\int_0^{t_0}dt_1\;\big(e^{-(t_0-t_1)(\mathrm{H}_0-V)}Ve^{-t_1\mathrm{H}_0}\big) \tag{7}$$

to obtain an asymptotic expansion $e^{-t\mathcal{D}_A^2} = \sum_{z=-4}^\infty a_z(\mathcal{D}_A^2)t^z$. After long calculation, we obtain the following spectral action

$$\begin{aligned} S = {}& \frac{\theta^4\chi_{-4}}{\Omega^4}+\frac{2\theta^2\chi_{-2}}{3\Omega^2}+\frac{52\chi_0}{45} \\ &+\frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4z\,\Big\{\big(\tfrac{(1-\Omega^2)^2}{2}-\tfrac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\big)\,(F^A_{\mu\nu}\star F_A^{\mu\nu}+F^B_{\mu\nu}\star F_B^{\mu\nu}) \\ &+\Big(\phi\star\bar{\phi}+\tfrac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_A\star\tilde{X}_{A\mu}-\frac{\chi_{-1}}{\chi_0}\Big)^2+\Big(\bar{\phi}\star\phi+\tfrac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_B\star\tilde{X}_{B\mu}-\frac{\chi_{-1}}{\chi_0}\Big)^2 \\ &-2\Big(\tfrac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_0\star\tilde{X}_{0\mu}-\frac{\chi_{-1}}{\chi_0}\Big)^2+2D^\mu\phi\star\overline{D_\mu\phi}\Big\}(z)+\mathcal{O}(\chi_1)\;, \end{aligned} \tag{8}$$

where $\chi_z = \int_0^\infty dt\,t^z\,\hat{\chi}(t)$. In (8), $\tilde{X}^\mu_A(x) := \frac{\tilde{x}^\mu}{2}+A^\mu(x)$ is a covariant coordinate with gauge transformation $X^\mu_A \mapsto u_A\star X^\mu_A\star\overline{u_A}$. Similarly for $\tilde{X}^\mu_B$. By $D_\mu\phi =$

$\partial_\mu\phi - iA \star \phi + i\phi \star B$ we denote the covariant coordinate of the Higgs field. The gauge transformation of the latter is $\phi \mapsto u_A \star \phi \star \overline{u_B}$.

Some conclusions and comments:

- The square of covariant derivatives combines with the Higgs field to a non-trivial potential. This was not noticed in [5, 6] where gauge theory induced by scalar fields was derived. We observe here a much deeper unification of the continuous geometry described by Yang-Mills fields and discrete geometry described by the Higgs field than previously in almost-commutative geometry. The distinction into discrete and continuous part is no longer possible in general noncommutative geometries. Therefore, the Higgs potential cannot be restricted to the Higgs field, it must include the gauge field, too.
- The coefficient in front of the Yang-Mills action is positive for all $\Omega \in [0, 1[$. In the bosonic model of [5, 6] there was only the analogue of the negative part, which leads to problems with the field equations.
- Most importantly, (8) is translation-invariant if we forget how $\tilde{X}$ is constructed: The transformation $\phi(x) \mapsto \phi(x+a)$ and $X_{G\mu}(x) \mapsto X_{G\mu}(x+a)$, for $G \in A, B, 0$, leaves the action invariant. Thus, a frequent objection against the renormalisable ϕ^4_4-models disappears for the Yang-Mills-Higgs spectral action.

We have found several solutions of the classical field equations resulting from the spectral action. For pure Yang-Mills theory (i.e. $\phi = 0$), there is an interesting radial solution in terms of modified Bessel and Struve functions:

$$(\tilde{X}^\mu \star \tilde{X}_\mu)(x) = \frac{\eta^2(1+\Omega^2)}{4\Omega^2}\Big(1 + \frac{\pi}{2}\big(I_1(\gamma\|x\|^2) - \mathbf{L}_{-1}(\gamma\|x\|^2)\big)\Big)\,, \tag{9}$$

where $\frac{\chi_{-1}}{\chi_0} = \eta^2$ and $\frac{4\sqrt{2}g\Omega^2}{\theta(1+\Omega^2)}$ for the coupling constant given by $\frac{1}{4g^2} := \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}$. For small distances $\|x\|$ we have $\tilde{X}^\mu \star \tilde{X}_\mu \sim \|x\|^2$, which provides the oscillator potential needed for renormalisation. At large distances, $\tilde{X}^\mu \star \tilde{X}_\mu$ approaches its asymptotic value where the Higgs type potential vanishes. The influence of this behaviour on renormalisation remains to be studied.

References

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