Renormalisation of noncommutative ϕ^4-theory to all orders

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(joint work with Harald Grosse)

In recent years there has been considerable interest in quantum field theories on the Moyal plane characterised by the $\star$-product (in D dimensions)

$$(a\star b)(x) := \int d^Dy \frac{d^Dk}{(2\pi)^D} a(x+\tfrac{1}{2}\theta\cdot k) b(x+y)\, \mathrm{e}^{\mathrm{i}ky}\,, \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R}\,. \tag{1}$$

The interest was to a large extent motivated by the observation that this kind of field theories arise in the zero-slope limit of open string theory in presence of a magnetic background field [1]. A few months later it was discovered [2] (first for scalar models) that these noncommutative field theories are not renormalisable beyond a certain loop order. The argument is that non-planar graphs are finite but their amplitude grows beyond any bound when the external momenta become exceptional. When inserted as subgraphs into bigger graphs, these exceptional momenta are attained in the loop integration and result in divergences for any number of external legs. This problem is called UV/IR-mixing.

The UV/IR-mixing contains a clear message: If we make the world noncommutative at very short distances, we must at the same time modify the physics at large distances. The required modification is, to the best of our knowledge, unique: It is given by an harmonic oscillator potential for the free field action. In fact, we can prove the following

Theorem 1 *The quantum field theory associated with the action*

$$S = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{\mu_0^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x)\,, \tag{2}$$

for $\tilde{x}_\mu := 2(\theta^{-1})_{\mu\nu}x^\nu$, ϕ-real, Euclidean metric, is perturbatively renormalisable to all orders in λ.

The proof is given in [3] and [4]. A summary of the main ideas and techniques can be found in [5].

Compared with the commutative ϕ^4-model, the bare action of relevant and marginal couplings contains necessarily an additional term: an harmonic oscillator potential for the free scalar field action. This is a result of the renormalisation proof. It entails a discrete spectrum of the corresponding differential operator: Renormalisation induces a compactification of the underlying noncommutative geometry.

Our proof rests on two concepts:

1. *The representation of the ϕ^4-action in the harmonic oscillator base of the Moyal plane.* Then, the action describes a matrix model the kinetic term of which is neither constant nor diagonal. We have derived a closed formula for the resulting propagator, using Meixner polynomials in an essential way.

2. *The renormalisation group approach for dynamical matrix models, the core of which is a flow equation for the effective action.* The renormalisation proof is now reduced to the verification that the flow equation—a non-linear first-order differential equation—admits a regular solution which depends on finitely many initial data. In the perturbative regime, the flow equation is solved by ribbon graphs drawn on Riemann surfaces.

We have proven a power-counting theorem which relates the power-counting behaviour of ribbon graphs to their topology and to the asymptotic scaling dimensions of the cut-off propagator. As a result, only planar graphs with two or four external legs can be relevant or marginal. These graphs are labelled by an infinite number of matrix indices. There exists a discrete Taylor expansion which decomposes the (infinite number of) planar two- and four-leg graphs into a linear combination of four relevant or marginal base functions and an irrelevant remainder. These four universal base functions have the same index dependence as the original action in matrix formulation, which implies the renormalisability of the model.

We have also computed in [6] the one-loop β-functions of the model which describe the dependence of the bare coupling constant and the bare oscillator frequency on the cut-off matrix size. It turned out that $\frac{\lambda}{\Omega^2}$ remains constant under the renormalisation flow. Starting from given small values for Ω_R, λ_R at an initial matrix size $\mathcal{N}_R$, the frequency Ω grows in a small region around $\ln \frac{\mathcal{N}}{\mathcal{N}_R} = \frac{48\pi^2}{\lambda_R}$ to $\Omega \approx 1$. The coupling constant approaches $\lambda_\infty = \frac{\lambda_R}{\Omega_R^2}$, which can be made small for sufficiently small λ_R.

References

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[3] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base," arXiv:hep-th/0401128, to appear in Commun. Math. Phys.

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