The cohomology of p-adic symmetric spaces

P.Schneider, U.Stuhler

In 1972 Mumford ([37]) exhibited a whole class of algebraic curves over a local field K which can be uniformized by a p-adic "upper half plane". This was taken up by Drinfeld in his study of the moduli varieties of elliptic modules ([16]). He showed that the spaces $\Omega^{(d+1)}$ which arise from projective d-space over K by removing all rational hyperplanes have a natural rigid analytic structure. Later on ([17]) he gave a new construction of $\Omega^{(d+1)}$ as a moduli space for formal groups. As an application he obtained a new proof of a result of Cherednik describing explicitly the p-adic uniformization of certain Shimura curves. Mustafin ([39]) proved that for arbitrary $d \geq 1$ the quotients of $\Omega^{(d+1)}$ by cocompact discrete groups are projective algebraic varieties. Rapoport and Zink ([41]) used the p-adic uniformization theory in their study of certain higher dimensional Shimura varieties.

It seems quite natural to view the spaces $\Omega^{(d+1)}$ as p-adic analogs of real symmetric spaces. But there is at least one essential difference. Although $\Omega^{(d+1)}$ is analytically simply connected ([40]) it still has many etale coverings and is far from being cohomologically trivial. Drinfeld in [16] computed explicitly the first etale cohomology group of $\Omega^{(2)}$. And in [17] he constructed a natural family of equivariant etale coverings of $\Omega^{(d+1)}$ expressing the fascinating hope that their cohomology realizes all discrete series representations of $GL_{d+1}(K)$.

In the first part of this paper we compute the cohomology of $\Omega^{(d+1)}$ for arbitrary d in any good cohomology theory. Our main result is that in degrees $\leq d$ the cohomology realizes certain explicitly described admissible representations of $GL_{d+1}(K)$; it vanishes in degrees $> d$. The specific cohomology theory only enters into the formula through its coefficients $A := H^0(point)$. For example the highest cohomology of $\Omega^{(d+1)}$ is the A-dual of the Steinberg representation. In case of etale cohomology the Galois action turns out to be pure and even by powers of the cyclotomic character.

In the second part we then use this result in order to study the cohomology of quotient varieties $X_\Gamma := \Gamma\backslash\Omega^{(d+1)}$ by cocompact discrete groups $\Gamma \subseteq PGL_{d+1}(K)$. We obtain that X_Γ has primitive cohomology only in middle degree. Furthermore this middle cohomology has a filtration the subquotients of which are pure and have dimensions which depend in a very simple way on a single invariant. This is the multiplicity of the Steinberg representation in the representation induced from the trivial character on Γ. Of course, this filtration has to be Deligne's monodromy filtration but we cannot prove this.

In the last part we develop a general formalism for constructing natural resolutions with good finiteness properties for smooth $GL_{d+1}(K)$-modules. This is needed as a second essential ingredient in the computation of the cohomology of the varieties X_Γ.

For the convenience of the reader we now describe the content of the individual Paragraphs in more detail. In the first Paragraph we discuss the rigid analytic structure of $\Omega^{(d+1)}$. We introduce certain neighbourhoods of a rational hyperplane in projective space over K and study the geometry of intersections of those neighbourhoods. The reason for considering them is that by the local compactness of the field

K already finitely many of them contain all infinitely many rational hyperplanes. One then can try to imitate classical techniques for computing the singular cohomology of a complex vector space with finitely many hyperplanes removed (compare [10]). We also prove that $\Omega^{(d+1)}$ is a Stein-space in the strong sense of Kiehl.

The second Paragraph contains general considerations from homological algebra. We introduce the notion of an abstract cohomology theory on the category of smooth rigid analytic varieties over K which is equipped axiomatically with four basic properties. The essential ones among them are the homotopy invariance and the validity of the usual formula for the cohomology of projective space. Those four properties will suffice to compute the cohomology of $\Omega^{(d+1)}$. Using results of Kiehl we show that the de Rham cohomology in characteristic 0 has the required properties. We then give a careful definition of the etale topology in the rigid analytic context. Here the necessary properties beyond any doubt hold true, too, but unfortunately there exist no published proofs for the two essential ones. At least there is unpublished work of Gabber which seems to lead to the wanted results. In the second half of the Paragraph we establish a general spectral sequence which allows to compute the cohomology of a pair $U \subseteq X$ in terms of the cohomology of the pairs $U_{i_0} \cup \ldots \cup U_{i_r} \subseteq X$ whenever U is the intersection of the U_i's.

In the third Paragraph we will use this spectral sequence in order to express the cohomology of $\Omega^{(d+1)}$ in terms of the relative cohomology of the projective space with respect to the complement of the intersection of finitely many hyperplane neighbourhoods. It turns out that our axioms together with the simple geometry of those hyperplane neighbourhoods lead to an explicit computation of the E_1-terms in this spectral sequence. A closer inspection of the d_1-differential then shows that the lines in the spectral sequence are closely related to the simplicial cohomology of certain generalized topological Tits buildings $\mathcal{T}.^{(s)}$ for $GL_{d+1}(K)$. We show that the $\mathcal{T}.^{(s)}$ have cohomology only in highest degree which implies the degeneration of our spectral sequence. From this it is easy to deduce our first main result (Theorem 1) that

$$H^s(\Omega^{(d+1)}) = \begin{cases} A & \text{if } s = 0 \ , \\ \mathrm{Hom}_{\mathbb{Z}}(\tilde{H}^{s-1}(|\mathcal{T}.^{(s)}|, \mathbb{Z}), A) & \text{if } 1 \le s \le d \ , \\ 0 & \text{if } s > d \ . \end{cases}$$

The building $\mathcal{T}.^{(d)}$ together with a computation of its cohomology already appears in [3].

In the fourth Paragraph we give two different explicit descriptions of the reduced cohomology groups $\tilde{H}^{s-1}(|\mathcal{T}.^{(s)}|, \mathbb{Z})$. Extending a technique in [3] we first construct certain filtrations on these groups the subquotients of which can be identified naturally with spaces of locally constant functions with compact support on cells in flag manifolds. In the case of de Rham cohomology we obtain in this way a natural map — a kind of p-adic Mellin transform — which associates with any closed s-form on $\Omega^{(d+1)}$ a K-valued distribution on the $\frac{s}{2} \cdot (2d+1-s)$-dimensional affine space over K (Corollary 6). Secondly we identify $H^s(\Omega^{(d+1)})$ with an explicitly defined subspace of the space of s-dimensional cochains on the Bruhat-Tits building $\mathcal{BT}$ of $SL_{d+1}(K)$ (Corollary 17). In the case $s = d$ the resulting subspace is the space of harmonic cochains in the sense of [19].

The fifth Paragraph contains the computation of the cohomology of the quotient varieties X_Γ. Of course we use a Hochschild-Serre type spectral sequence for the covering $\Omega^{(d+1)} \to X_\Gamma$. The E_2-terms can be transformed into Ext-groups in the category of smooth $PGL_{d+1}(K)$-representations. In order to compute these Ext-groups we generalize ideas of Casselman ([12], [13]) which he used in his representation theoretic proof of the Garland vanishing theorem ([19]). There are two additional features: One is that for the representations which appear in the E_2-terms we need resolutions with nice finiteness properties with respect to the Γ-action. Their construction is postponed to Paragraph 6 (Proposition 16). Second we establish a general cohomological duality (Duality Theorem) in the category of smooth representations in which the role of the dualizing object is played by the Steinberg representation. This result resembles very much the Borel/Serre duality for discrete groups ([3]) but it does not seem to be a formal consequence of it. The final results (Theorems 4 and 5) were already explained above. In the rest of the Paragraph we deduce from our computation of $H^1_{DR}(\Omega^{(2)})$ a proof of the p-adic Shimura isomorphism between modular forms and group cohomology (a corrected version of the corresponding statement in [44]). We do this by a careful analysis of certain p-adic representations of $SL_2(K)$. In particular, we study the nonexistence of Γ-invariant continuous linear forms on Morita's p-adic principal series of $SL_2(K)$ (Theorem 6). As another application of this we will see the surprising fact that a cocompact discrete subgroup $\Gamma \subseteq SL_2(K)$ has no nonvanishing automorphic forms of weight 1 (Corollary 13). A much more direct proof of the p-adic Shimura isomorphism was given by de Shalit ([46]); he also pointed out that the original form of the statement in [44] does not hold.

The Bernstein-Borel-Matsumoto theory of smooth representations generated by their Iwahori-fixed vectors (see, e.g., [2]) would provide us rather easily with (noncanonical) resolutions of the type which we needed in Paragraph 5 if it would be sufficient for our purposes to work with $\mathbb{Q}$-coefficients. But the $GL_{d+1}(K)$-modules we have to study have $\mathbb{Z}$- or at least $\mathbb{Z}[\frac{1}{p}]$-coefficients. The theory unfortunately breaks down in this situation. Instead we start in the last Paragraph with the observation that any smooth $GL_{d+1}(K)$-module in a completely natural way gives rise to a coefficient system on the Bruhat-Tits building $\mathcal{BT}$. The chain complex of this coefficient system is a good candidate for the required resolution. For coefficient systems of a certain type on an arbitrary contractible simplicial complex $\mathcal{K}$ we develop a criterion which reduces the homological triviality of the coefficient system to the question whether certain subcomplexes of $\mathcal{K}$ (defined in terms of the coefficient system) are contractible (Proposition 1). In the concrete situation on $\mathcal{BT}$ to which we want to apply this criterion we explicitly describe the respective subcomplexes as unions of certain apartments in $\mathcal{BT}$. This makes it possible to apply the geodesic action on $\mathcal{BT}$ to show their contractibility. In this way we prove the chain complexes in question to be exact (Theorems 7 and 8).

We are especially grateful to P.Deligne. He pointed out an erroneous argument in our proof of §3 Proposition 5 and he suggested the present version of the proof. We also want to thank O.Gabber and M.Rapoport for several interesting conversations.

§1 The rigid analytic space $\Omega^{(d+1)}$

Once and for all we fix a local field K, that is a field K which is locally compact with respect to a non-trivial non-archimedean valuation $|\ |$. We always will assume the valuation $|\ |$ to be normalized. Let $p > 0$ be the characteristic of the finite residue class field of K, o be the valuation ring of K, and $\hat{\bar{K}}$ be the completion of a fixed algebraic closure $\bar{K}$ of K. We also fix a natural number d and denote by $\mathbb{P}^d_{/K}$, as usual, the d-dimensional projective space viewed as a K-analytic variety.

If $\mathcal{H}$ is the set of all K-rational hyperplanes in $\mathbb{P}^d_{/K}$ then our object of interest is

$$\Omega^{(d+1)} := \mathbb{P}^d_{/K} \setminus \bigcup_{H \in \mathcal{H}} H \quad .$$

Proposition 1:

$\Omega^{(d+1)}$ is an admissible open subset and consequently an open analytic subvariety of $\mathbb{P}^d_{/K}$.

We indicate two proofs and explicitly give a third one.

A) The first proof is taken from [16]. As this will be of use later on, we give some of the ideas behind it. Of central importance is the close relationship between $\Omega^{(d+1)}$ and the Bruhat-Tits building $\mathcal{BT}$ of the group $SL_{d+1}(K)$.

Definition: ([9])

$\mathcal{BT}$ is the simplicial complex whose vertices are the similarity classes $[L]$ of o-lattices in the vector space K^{d+1} and whose q-simplices are given by families $\{[L_0], ..., [L_q]\}$ of similarity classes such that

$$L_0 \underset{\neq}{\subset} L_1 \underset{\neq}{\subset} ... \underset{\neq}{\subset} L_q \underset{\neq}{\subset} \pi^{-1} L_0$$

where π is a uniformizing element in K.

Remarks:

i. $\mathcal{BT}$ is contractible.
ii. $GL_{d+1}(K)$ in a natural way acts on $\mathcal{BT}$.
iii. ([21]) The topological space of similarity classes of real norms on K^{d+1} can be identified $GL_{d+1}(K)$ - equivariantly with the geometric realization $|\mathcal{BT}|$ of the simplicial complex $\mathcal{BT}$.

On the other hand, any point $z = [z_0 : ... : z_d] \in \Omega^{(d+1)}(\hat{\bar{K}})$ defines a similarity class of norms $\| \ \|_{\rho(z)}$ by

$$\|w\|_{\rho(z)} := |\sum_{i=0}^{d} w_i z_i| \quad \text{for} \quad w = (w_0, ..., w_d) \in K^{d+1} \quad .$$

In this way we obtain a $GL_{d+1}(K)$-equivariant map

$$\rho : \Omega^{(d+1)} \longrightarrow |\mathcal{BT}| \quad .$$

With the help of this map Drinfel'd constructs an explicit family $(U_i)_{i \in I}$ of open affinoid subvarieties of $\mathbb{P}^d_{/K}$ such that

a. $\bigcup_{i \in I} U_i = \Omega^{(d+1)}$ and

b. any K-morphism $f : Y \longrightarrow \mathbb{P}^d$ from a K-affinoid variety into $\mathbb{P}^d_{/K}$ with $f(Y) \subseteq \Omega^{(d+1)}$ factorizes through some U_i.

This in particular shows (compare [6] 9.1.4 Prop. 2) that $\Omega^{(d+1)}$ is an admissible open subspace of $\mathbb{P}^d_{/K}$.

B) In the second proof one constructs a formal scheme $\hat{\Omega}^{(d+1)}$ over $Spf(o)$ of which $\Omega^{(d+1)}$ is the "generic fibre". For details see [17], [37], [39], [30], [42], [32].

C) The third proof proceeds again by constructing a certain explicit family of admissible open subvarieties of $\mathbb{P}^d_{/K}$ with the properties a. and b. above. Precisely this covering of $\Omega^{(d+1)}$ later on will be the main technical tool in our computations. First we introduce the following convention: If not indicated otherwise any coordinate representation $z = [z_0 : ... : z_d]$ of a point $z \in \mathbb{P}^d(\hat{\bar{K}})$ is assumed to be unimodular, i.e., such that $|z_i| \leq 1$ for $0 \leq i \leq d$ and $|z_i| = 1$ for some i. Similarly, for any hyperplane $H \in \mathcal{H}$, we let $\ell_H \in L_0^*$ with $L_0 := o^{d+1}$ always be a unimodular vector (determined up to a unit in o) such that $H(\hat{\bar{K}})$ is its zero set

$$H(\hat{\bar{K}}) = \{z \in \mathbb{P}^d(\hat{\bar{K}}) : \ell_H(z) = 0\} \quad .$$

Definition:

If $\epsilon > 0$ is a rational number the set

$$H(\epsilon) := \{z \in \mathbb{P}^d(\hat{\bar{K}}) : |\ell_H(z)| \leq \epsilon\}$$

is called the ϵ-neighbourhood of the hyperplane $H \in \mathcal{H}$.

Because of our above convention this definition obviously is independent of the particular choices of ℓ_H and of the coordinate representation for z. For simplicity, we let

$H(\epsilon)$ also denote the subset of $\mathbb{P}^d_{/K}$ which we get by identifying (over K) conjugate points. Usually ϵ will be of the form $\epsilon = |\pi|^n$ for some $n \in \mathbb{N}$ where π is a fixed uniformizing element of o.

Definition:

Two hyperplanes $H, H' \in \mathcal{H}$ are called congruent $\mathrm{mod}(\pi^n)$ *for some $n \in \mathbb{N}$ if appropriate representing vectors ℓ_H and $\ell_{H'}$ satisfy the congruence*

$$\ell_H \equiv \ell_{H'} \bmod \pi^n L_0^* \quad .$$

If $\mathcal{H}_n$ denotes the set of equivalence classes of hyperplanes $H \in \mathcal{H} \mod(\pi^n)$ then we have $\mathcal{H}_n = \mathbb{P}(L_0^*/\pi^n L_0^*)$ and $\mathcal{H} = \varprojlim \mathcal{H}_n$.

Lemma 2:

For two hyperplanes $H, H' \in \mathcal{H}$ we have $H(|\pi|^n) = H'(|\pi|^n)$ if and only if H and H' are congruent $\mathrm{mod}(\pi^n)$.

Proof: Since ℓ_H and $\ell_{H'}$ are unimodular they induce surjective linear maps

$$\bar{\ell}_H \quad \text{and} \quad \bar{\ell}_{H'} : L_0/\pi^n L_0 \longrightarrow o/\pi^n o \quad .$$

If we assume that $H(|\pi|^n) = H'(|\pi|^n)$ then $\bar{\ell}_H$ and $\bar{\ell}_{H'}$ have the same kernel. This implies that $\bar{\ell}_{H'} = \alpha \cdot \bar{\ell}_H$ for some $\alpha \in (o/\pi^n o)^\times$. The reverse implication easily follows from the inequality $|\ell_H(z)| \leq \max\{|\ell_H(z) - \ell_{H'}(z)|, |\ell_{H'}(z)|\}$.

Lemma 3:

$\bigcup\limits_{H \in \mathcal{H}_n} H(|\pi|^n) \supseteq \bigcup\limits_{H \in \mathcal{H}} H.$

Proof: Clear from Lemma 2.

We now consider the subsets

$$\Omega_n := \Omega(|\pi|^n) := \mathbb{P}^d_{/K} \setminus \bigcup_{H \in \mathcal{H}_n} H(|\pi|^n)$$

in $\Omega^{(d+1)}$. They are admissible open in $\mathbb{P}^d_{/K}$. Each Ω_n is a finite intersection of subsets of the form $\mathbb{P}^d_{/K} \setminus H(|\pi|^n)$. It therefore suffices to show that those subsets are admissible open. But up to isomorphism $\mathbb{P}^d_{/K} \setminus H(|\pi|^n)$ is an open polydisc in the affine d-space $\mathbb{P}^d_{/K} \setminus H$. The family $\{\mathbb{P}^d_{/K} \setminus H(|\pi|^n)\}_{n \in \mathbb{N}}$ even is an admissible covering of $\mathbb{P}^d_{/K} \setminus H$. Therefore, if $f : Y \to \mathbb{P}^d$ is any K-morphism from a K-affinoid variety Y into $\mathbb{P}^d$ such that $f(Y) \subseteq \mathbb{P}^d \backslash H$ then there exists a $n(H) \in \mathbb{N}$ such that $f(Y) \subseteq \mathbb{P}^d \backslash H(|\pi|^{n(H)})$. If we apply this to a morphism f such that $f(Y) \subseteq \Omega^{(d+1)}$ we see that

$$f(Y) \subseteq \mathbb{P}^d \backslash \bigcup_{H \in \mathcal{H}} H(|\pi|^{n(H)}) \quad .$$

Because of Lemma 2 the sets $\{H' \in \mathcal{H} : H' \subseteq H(|\pi|^{n(H)})\}$ are open in $\mathcal{H} = \underleftarrow{\lim}\, \mathcal{H}_n$. Since $\mathcal{H}$ is compact we conclude that we find finitely many hyperplanes $H_1, ..., H_r \in \mathcal{H}$ and numbers $n_1, ..., n_r \in \mathbb{N}$ such that

$$\bigcup_{H \in \mathcal{H}} H \subseteq H_1(|\pi|^{n_1}) \cup ... \cup H_r(|\pi|^{n_r}) \quad .$$

As a consequence, for $n := \max n_j$, we get that

$$\bigcup_{H \in \mathcal{H}} H \subseteq \bigcup_{H \in \mathcal{H}_n} H(|\pi|^n) \subseteq \bigcup_j H_j(|\pi|^{n_j})$$

which, in particular, means that $f(Y) \subseteq \Omega_n$. We thus have established Proposition 1 and have shown that the increasing sequence of admissible open subvarieties Ω_n forms an admissible covering of $\Omega^{(d+1)}$.

Proposition 4:

$\Omega^{(d+1)}$ *is a Stein-space.*

Proof: (See [29] §2 for the notion of Stein-space.) Similarly as before one shows that the increasing sequence of subsets

$$\bar{\Omega}_n := \{z \in \mathbb{P}^d(\hat{\bar{K}}) : |\ell_H(z)| \geq |\pi|^n \text{ for all } H \in \mathcal{H}\}$$

in $\Omega^{(d+1)}$ forms an admissible covering by open affinoid subvarieties. For any pair H, H' in $\mathcal{H}$ we have the analytic function

$$f_{H,H'} := \frac{\ell_H}{\ell_{H'}} \quad \text{on} \quad \Omega^{(d+1)} \quad .$$

For each n we choose a set $\bar{\mathcal{H}}_n$ of representatives for the equivalence classes of hyperplanes in $\mathcal{H}_{n+1}$ in such a way that it contains the coordinate hyperplanes $H_i = \{z_i = 0\}$ for $0 \leq i \leq d$. It is then easy to see that

$$\begin{aligned}\bar{\Omega}_n &= \{z \in \Omega^{(d+1)} : |f_{H,H'}(z)| \leq |\pi|^{-n} \text{ for all } H, H' \in \bar{\mathcal{H}}_n\} \\ &= \{z \in \Omega^{(d+1)} : |f_{H_i,H'}(z)| \leq |\pi|^{-n} \text{ for all } 0 \leq i \leq d \text{ and } H' \in \bar{\mathcal{H}}_n\} \\ &= \{z \in \bar{\Omega}_{n+1} : |\pi^{n+1} f_{H,H'}(z)| \leq |\pi| \text{ for all } H, H' \in \bar{\mathcal{H}}_{n+1}\} \ .\end{aligned}$$

The last description shows that in order to establish our assertion it suffices to prove that the functions $\pi^n f_{H,H'}$ for $H, H' \in \bar{\mathcal{H}}_n$ form a system of affinoid generators of $\mathcal{O}(\bar{\Omega}_n)$. This will be done by explicitly determining $\mathcal{O}(\bar{\Omega}_n)$. Define the affinoid K-algebra A_n to be the free Tate algebra over K in the indeterminates $T_{H,H'}$ for $H, H' \in \bar{\mathcal{H}}_n$ divided by the (closed) ideal generated by

$$T_{H,H} - \pi^n \quad \text{for} \quad H \in \bar{\mathcal{H}}_n \ ,$$

$$T_{H,H'} \cdot T_{H',H''} - \pi^n T_{H,H''} \quad \text{for} \quad H, H', H'' \in \bar{\mathcal{H}}_n \ , \quad \text{and}$$

$$T_{H,H_j} - \sum_{i=0}^{d} \lambda_i T_{H_i,H_j} \quad \text{if} \quad \ell_H(z) = \sum_{i=0}^{d} \lambda_i z_i \quad \text{for} \quad H \in \bar{\mathcal{H}}_n \quad \text{and} \quad 0 \le j \le d \ .$$

We then have the K-morphisms

$$\phi_n : \bar{\Omega}_n \longrightarrow Sp(A_n) \quad \text{given by} \quad \begin{array}{c} A_n \longrightarrow \mathcal{O}(\bar{\Omega}_n) \\ T_{H,H'} \longmapsto \pi^n f_{H,H'} \end{array}$$

and

$$\begin{aligned} \tilde{\psi}_n : Sp(A_n) &\longrightarrow \mathbb{P}^d \\ x &\longmapsto [T_{H_0,H_j}(x) : \ldots : T_{H_d,H_j}(x)] \ \text{(not necessarily unimodular)} \end{aligned}$$

the latter being independent of the particular choice of $0 \le j \le d$. We leave it to the reader to check that the image of $\tilde{\psi}_n$ is contained in $\bar{\Omega}_n$ so that it factorizes through a K-morphism $\psi_n : Sp(A_n) \to \bar{\Omega}_n$ and furthermore that ψ_n and ϕ_n are inverse to each other.

For later computations it is necessary to look more closely at the geometric nature of the admissible open subvarieties in $\mathbb{P}^d_{/K}$ of the form

$$\mathbb{P}^d_{/K} \backslash (H_0(|\pi|^n) \cap \ldots \cap H_r(|\pi|^n))$$

where $H_0, \ldots, H_r \in \mathcal{H}$ are finitely many hyperplanes. We consider the o-module

$$M := \sum_{i=0}^{r} o\ell_{H_i} \subseteq L_0^* \quad .$$

Obviously we have

$$H_0(|\pi|^n) \cap \ldots \cap H_r(|\pi|^n) = \{z \in \mathbb{P}^d(\hat{\bar{K}}) : |\ell(z)| \le |\pi|^n \quad \text{for all} \quad \ell \in M\} \quad .$$

By the elementary divisor theorem we find a basis $\ell_0, \ldots, \ell_d$ in L_0^* such that $\pi^{\alpha_0}\ell_0, \ldots, \pi^{\alpha_m}\ell_m$ with appropriate integers $0 \le m \le d$ and $0 \le \alpha_0 \le \ldots \le \alpha_m$ is a basis of M. Therefore, up to a linear transformation, we can assume that

$$H_0(|\pi|^n) \cap \ldots \cap H_r(|\pi|^n) = \{[z_0 : \ldots : z_d] \in \mathbb{P}^d(\hat{\bar{K}}) : |\pi^{\alpha_i} z_i| \le |\pi|^n \quad \text{for} \quad 0 \le i \le m\} \ .$$

Since M contains a unimodular vector we must have $\alpha_0 = 0$. On the other hand, the condition $|\pi^{\alpha_i} z_i| < |\pi|^n$ is automatically fulfilled if $\alpha_i \ge n$. Writing $\beta_i := n - \alpha_i$ we can state the following result.

Lemma 5:

There are integers $0 \le s \le d$ and $n = \beta_0 \ge \beta_1 \ge ... \ge \beta_s > 0$ such that, up to K-linear isomorphism,

$$H_0(|\pi|^n) \cap ... \cap H_r(|\pi|^n) = \{[z_0 : ... : z_d] \in \mathbb{P}^d(\hat{\bar{K}}) : |z_i| \le |\pi|^{\beta_i} \quad for \quad 0 \le i \le s\} \quad .$$

The integer s in the above assertion can be intrinsically characterized in the following way: Put

$$\bar{M} := \sum_{i=0}^{r} (o/\pi^n o)\ell_{H_i} \subseteq L_0^*/\pi^n L_0^* \quad .$$

The minimal number of generators of $\bar{M}$ as an o-module is called the rank of $\bar{M}$. We have

$$\mathrm{rank}\, \bar{M} = s + 1 \quad .$$

Proposition 6:

We have a (in the rigid sense) locally trivial fibration

$$\mathbb{P}^d \backslash (H_0(|\pi|^n) \cap ... \cap H_r(|\pi|^n)) \longrightarrow \mathbb{P}^s$$

over K with fibers open polydiscs in $\mathbf{A}^{d-s}$.

Proof: By Lemma 5 it suffices to consider the case

$$X := \mathbb{P}^d_{/K} \backslash \{z : |z_i| \le |\pi|^{\beta_i} \quad \text{for} \quad 0 \le i \le s\} \quad .$$

The projection

$$\begin{array}{cccccl} pr : & X & \longrightarrow & \mathbb{P}^s & \\ & [z_0 : ... : z_d] & \longmapsto & [z_0 : ... : z_s] & \text{(not necessarily unimodular)} \end{array}$$

is well-defined because at least for one i with $0 \le i \le s$ we have $|z_i| > |\pi|^{\beta_i}$. The subsets

$$U_j := \{[z_0 : ... : z_s] \in \mathbb{P}^s(\hat{\bar{K}}) : \frac{|z_j|}{|\pi|^{\beta_j}} \ge \frac{|z_i|}{|\pi|^{\beta_i}} \quad \text{for} \quad 0 \le i \le s\}$$

for $0 \le j \le s$ form an admissible K-affinoid covering of $\mathbb{P}^s_{/K}$. We now consider, for a fixed j, the K-morphism

$$pr^{-1}(U_j) \xrightarrow{(pr,\phi)} U_j \times \mathbf{A}^{d-s}$$

where ϕ is defined by

$$\phi([z_0 : ... : z_d]) := (\frac{z_{s+1}}{z_j}, ..., \frac{z_d}{z_j}) \quad .$$

The above argument for pr being well-defined also shows that on $pr^{-1}(U_j)$ we have $\frac{|z_j|}{|\pi|^{\beta_j}} > 1$. Therefore the image of ϕ is contained in the open polydisc

$$D_j := \{(w_{s+1}, ..., w_d) \in \mathbf{A}^{d-s}(\hat{\bar{K}}) : |w_t| < |\pi|^{-\beta_j} \text{ for } s < t \leq d\} \quad .$$

We claim that

$$(pr, \phi) : pr^{-1}(U_j) \longrightarrow U_j \times D_j$$

is a K-analytic isomorphism. But an inverse morphism is given by

$$([z_0 : ... : z_s], (w_{s+1}, ..., w_d)) \longmapsto [z_0 : ... : z_s : z_j w_{s+1} : ... : z_j w_d]$$

where we may assume that the projective coordinates on the right hand side (but not necessarily the ones on the left hand side) are unimodular. If $|z_i| = 1$ for some $0 \leq i \leq s$ we trivially have $|z_i| > |\pi|^{\beta_i}$ and if $|z_j w_t| = 1$ for some $s < t \leq d$ we get $|z_j| = |w_t|^{-1} > |\pi|^{\beta_j}$. In any case we see that the right hand side lies in X. q.e.d.

We like to consider such fibrations as in Proposition 6 as “homotopy equivalences”. At least they should induce isomorphisms in any reasonable cohomology theory.

§2 Abstract cohomology theory

Since our later computations will be valid in at least two different interesting cohomology theories we proceed in an axiomatic way. Let $\mathcal{V}$ be the category of smooth separated (rigid) analytic varieties over K equipped with a fixed Grothendieck topology which we assume to be finer than the analytic topology. Let $\mathcal{F}$ be an object in the derived category $D^{\geq 0}(\mathcal{V})$ of complexes of sheaves on $\mathcal{V}$ in nonnegative degrees. For any variety X in $\mathcal{V}$ we put

$$H^*(X) := H^*(X, \mathcal{F}) \quad ;$$

if $U \subseteq X$ is an (admissible) open subvariety we also will use the relative cohomology

$$H^*(X, U) := H^*(X, U; \mathcal{F}) \quad .$$

We recall that relative cohomology of the pair (X, U) is the derived functor of the functor “sections on X which vanish on U”.

Remark:

In algebraic geometry relative cohomology usually is denoted by $H^*_Z(X, .)$ with $Z := X \backslash U$. Since in our context Z rarely is a closed subvariety of X the above notation seems to be more appropriate.

These groups constitute our "abstract" cohomology theory. In order to make this theory interesting we require the following properties to be fulfilled:

I) (Homotopy invariance)

If D denotes the 1-dimensional open unit disc then, for any affinoid variety X in $\mathcal{V}$, the projection $X \times D \to X$ induces an isomorphism

$$H^*(X) \xrightarrow{\cong} H^*(X \times D)$$

in cohomology.

II) (Product structure)

There are homomorphisms

$$\cup : \mathcal{F} \overset{L}{\underset{\mathbb{Z}}{\otimes}} \mathcal{F} \longrightarrow \mathcal{F} \quad \text{and} \quad e : \mathbb{Z} \longrightarrow \mathcal{F}$$

in $D^+(\mathcal{V})$ such that $\cup$ is associative and (graded) commutative with unit e.

III) (Cohomology of the point)

We have $H^s(Sp(K)) = 0$ for $s \geq 1$. Furthermore, the ring

$$A := H^0(Sp(K))$$

is artinian.

IV) (Cohomology of projective space)

We have $H^s(\mathbb{P}^n_{/K}) = 0$ for odd s or $s > 2n$. Furthermore, there is a homomorphism

$$c : \mathbf{G}_m[-1] \longrightarrow \mathcal{F}$$

in $D^{\geq 0}(\mathcal{V})$ such that, for $0 \leq s \leq n$, the map

$$\pi^*(\ ) \cup \xi^s : A = H^0(Sp(K)) \xrightarrow{\cong} H^{2s}(\mathbb{P}^n_{/K})$$

is an isomorphism where $\pi : \mathbb{P}^n_{/K} \to Sp(K)$ is the structure morphism and ξ is the image of the canonical line bundle, i.e.,

$$\begin{array}{ccc} H^1(\mathbb{P}^n_{/K}, \mathcal{O}^\times) & \xrightarrow{c} & H^2(\mathbb{P}^n_{/K}) \\ \mathcal{O}(1) & \longmapsto & \xi \end{array} .$$

Our abstract cohomology theory consequently takes values in the category of A-modules. For future reference let us state two further simple consequences of these properties.

Lemma 1:

Any locally trivial fibration $Y \to X$ in $\mathcal{V}$ with fibers open polydiscs in affine space induces an isomorphism $H^*(X) \xrightarrow{\cong} H^*(Y)$ in cohomology.

Proof: This is a well-known formal consequence of the property I) above.

Lemma 2:

Let $m \leq n$ be natural numbers and let α be a K-linear automorphism of $\mathbb{P}^n$. For the morphism $f : \mathbb{P}^m \to \mathbb{P}^n$ given by $[z_0 : ... : z_m] \longmapsto \alpha([z_0 : ... : z_m : 0 : ... : 0])$ and any $0 \leq s \leq m$ we have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\cong} & H^{2s}(\mathbb{P}^n_{/K}) \\ \| & & \downarrow f^* \\ A & \xrightarrow{\cong} & H^{2s}(\mathbb{P}^m_{/K}) \end{array}$$

where the horizontal arrows represent the isomorphisms in property IV) above.

Proof: f^* respects the canonical line bundles.

There are two basic examples for such a cohomology theory. The first one is the de Rham cohomology. We assume that our base field K has characteristic 0 and we equip $\mathcal{V}$ with the analytic topology. Let $\mathcal{F}$ be the complex

$$\Omega^{\cdot} : \mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} ...$$

of sheaves of holomorphic differential forms. The de Rham cohomology of a variety X in $\mathcal{V}$ is defined to be the hypercohomology

$$H^*_{DR}(X) := H^*(X, \Omega^{\cdot}) \quad .$$

It has the four properties which we have required above:

The product structure is given by the usual exterior multiplication of differential forms. The cohomology of the point is obvious. The cohomology of projective space can be computed as in [23] (last paragraph of §7.1) using in addition the GAGA-principle ([28]) that analytic and algebraic coherent sheaf cohomology of $\mathbb{P}^n$ are equal.

For the homotopy invariance we imitate the argument in [23] (Prop. 7.1). If X is affinoid then X and $X \times D$ have trivial coherent sheaf cohomology so that in each case the de Rham cohomology is computed by the complex of global sections of $\Omega^{\cdot}$. We therefore have to show that

$$\Omega^{\cdot}(X) \longrightarrow \Omega^{\cdot}(X \times D)$$

is a quasi-isomorphism. The injectivity on cohomology groups as well as on the level of complexes is obvious since the projection $X \times D \to X$ has a section. It remains to establish the surjectivity on cohomology groups. Each $\omega \in \Omega^*(X \times D)$ is of the form

$$\omega = \sum_{i \geq 0} \omega_i T^i + \sum_{i \geq 0} \eta_i T^i dT \quad \text{with} \quad \omega_i \in \Omega^*(X), \eta_i \in \Omega^{*-1}(X)$$

$$\text{such that} \quad \limsup \sqrt[i]{\|\omega_i\|} \leq 1, \ \limsup \sqrt[i]{\|\eta_i\|} \leq 1 \quad .$$

Observing that

$$\limsup \sqrt[i]{\frac{\|\eta_{i-1}\|}{i}} \leq 1 \quad \text{and}$$

$$d(\sum_{i \geq 1} \frac{\eta_{i-1}}{i} T^i) = \sum_{i \geq 1} \frac{d\eta_{i-1}}{i} T^i + \sum_{i \geq 0} \eta_i T^i dT$$

we see that modulo exact forms any ω has the form

$$\omega = \sum_{i \geq 0} \omega_i T^i \quad \text{with} \quad \omega_i \in \Omega^*(X), \limsup \sqrt[i]{\|\omega_i\|} \leq 1 \quad .$$

We then have

$$d\omega = \sum_{i \geq 0} d\omega_i \cdot T^i + \sum_{i \geq 1} \omega_i \cdot i \cdot T^{i-1} dT \quad .$$

If ω is closed it follows that $\omega = \omega_0 \in \Omega^*(X)$.

Proposition 3:

We have

$$H^*_{DR}(\Omega^{(d+1)}) = \{\omega \in \Omega^*(\Omega^{(d+1)}) : d\omega = 0\} / d\Omega^{*-1}(\Omega^{(d+1)}) \quad .$$

Proof: $\Omega^{(d+1)}$ is a Stein-space according to §1 Proposition 4. But Theorem B ([29] Satz 2.4) says that Stein-spaces have trivial coherent sheaf cohomology.

The second example is the etale cohomology over $\hat{\bar{K}}$ with coefficients in a finite ring whose order is prime to the characteristic of the residue class field of K.

Definition:

A morphism $f : X \to Y$ of K-analytic varieties is called etale if, for any $x \in X$, the induced homomorphism of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat and unramified.

Over an algebraically closed base field this definition is equivalent to the one given in [18] V.3. Due to the fact that the image of an etale morphism need not to be an admissible open subvariety the notion of an etale covering of analytic varieties is not as straightforward as the corresponding notion in algebraic geometry.

Remark:

If $f : X \to Y$ is a flat morphism between K-affinoid varieties then $f(X)$ is a finite union of K-affinoid open subvarieties of Y and, in particular, is an admissible open subvariety of Y.

Proof: [32] 3.4.8.

A family of etale morphisms $f_i : X_i \to Y$ between K-affinoid varieties is called an etale covering if the $f_i(X_i)$ form an admissible covering of Y. Because of the above Remark it is clear that in this way the category of K-affinoid varieties is equipped with a Grothendieck topology. If Y is an arbitrary K-analytic variety, $Y_j \overset{\subseteq}{\to} Y$ is an admissible covering by K-affinoid open subvarieties, and $X_{ij} \to Y_j$ for each j is an etale covering of K-affinoid varieties then we call the family $X_{ij} \to Y$ a special etale covering. An arbitrary family of etale morphisms $f_i : X_i \to Y$ in the category of K-analytic varieties is called an etale covering if it can be refined to a special etale covering. This defines a Grothendieck topology on the category of K-analytic varieties (compare SGA 3 exp. IV §6.2) which restricts to the previously defined topology on the subcategory of K-affinoid varieties. We call it as well as its restriction to $\mathcal{V}$ the etale topology and denote by $\mathcal{V}_{et}$ the corresponding site.

Let $\bar{\mathcal{V}}$ be the category of smooth separated analytic varieties over $\hat{\bar{K}}$, and let A be a fixed finite ring of order m prime to the characteristic of K viewed as a constant sheaf on $\bar{\mathcal{V}}_{et}$. The extension of ground field functor $X \mapsto \bar{X}$ ([6] 9.3.6) induces a morphism of sites $e : \bar{\mathcal{V}}_{et} \to \mathcal{V}_{et}$. The complex $\mathcal{F}$ on $\mathcal{V}_{et}$ then is defined to be the total direct image

$$\mathcal{F} := Re_* A$$

under e of the sheaf A; the corresponding cohomology of a variety X in $\mathcal{V}$ is

$$H^*(X) = H^*_{et}(X, \mathcal{F}) = H^*_{et}(\bar{X}, A) \quad .$$

Among our four properties the cohomology of the point is obvious and the product structure is induced by the ring multiplication on A. The other two properties are consequences of general results of O.Gabber (unpublished). In the formula for the cohomology of projective space one has

$$H^{2s}_{et}(\mathbb{P}^n_{/\hat{\bar{K}}}, A) = A(s) \quad \text{for} \quad 0 \le s \le n$$

where $A(s)$ is the s-th Tate twist of A. For simplicity we use a once and for all fixed primitive m-th root of unity in $\hat{\bar{K}}$ in order to identify $A(s)$ and A. The homotopy invariance only holds under the assumption that the order of A is prime to the residue class field characteristic of K.

Apart from the axiomatically given properties and their consequences we will need two results from general cohomology theory. The first result concerns the behaviour of cohomology with respect to certain direct limits of varieties. Let X be a variety in $\mathcal{V}$ and let $U \subseteq X$ be an open subvariety which possesses an admissible covering by an increasing family

$$\ldots \subseteq U_n \subseteq U_{n+1} \subseteq \ldots \quad \text{for} \quad n \in \mathbb{N}$$

of open subvarieties.

Proposition 4:

There is a natural exact sequence

$$0 \to \varprojlim{}^{(1)} H^{*-1}(X, U_n) \to H^*(X, U) \to \varprojlim H^*(X, U_n) \to 0 \quad .$$

Proof: (See [27] §1-2 for the basic properties of the derived functors of $\varprojlim$.) We first show that, for an arbitrary injective sheaf I on $\mathcal{V}$, we have

$$H^0(X, U; I) = \varprojlim H^0(X, U_n; I) \;\; \text{and} \;\; \varprojlim{}^{(r)} H^0(X, U_n; I) = 0 \;\; \text{for} \;\; r \geq 1 \, .$$

The sequence of projective systems (with respect to n)

$$0 \longrightarrow H^0(X, U_n; I) \longrightarrow H^0(X, I) \longrightarrow H^0(U_n, I) \longrightarrow 0$$

is exact since the right arrows are surjective by the injectivity of I (SGA 4 V 4.7). Furthermore, the sheaf property of I implies $H^0(U, I) = \varprojlim H^0(U_n, I)$. Therefore in the projective limit we get the sequence

$$0 \longrightarrow H^0(X, U; I) \longrightarrow H^0(X, I) \longrightarrow H^0(U, I) \longrightarrow 0$$

which is exact for the same reason as before. From that follows the first part of our claim and also the second part once we observe that the projective systems $\{H^0(X, I)\}_n$ and $\{H^0(U_n, I)\}_n$ are acyclic, the first one trivially and the second one because of the surjectivity of the transition maps.

Now let $\mathcal{F} \xrightarrow{\sim} I^{\cdot}$ be an injective resolution. We consider the two hypercohomology spectral sequences

$$E_1^{r,s} = \varprojlim{}^{(s)} H^0(X, U_n; I^r) \Longrightarrow R^{r+s} \varprojlim (H^0(X, U_n; I^{\cdot})) \;\; \text{and}$$
$$E_2^{r,s} = \varprojlim{}^{(r)} H^s(H^0(X, U_n; I^{\cdot})) \Longrightarrow R^{r+s} \varprojlim (H^0(X, U_n; I^{\cdot})) \quad .$$

The first one, by the above established facts, degenerates and gives

$$H^s(H^0(X, U; I^{\cdot})) = H^s(\varprojlim H^0(X, U_n; I^{\cdot})) = R^s \varprojlim (H^0(X, U_n; I^{\cdot})) \quad .$$

Because of $\varprojlim{}^{(r)} = 0$ for $r \geq 2$ the second one then splits into the short exact sequences in our assertion.

Corollary 5:

If the A-modules in the projective system $\{H^{s-1}(X, U_n)\}_n$ are finitely generated then we have $H^s(X,U) = \varprojlim H^s(X, U_n)$.

Proof: Use [27] Cor. 7.2.

For the second result let $U_1, ..., U_m \subseteq X$ be a finite family of open subvarieties of the variety X in $\mathcal{V}$ and put $U := U_1 \cap ... \cap U_m$. We then want to construct a strongly convergent spectral sequence

$$(*) \qquad E_1^{r,s} = \bigoplus_{1 \leq i_0, ..., i_{-r} \leq m} H^s(X, U_{i_0} \cup ... \cup U_{i_{-r}}) \Longrightarrow H^{r+s}(X, U) \quad .$$

This is based on the following observation about simplicial abelian groups. Let $G_1, ..., G_m$ be a finite family of subgroups of some abelian group G. We then consider the simplicial abelian group

$$\bigoplus_i G_i \;\; \leftleftarrows \;\; \bigoplus_{i_0, i_1} G_{i_0} \cap G_{i_1} \;\; \leftleftarrows\!\!\leftarrow \;\; \bigoplus_{i_0, i_1, i_2} G_{i_0} \cap G_{i_1} \cap G_{i_2} \;\; \leftleftarrows\!\!\leftleftarrows \;\; ...$$

with the obvious face and degeneracy maps. Let $C(G_1, ..., G_m)$ denote the associated (homological) complex of abelian groups (where the differential is given by the alternating sum of the face maps).

Proposition 6:

Suppose that

$$\Big(\sum_{i \in V} G_i\Big) \cap \Big(\bigcap_{j \in W} G_j\Big) = \sum_{i \in V} \Big(G_i \cap \Big(\bigcap_{j \in W} G_j\Big)\Big)$$

holds true for all subsets $V, W \subseteq \{1, ..., m\}$. Then $C(G_1, ..., G_m)$ is an acyclic resolution of $\sum\limits_i G_i$.

Proof: We first establish that the subcomplex

$$C^+(G_1, ..., G_m) : \bigoplus_i G_i \longleftarrow \bigoplus_{i_0 < i_1} G_{i_0} \cap G_{i_1} \longleftarrow \bigoplus_{i_0 < i_1 < i_2} G_{i_0} \cap G_{i_1} \cap G_{i_2} \longleftarrow ...$$

is an acyclic resolution of $\sum\limits_i G_i$. For that we consider the short exact sequence of augmented strict-simplicial abelian groups

$$\begin{array}{ccccccc}
0\to & \sum\limits_{i\neq 1} G_i & \to & \sum\limits_i G_i & \to & G_1/G_1\cap\sum\limits_{i\neq 1}G_i & \to 0\\
& \uparrow & & \uparrow & & \uparrow & \\
0\to & \bigoplus\limits_{i\neq 1} G_i & \to & \bigoplus\limits_i G_i & \to & G_1 & \to 0\\
& \uparrow\uparrow & & \uparrow\uparrow & & \circ\uparrow\uparrow & \\
0\to & \bigoplus\limits_{1<i_0<i_1} G_{i_0}\cap G_{i_1} & \to & \bigoplus\limits_{i_0<i_1} G_{i_0}\cap G_{i_1} & \to & \bigoplus\limits_{1<i_1} G_1\cap G_{i_1} & \to 0\\
& \uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow & & \circ\uparrow\uparrow\uparrow & \\
0\to & \bigoplus\limits_{1<i_0<i_1<i_2} G_{i_0}\cap G_{i_1}\cap G_{i_2} & \to & \bigoplus\limits_{i_0<i_1<i_2} G_{i_0}\cap G_{i_1}\cap G_{i_2} & \to & \bigoplus\limits_{1<i_1<i_2} G_1\cap G_{i_1}\cap G_{i_2} & \to 0\\
& \uparrow\uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow\uparrow & & \circ\uparrow\uparrow\uparrow\uparrow & .
\end{array}$$

Since the left vertical arrows in the last column represent, as indicated, the zero maps this can be rewritten as a short exact sequence of augmented complexes

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \uparrow & & \uparrow & & \uparrow & \\
0\ \to & \sum\limits_{i\neq 1} G_i & \to & \sum\limits_i G_i & \to & G_1/G_1\cap\sum\limits_{i\neq 1} G_i & \to\ 0\\
 & \uparrow & & \uparrow & & \uparrow & \\
0\ \to & C^+(G_2,\ldots,G_m) & \to & C^+(G_1,\ldots,G_m) & \to & \left\{\begin{array}{c} G_1\\ \uparrow\\ C^+(G_1\cap G_2,\ldots,G_1\cap G_m)\end{array}\right\} & \to\ 0.
\end{array}$$

We now argue by induction with respect to m. There is nothing to prove for $m=1$. Obviously, with $G_1,\ldots,G_m$, also the families $G_2,\ldots,G_m$ and $G_1\cap G_2,\ldots,G_1\cap G_m$ fulfill the assumption of our assertion. Therefore, by induction, we can assume that $C^+(G_2,\ldots,G_m)$, resp. $C^+(G_1\cap G_2,\ldots,G_1\cap G_m)$, is an acyclic resolution of $\sum\limits_{i\neq 1} G_i$, resp. $\sum\limits_{i\neq 1} G_1\cap G_i = G_1\cap\sum\limits_{i\neq 1} G_i$. The above short exact sequence then shows that also $C^+(G_1,\ldots,G_m)$ is an acyclic resolution of $\sum\limits_i G_i$.

It remains to prove that the inclusion $C^+(G_1,\ldots,G_m)\subseteq C(G_1,\ldots,G_m)$ is a homotopy equivalence. In case $G=G_1=\ldots=G_m=\mathbb{Z}$ we view $\mathbb{Z}N.=C(\mathbb{Z},\ldots,\mathbb{Z})$ as the complex associated with the free abelian group on the simplicial set

$$N \ \leftleftarrows\ N\times N \ \substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\ N\times N\times N \ \substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\ \ldots\ ,\quad N:=\{1,\ldots,m\}\quad .$$

It is well-known (compare, for example, [6] p. 322/323) that the homomorphisms

$$\begin{array}{ccc} \mathbb{Z}N_q & \longrightarrow & \mathbb{Z}N_q \\ (i_0,...,i_q) & \longmapsto & \begin{cases} (sgn\ \pi)(i_{\pi(0)},...,i_{\pi(q)}) & \text{if there is a permutation} \\ & \pi \text{ such that } i_{\pi(0)}<...<i_{\pi(q)}\ , \\ 0 & \text{otherwise} \end{cases} \end{array}$$

form an endomorphism of the complex $\mathbb{Z}N.$ which is homotopic to the identity; furthermore the homotopy $\mathbb{Z}N. \to \mathbb{Z}N._{+1}$ can be chosen in such a way that

$$(i_0,...,i_q) \longmapsto \sum_{j_0,...,j_q \in \{i_0,...,i_q\}} c^{i_0...i_q}_{j_0...j_q} \cdot (i_0,j_0,...,j_q)$$

with appropriate integers $c^{...}_{...} \in \mathbb{Z}$. Coming back to our general situation we define an endomorphism ϕ of $C(G_1,...,G_m)$ by

$$g \in G_{i_0} \cap ... \cap G_{i_q} \longmapsto \begin{cases} (sgn\ \pi)\cdot g \in G_{i_{\pi(0)}} \cap ... \cap G_{i_{\pi(q)}} & \text{if there is a permutation} \\ & \pi \text{ such that} \\ & i_{\pi(0)}<...<i_{\pi(q)}\ , \\ 0 & \text{otherwise} \end{cases}$$

and a homotopy h on $C(G_1,...,G_m)$ by

$$g \in G_{i_0} \cap ... \cap G_{i_q} \longmapsto \bigoplus_{j_0,...,j_q \in \{i_0,...,i_q\}} c^{i_0...i_q}_{j_0...j_q} \cdot g \in G_{i_0} \cap G_{j_0} \cap ... \cap G_{j_q} \quad .$$

It is clear that h is a homotopy between ϕ and id; furthermore we have $\phi | C^+(G_1,...,G_m) = id$ and $im(\phi) = C^+(G_1,...,G_m)$. q.e.d.

We are going to use this general result in the following particular situation.

Lemma 7:

For any injective sheaf I on $\mathcal{V}$ we have (where all groups are considered as subgroups of $I(X)$):
i. $\sum\limits_i H^0(X,U_i;I) = H^0(X,U;I)$;
ii. $(\sum\limits_{i\in V} H^0(X,U_i;I)) \cap (\bigcap\limits_{j\in W} H^0(X,U_j;I)) = \sum\limits_{i\in V}(H^0(X,U_i;I) \cap (\bigcap\limits_{j\in W} H^0(X,U_j;I)))$
for any two subsets $V,W \subseteq \{1,...,m\}$.

Proof: i. Assume first that $m=2$ and let $s \in I(X)$ be a section with $s|U_1 \cap U_2 = 0$. By the sheaf property we find a section $s_1' \in I(U_1 \cup U_2)$ such that $s_1'|U_1 = 0$ and $s_1'|U_2 = s|U_2$, and by the injectivity of I the section s_1' extends to a section $s_1 \in I(X)$. If we put $s_2 := s - s_1$ then we obviously have $s_2|U_2 = 0$. The general case follows by induction.

ii. Because of $\bigcap_{j\in W} H^0(X,U_j;I) = H^0(X, \bigcup_{j\in W} U_j;I)$ we can assume that, say, $W = \{1\}$. But, using i., we compute

$$\begin{aligned}
&(\sum_{i\in V} H^0(X,U_i;I)) \cap H^0(X,U_1;I) = H^0(X, \bigcap_{i\in V} U_i;I) \cap H^0(X,U_i;I) \\
&= H^0(X,(\bigcap_{i\in V} U_i) \cup U_1;I) = H^0(X, \bigcap_{i\in V}(U_i \cup U_1);I) \\
&= \sum_{i\in V} H^0(X,U_i \cup U_1;I) = \sum_{i\in V}(H^0(X,U_i;I) \cap H^0(X,U_1;I)) \quad .
\end{aligned}$$

q.e.d.

Now, let $\mathcal{F} \xrightarrow{\sim} I^\cdot$ be an injective resolution. We then have the augmented double complex

$$\begin{array}{c} H^0(X,U;I^\cdot) \\ \uparrow \\ C(H^0(X,U_1;I^\cdot),...,H^0(X,U_m;I^\cdot)) \end{array}$$

in which the columns are acyclic according to Proposition 6 and Lemma 7. Consequently, the homology of the total complex of this double complex is equal to the homology of the augmenting complex which is $H^*(X,U)$. On the other hand, the homology of the r-th row in the double complex, for trivial reasons, is equal to $\bigoplus_{i_0,...,i_r} H^*(X,U_{i_0} \cup ... \cup U_{i_r})$. Therefore $(*)$ simply is the second spectral sequence of this double complex. It is strongly convergent since, as we have seen in the proof of Proposition 6, the above double complex is homotopy equivalent to the subcomplex $C^+(H^0(X,U_1;I^\cdot),...,H^0(X,U_m;I^\cdot))$ whose r-th row is zero for $r \geq m$.

§3 The cohomology of $\Omega^{(d+1)}$ - Connection to the Tits building

In this Paragraph we want to compute the abstract cohomology of $\Omega^{(d+1)}$ in terms of the cohomology of certain p-adically topologized simplicial complexes which naturally arise in the theory of the Tits building for GL_{d+1}. Because of the relative cohomology sequence

$$... \to H^s(\mathbb{P}^d) \to H^s(\Omega^{(d+1)}) \to H^{s+1}(\mathbb{P}^d, \Omega^{(d+1)}) \to H^{s+1}(\mathbb{P}^d) \to ...$$

and our axiomatic knowledge of the cohomology of $\mathbb{P}^d$ we equivalently have to determine the relative cohomology $H^*(\mathbb{P}^d, \Omega^{(d+1)})$. In §1 we have seen that the open subvarieties

$$\Omega_n = \mathbb{P}^d \setminus \bigcup_{H\in\mathcal{H}_n} H(|\pi|^n)$$

form an increasing admissible covering of $\Omega^{(d+1)}$. Later on we will see that we can apply §2 Corollary 5 and calculate $H^*(\mathbb{P}^d, \Omega^{(d+1)})$ as the projective limit of the groups $H^*(\mathbb{P}^d, \Omega_n)$. But Ω_n can be written as an intersection

$$\Omega_n = \bigcap_{H \in \mathcal{H}_n} U(H;n)$$

of finitely many open subvarieties in $\mathbb{P}^d$ of the form

$$U(H;n) := \mathbb{P}^d \backslash H(|\pi|^n) \quad .$$

Therefore the spectral sequence $(*)$ which we have constructed in §2 is at our disposal:

$$E_1^{-r,s} = \bigoplus_{\substack{(H_0,\ldots,H_r) \\ \in \mathcal{H}_n^{r+1}}} H^s(\mathbb{P}^d, U(H_0;n) \cup \ldots \cup U(H_r;n)) \Longrightarrow H^{s-r}(\mathbb{P}^d, \Omega_n) \quad .$$

We see that the central problem is to understand this spectral sequence. For that it is useful to introduce the notation

$$rk(H_0, \ldots, H_r) := \operatorname{rank} \bar{M}$$

where

$$\bar{M} := \sum_{i=0}^{r} (o/\pi^n o)\ell_{H_i} \subseteq L_0^*/\pi^n L_0^* \quad .$$

We observe that $rk(H_0, \ldots, H_r) \geq 1$ since the ℓ_{H_i} are unimodular.

Lemma 1:

$$H^s(\mathbb{P}^d, U(H_0;n) \cup \ldots \cup U(H_r;n)) = \begin{cases} A & \textit{if } s \textit{ is even with } rk(H_0, \ldots, H_r) \leq \frac{s}{2} \leq d, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: According to §1 Lemma 5 and Proposition 6 we have, after a suitable K-linear automorphism of $\mathbb{P}^d$, the diagram

$$\begin{tikzcd} \mathbb{P}^d & U(H_0;n) \cup \ldots \cup U(H_r;n) \arrow[l, hook', "\supseteq"'] \arrow[d, shift left, "pr"] & = & \mathbb{P}^d \backslash \bigcap_{i=0}^{r} H_i(|\pi|^n) \\ & \mathbb{P}^m \arrow[u, dashed, shift left] & & \end{tikzcd}$$

with $m := rk(H_0, \ldots, H_r) - 1$ where pr is the projection $[z_0 : \ldots : z_d] \mapsto [z_0 : \ldots : z_m]$ and the dotted arrow represents the section $[z_0 : \ldots : z_m] \mapsto [z_0 : \ldots : z_m : 0 : \ldots : 0]$; we furthermore know that pr is a locally trivial fibration with polydiscs as fibers. By §2 Lemma 1 pr induces an isomorphism in cohomology. If then follows from §2 Lemma 2 that the restriction map

$$H^s(\mathbb{P}^d) \longrightarrow H^s(U(H_0;n) \cup \ldots \cup U(H_r;n))$$

is an isomorphism for $s < 2rk(H_0, \ldots, H_r)$ and is surjective and the zero map otherwise. Using the relative cohomology sequence this is easily translated into our assertion. q.e.d.

Inserting this into our spectral sequence we get

$$E_1^{-r,s} = \begin{cases} \bigoplus\limits_{\substack{(H_0,\ldots,H_r)\in\mathcal{H}_n^{r+1} \\ rk(H_0,\ldots,H_r)\le\frac{s}{2}}} A & \text{if } s \text{ is even and } 2\le s\le 2d, \\ 0 & \text{otherwise} \quad . \end{cases}$$

In order to keep track of the d_1-differential in this spectral sequence we introduce the simplicial sets $Y.^{(n,k)}$, for $1 \le k \le d$, given by

$$Y_r^{(n,k)} := \text{ set of all } (H_0,...,H_r) \in \mathcal{H}_n^{r+1} \text{ such that } rk(H_0,...,H_r) \le k$$

with face, resp. degeneracy, maps given by omitting, resp. doubling, one hyperplane in a tuple. If $C(Y.^{(n,k)}, A)$ denotes the chain complex on $Y.^{(n,k)}$ with coefficients in the abelian group A (viewed as a cohomological complex in negative degrees) we have

$$E_1^{-r,s} = \begin{cases} C(Y_r^{(n,\frac{s}{2})}, A) & \text{if } s \text{ is even and } 2\le s\le 2d \ , \\ 0 & \text{otherwise} \ ; \end{cases}$$

furthermore, the d_1-differential is induced from the chain differentials in the complexes $C(Y.^{(n,k)}, A)$. The corresponding E_2-spectral sequence therefore reads

$$E_2^{-r,s} = \left\{ \begin{array}{ll} H_r(Y.^{(n,\frac{s}{2})}, A) & \text{if } s \text{ is even and} \\ & 2\le s\le 2d \\ 0 & \text{otherwise} \end{array} \right\} \Longrightarrow H^{s-r}(\mathbb{P}^d, \Omega_n) \quad .$$

Remarks:

1) Because of $Y_r^{(n,k)} = \mathcal{H}_n^{r+1}$ *for* $r < k$ *we have*

$$H_r(Y.^{(n,k)}, \mathbb{Z}) = 0 \quad \textit{for} \quad 0 < r < k-1 \quad .$$

2) Since the $C(Y.^{(n,k)}, A)$ *are complexes of finitely generated modules over the artinian ring* A *the spectral sequence shows that the relative cohomology groups* $H^s(\mathbb{P}^d, \Omega_n)$ *are finitely generated* A*-modules.*

Instead of trying to compute the E_2-terms further we will pass at this point to the projective limit with respect to n. It is easy to check that there is a homomorphism of spectral sequences

$$\begin{array}{ccc} H_r(Y.^{(n+1,\frac{s}{2})}, A) \text{ (resp. } 0) & \Longrightarrow & H^{s-r}(\mathbb{P}^d, \Omega_{n+1}) \\ \downarrow & & \downarrow \\ H_r(Y.^{(n,\frac{s}{2})}, A) \text{ (resp. } 0) & \Longrightarrow & H^{s-r}(\mathbb{P}^d, \Omega_n) \end{array}$$

where the left arrow is induced from the obvious map of simplicial sets $Y.^{(n+1,k)} \rightarrow Y.^{(n,k)}$ and the right arrow is induced from the inclusion $\Omega_n \hookrightarrow \Omega_{n+1}$. According to the above Remark we therefore have a projective system of spectral sequences of finitely generated modules over the artinian ring A so that passing to the projective limit still gives a spectral sequence ([27] Cor. 7.2). The same Remark together with §2 Corollary 5 implies that in the abutment we get

$$\varprojlim H^*(\mathbb{P}^d, \Omega_n) = H^*(\mathbb{P}^d, \Omega^{(d+1)}) \quad .$$

The limit spectral sequence consequently reads

$$E_2^{-r,s} = \varprojlim H_r(Y.^{(n,\frac{s}{2})}, A) \text{ (resp. 0)} \Longrightarrow H^{s-r}(\mathbb{P}^d, \Omega^{(d+1)}) \quad .$$

In the next step we will express the E_2-terms above in terms of the simplicial profinite sets

$$Y.^{(k)} := \varprojlim Y.^{(n,k)} \quad .$$

For any tuple $(H_0, ..., H_r) \in \mathcal{H}^{r+1}$ we have

$$\operatorname{rank}(\sum_{i=0}^{r} (o/\pi^n o)\ell_{H_i}) \leq k \text{ for all } n \in \mathbb{N}$$

if and only if

$$\dim_K(\sum_{i=0}^{r} K\ell_{H_i}) \leq k \quad .$$

We therefore can interprete $Y.^{(k)}$ as the simplicial profinite set given by

$$Y_r^{(k)} = \text{set of all } (H_0, ..., H_r) \in \mathcal{H}^{r+1} \text{ such that}$$
$$\dim_K(\sum_{i=0}^{r} K\ell_{H_i}) \leq k$$

with face, resp. degeneracy, maps given by omitting, resp. doubling, one hyperplane in a tuple; the topology on $Y_r^{(k)}$ is induced from the obvious topology on $\mathcal{H} = \mathbb{P}((K^{d+1})^*) = \mathbb{P}^d(K)$ so that $Y_r^{(k)}$ is a closed subset of $\mathcal{H}^{r+1}$.

Let $|Y.^{(k)}|$ be the topological realization of $Y.^{(k)}$. We recall that $|Y.^{(k)}|$ is defined to be the quotient of $\bigcup_{r\geq 0} Y_r^{(k)} \times \Delta_r$ with respect to the equivalence relation given by the face and degeneracy maps; here the topology on $Y_r^{(k)} \times \Delta_r$ is the product of the profinite topology on $Y_r^{(k)}$ and the usual topology on the topological standard r-simplex Δ_r (compare [45] §1).

Lemma 2:

There are natural exact sequences

$$\begin{array}{llll} 0 & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(H^{*+1}(|Y.^{(k)}|,\mathbb{Z}),A) & \longrightarrow \quad \varprojlim H_*(Y.^{(n,k)},A) \\ & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(H^*(|Y.^{(k)}|,\mathbb{Z}),A) & \longrightarrow \quad 0. \end{array}$$

Proof: (Cohomology of a topological space always is (constant) sheaf cohomology!) By the universal coefficient theorem ([47] Th. 5.5.12) we have natural exact sequences

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H^{*+1}(Y.^{(n,k)},\mathbb{Z}),A) \to H_*(Y.^{(n,k)},A) \to \mathrm{Hom}_{\mathbb{Z}}(H^*(Y.^{(n,k)},\mathbb{Z}),A) \to 0.$$

Since all groups involved are finitely generated modules over the artinian ring A passing to the projective limit gives the exact sequences

$$\begin{array}{llll} 0 & \longrightarrow & \mathrm{Ext}^1(\varinjlim H^{*+1}(Y.^{(n,k)},\mathbb{Z}),A) & \longrightarrow \quad \varprojlim H_*(Y.^{(n,k)},A) \\ & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}(\varinjlim H^*(Y.^{(n,k)},\mathbb{Z}),A) & \longrightarrow \quad 0 \end{array}$$

([27] Th. 4.2 and Cor. 7.2). It remains to show that we have

$$H^*(|Y.^{(k)}|,\mathbb{Z}) = \varinjlim H^*(Y.^{(n,k)},\mathbb{Z}) \quad .$$

Using the standard spectral sequence

$$H^s(Y_r^{(k)},\mathbb{Z}) \Longrightarrow H^{s+r}(|Y.^{(k)}|,\mathbb{Z})$$

([45] Prop. 5.1 together with [47] Th. 6.6.5) and analogous spectral sequences for the $Y.^{(n,k)}$ this follows from the continuity property of cohomology ([47] p. 319) which implies that

$$H^*(Y_r^{(k)},\mathbb{Z}) = \varinjlim H^*(Y_r^{(n,k)},\mathbb{Z})$$

holds true for each $r \geq 0$.

Remark:

Let $S.$ be an arbitrary simplicial profinite set. If $C^\infty(S_r,\mathbb{Z})$ denotes the group of $\mathbb{Z}$-valued locally constant functions on S_r then we have the (cohomological) complex

$$\ldots \longrightarrow C^\infty(S_r,\mathbb{Z}) \longrightarrow C^\infty(S_{r+1},\mathbb{Z}) \longrightarrow \ldots$$

where the differential is given as the alternating sum of the maps induced from the face maps in $S.$. Since

$$H^s(S_r,\mathbb{Z}) = \begin{cases} C^\infty(S_r,\mathbb{Z}) & \textit{if} \quad s=0 \\ 0 & \textit{if} \quad s>0 \end{cases} \quad ,$$

by the continuity property of cohomology the same spectral sequence as in the above proof shows that the complex $C^\infty(S., \mathbb{Z})$ computes the cohomology of $|S.|$, i.e., that

$$H^*(|S.|, \mathbb{Z}) = H^*(C^\infty(S., \mathbb{Z})) \quad .$$

We now define simplicial profinite sets $\mathcal{T}.^{(k)}$, for $1 \leq k \leq d$, by

$$\begin{aligned} \mathcal{T}_r^{(k)} := & \text{ set of all flags } W_0 \subseteq W_1 \subseteq \ldots \subseteq W_r \text{ of } K-\text{subspaces} \\ & \text{in } (K^{d+1})^* \text{ such that } 1 \leq \dim_K W_i \leq k \end{aligned}$$

with face, resp. degeneracy, maps given by omitting, resp. doubling, one subspace in a flag; furthermore, the topology on $\mathcal{T}_r^{(k)}$ is given in the following way: The group $GL_{d+1}(K)$ in a natural way acts on $\mathcal{T}_r^{(k)}$ and the stabilizer of a flag $\tau \in \mathcal{T}_r^{(k)}$ is a parabolic subgroup $P_\tau(K)$. We take the finest topology on $\mathcal{T}_r^{(k)}$ whose restriction on each orbit

$$GL_{d+1}(K) \cdot \tau = GL_{d+1}(K)/P_\tau(K)$$

is the obvious topology on the right hand side (induced from the valuation on K). (Remember that $\mathcal{T}.^{(d)}$ considered only as a simplicial set is the Tits building of GL_{d+1} over K. This has to be well distinguished from the Bruhat-Tits building $\mathcal{BT}$ which we introduced earlier. In case $d = 1$, for example, $\mathcal{BT}$ is the familiar tree whereas the Tits building is just a set of points which by the way can be viewed as the set of ends of the tree. An excellent introductory text to the theory of buildings is [8].) We will show that $|Y.^{(k)}|$ and $|\mathcal{T}.^{(k)}|$ have naturally isomorphic cohomology.

Lemma 3:

For any subspace $W \subseteq (K^{d+1})^*$ such that $1 \leq \dim_K W \leq k$ let $Y.^{(k)}(W) \subseteq Y.^{(k)}$, resp. $\mathcal{T}.^{(k)}(W) \subseteq \mathcal{T}.^{(k)}$, be the simplicial profinite subset defined by

$$\begin{aligned} Y_r^{(k)}(W) &:= \{(H_0, \ldots, H_r) \in Y_r^{(k)} : \sum_{i=0}^{r} K\ell_{H_i} \subseteq W\} \;, \quad \textit{resp.} \\ \mathcal{T}_r^{(k)}(W) &:= \{(W_0 \subseteq \ldots \subseteq W_r) \in \mathcal{T}_r^{(k)} : W \subseteq W_0\} \quad . \end{aligned}$$

Then the augmented complexes

$$\mathbb{Z} \longrightarrow C^\infty(Y.^{(k)}(W), \mathbb{Z}) \quad \textit{and} \quad \mathbb{Z} \longrightarrow C^\infty(\mathcal{T}.^{(k)}(W), \mathbb{Z})$$

are acyclic.

Proof: This is a standard fact in the context of simplicial sets. We only have to observe in addition that the maps

$$\begin{array}{ccc} Y_{r-1}^{(k)}(W) & \longrightarrow & Y_r^{(k)}(W) \\ (H_0, \ldots, H_{r-1}) & \longmapsto & (H, H_0, \ldots, H_{r-1}) \end{array}$$

where $H \in \mathcal{H}$ is some fixed hyperplane such that $\ell_H \in W$ and

$$\begin{array}{ccc} \mathcal{T}^{(k)}_{r-1}(W) & \longrightarrow & \mathcal{T}^{(k)}_r(W) \\ (W_0 \subseteq ... \subseteq W_{r-1}) & \longmapsto & (W \subseteq W_0 \subseteq ... \subseteq W_{r-1}) \end{array}$$

are continuous so that they also induce contracting homotopies in our profinite setting.

In order to relate $Y.^{(k)}$ and $\mathcal{T}.^{(k)}$ we introduce the bisimplicial profinite set $Z..^{(k)}$ defined by

$$Z^{(k)}_{rs} := \{(W_0 \subseteq ... \subseteq W_r; H_0, ..., H_s) \in \mathcal{T}^{(k)}_r \times Y^{(k)}_s : \sum_{i=0}^{s} K\ell_{H_i} \subseteq W_0\}$$

together with the obvious face and degeneracy maps. It will follow from Lemma 3 and a simplified version of the base change formalism (compare [20] II.4.17) that the projection maps

$$\begin{array}{ccc} & & Y.^{(k)} \\ & & \uparrow \\ \mathcal{T}.^{(k)} & \longleftarrow & Z..^{(k)} \end{array}$$

induce cohomology isomorphisms between the corresponding topological realizations.

Remark:

Let $T \subseteq S$ be a closed subset in a profinite set S. We have:
i. The restriction map $C^\infty(S, \mathbb{Z}) \to C^\infty(T, \mathbb{Z})$ is surjective;
ii. if $f : S' \to S$ is a continuous map between profinite sets and if U runs through the compact open neighbourhoods of T in S then

$$C^\infty(f^{-1}(T), \mathbb{Z}) = \varinjlim C^\infty(f^{-1}(U), \mathbb{Z}) \quad .$$

Proof: i. Since the compact open subsets in S form a base of the topology it is easy to see that for any compact open subset V in T there is a compact open subset U in S such that $U \cap T = V$. ii. From i. we see that

$$C^\infty(f^{-1}(T), \mathbb{Z}) = \varinjlim C^\infty(U', \mathbb{Z})$$

where U' runs through the compact open neighbourhoods of $f^{-1}(T)$ in S'. But any U' contains a $f^{-1}(U)$ (compare [20] p.202).

Lemma 4:

Let $f. : S. \to T$ be a map from a simplicial profinite set $S.$ into a (constant simplicial) profinite set T. If the augmented complexes

$$\mathbb{Z} \xrightarrow{f_0^*} C^\infty(f.^{-1}(t), \mathbb{Z}) \quad \textit{for} \quad t \in T$$

are acyclic then the augmented complex

$$C^\infty(T, \mathbb{Z}) \xrightarrow{f_0^*} C^\infty(S., \mathbb{Z})$$

is acyclic, too.

Proof: We consider the complex of sheaves on T

$$\mathbb{Z} \longrightarrow f_{0*}\mathbb{Z} \longrightarrow \dots \longrightarrow f_{r*}\mathbb{Z} \longrightarrow \dots$$

From the above Remark together with [20] II.3.3 Cor. 1 we conclude that the sheaves $f_{r*}\mathbb{Z}$ are soft. Part ii. of the above Remark also shows that

$$\mathbb{Z} \longrightarrow C^\infty(f.^{-1}(t), \mathbb{Z})$$

is the associated complex of stalks in t. Our complex therefore is an acyclic complex of soft sheaves. According to [20] II Th.3.5.4 the associated complex of global sections

$$C^\infty(T, \mathbb{Z}) \longrightarrow C^\infty(S., \mathbb{Z})$$

then is acyclic, too.

Proposition 5:

For any $1 \le k \le d$ we have a natural isomorphism

$$H^*(|\mathcal{T}.^{(k)}|, \mathbb{Z}) \cong H^*(|Y.^{(k)}|, \mathbb{Z}) \quad .$$

Proof: Consider the biaugmented double complex

$$\begin{array}{ccc} & & C^\infty(Y.^{(k)}, \mathbb{Z}) \\ & & \downarrow \\ C^\infty(\mathcal{T}.^{(k)}, \mathbb{Z}) & \longrightarrow & C^\infty(Z..^{(k)}, \mathbb{Z}) \end{array} \quad .$$

Because of Lemma 3 we can apply Lemma 4 to each row and each column of this double complex and we get that all rows and columns are acyclic. Therefore both augmentation maps are quasi-isomorphisms.

Remark:

Define a third simplicial profinite set $X.^{(k)}$ *by*

$$X_r^{(k)} := \text{set of all flags } M_0 \subseteq M_1 \subseteq ... \subseteq M_r \text{ of o-submodules in } L_0^* \\ \text{such that } M_0 \text{ contains an unimodular vector and } \operatorname{rank} M_r \leq k.$$

The cohomology of $|X.^{(k)}|$ *also is naturally isomorphic to the cohomology of* $|Y.^{(k)}|$.

Proposition 6:

i. $\varprojlim H_r(Y.^{(n,k)}, A) = \operatorname{Hom}_{\mathbb{Z}}(H^r(|\mathcal{T}.^{(k)}|, \mathbb{Z}), A)$;

ii. $H^r(|\mathcal{T}.^{(k)}|, \mathbb{Z}) = 0$ *for* $r \neq 0, k-1$;

iii. $H^0(|\mathcal{T}.^{(k)}|, \mathbb{Z}) = \begin{cases} C^\infty(\mathbb{P}((K^{d+1})^*), \mathbb{Z}) & \text{if} \quad k = 1 \quad , \\ \mathbb{Z} & \text{if} \quad k > 1 \quad . \end{cases}$

Proof: Let $N\mathcal{T}_r^{(k)}$ denote the open and closed subset of "nondegenerate" flags $W_0 \underset{\neq}{\subset} ... \underset{\neq}{\subset} W_r$ in $\mathcal{T}_r^{(k)}$. By the cosimplicial version of the normalization theorem (compare [31] VIII. 6) the inclusion of complexes

$$C^\infty(N\mathcal{T}.^{(k)}, \mathbb{Z}) \subseteq C^\infty(\mathcal{T}.^{(k)}, \mathbb{Z})$$

is a homotopy equivalence. But we have $N\mathcal{T}_r^{(k)} = \phi$ for $r \geq k$ which implies $H^r(C^\infty(\mathcal{T}.^{(k)}, \mathbb{Z})) = 0$ and consequently $H^r(|\mathcal{T}.^{(k)}|, \mathbb{Z}) = 0$ for $r \geq k$. On the other hand we already know from the Remark after Lemma 1 that $H_r(Y.^{(n,k)}, \mathbb{Z}) = 0$ for $0 < r < k-1$ and $n \in \mathbb{N}$. By the universal coefficient theorem ([47] Th. 5.5.3) we then also have $H^r(Y.^{(n,k)}, \mathbb{Z}) = 0$ for $0 < r < k-1$ and $n \in \mathbb{N}$. Using Proposition 5 and the argument in the proof of Lemma 2 we see that $H^r(|\mathcal{T}.^{(k)}|, \mathbb{Z}) = H^r(|Y.^{(k)}|, \mathbb{Z}) = \varinjlim H^r(Y.^{(n,k)}, \mathbb{Z}) = 0$ for $0 < r < k-1$. This establishes our second assertion. In order to prove the first assertion we consider the exact sequence

$$\begin{array}{cccc} 0 & \longrightarrow & \operatorname{Ext}^1_{\mathbb{Z}}(H^{r+1}(|\mathcal{T}.^{(k)}|, \mathbb{Z}), A) & \longrightarrow & \varprojlim H_r(Y.^{(n,k)}, A) \\ & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(H^r(|\mathcal{T}.^{(k)}|, \mathbb{Z}), A) & \longrightarrow & 0 \end{array}$$

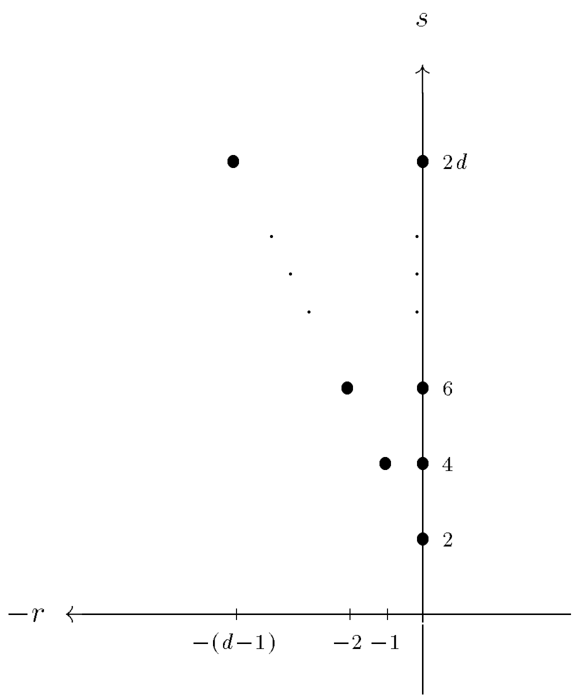
which arises from Lemma 2 and Proposition 5. By what we have just seen, for $r \neq k-2$, the first term vanishes and, for $r = k-2$, even the middle term vanishes. Finally, our third assertion is clear since $|\mathcal{T}.^{(k)}|$ obviously is connected for $k > 1$ and since, on the other hand, we have $N\mathcal{T}_0^{(1)} = \mathcal{T}_0^{(1)} = \mathbb{P}((K^{d+1})^*)$. q.e.d.

In the next Paragraph we will compute the groups $H^{k-1}(|\mathcal{T}.^{(k)}|, \mathbb{Z})$. Here we go back to our spectral sequence

$$\varprojlim H_r(Y.^{(n, \frac{s}{2})}, A) \quad (\text{resp. } 0) \Longrightarrow H^{s-r}(\mathbb{P}^d, \Omega^{(d+1)}) \quad .$$

Because of Proposition 6 we can rewrite it in the form

$$E_2^{-r,s} = \left\{ \begin{array}{ll} \mathrm{Hom}_{\mathbb{Z}}(H^r(|\mathcal{T}.^{(\frac{s}{2})}|, \mathbb{Z}), A) & \text{if } s \text{ is even,} \\ & 2 \leq s \leq 2d, \\ & \text{and } r = 0 \text{ or } \frac{s}{2} - 1 \\ 0 & \text{otherwise} \end{array} \right\} \Longrightarrow H^{s-r}(\mathbb{P}^d, \Omega^{(d+1)}).$$



Lemma 7:

The composed map

$$E_2^{0,s} \longrightarrow H^s(\mathbb{P}^d, \Omega^{(d+1)}) \longrightarrow H^s(\mathbb{P}^d) \ ,$$

where the first arrow is the edge homomorphism in the above spectral sequence, is an isomorphism for $s > 2$ and is surjective for $s = 2$.

Proof: We can assume that s is even with $2 \leq s \leq 2d$ since otherwise both terms are zero. The edge homomorphism

$$E_1^{0,s} = \bigoplus_{H \in \mathcal{H}_n} H^s(\mathbb{P}^d, U(H; n)) \longrightarrow H^s(\mathbb{P}^d, \Omega_n)$$

in our original spectral sequence is, of course, the natural homomorphism induced by the inclusions $\Omega_n \subseteq U(H; n)$. In the proof of Lemma 1 we have seen that the natural map

$$\begin{array}{ccc} \bigoplus\limits_{H\in\mathcal{H}_n} H^s(\mathbb{P}^d, U(H;n)) & \longrightarrow & H^s(\mathbb{P}^d) \\ \| & & \| \\ \bigoplus\limits_{H\in\mathcal{H}_n} A & \xrightarrow{\sum} & A \end{array}$$

can be identified with the sum homomorphism. If π denotes the set of connected components of $|Y.^{(n,\frac{s}{2})}|$ we therefore get the following commutative diagram

$$\begin{array}{ccccccc} E_1^{0,s} & \longrightarrow & E_2^{0,s} & \longrightarrow & H^s(\mathbb{P}^d, \Omega_n) & \longrightarrow & H^s(\mathbb{P}^d) \\ & & \| & & & & \\ \| & & H_0(Y.^{(n,\frac{s}{2})}, A) & & & & \| \\ & & \| & & & & \\ \bigoplus\limits_{H\in\mathcal{H}_n} A & \xrightarrow[\bigoplus\limits_{C}\sum\limits_{H\in C}]{} & \bigoplus\limits_{C\in\pi} A & & \xrightarrow[\sum]{\hspace{6em}} & & A \end{array} .$$

It remains to observe that $|Y.^{(n,\frac{s}{2})}|$ is connected for $s > 2$. q.e.d.

This Lemma implies that $E_2^{0,s} = E_\infty^{0,s}$ is canonically a direct summand of $H^s(\mathbb{P}^d, \Omega^{(d+1)})$. Since, on the other hand, there can be no differentials between terms on the line $s = 2 - 2 \cdot (-r)$ we see that our spectral sequence degenerates and gives canonical isomorphisms

$$H^s(\mathbb{P}^d, \Omega^{(d+1)}) = \begin{cases} E_2^{-(s-2),2(s-1)} \oplus E_2^{0,s} & \text{if } s > 2 \text{ is even,} \\ E_2^{-(s-2),2(s-1)} & \text{if } s > 1 \text{ is odd,} \\ E_2^{0,2} & \text{if } s = 2, \\ 0 & \text{if } s = 0, 1 \ . \end{cases}$$

In order to pass to the cohomology of $\Omega^{(d+1)}$ we use the relative cohomology sequence which, in the light of our axiom about the cohomology of $\mathbb{P}^d$, breaks up into exact sequences

$$\begin{array}{lllll} 0 \ \to & H^{2t-1}(\Omega^{(d+1)}) & \to \ H^{2t}(\mathbb{P}^d, \Omega^{(d+1)}) & \to & H^{2t}(\mathbb{P}^d) = A \text{ (resp. 0 for } t > d) \\ \quad \to & H^{2t}(\Omega^{(d+1)}) & \to \ H^{2t+1}(\mathbb{P}^d, \Omega^{(d+1)}) & \to & 0 \ . \end{array}$$

Inserting the above table into these sequences and applying Lemma 7 once more we derive

$$H^s(\Omega^{(d+1)}) = \begin{cases} E_2^{-(s-1),2s} & \text{if } s \geq 2 \ , \\ \ker(E_2^{0,2} \to H^2(\mathbb{P}^d)) & \text{if } s = 1 \ , \\ H^0(\mathbb{P}^d) & \text{if } s = 0 \ . \end{cases}$$

We finally have established our first main result.

Theorem 1:

$$H^s(\Omega^{(d+1)}) = \begin{cases} A & \textit{if } s = 0 \ , \\ \ker(\mathrm{Hom}_{\mathbb{Z}}(C^\infty(\mathbb{P}((K^{d+1})^*), \mathbb{Z}), A) \to A) & \textit{if } s = 1 \ , \\ \mathrm{Hom}_{\mathbb{Z}}(H^{s-1}(|\mathcal{T}.^{(s)}|, \mathbb{Z}), A) & \textit{if } 2 \leq s \leq d \ , \\ 0 & \textit{if } s > d \ . \end{cases}$$

Remarks:

1) As we have seen in the proof of Lemma 7 the map $\mathrm{Hom}_{\mathbb{Z}}(C^\infty(\mathbb{P}((K^{d+1})^*), \mathbb{Z}), A) \to A$ *in the above statement is given by evaluation on the constant function on* $\mathbb{P}((K^{d+1})^*)$ *with value* 1.

2) The isomorphism in Theorem 1 is equivariant with respect to the natural actions of $GL_{d+1}(K)$ *on both sides. This is not entirely obvious since in its construction we have used unimodular coordinates in order to define the* Ω_n. *But if we fix a* $g \in GL_{d+1}(K)$ *and choose an* $a \geq 0$ *such that* $\pi^a L_0 \subseteq gL_0 \subseteq \pi^{-a} L_0$ *then it is easy to see that* g *induces a homomorphism from the spectral sequence for* Ω_{n+2a} *into the corresponding spectral sequence for* Ω_n. *Therefore* $GL_{d+1}(K)$ *acts on the limit spectral sequence (after Prop. 6).*

3) If the cohomology theory is etale cohomology then the isomorphism in Theorem 1 is $Aut(\bar{K}|K)$*-equivariant if the left hand side is replaced by the Tate twist* $H^s_{et}(\overline{\Omega^{(d+1)}}, A(s))$.

§4 The cohomology of $\Omega^{(d+1)}$ - Distributions and harmonic cochains

In this Paragraph we will discuss two ways of computing the cohomology groups $H^{k-1}(|\mathcal{T}.^{(k)}|, \mathbb{Z})$. As a consequence we will obtain two explicit expressions for the cohomology of $\Omega^{(d+1)}$ one in terms of distributions on flag manifolds and another one in terms of generalized harmonic cochains on the Bruhat-Tits building.

We have seen in the proof of §3 Proposition 6 that there is a canonical exact sequence

$$C^\infty(N\mathcal{T}^{(k)}_{k-2}, \mathbb{Z}) \xrightarrow{d} C^\infty(N\mathcal{T}^{(k)}_{k-1}, \mathbb{Z}) \longrightarrow H^{k-1}(|\mathcal{T}.^{(k)}|, \mathbb{Z}) \longrightarrow 0$$

where d is the alternating sum of the maps induced from the face maps

$$N\mathcal{T}^{(k)}_{k-1} \quad \begin{matrix} \longrightarrow \\ \vdots \\ \longrightarrow \end{matrix} \quad N\mathcal{T}^{(k)}_{k-2} \quad .$$

In order to reinterprete this map in purely group theoretic terms we have to recall very briefly the theory of parabolic subgroups. Set $G := GL_{d+1}(K)$ and let $e_1^*, ..., e_{d+1}^*$ be the standard basis of $(K^{d+1})^*$. For every subset $I \subseteq \Delta := \{1, ..., d\}$ we have the standard flag

$$\begin{aligned} \tau_I &:= \left(\sum_{i=i_r+1}^{d+1} Ke_i^* \subsetneqq ... \subsetneqq \sum_{i=i_1+1}^{d+1} Ke_i^* \subsetneqq \sum_{i=i_0+1}^{d+1} Ke_i^* \right) \\ &= \left(\sum_{i=1}^{i_0} Ke_i \subsetneqq \sum_{i=1}^{i_1} Ke_i \subsetneqq ... \subsetneqq \sum_{i=1}^{i_r} Ke_i \right)^* \end{aligned}$$

where $\Delta \backslash I = \{i_0 < i_1 < ... < i_r\}$ and its stabilizer P_I in G. Then the map

$$\begin{array}{ccc} \text{subsets of } \Delta & \longrightarrow & \text{parabolic subgroups of } G \\ & & \text{which contain } P_\phi \\ I & \longmapsto & P_I \end{array}$$

is an inclusion preserving bijection. Furthermore, for $0 \leq r < k \leq d$, we have the homeomorphism

$$\begin{array}{ccc} \dot{\bigcup\limits_{\substack{|\Delta \backslash I| = r+1 \\ \Delta \backslash I \subseteq \{d+1-k, ..., d\}}}} G/P_I & \xrightarrow{\sim} & N\mathcal{T}^{(k)}_r \\ gP_I & \longmapsto & g(\tau_I) \end{array} \quad .$$

Of particular interest for us are the homeomorphisms

$$G/P_{\{1,...,d-k\}} \xrightarrow{\sim} N\mathcal{T}^{(k)}_{k-1}$$

and, if $k > 1$,

$$\dot{\bigcup_{d-k<i\leq d}} G/P_{\{1,\ldots,d-k,i\}} \xrightarrow{\sim} N\mathcal{T}^{(k)}_{k-2} \quad .$$

The face maps $N\mathcal{T}^{(k)}_{k-1} \to N\mathcal{T}^{(k)}_{k-2}$ obviously correspond under those identifications to the projections

$$G/P_{\{1,\ldots,d-k\}} \longrightarrow G/P_{\{1,\ldots,d-k,j\}} \subseteq \dot{\bigcup_{d-k<i\leq d}} G/P_{\{1,\ldots,d-k,i\}} \quad .$$

Lemma 1:

$\tilde{H}^{k-1}(|\mathcal{T}.^{(k)}|, \mathbb{Z}) = C^{\infty}(G/P_{\{1,\ldots,d-k\}}, \mathbb{Z}) / \sum_{i=d-k+1}^{d} C^{\infty}(G/P_{\{1,\ldots,d-k,i\}}, \mathbb{Z})$.

Proof: ($\tilde{H}^{\cdot}(\ )$ denotes reduced cohomology.) Obvious.

In order to further analyze the quotient group on the right hand side in the above Lemma we use the theory of the Bruhat decomposition. Let $W \subseteq G$ be the subgroup of permutation matrices which we identify with the symmetric group on the $d+1$ letters $1, \ldots, d+1$. The injection

$$\begin{array}{ccc} \Delta & \longrightarrow & W \\ i & \longmapsto & s_i := (i\ i+1) \end{array}$$

defines a set of generators for W; if we speak about the length of an element in W this is always meant with respect to this set of generators. For $I \subseteq \Delta$ we set

$$W_I := \text{ subgroup of } W \text{ generated by } \{s_i : i \in I\} \quad .$$

We then have

$$P_I = P_\phi W_I P_\phi$$

and the Bruhat decomposition

$$G/P_I = \dot{\bigcup_{w \in W/W_I}} C_I(w)$$

of G/P_I as a disjoint union of the subsets

$$C_I(w) := P_\phi w P_I / P_I \quad .$$

Lemma 2:

In each coset wW_I there is a unique element $\tilde{w}$ of minimal length; it is also the unique element in wW_I such that the projection map $C_\phi(\tilde{w}) \to C_I(\tilde{w})$ is injective (and then even a homeomorphism).

Proof: [4] Prop. (3.9) and (3.16).

Let $W^I \subseteq W$ denote the subset of those elements w which are of minimal length in their coset wW_I. The Bruhat decomposition then reads

$$G/P_I = \dot{\bigcup_{w \in W^I}} C_I(w) \quad .$$

In terms of permutations we can explicitly describe W^I as being the set of permutations w such that $w(i) < w(i+1)$ for all $i \in I$ ([7] Chap. IV§1 Exerc. 3 and 4). We put

$$V^I := W^I \backslash \bigcup_{i \in \Delta \backslash I} W^{I \cup \{i\}} \quad .$$

Using Lemma 2 we see that

> V^I is the set of all $w \in W^I$ such that none of the projection maps $C_I(w) \to C_{I \cup \{i\}}(w)$ for $i \in \Delta \backslash I$ is a homeomorphism.

The topological properties of the Bruhat decomposition are described by the Bruhat order on W. We write $w \to tw$ for $w \in W$ and any transposition $t \in W$ such that $\mathrm{length}(tw) = \mathrm{length}(w) + 1$. The Bruhat order $\leq$ is defined to be the transitive closure of $\to$ (compare [24] I §6).

Proposition 3:

For any $w \in W^I$ the closure of $C_I(w)$ in G/P_I is

$$\overline{C_I(w)} = \bigcup_{\substack{v \in W^I \\ v \leq w}} C_I(v) \quad .$$

Proof: In case $I = \phi$ the assertion is shown in [4] Cor. 3.15. The general case can easily be derived from that using that the projection maps $G/P_\phi \to G/P_I$ are closed and that $w \leq ww'$ holds true for all $w \in W^I$ and $w' \in W_I$.

We now fix an enumeration $W^I = \{w_1, w_2, ..., w_N\}$ of the elements in W^I in such a way that

$$a \leq b \ \text{ if } \ w_a \leq w_b \quad .$$

On the group $C^\infty(G/P_I, \mathbb{Z})$ we then have the descending filtration

$$C^\infty(G/P_I, \mathbb{Z}) = F_I^0 \supseteq F_I^1 \supseteq ... \supseteq F_I^N = \{0\}$$

defined by

$$F_I^a := \{f \in C^\infty(G/P_I, \mathbb{Z}) : f|C_I(w_\alpha) = 0 \text{ for all } 1 \leq \alpha \leq a\} \ .$$

From Proposition 3 we easily deduce that, for any $0 \leq a < b \leq N$, the restriction of functions induces an isomorphism

$$F_I^a / F_I^b = C_c^\infty\Big(\bigcup_{a < \beta \leq b} C_I(w_\beta), \mathbb{Z}\Big) \quad .$$

Here $C_c^\infty(S, \mathbb{Z})$ denotes the group of $\mathbb{Z}$-valued locally constant functions with compact support on the topological space S. In the following we want to examine more closely the filtration $\bar{F}_I^\cdot$ on the quotient group

$$C^\infty(G/P_I, \mathbb{Z}) / \sum_{i \in \Delta \setminus I} C^\infty(G/P_{I \cup \{i\}}, \mathbb{Z})$$

which is induced by $F_I^\cdot$.

Proposition 4:

i. If $w_a \in W^I \setminus V^I$ then $\bar{F}_I^a = \bar{F}_I^{a-1}$;

ii. if $w_\beta \in V^I$ for all $a < \beta \leq b$ then $\bar{F}_I^a / \bar{F}_I^b = C_c^\infty\big(\bigcup_{a < \beta \leq b} C_I(w_\beta), \mathbb{Z}\big)$.

Proof: i. Let f be any function in F_I^{a-1} and put

$$f_0 := f | C_I(w_a) \in C_c^\infty(C_I(w_a), \mathbb{Z}) \quad .$$

Since w_a is not in V^I there exists an $i \in \Delta \setminus I$ such that the projection map induces a homeomorphism $C_I(w_a) \xrightarrow{\sim} C_{I \cup \{i\}}(w_a)$. Define $g_0 \in C_c^\infty(C_{I \cup \{i\}}(w_a), \mathbb{Z})$ to be the function which corresponds to f_0 under the induced isomorphism

$$C_c^\infty(C_{I \cup \{i\}}(w_a), \mathbb{Z}) \xrightarrow{\cong} C_c^\infty(C_I(w_a), \mathbb{Z}) \quad .$$

According to Proposition 3 the set $C_{I \cup \{i\}}(w_a)$ is open in the closed subset $\bigcup_{1 \leq \alpha \leq a} C_{I \cup \{i\}}(w_\alpha)$ of $G/P_{I \cup \{i\}}$. We therefore find a function $g \in C^\infty(G/P_{I \cup \{i\}}, \mathbb{Z})$ such that

$$g | \bigcup_{1 \leq \alpha \leq a} C_{I \cup \{i\}}(w_\alpha) = \text{extension by zero of } g_0 \quad .$$

By construction we have $f - g \in F_I^a$. ii. We have to show that

$$F_I^a \cap \sum_{i \in \Delta \setminus I} C^\infty(G/P_{I \cup \{i\}}, \mathbb{Z}) \subseteq F_I^b \quad .$$

Let f be any function in F_I^a which can be written as a sum

$$f = \sum_{i \in \Delta \setminus I} f_i$$

of functions $f_i \in C^\infty(G/P_{I\cup\{i\}}, \mathbb{Z})$. In a first step we show that then f also can be written as a sum

$$f = \sum_{i \in \Delta \setminus I} g_i$$

of functions $g_i \in F_I^a \cap C^\infty(G/P_{I\cup\{i\}}, \mathbb{Z})$. By induction we may assume that the f_i already lie in F_I^{a-1}. For any $i \in \Delta\setminus I$ such that $w_a \notin W^{I\cup\{i\}}$ we find an $\alpha_i \leq a-1$ such that

$$C_{I\cup\{i\}}(w_a) = C_{I\cup\{i\}}(w_{\alpha_i}) \quad .$$

Therefore for those i the function $g_i := f_i$ must be contained in F_I^a. If there is at most one $i \in \Delta\setminus I$ with $w_a \in W^{I\cup\{i\}}$ the sum relation implies that the corresponding f_i also is contained in F_I^a so that we again define $g_i := f_i$. Otherwise there are $i, j \in \Delta\setminus I, i \neq j$, such that $w_a \in W^{I\cup\{i\}} \cap W^{I\cup\{j\}} = W^{I\cup\{i,j\}}$. The projection map then induces a homeomorphism

$$C_{I\cup\{i\}}(w_a) \xrightarrow{\sim} C_{I\cup\{i,j\}}(w_a)$$

and similarly as in the proof of part i. we find a function

$$h \in F_I^{a-1} \cap C^\infty(G/P_{I\cup\{i,j\}}, \mathbb{Z})$$

such that $f_i - h \in F_I^a$. Replacing f_i by $g_i := f_i - h$ and f_j by $f_j + h$ we get a new sum representation for f. It is clear that in this manner we can construct inductively a sum representation for f of the wanted form. In the second step it remains to observe that $F_I^a \cap C^\infty(G/P_{I\cup\{i\}}, \mathbb{Z}) \subseteq F_I^b$ for any $i \in \Delta\setminus I$. By assumption we namely have that, for any $a < \beta \leq b$,

$$C_{I\cup\{i\}}(w_\beta) = C_{I\cup\{i\}}(w_\alpha)$$

with an appropriate $\alpha \leq a$ depending on i and β.

Corollary 5:

$H^{k-1}(|\mathcal{T}.^{(k)}|, \mathbb{Z})$ *is* $\mathbb{Z}$*-free.*

Proof: According to the above Proposition the group in question has a finite filtration the quotients of which are of the form $C_c^\infty(S, \mathbb{Z})$ with some locally compact totally disconnected and metrizable space S. Such groups are $\mathbb{Z}$-free as is shown in [3] 2.2.

Corollary 6:

For $0 \leq s \leq d$ *define* $\ell(s) := \frac{(2d+1-s)\cdot s}{2}$. *There is a natural surjective homomorphism*

$$H^s(\Omega^{(d+1)}) \longrightarrow \mathrm{Dist}(\mathbf{A}^{\ell(s)}(K), A)$$

where the right hand side denotes the group of A*-valued distributions on* $\mathbf{A}^{\ell(s)}(K)$*; for* $s = 0$ *or* d *this even is an isomorphism.*

Proof: (We recall that a distribution is a finitely additive function on the family of compact open subsets.) Let w_I denote the unique element of maximal length in W_I. For any enumeration of W^I we then have $w_N = w_\Delta w_I$. Therefore the subspace $\bar{F}_I^{N-1}$ in $C^\infty(G/P_I, \mathbb{Z})/\sum_{i\in\Delta\setminus I} C^\infty(G/P_{I\cup\{i\}}, \mathbb{Z})$ is defined independently of the particular choice of the enumeration. Since furthermore $w_\Delta w_I \in V^I$ the above Proposition implies that $\bar{F}_I^{N-1} = C_c^\infty(C_I(w_N), \mathbb{Z})$ is a direct summand. On the other hand it is well-known (compare [4]) that if U_I denotes the unipotent radical of P_I and U_I^- its transpose then the map

$$\begin{array}{ccc} U_I^- & \longrightarrow & C_I(w_\Delta w_I) \\ u & \longmapsto & w_\Delta u P_I/P_I \end{array}$$

is a homeomorphism and that

$$U_I = \mathbf{A}^{\ell(I)}(K)$$

with $\ell(I) := \mathrm{length}(w_\Delta w_I) = \frac{d(d+1)}{2} - \mathrm{length}(w_I)$. Combining these facts and dualizing we obtain a natural epimorphism

$$\mathrm{Hom}_{\mathbb{Z}}(C^\infty(G/P_I, \mathbb{Z})/\sum_{i\in\Delta\setminus I} C^\infty(G/P_{I\cup\{i\}}, \mathbb{Z}), A) \longrightarrow \mathrm{Dist}(\mathbf{A}^{\ell(I)}(K), A)\ .$$

In case $I = \phi$ we have $V^I = \{w_\Delta\}$ so that, by the above Proposition again, this map even is an isomorphism. For $I = \{1, ..., d-s\}$ the left hand side is canonically isomorphic to $H^s(\Omega^{(d+1)})$ and $\mathrm{length}(w_I) = \frac{(d-s)(d-s+1)}{2}$.

In particular, if our cohomology theory is the de Rham cohomology we have constructed in this way surjections

$$\begin{array}{c} \text{closed } s\text{-forms} \\ \text{on } \Omega^{(d+1)} \end{array} \longrightarrow \begin{array}{c} K\text{-valued distributions} \\ \text{on } \mathbf{A}^{\ell(s)}(K) \end{array}$$

for $0 \leq s \leq d$ which can be viewed as some kind of p-adic Mellin transforms. An explicit version of this map in case $d = s = 1$ was given in [44].

Remarks:

1) In case $k = d$ our Lemma 1 and Proposition 4 already are contained in [3] §1-3. In fact, the topological realization of our simplicial profinite set $\mathcal{T}.^{(d)}$ appears as space Y_t in loc. cit. Our proof of Proposition 4 is a slight extension of the method proposed in 3.6 of loc. cit.

2) The filtration on $H^{k-1}(|\mathcal{T}.^{(k)}|, \mathbb{Z})$ which corresponds to $\bar{F}^{\cdot}_{\{1,\ldots,d-k\}}$ depends on an enumeration of $W^{\{1,\ldots,d-k\}}$. There seems to be no distinguished choice of such an enumeration which leads to a filtration of minimal length. Consider, for example, the case $d = 3$ and $k = 2$. The set $W^{\{1\}}$ then has 12 elements. In any enumeration of $W^{\{1\}}$ which is compatible with the Bruhat order the first 8 elements lie in $W^{\{1\}} \backslash V^{\{1\}}$. The remaining 4 elements are

$$\begin{array}{ccccc} (24) & \longrightarrow & (124) & \longrightarrow & (1324) \\ & \searrow & & \nearrow & \\ & & (13)(24) & & \end{array}$$

of which the three upper ones form the set $V^{\{1\}}$. We see that either $w_{10} \notin V^{\{1\}}$ or $w_{11} \notin V^{\{1\}}$. This means that our construction gives two different filtrations of minimal length 2 on $H^1(|\mathcal{T}.^{(2)}|, \mathbb{Z})$.

We now turn to our second computation in terms of generalized harmonic cochains. Let $B \subseteq GL_{d+1}(o)$ denote the standard Iwahori subgroup, i.e., the subgroup of all matrices which are upper triangular modulo π, and, for simplicity, put $P := P_\phi$. Let χ be the characteristic function of the compact open subset $BP/P \subseteq G/P$. We will study the homomorphism of G-modules

$$\begin{array}{rcl} H : C_c^\infty(G/B, \mathbb{Z}) & \longrightarrow & C^\infty(G/P, \mathbb{Z}) \\ \varphi & \longmapsto & \varphi * \chi := \sum\limits_{g \in G/B} \varphi(g) \cdot g(\chi) \quad . \end{array}$$

The theory of Bernstein, Borel, and Casselman of representations generated by its Iwahori fixed vectors cannot be applied since we are working with integral coefficients. Instead we observe that the properties of H obviously reflect properties of the family $\{gBP/P : g \in G\}$ of compact open subsets in G/P so that we are going to explore this family. Define the semigroup T^{++} by

$$T^{++} := \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_{d+1} \end{pmatrix} \in G : 1 \geq |t_1| \geq \ldots \geq |t_{d+1}| \right\} \quad .$$

In particular, we put

$$t := \begin{pmatrix} 1 & & & 0 \\ & \pi & & \\ & & \ddots & \\ 0 & & & \pi^d \end{pmatrix} \quad \text{and} \quad y_j := \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \pi & & \\ & & & & \ddots & \\ 0 & & & & & \pi \end{pmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} j \\ \\ \\ \\ \end{array}$$

for $0 \leq j \leq d$.

Proposition 7:

If the sets $byBP$ and $\tilde{b}yBP$ with $b, \tilde{b} \in GL_{d+1}(o)$ and $y \in T^{++}$ are not disjoint then they are equal and we have $byB = \tilde{b}yB$ (and, in particular, $bB = \tilde{b}B$).

Proof: Obviously it suffices to treat the case that $byBP$ and yBP are not disjoint. From the Iwahori decomposition

$$B = (B \cap U^-)(B \cap P) \quad \text{with} \quad U^- := w_\Delta U_\phi w_\Delta$$

we deduce that

$$\begin{aligned} byBP &= by(B \cap U^-)y^{-1}P \quad \text{and} \\ yBP &= y(B \cap U^-)y^{-1}P \quad . \end{aligned}$$

Because of

$$y(B \cap U^-)y^{-1} \subseteq B \cap U^-$$

we then have $bBP \cap BP \neq \phi$ so that there exist $b_0, b_1 \in B$ and $p \in P$ with $bb_0 = b_1 p$. This implies $p \in GL_{d+1}(o) \cap P = B \cap P$ and hence $b \in B$. Again by the Iwahori decomposition we find a $b' \in B \cap U^-$ such that $b^{-1}b' \in B \cap P$ and we see using

$$y^{-1}(B \cap P)y \subseteq B \cap P$$

that

$$byBP = b'yBP \quad .$$

Our assumption becomes

$$b'y(B \cap U^-)y^{-1}P \cap y(B \cap U^-)y^{-1}P \neq \phi \quad .$$

Since b' and $y(B \cap U^-)y^{-1}$ are contained in $B \cap U^-$ already the intersection $b'y(B \cap U^-)y^{-1} \cap y(B \cap U^-)y^{-1}$ must be nonempty which means that $b' \in y(B \cap U^-)y^{-1}$, resp. that $b \in y(B \cap U^-)y^{-1}(B \cap P)$. Our assertions are easily derived from this.

Proposition 8:

Any compact open subset in G/P can be written, for any $n \geq 0$ big enough, as a finite disjoint union of subsets of the form bt^nBP/P with $b \in GL_{d+1}(o)$.

Proof: We have

$$\begin{aligned} t^nBP &= t^n(B \cap U^-)t^{-n}P \\ &= \bigcup_x xt^{n+1}(B \cap U^-)t^{-n-1}P \\ &= \bigcup_x xt^{n+1}BP \end{aligned}$$

where x runs through the left cosets of $t^{n+1}(B\cap U^-)t^{-n-1}$ in $t^n(B\cap U^-)t^{-n}$. In the light of the above Proposition we therefore only have to show that the sets bt^nBP/P for $b \in GL_{d+1}(o)$ and $n \geq 0$ form a basis of the topology of G/P. Because of the Iwasawa decomposition $G = GL_{d+1}(o) \cdot P$ it furthermore suffices to show that the sets t^nBP/P for $n \geq 0$ form a fundamental system of neighbourhoods of the trivial coset. But we have

$$t^nBP = t^n(B \cap U^-)t^{-n}P \subseteq B^{(n)}P$$

where the compact open subgroups

$$B^{(n)} := \{b \in B : b \equiv 1 \bmod \pi^n\}$$

in G form a fundamental system of neighbourhoods of the unit element.

Corollary 9:

$C^\infty(G/P, \mathbb{Z})$ as a G-module is generated by the characteristic function χ; in particular, the homomorphism H is surjective.

In order to determine the kernel of H we recall that $C_c^\infty(B\backslash G/B, \mathbb{Z})$ is an associative ring with unit (the Hecke ring of B) via the convolution product

$$\varphi * \psi := \sum_{g \in G/B} \varphi(g) \cdot g(\psi) \quad .$$

It acts via convolution from the right on $C_c^\infty(G/B, \mathbb{Z})$. We denote by $\mathcal{A}$ the subring which is generated by the characteristic functions χ_y of the double cosets ByB for $y \in T^{++}$. More generally let χ_M denote the characteristic function of a subset $M \subseteq G$. The following result is well-known but we have not found an appropriate reference.

Lemma 10:

$\mathcal{A}$ is a polynomial ring over $\mathbb{Z}$ in the variables $\chi_{y_0}, ..., \chi_{y_d}$; we have $\chi_y * \chi_{\tilde{y}} = \chi_{y\tilde{y}}$ for $y, \tilde{y} \in T^{++}$.

Proof: By definition we have

$$\chi_y * \chi_{\tilde{y}} = \sum_{g \in ByB/B} g(\chi_{\tilde{y}}) = \sum_{g \in ByB/B} \chi_{gB\tilde{y}B} \quad .$$

Using once more the formulae

$$\begin{aligned} &B = (B \cap U^-)(B \cap P) \quad \text{and} \\ &y^{-1}(B \cap P)y \subseteq B \cap P \quad , \quad y(B \cap U^-)y^{-1} \subseteq B \cap U^- \end{aligned}$$

we get

$$ByB = (B \cap U^-)yB = \dot{\bigcup_x} xyB$$

where x runs through the left cosets of $y(B \cap U^-)y^{-1}$ in $B \cap U^-$ (the union is disjoint). We also get

$$\begin{aligned} xyB\tilde{y}B &= xy(B \cap U^-)(B \cap P)\tilde{y}B = xy(B \cap U^-)\tilde{y}B \\ &= x(y(B \cap U^-)y^{-1})y\tilde{y}B \subseteq By\tilde{y}B \quad . \end{aligned}$$

Combining these computations we obtain

$$\begin{aligned} \chi_y * \chi_{\tilde{y}} &= \sum_x \chi_{xyB\tilde{y}B} \quad \text{and} \\ By\tilde{y}B &= \bigcup_x xyB\tilde{y}B \quad . \end{aligned}$$

Next we have to show that the above union is disjoint. Assume therefore that

$$xyB\tilde{y}B \cap yB\tilde{y}B \neq \phi \quad .$$

We then find $b_0, b_1 \in y(B \cap U^-)y^{-1}$ such that $xb_0y\tilde{y}B = b_1y\tilde{y}B$. Since xb_0 and b_1 are contained in $B \cap U^-$ it follows that

$$xb_0(y\tilde{y}(B \cap U^-)\tilde{y}^{-1}y^{-1}) = b_0(y\tilde{y}(B \cap U^-)\tilde{y}^{-1}y^{-1})$$

and hence that $x \in y(B \cap U^-)y^{-1}$. Therefore the above union is disjoint and we have established that

$$\chi_y * \chi_{\tilde{y}} = \chi_{y\tilde{y}} \quad .$$

But then

$$\begin{array}{lcl} \mathbb{Z}[X_0, ..., X_d] & \longrightarrow & \mathcal{A} \\ X_j & \longmapsto & \chi_{y_j} \end{array}$$

defines a surjective ring homomorphism. It is also injective since, by the Cartan decomposition of G, for any $y \in T^{++}$ there are unique integers $n_0, ..., n_d \geq 0$ such that

$$ByB = By_0^{n_0} \cdot ... \cdot y_d^{n_d} B \quad .$$

q.e.d.

In the following we always view $\mathbb{Z}$ as an $\mathcal{A}$-module through the ring homomorphism

$$\begin{array}{c} \mathcal{A} \longrightarrow \mathbb{Z} \\ \chi_{y_j} \longmapsto 1 \quad . \end{array}$$

Proposition 11:

H induces a G-isomorphism $C_c^\infty(G/B, \mathbb{Z}) \underset{\mathcal{A}}{\otimes} \mathbb{Z} \xrightarrow{\cong} C^\infty(G/P, \mathbb{Z})$.

Proof: $C_c^\infty(G/B, \mathbb{Z})$ as a G-module is generated by χ_1. In order to see that H factorizes through $C_c^\infty(G/B, \mathbb{Z}) \underset{\mathcal{A}}{\otimes} \mathbb{Z}$ it therefore suffices to show that

$$H(\chi_1 * (\chi_y - \chi_1)) = H(\chi_y - \chi_1) = 0 \quad \text{for any} \quad y \in T^{++} \quad .$$

According to Proposition 7 the union in

$$ByBP = \dot{\bigcup_{g \in ByB/B}} gBP$$

is disjoint. We get

$$H(\chi_y - \chi_1) = (\sum_{g \in ByB/B} \chi_{gBP}) - \chi_{BP} = \chi_{ByBP} - \chi_{BP} \quad .$$

It remains to notice that

$$ByBP = By(B \cap U^-)y^{-1}P = BP \quad .$$

The induced map

$$(*) \qquad C_c^\infty(G/B, \mathbb{Z}) \underset{\mathcal{A}}{\otimes} \mathbb{Z} \longrightarrow C^\infty(G/P, \mathbb{Z})$$

clearly is surjective. For the proof of its injectivity we need the following fact.

Lemma 12:

For any $g \in G$ there is a $y \in T^{++}$ such that $gByB \subseteq GL_{d+1}(o)T^{++}B$.

Proof: Let T be the subgroup of diagonal matrices in G and let $N(T)$ denote its normalizer. Because of the Cartan decomposition $G = GL_{d+1}(o)TB$ we can assume $g \in T$. As in any generalized Tits system (compare [26]) we then find finitely many $g_\alpha \in N(T)$ such that

$$gBhB \subseteq \bigcup_\alpha Bg_\alpha hB \quad \text{for any} \quad h \in N(T) \quad .$$

Since $N(T) = WT$ we can write $g_\alpha = w_\alpha y_\alpha$ with $w_\alpha \in W$ and $y_\alpha \in T$. Choosing a $y \in T^{++}$ such that $y_\alpha y \in T^{++}$ for all α we then have

$$gByB \subseteq \bigcup_\alpha Bw_\alpha y_\alpha yB \subseteq GL_{d+1}(o)T^{++}B \quad .$$

q.e.d.

Because of

$$\chi_{gB} + \chi_{gB} * (\chi_y - \chi_1) = \chi_{gB} * \chi_y = \chi_{gByB}$$

the above Lemma says that the composed map

$$C_c^\infty(GL_{d+1}(o)T^{++}B/B, \mathbb{Z}) \xrightarrow{\subseteq} C_c^\infty(G/B, \mathbb{Z}) \longrightarrow C_c^\infty(G/B, \mathbb{Z}) \otimes_{\mathcal{A}} \mathbb{Z}$$

is surjective. Given $y \in T^{++}$ and $n \geq 0$ big enough we find a $\tilde{y} \in T^{++}$ such that $y\tilde{y} = t^n$. In the proof of Lemma 10 we have seen that $yB\tilde{y}B \subseteq Bt^nB$ so that

$$\chi_{yB} + \chi_{yB} * (\chi_{\tilde{y}} - \chi_1) = \chi_{yB\tilde{y}B} \in C_c^\infty(Bt^nB/B, \mathbb{Z}) \quad .$$

Therefore even the composed map

$$\bigcup_{n \geq 0} C_c^\infty(GL_{d+1}(o)t^nB/B, \mathbb{Z}) \xrightarrow{\subseteq} C_c^\infty(G/B, \mathbb{Z}) \longrightarrow C_c^\infty(G/B, \mathbb{Z}) \otimes_{\mathcal{A}} \mathbb{Z}$$

is surjective. But as an immediate consequence of Proposition 7 we have the following result which then implies the injectivity of $(*)$.

Lemma 13:

For any $y \in T^{++}$ the restriction of H to the subspace $C_c^\infty(GL_{d+1}(o)yB/B, \mathbb{Z})$ is injective.

In the next step we want to identify the preimages under the map H of the G-submodules $C^\infty(G/P_I, \mathbb{Z})$ in $C^\infty(G/P, \mathbb{Z})$. In $C_c^\infty(G/B, \mathbb{Z})$ we have the G-submodules $C_c^\infty(G/B_I, \mathbb{Z})$ where the subgroups B_I in G are given by the inclusion preserving bijection

$$\begin{array}{ccl} \text{subsets of } \Delta & \longrightarrow & \text{subgroups of } GL_{d+1}(o) \\ & & \text{which contain } B \\ I & \longmapsto & B_I := BW_IB \end{array} \quad .$$

Any parahoric subgroup of G is conjugate to some (possibly several) B_I.

Lemma 14:

i. $B_I = (B \cap U_I^-)(B_I \cap P_I)$ *with* $U_I^- :=$ *transpose of* U_I*;*
ii. $B_I P_I = BP_I = B_I P$.

Proof: i. In case $I = \phi$ this is the Iwahori decomposition of B which also holds in the form

$$B = (B \cap U_I^-)(B \cap P_I) \quad \text{for any} \quad I \subseteq \Delta \quad .$$

In the general case we first notice that, for any $w \in W$, the Iwahori decomposition together with the fact that

$$w(B \cap U^-) \subseteq Bw$$

implies that

$$BwB = Bw(B \cap P) \quad .$$

We then compute

$$\begin{aligned} B_I = \bigcup_{w \in W_I} BwB &= \bigcup_{w \in W_I} (B \cap U_I^-)(B \cap P_I)w(B \cap P) \\ &= (B \cap U_I^-)(B_I \cap P_I) \quad . \end{aligned}$$

ii. The first equality immediately follows from the assertion i. For the second equality we use the fact that

$$wPs_i \subseteq BwP \cup Bws_iP$$

holds true for any $w \in W$ and $i \in \Delta$. This can be shown by adapting the proof in [7] Chap. IV §2.2. We then see that

$$P_I = PW_IP \subseteq BW_IP \subseteq B_IP$$

and therefore that $B_I P_I = B_I P$.

Proposition 15:

H induces a surjective G-homomorphism $C_c^\infty(G/B_I,\mathbb{Z}) \to C^\infty(G/P_I,\mathbb{Z})$; its kernel as a G-module is generated by the functions $\chi_{By_jB_I} - \chi_{B_I}$ for $0 \leq j \leq d$.

Proof: By Lemma 14 ii and Proposition 7 we have

$$B_IP_I = B_IP = \dot{\bigcup_{g\in B_I/B}} gBP$$

where the union is disjoint. We therefore obtain

$$H(\chi_{B_I}) = \sum_{g\in B_I/B} \chi_{gBP} = \chi_{B_IP_I}$$

which implies $H(C_c^\infty(G/B_I,\mathbb{Z})) \subseteq C^\infty(G/P_I,\mathbb{Z})$ since $C_c^\infty(G/B_I,\mathbb{Z})$ as a G-module is generated by χ_{B_I}. In order to establish the surjectivity we introduce the semigroup

$$T_I^{++} := \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_{d+1} \end{pmatrix} \in T^{++} : |t_i| = |t_{i+1}| \quad \text{for} \quad i \in I \right\}$$
$$\text{and} \quad t_I := \prod_{j\in\Delta\setminus I} y_j \in T_I^{++} \quad .$$

Exactly the same arguments as in the proof of the Propositions 7 and 8 based now on Lemma 14 i and on the obvious inclusions

$$y(B\cap U_I^-)y^{-1} \subseteq B\cap U_I^- \quad \text{and} \quad y^{-1}(B_I\cap P_I)y \subseteq B_I\cap P_I \quad \text{for} \quad y\in T_I^{++}$$

then lead to the following generalization of those Propositions.

Proposition 8':

i. If the sets byB_IP_I and $\tilde{b}yB_IP_I$ with $b,\tilde{b} \in GL_{d+1}(o)$ and $y \in T_I^{++}$ are not disjoint then they are equal and we have $byB_I = \tilde{b}yB_I$ and $bB_I = \tilde{b}B_I$;
ii. any compact open subset in G/P_I can be written, for any $n \geq 0$ big enough, as a finite disjoint union of subsets of the form $bt_I^nB_IP_I/P_I$ with $b \in GL_{d+1}(o)$.

Corollary 9':

$C^\infty(G/P_I,\mathbb{Z})$ as a G-module is generated by $H(\chi_{B_I}) = \chi_{B_IP_I}$.

Since $C_c^\infty(G/B_I,\mathbb{Z})$ is not a right $\mathcal{A}$-submodule of $C_c^\infty(G/B,\mathbb{Z})$ the computation of the kernel becomes slightly more complicated. Let $M \subseteq C_c^\infty(G/B_I,\mathbb{Z})$ denote the

G-submodule generated by $\chi_{By_jB_I} - \chi_{B_I}$ for $0 \leq j \leq d$. Using Lemma 14 i we see that with x running through the left cosets of $y_j(B \cap U_I^-)y_j^{-1}$ in $B \cap U_I^-$ we have

$$\begin{aligned} H(\chi_{B_I}) = \chi_{B_I P_I} = \chi_{(B\cap U_I^-)P_I} &= \sum_x \chi_{xy_j(B\cap U_I^-)y_j^{-1}P_I} \\ &= \sum_x \chi_{xy_jB_IP_I} = H(\sum_x \chi_{xy_jB_I}) \quad . \end{aligned}$$

Lemma 16:

$y_j^{-1}(B \cap P_I)y_j \subseteq B_I \cap P_I$ for $0 \leq j \leq d$.

Proof: Exercise.

From Lemma 14 i and Lemma 16 we now deduce that

$$\begin{aligned} B \cap y_jB_Iy_j^{-1} &= (B\cap U_I^-)(B\cap P_I) \cap [y_j(B\cap U_I^-)y_j^{-1}][y_j(B_I \cap P_I)y_j^{-1}] \\ &= y_j(B\cap U_I^-)y_j^{-1}(B\cap P_I) \end{aligned}$$

and consequently that

$$B/B\cap y_jB_Iy_j^{-1} = (B\cap U_I^-)/y_j(B\cap U_I^-)y_j^{-1} \quad .$$

Since By_jB_I is the disjoint union of the cosets $x'y_jB_I$ where x' runs through the left cosets of $B \cap y_jB_Iy_j^{-1}$ in B we obtain

$$H(\chi_{B_I}) = H(\sum_x \chi_{xy_jB_I}) = H(\sum_{x'} \chi_{x'y_jB_I}) = H(\chi_{By_jB_I}) \quad .$$

Therefore M is contained in the kernel of H. On the other hand we get

$$\begin{aligned} \chi_{yB_I} &= \chi_{yBy_jB_I} - y(\chi_{By_jB_I} - \chi_{B_I}) \\ &\in C_c^\infty(Byy_jB_I/B_I, \mathbb{Z}) + M \quad \text{for any} \quad y \in T^{++} \end{aligned}$$

since $yBy_jB_I \subseteq Byy_jB_I$. It follows inductively that

$$\chi_{B_I} \in C_c^\infty(ByB_I/B_I, \mathbb{Z}) + M \quad \text{for any} \quad y \in T^{++}$$

which by Lemma 12 implies that the projection map

$$C_c^\infty(GL_{d+1}(o)T^{++}B_I/B_I, \mathbb{Z}) \longrightarrow C_c^\infty(G/B_I, \mathbb{Z})/M$$

is surjective. Let us fix a $y \in T^{++}$ for the moment. For any $z = wy_jw^{-1}$ with $w \in W_I$ we have

$$\begin{aligned}\chi_{(B\cap U_I^-)zB_I} - \chi_{B_I} &= \chi_{w(B\cap U_I^-)y_jB_I} - \chi_{B_I} \\ &= w(\chi_{By_jB_I} - \chi_{B_I}) \in M \quad ;\end{aligned}$$

it suffices to observe that $w^{-1}(B\cap U_I^-)w = B\cap U_I^-$ and that $By_jB_I = (B\cap U_I^-)y_jB_I$ by Lemma 16. This shows that

$$\begin{aligned}\chi_{yB_I} &= \chi_{y(B\cap U_I^-)zB_I} - y(\chi_{(B\cap U_I^-)zB_I} - \chi_{B_I}) \\ &\in C_c^\infty((B\cap U_I^-)yzB_I/B_I, \mathbb{Z}) + M \quad .\end{aligned}$$

But for any $n \geq 0$ big enough we find a sequence $z_1, ..., z_a$ of elements in the set $\{wy_jw^{-1} : 0 \leq j \leq d, w \in W_I\}$ such that

$$yz_1, yz_1z_2, ..., yz_1 \cdot ... \cdot z_a \in T^{++} \quad \text{and} \quad yz_1 \cdot ... \cdot z_a = t_I^n \quad .$$

We therefore obtain inductively that

$$\chi_{yB_I} \in C_c^\infty((B\cap U_I^-)t_I^nB_I/B_I, \mathbb{Z}) + M \quad .$$

This means that even the projection map

$$\bigcup_{n\geq 0} C_c^\infty(GL_{d+1}(o)t_I^nB_I/B_I, \mathbb{Z}) \longrightarrow C_c^\infty(G/B_I, \mathbb{Z})/M$$

is surjective. But by Proposition 8'i the restriction of H to $C_c^\infty(GL_{d+1}(o)t_I^nB_I/B_I, \mathbb{Z})$ is injective. Therefore the kernel of H restricted to $C_c^\infty(G/B_I, \mathbb{Z})$ is contained in M. This finishes the proof of Proposition 15.

Corollary 17:

For $0 \leq s \leq d$ there is a $GL_{d+1}(K)$-equivariant isomorphism

$$H^s(\Omega^{(d+1)}) = \mathrm{Hom}_{\mathbb{Z}}(C_c^\infty(G/B_I, \mathbb{Z})/R_I, A)$$

where $I = \{1, \ldots, d-s\}$ and where R_I is the $GL_{d+1}(K)$-submodule generated by

$$\chi_{By_jB_I} - \chi_{B_I} \quad \textit{for } 0 \leq j \leq d \quad \textit{and } \chi_{B_Is_iB_I} + \chi_{B_I} \quad \textit{for } d-s < i \leq d\ .$$

Proof: The only necessary additional observation is that for subsets of Δ of the form $I = \{1, \ldots, d-s\}$ we have $W_{I\cup\{i\}} = W_I \cup W_Is_iW_I$. Quite generally $B_{I\cup\{i\}}$ is the disjoint union of the double cosets B_IwB_I with w running through $W_I\backslash W_{I\cup\{i\}}/W_I$. It then follows that for $I = \{1, \ldots, d-s\}$ and $d-s < i \leq d$ the relation $\chi_{B_{I\cup\{i\}}} = \chi_{B_Is_iB_I} + \chi_{B_I}$ holds true.

Remarks:

1) In case $I = \phi$ Proposition 15 gives a natural isomorphism

$$C_c^\infty(G/B, \mathbb{Z})/R_\phi = \tilde{H}^{d-1}(|\mathcal{T}.^{(d)}|, \mathbb{Z}) \quad .$$

A straightforward computation shows that the left hand side is nothing else than the highest cohomology with compact support $H_c^d(|\mathcal{BT}|, \mathbb{Z})$ of the Bruhat-Tits building $\mathcal{BT}$. In this way we obtain a new proof of (part of) a result of Borel/Serre ([3] Th. 5.6) which avoids the use of their compactification theory for $\mathcal{BT}$.

2) It is clear that the space on the right hand side of the statement of Corollary 17 can be viewed as a space of certain cochains on the Bruhat-Tits building $\mathcal{BT}$. In case $I = \phi$ it is precisely the space of harmonic cochains in the sense of [19].

3) In the case $s = d = 1$ with etale cohomology as the underlying cohomology theory Corollary 17 was proved by Drinfeld ([16]) in an entirely different manner. If K has characteristic 0 his proof even works for arbitrary coefficients.

§5 The cohomology of quotient varieties

Let $\Gamma \subseteq PGL_{d+1}(K)$ be a cocompact discrete subgroup which we always assume to act without fixed points on $\Omega^{(d+1)}$. We want to apply our results to the study of the cohomology of the quotient

$$X_\Gamma := \Gamma \backslash \Omega^{(d+1)} \quad .$$

Theorem 2:

X_Γ in a natural way is a proper and smooth (rigid) analytic variety over K. The projection map $pr: \Omega^{(d+1)} \to X_\Gamma$ is an etale covering.

The proof which we will sketch in the following is extracted from [16]. For torsionfree Γ a corresponding result in the context of formal schemes is shown in [39]. X_Γ becomes a G-ringed space (in the terminology of [6]) over K in the following way: The topology is the quotient topology, i.e., a subset $U \subseteq X_\Gamma$ is called admissible open if $pr^{-1}(U)$ is admissible open in $\Omega^{(d+1)}$ and similarly for admissible coverings; the structure sheaf on X_Γ is given by

$$\mathcal{O}_{X_\Gamma}(U) := \mathcal{O}_{\Omega^{(d+1)}}(pr^{-1}(U))^\Gamma \quad .$$

Obviously, $pr: \Omega^{(d+1)} \to X_\Gamma$ then is a morphism of G-ringed spaces over K. In order to analyze the situation more closely we use the open affinoid subvarieties $U_\sigma^a \subseteq \Omega^{(d+1)}$, for any simplex σ in $\mathcal{BT}$ and any rational number $0 < a < 1$, which were constructed in [16] Prop. 6.1 and which have the following properties:

1) The U_σ^a, for fixed a and varying σ, form an admissible covering of $\Omega^{(d+1)}$ whose nerve is the barycentric subdivision of $\mathcal{BT}$;
2) $g(U_\sigma^a) = U_{g\sigma}^a$ for any $g \in PGL_{d+1}(K)$;
3) $U_\sigma^a \underset{K}{\subset\subset} U_\sigma^b$ for $0 < a < b < 1$ (for the notation see [6] 9.6.2).

Drinfeld first constructs subsets $V_\sigma^a \subseteq |\mathcal{BT}|$ with corresponding properties and then defines $U_\sigma^a := \rho^{-1}(V_\sigma^a)$ where $\rho: \Omega^{(d+1)} \to |\mathcal{BT}|$ is the map which we described in the first Paragraph.

Let us fix σ and a for a moment. The subgroup $\Gamma_\sigma := \{g \in \Gamma : g\sigma = \sigma\}$ is finite; furthermore σ and $g\sigma$ for $g \in \Gamma \setminus \Gamma_\sigma$ never are neighbouring vertices in the barycentric subdivision of $\mathcal{BT}$. By 1) and 2) this implies that the open affinoid subsets $U_{g\sigma}^a = g(U_\sigma^a)$ for $g \in \Gamma/\Gamma_\sigma$ are pairwise disjoint. Put

$$U_{\Gamma\sigma}^a := \dot{\bigcup_{g \in \Gamma/\Gamma_\sigma}} U_{g\sigma}^a = \dot{\bigcup_{g \in \Gamma/\Gamma_\sigma}} g(U_\sigma^a)$$

and

$$X^a_{\Gamma,\sigma} := pr(U^a_{\Gamma\sigma}) \quad .$$

Using 1) and the fact that $\mathcal{BT}$ is locally finite we find for any simplex τ in $\mathcal{BT}$ finitely many $g_1, \ldots, g_r \in \Gamma$ such that

$$U^a_\tau \cap U^a_{\Gamma\sigma} = \bigcup_{i=1}^{r} (U^a_\tau \cap U^a_{g_i\sigma}) \quad .$$

This shows that $U^a_{\Gamma\sigma}$ is admissible open in $\Omega^{(d+1)}$ and consequently that $X^a_{\Gamma,\sigma}$ is admissible open in X_Γ. Moreover we see that pr induces an isomorphism of G-ringed spaces

$$\Gamma_\sigma \backslash U^a_\sigma \xrightarrow{\cong} X^a_{\Gamma,\sigma} \quad .$$

But by [16] Prop. 6.3 (or [6] 6.3.3) the left hand side is an affinoid variety over K. Since Γ is cocompact there are simplices $\sigma_1, \ldots, \sigma_m$ such that any simplex in $\mathcal{BT}$ is a Γ-translate of one of them. The $X^a_{\Gamma,\sigma_1}, \ldots, X^a_{\Gamma,\sigma_m}$ then form an admissible covering of X_Γ by affinoid varieties over K; the intersections $X^a_{\Gamma,\sigma_i} \cap X^a_{\Gamma,\sigma_j}$ are easily checked to be affinoid, too. Therefore it follows that X_Γ is a separated analytic variety over K. By 3) and [16] Prop. 6.4 we have $X^a_{\Gamma,\sigma} \underset{K}{\subset\subset} X^b_{\Gamma,\sigma}$ for $0 < a < b < 1$ which implies that X_Γ is proper. Finally our assumption that Γ acts without fixed points on $\Omega^{(d+1)}$ guarantees that pr is an etale covering and that X_Γ is smooth.

We also obtain that

$$\begin{array}{ccc} \coprod\limits_{g\in\Gamma} \Omega^{(d+1)} & \xrightarrow{\cong} & \Omega^{(d+1)} \underset{X_\Gamma}{\times} \Omega^{(d+1)} \\ z \text{ in component } g & \longmapsto & (gz, z) \end{array}$$

is an isomorphism (compare [38] Prop. 2 on p. 70). If Γ is torsionfree so that the subgroups Γ_σ are trivial we see that pr even is an analytic covering in the sense that the natural sequence

$$\mathcal{G}(X_\Gamma) \longrightarrow \mathcal{G}(\Omega^{(d+1)}) \rightrightarrows \mathcal{G}(\Omega^{(d+1)} \underset{X_\Gamma}{\times} \Omega^{(d+1)})$$

is exact for any sheaf $\mathcal{G}$ in the analytic topology.

Lemma 1:

i. For any injective sheaf $\mathcal{G}$ on $\mathcal{V}$ the Γ-module $\mathcal{G}(\Omega^{(d+1)})$ is injective;
ii. if either the given topology is finer than the etale topology or Γ is torsionfree then we have $\mathcal{G}(X_\Gamma) = \mathcal{G}(\Omega^{(d+1)})^\Gamma$ for any sheaf $\mathcal{G}$ on $\mathcal{V}$.

Proof: i. The functor

$$\begin{array}{ccc} \text{sheaves on } \mathcal{V} & \longrightarrow & \Gamma\text{-modules} \\ \mathcal{G} & \longmapsto & \mathcal{G}(\Omega^{(d+1)}) \end{array}$$

respects injective objects since it has an exact left adjoint functor $M \to \mathcal{M}$ which is given in the following way: For any variety Y in $\mathcal{V}$ put

$$M(Y) := \Big(\bigoplus_{x \in \Omega^{(d+1)}(Y)} M \Big) \text{ modulo the subgroup generated}$$

by all elements of the form

$(m$ in component $x)$ - $(gm$ in component $gx)$

with $g \in \Gamma$.

This is a presheaf on $\mathcal{V}$; $\mathcal{M}$ is defined to be the associated sheaf.

ii. This follows easily from the two observations preceding this Lemma (compare [33] II.1.4).

Proposition 2:

We have the spectral sequence

$$H^r(\Gamma, H^s(\Omega^{(d+1)})) \Longrightarrow H^{r+s}(X_\Gamma)$$

in each of the following cases:

1) *Γ is torsionfree;*
2) *the given topology on $\mathcal{V}$ is finer than the etale topology;*
3) *the given cohomology theory is de Rham cohomology.*

Proof: In the cases 1) and 2) this is an immediate consequence of Lemma 1. But then also the assertion in the case 3) follows since de Rham cohomology can be computed on the small etale site on X_Γ. This can be seen in much the same way as the corresponding statement in algebraic geometry: [33] III.3.7 and II.1.6 based on [32] 4.1.8 and on the following fact.

Lemma 3:

For any etale morphism $f: X \to Y$ of K-analytic varieties we have

$$f^{-1}\Omega^1_{Y/K} \underset{f^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_X = \Omega^1_{X/K} \quad .$$

Proof: Obviously there is a natural map $f^{-1}\Omega^1_{Y/K} \otimes \mathcal{O}_X \to \Omega^1_{X/K}$. Since both sides are coherent $\mathcal{O}_X$-modules it can be checked to be an isomorphism locally in the neighbourhood of each point of X. Furthermore we may assume that K is algebraically closed. The fibers of any etale morphism between K-affinoid varieties are finite. As the analytic topology is Hausdorff it follows that f is locally injective at each point. But then SGA 1 exp. I Cor. 4.4 and [6] 7.3.3 Prop. 5 imply that f even is locally an isomorphism.

Let us now assume that our cohomology theory has the properties I)–IV) in Paragraph 2 and is such that the spectral sequence in Proposition 2 is at our disposal. Preserving the notations introduced in Paragraph 4 we put, for any subset $I \subseteq \Delta$ and any abelian group M,

$$\begin{aligned} V_I(M) :=& C^\infty(G/P_I, M) / \sum_{i \in \Delta \setminus I} C^\infty(G/P_{I \cup \{i\}}, M) \\ =& V_I(\mathbb{Z}) \otimes M \quad . \end{aligned}$$

It is known ([5] X.4.6 and 4.11) that in case M is a field of characteristic 0 the $V_I(M)$ are the irreducible constituents of the $PGL_{d+1}(K)$-module $C^\infty(G/P, M)$. Our main result says that $H^s(\Omega^{(d+1)}) = 0$ for $s > d$ and that

$$\begin{aligned} H^s(\Omega^{(d+1)}) &= \mathrm{Hom}_{\mathbb{Z}}(V_{\{1,\ldots,d-s\}}(\mathbb{Z}), A) \\ &= \mathrm{Hom}_A(V_{\{1,\ldots,d-s\}}(A), A) \quad \text{for } 0 \le s \le d \quad . \end{aligned}$$

Since $V_{\{1,\ldots,d-s\}}(A)$ is a free A-module by §4 Corollary 5 we get

$$\begin{aligned} H^r(\Gamma, H^s(\Omega^{(d+1)})) &= H^r(\Gamma, \mathrm{Hom}_A(V_{\{1,\ldots,d-s\}}(A), A)) \\ &= \mathrm{Ext}^r_{A[\Gamma]}(V_{\{1,\ldots,d-s\}}(A), A) \quad \text{for } 0 \le s \le d \end{aligned}$$

so that the spectral sequence in Proposition 2 becomes

$$E_2^{r,s} = \left\{ \begin{array}{ll} \mathrm{Ext}^r_{A[\Gamma]}(V_{\{1,\ldots,d-s\}}(A), A) & \text{if } 0 \le s \le d \\ 0 & \text{otherwise} \end{array} \right\} \Longrightarrow H^{r+s}(X_\Gamma) \quad .$$

We therefore have the task of studying the groups $\mathrm{Ext}^r_{A[\Gamma]}(V_I(A), A)$.

Proposition 4:

i. $V_I(\mathbb{Z})$ *has a projective resolution by finitely generated free* $\mathbb{Z}[\Gamma]$*-modules;*
ii. $\mathrm{Ext}^r_{\mathbb{Z}[\Gamma]}(V_I(\mathbb{Z}), \mathbb{Z})$ *is finitely generated for any* $r \ge 0$*;*
iii. there is a natural exact sequence

$$0 \to \mathrm{Ext}^r_{\mathbb{Z}[\Gamma]}(V_I(\mathbb{Z}), \mathbb{Z}) \underset{\mathbb{Z}}{\otimes} A \to \mathrm{Ext}^r_{A[\Gamma]}(V_I(A), A) \to \mathrm{Tor}_{\mathbb{Z}}(\mathrm{Ext}^{r+1}_{\mathbb{Z}[\Gamma]}(V_I(\mathbb{Z}), \mathbb{Z}), A) \to 0$$

for any $r \ge 0$.

Proof: i. The proof of this result is quite complicated and will be given in the last Paragraph (Proposition 16). ii. This is an immediate consequence of the first assertion. iii. Let $F. \to V_I(\mathbb{Z})$ be a projective resolution by finitely generated free $\mathbb{Z}[\Gamma]$-modules. Then $F. \underset{\mathbb{Z}}{\otimes} A \to V_I(A)$ is a projective resolution by finitely generated free $A[\Gamma]$-modules since $V_I(\mathbb{Z})$ is $\mathbb{Z}$-free by the argument in the proof of §4 Corollary

5. Therefore $\mathrm{Ext}^*_{\mathbb{Z}[\Gamma]}(V_I(\mathbb{Z}),\mathbb{Z})$, resp. $\mathrm{Ext}^*_{A[\Gamma]}(V_I(A),A)$, can be computed from the complex $\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(F.,\mathbb{Z})$, resp. $\mathrm{Hom}_{A[\Gamma]}(F.\otimes_{\mathbb{Z}} A,A)$. The assertion follows now from the universal coefficient theorem applied to the complex

$$\mathrm{Hom}_{A[\Gamma]}(F.\otimes_{\mathbb{Z}} A,A)=\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(F.,\mathbb{Z})\otimes_{\mathbb{Z}} A \quad .$$

This Proposition in particular implies that we may pass in the corresponding spectral sequences for $H^*_{et}(\overline{X}_\Gamma,\mathbb{Z}/\ell^\nu\mathbb{Z})$ with $\ell\neq p$ to the projective limit with respect to ν and obtain the Galois equivariant spectral sequence

$$E_2^{r,s}=\left\{\begin{array}{ll}\mathrm{Ext}^r_{\mathbb{Q}_\ell[\Gamma]}(V_{\{1,\ldots,d-s\}}(\mathbb{Q}_\ell),\mathbb{Q}_\ell)(-s) & \\ & \text{if } 0\leq s\leq d\\ 0 & \text{otherwise}\end{array}\right\}\Longrightarrow H^{r+s}_{et}(\overline{X}_\Gamma,\mathbb{Q}_\ell)$$

for $\mathbb{Q}_\ell$-adic cohomology. In the following we will compute the groups $\mathrm{Ext}^r_{A[\Gamma]}(V_I(A),A)$ under the assumptions that

— Γ is a discrete cocompact subgroup in $PGL_{d+1}(K)$, and

— A is a field of characteristic 0.

Because of the above Proposition this amounts to the computation of

$$\dim_{\mathbb{C}}\mathrm{Ext}^r_{\mathbb{C}[\Gamma]}(V_I(\mathbb{C}),\mathbb{C}) \quad .$$

Since Γ is cocompact the $PGL_{d+1}(K)$-representation

$$\mathrm{Ind}_\Gamma:=C^\infty(PGL_{d+1}(K)/\Gamma,\mathbb{C})$$

is admissible. By Shapiro's lemma ([13] A.8) we have

$$\mathrm{Ext}^*_{\mathbb{C}[\Gamma]}(V_I(\mathbb{C}),\mathbb{C})=\mathrm{Ext}^*(V_I(\mathbb{C}),\mathrm{Ind}_\Gamma)$$

where Ext^* (without a subscript) always denotes the Ext-functor on the category of smooth $PGL_{d+1}(K)$-representations. In order to understand these Ext-groups on the right hand side we use the ideas in the proof of the Garland-Casselman theorem in [12], [13], and [5]. The representation Ind_Γ is unitary and therefore completely reducible (compare [12] and [11] 2.1.14). In addition the admissibility implies that only finitely many of its irreducible constituents can have a nonzero vector fixed under the Iwahori subgroup B. We therefore obtain a decomposition

$$\mathrm{Ind}_\Gamma\cong V_0\oplus V_1\oplus\ldots\oplus V_m$$

into admissible unitary representations V_j such that $V_0^B = 0$ and V_j is irreducible with $V_j^B \neq 0$ for $1 \leq j \leq m$. In particular we get

$$\mathrm{Ext}^*(V_I(\mathbb{C}), \mathrm{Ind}_\Gamma) \cong \bigoplus_{j=0}^{m} \mathrm{Ext}^*(V_I(\mathbb{C}), V_j) \quad .$$

Proposition 5:

Let V and V' be smooth $PGL_{d+1}(K)$-representations such that $V^B = 0$ but V' is generated as a $PGL_{d+1}(K)$-module by $(V')^B$. Then we have

$$\mathrm{Ext}^*(V', V) = 0 \quad .$$

Proof: Consider any Yoneda extension of smooth $PGL_{d+1}(K)$-representations

$$E \;:\; 0 \to V = E_0 \to E_1 \to \ldots \to E_{r+1} = V' \to 0 \quad .$$

In any E_j we have the subrepresentation $\widehat{E}_j$ generated by E_j^B. In this way we obtain the commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \to & V & = & E_0 & \to & E_1 & \to & \ldots & \to & E_{r+1} & = & V' & \to & 0 \\ & & & & \uparrow\subseteq & & \uparrow\subseteq & & & & \| & & & & \\ 0 & \to & 0 & = & \widehat{E}_0 & \to & \widehat{E}_1 & \to & \ldots & \to & \widehat{E}_{r+1} & = & V' & \to & 0 \end{array} \quad .$$

If the lower row would be exact, too, then obviously the extension class of E would be trivial. That this indeed holds true is a consequence of the following fact: If a smooth $PGL_{d+1}(K)$- representation is generated by its Iwahori fixed vectors then all of its subrepresentations have the same property. This is proved in [2] 4.4 with B replaced by a principal congruence subgroup. But the same proof works for B if one takes into account the following additional observation: Let $J \subseteq \Delta$ be any subset and let L_J be the standard Levi component of P_J, i.e., the intersection of P_J with its transpose; for any $w \in (W^J)^{-1}$ we then have

$$wBw^{-1} \cap L_J = B \cap L_J \quad .$$

Proposition 6:

If V is an irreducible (admissible) $PGL_{d+1}(K)$-representation such that $V^B \neq 0$ and $V \not\cong V_J(\mathbb{C})$ for all $J \subseteq \Delta$ then $\mathrm{Ext}^*(V_I(\mathbb{C}), V) = 0$.

Proof: Let L denote the subgroup of diagonal matrices in $PGL_{d+1}(K)$, and let $\delta : P \to \mathbb{C}^\times$ be the modulus character. Our assumptions guarantee that there is an unramified character $\chi : L \to \mathbb{C}^\times$ with

$$\chi \neq (w\delta^{-1/2})\delta^{1/2} =: \delta_w \quad \text{for all} \quad w \in W$$

such that V is a constituent of the unramified principal series representation

$$\mathrm{Ind}(\chi) := \{f \in C^\infty(G, \mathbb{C}) : f(hg) = \chi(h) f(g) \text{ for all } h \in P, g \in G\}$$

where we view χ as a character of P in the obvious way ([5] X.3.2). By an easy induction argument (compare the proof of X.4.3 in [5]) it suffices to prove that

$$\mathrm{Ext}^*(V_I(\mathbb{C}), \mathrm{Ind}(\chi)) = 0 \quad .$$

From [13] A.12 and 1.3 we get

$$\begin{aligned} \mathrm{Ext}^*(V_I(\mathbb{C}), \mathrm{Ind}(\chi)) &= \mathrm{Ext}^*_L(V_I(\mathbb{C})_U, \chi) \\ &= \bigoplus_{w \in V^I} \mathrm{Ext}^*_L(\delta_w, \chi) \quad ; \end{aligned}$$

here U denotes the unipotent radical of P and Ext^*_L is the Ext-functor on the category of smooth L-representations. Because of $\delta_w \neq \chi$ the same argument as in [5] IX.1.9 shows that those last groups vanish.

The following is the main result of Casselman ([13] 2.1 or [5] XI.4.5).

Proposition 7:

If $V_J(\mathbb{C})$ is a constituent of Ind_Γ *for some $J \subseteq \Delta$ then $J = \phi$ or Δ.*

If we apply these three Propositions to our above decomposition of Ind_Γ then we obtain

$$\begin{aligned} &\mathrm{Ext}^*(V_I(\mathbb{C}), \mathrm{Ind}_\Gamma) \\ &= \mathrm{Ext}^*(V_I(\mathbb{C}), \mathbb{C}) \oplus \left[\mathrm{Hom}_G(V_\phi(\mathbb{C}), \mathrm{Ind}_\Gamma) \otimes \mathrm{Ext}^*(V_I(\mathbb{C}), V_\phi(\mathbb{C}))\right] \quad . \end{aligned}$$

Observe that $\mathrm{Hom}_G(V_\Delta(\mathbb{C}), \mathrm{Ind}_\Gamma) = \mathbb{C}$.

Proposition 8:

$$\mathrm{Ext}^r(V_I(\mathbb{C}), \mathbb{C}) \cong \begin{cases} \mathbb{C} & \text{if } r = \#\Delta \setminus I \ , \\ 0 & \text{otherwise} \quad . \end{cases}$$

Proof: Put $\overline{I} := \{d + 1 - i : i \in I\}$. The considerations on p. 915 in [13] imply that $V_{\overline{I}}(\mathbb{C})$ is isomorphic to the smooth contragredient of $V_I(\mathbb{C})$. By [12] A.11 we then have

$$\mathrm{Ext}^*(V_I(\mathbb{C}), \mathbb{C}) \cong \mathrm{Ext}^*(\mathbb{C}, V_{\overline{I}}(\mathbb{C})) \quad .$$

But those last groups were computed by Casselman ([13] A.13 or [5] X.4.7).

In order to determine the remaining terms we need the following general duality statement. Let $H_*(.) = H_*(PGL_{d+1}(K),.)$ denote the homology functor on the category of smooth $PGL_{d+1}(K)$-representations (see [13] p. 925).

Duality Theorem:

There is a natural isomorphism

$$\mathrm{Ext}^{d-*}(V_\phi(\mathbb{C}),.) \cong H_*(.) \quad .$$

Proof: We put

$$\mathcal{H} := \text{space of all } \mathbb{C}\text{-valued locally constant functions with compact support on } PGL_{d+1}(K) \ .$$

Via convolution (fixing once and for all a Haar measure on $PGL_{d+1}(K)$) $\mathcal{H}$ is an associative ring which acts in a natural way on any smooth $PGL_{d+1}(K)$-representation V. The space $\mathcal{H}$ itself is a smooth representation via left translations. More generally we have the smooth representation $\mathcal{H} \underset{\mathbb{C}}{\otimes} V$ with $PGL_{d+1}(K)$ acting only on the first factor. The map

$$\rho_V : \mathcal{H} \underset{\mathbb{C}}{\otimes} V \longrightarrow V$$
$$\varphi \otimes v \longmapsto \varphi v$$

then is a $PGL_{d+1}(K)$-equivariant epimorphism. By a result of Blanc (see [13] A.4) $\mathcal{H} \underset{\mathbb{C}}{\otimes} V$ is a projective object in the category of smooth representations. The maps $\rho_V, \rho_{\ker(\rho_V)}, \ldots$ therefore constitute a functorial projective resolution of V which we will use later on. We also observe that the map $\rho_{\mathbb{C}} \otimes id_V$ induces a natural $\mathbb{C}$-linear isomorphism

$$H_0(\mathcal{H} \underset{\mathbb{C}}{\otimes} V) \cong V \quad .$$

In order to understand the groups $\mathrm{Ext}^*(V_\phi(\mathbb{C}), V)$ we use the following explicit resolution of $V_\phi(\mathbb{C})$. Let $\mathcal{BT}_{(q)}$ be the set of oriented q-simplices of $\mathcal{BT}$; for $\sigma \in \mathcal{BT}_{(q)}$ with $q \geq 1$ we denote by $\overline{\sigma}$ that oriented simplex with the same underlying simplex as σ but with the opposite orientation. Let $C_c^{or}(\mathcal{BT}_{(q)}, \mathbb{C})$ denote the space of $\mathbb{C}$-valued oriented q-cochains with finite support on $\mathcal{BT}$. On the one hand we can view $C_c^{or}(\mathcal{BT}_{(q)}, \mathbb{C})$ also as the space of oriented q-chains so that we have the boundary map

$$\partial_q : C_c^{or}(\mathcal{BT}_{(q+1)}, \mathbb{C}) \longrightarrow C_c^{or}(\mathcal{BT}_{(q)}, \mathbb{C}) \quad .$$

Because of the contractibility of $\mathcal{BT}$ the complex $(C_c^{or}(\mathcal{BT}_{(.)}, \mathbb{C}), \partial.)$ is a resolution of $\mathbb{C}$. On the other hand since $\mathcal{BT}$ is locally finite the coboundary map restricts to a map

$$d^q : C_c^{or}(\mathcal{BT}_{(q)}, \mathbb{C}) \longrightarrow C_c^{or}(\mathcal{BT}_{(q+1)}, \mathbb{C}) \quad .$$

By [3] 3.3 and 5.6 we have the exact sequence of smooth representations

$$0 \to C_c^{or}(\mathcal{BT}_{(0)}, \mathbb{C}) \xrightarrow{d^0} \dots \xrightarrow{d^{d-1}} C_c^{or}(\mathcal{BT}_{(d)}, \mathbb{C}) \to V_\phi(\mathbb{C}) \to 0 \quad .$$

We want to show that $C_c^{or}(\mathcal{BT}_{(q)}, \mathbb{C})$ is a projective representation. Fix a $\sigma \in \mathcal{BT}_{(q)}$ and define $\omega_\sigma \in C_c^{or}(\mathcal{BT}_{(q)}, \mathbb{C})$ by

$$\omega_\sigma(\tau) := \begin{cases} +1 & \text{if } \tau = \sigma, \\ -1 & \text{if } \tau = \overline{\sigma}, \\ 0 & \text{otherwise}; \end{cases}$$

let $C(\omega_\sigma) \subseteq C_c^{or}(\mathcal{BT}_{(q)}, \mathbb{C})$ be the subrepresentation generated by ω_σ. Obviously $C_c^{or}(\mathcal{BT}_{(q)}, \mathbb{C})$ is a finite direct sum of subrepresentations of the form $C(\omega_\sigma)$ so that it suffices to show that $C(\omega_\sigma)$ is projective. Let $B_\sigma \subseteq G$ denote the stabilizer of σ; the image of B_σ in $PGL_{d+1}(K)$ is a compact open subgroup. In case there is no $g \in G$ such that $g\sigma = \overline{\sigma}$ we have

$$C(\omega_\sigma) \cong C_c^\infty(G/B_\sigma, \mathbb{C}) \quad .$$

Since $\mathrm{Hom}_G(C_c^\infty(G/B_\sigma, \mathbb{C}), V) = V^{B_\sigma}$ is an exact functor on smooth $PGL_{d+1}(K)$-representations V ([13]App.) we see that $C(\omega_\sigma)$ is projective. Let us now assume that there is a $h \in G$ such that $h\sigma = \overline{\sigma}$. Then h normalizes B_σ with $h^2 \in B_\sigma$; we have

$$C(\omega_\sigma) \cong \{\chi \in C_c^\infty(G/B_\sigma, \mathbb{C}) : \chi(gh) = -\chi(g) \ \text{ for all } \ g \in G\}$$

and consequently

$$\mathrm{Hom}_G(C(\omega_\sigma), V) \cong (V^{B_\sigma})^{h=-1}$$

which again is an exact functor. Therefore $C(\omega_\sigma)$ is projective in this case, too. Using this explicit projective resolution of $V_\phi(\mathbb{C})$ we see that the groups $\mathrm{Ext}^*(V_\phi(\mathbb{C}), V)$ can be computed as the homology groups of the complex

$$\mathrm{Hom}_G((C_c^{or}(\mathcal{BT}_{(.)}, \mathbb{C}), d^{\cdot}), V) \quad .$$

As an immediate consequence we obtain that

$$\text{(1)} \qquad \mathrm{Ext}^r(V_\phi(\mathbb{C}), V) = 0 \ \text{ for } \ r > d \quad .$$

For $V = \mathcal{H}$ a straightforward computation shows that

$$\begin{array}{ccc} \mathrm{Hom}_G((C_c^{or}(\mathcal{BT}_{(.)}, \mathbb{C}), d^{\cdot}), \mathcal{H}) & \xrightarrow{\cong} & (C_c^{or}(\mathcal{BT}_{(.)}, \mathbb{C}), \partial.) \\ f & \longmapsto & (\sigma \mapsto f(\omega_\sigma)(1)) \end{array}$$

is an isomorphism of complexes. This implies, more generally, that

$$\begin{aligned}\mathrm{Hom}_G((C_c^{or}(\mathcal{BT}_{(.)},\mathbb{C}),d^{\cdot}),\mathcal{H}\underset{\mathbb{C}}{\otimes}V) &\cong \mathrm{Hom}_G((C_c^{or}(\mathcal{BT}_{(.)},\mathbb{C}),d^{\cdot}),\mathcal{H})\underset{\mathbb{C}}{\otimes}V \\ &\cong (C_c^{or}(\mathcal{BT}_{(.)},\mathbb{C}),\partial.)\underset{\mathbb{C}}{\otimes}V \quad .\end{aligned}$$

Since $\mathcal{BT}$ is contractible the last complex is a resolution of V which in particular means that

(2) $$\mathrm{Ext}^r(V_\phi(\mathbb{C}),\mathcal{H}\underset{\mathbb{C}}{\otimes}V)=0 \quad \text{for} \quad r<d \quad .$$

From the facts (1) and (2) we deduce (compare [22] I.7.4) that $\mathrm{Ext}^*(V_\phi(\mathbb{C}),.)$ is the left derived functor of $\mathrm{Ext}^d(V_\phi(\mathbb{C}),.)$. It remains to exhibit a natural isomorphism

$$\mathrm{Ext}^d(V_\phi(\mathbb{C}),V)\cong H_0(V) \quad .$$

For that we consider the natural transformation

$$\begin{array}{ccc}\mathrm{Hom}_G(C_c^{or}(\mathcal{BT}_{(0)},\mathbb{C}),V) & \longrightarrow & H_0(V)=V/\ldots \\ f & \longmapsto & f(\omega_\sigma)\,\mathrm{mod}\ldots\end{array}$$

where σ is some fixed vertex of $\mathcal{BT}$. Since G acts transitively on $\mathcal{BT}_{(0)}$ this map actually is independent of the choice of σ. It is surjective since it identifies with the projection map

$$\mathrm{Hom}_G(C_c^{or}(\mathcal{BT}_{(0)},\mathbb{C}),V)=V^{B_\sigma}\xrightarrow{pr}H_0(V)$$

and since the functor $V\mapsto V^{B_\sigma}$ is exact.

Next we claim that

$$\mathrm{im}\,d^0\subseteq\ker(C_c^{or}(\mathcal{BT}_{(1)},\mathbb{C})\xrightarrow{pr}H_0(C_c^{or}(\mathcal{BT}_{(1)},\mathbb{C})))$$

holds true. We have $d^0(\omega_\sigma)=\sum\limits_\tau\omega_{<\tau,\sigma>}$ where the sum ranges over all vertices τ which are neighbours of σ; $<\tau,\sigma>$ then denotes the corresponding oriented 1-simplex. For any τ there is a $g\in G$ such that $\tau=g\sigma$; then also $g^{-1}\sigma$ is a neighbouring vertex of σ. We get

$$pr(\omega_{<\tau,\sigma>})=pr(\omega_{<g\sigma,\sigma>})=pr(\omega_{<\sigma,g^{-1}\sigma>})=-pr(\omega_{<g^{-1}\sigma,\sigma>})$$

and consequently

$$pr(\omega_{<\tau,\sigma>}+\omega_{<g^{-1}\sigma,\sigma>})=0 \quad .$$

If $g^{-1}\sigma=\tau$ then already $pr(\omega_{<\tau,\sigma>})=0$ must hold. We see that $pr(d^0(\omega_\sigma))=0$.

Therefore our above natural transformation vanishes on the image of $\mathrm{Hom}_G(d^0,V)$ and factorizes through a surjective natural transformation

$$\mathrm{Ext}^d(V_\phi(\mathbb{C}),V)\longrightarrow H_0(V) \quad .$$

In order to establish injectivity it suffices to consider the case $V=\mathcal{H}$ where both sides are easily checked to be 1-dimensional over $\mathbb{C}$.

Remark:

Similar considerations as in the above proof show that

$$\mathrm{Ext}^r(\mathbb{C},\mathcal{H}) \cong \begin{cases} V_\phi(\mathbb{C}) & \textit{if } r=d \ , \\ 0 & \textit{otherwise} \ . \end{cases}$$

Proposition 9:

$$\mathrm{Ext}^r(V_I(\mathbb{C}),V_\phi(\mathbb{C})) \cong \begin{cases} \mathbb{C} & \textit{if } r=\#I \ , \\ 0 & \textit{otherwise} \ . \end{cases}$$

Proof: By [13] A.11 and the above duality we have

$$\begin{aligned} \mathrm{Ext}^*(V_I(\mathbb{C}),V_\phi(\mathbb{C})) &\cong \mathrm{Ext}^*(V_\phi(\mathbb{C}),V_{\overline{I}}(\mathbb{C})) \\ &\cong H_{d-*}(V_{\overline{I}}(\mathbb{C})) \quad . \end{aligned}$$

By [13] A.10 the $\mathbb{C}$-dual of this last group is isomorphic to $\mathrm{Ext}^{d-*}(\mathbb{C},V_I(\mathbb{C}))$ so that we again are reduced to Casselman's computation.

Definition:

$\mu(\Gamma) :=$ *multiplicity of the Steinberg representation* $V_\phi(\mathbb{C})$ *in* Ind_Γ.

Altogether we have proved now the following result.

Theorem 3:

If $\Gamma \subseteq PGL_{d+1}(K)$ *is a cocompact discrete subgroup and* A *is a field of characteristic* 0 *then we have*

$$\mathrm{Ext}^r_{A[\Gamma]}(V_I(A),A) \cong \begin{cases} A & \textit{if } \#I=d-r\neq \frac{d}{2} \ , \\ A^{\mu(\Gamma)} & \textit{if } \#I=r\neq\frac{d}{2} \ , \\ A^{\mu(\Gamma)+1} & \textit{if } \#I=r=\frac{d}{2} \ , \\ 0 & \textit{otherwise} \ . \end{cases}$$

In particular we obtain that in our spectral sequence nonvanishing $E_2^{r,s}$-terms only occur on the lines $r=s$ and $r+s=d$. If d is even then all differentials in the spectral sequence automatically must be zero. The only interesting cohomology group of X_Γ is $H^d(X_\Gamma)$. Let

$$H^d(X_\Gamma)=F^0H^d(X_\Gamma)\supseteq F^1H^d(X_\Gamma)\supseteq ...\supseteq F^{d+1}H^d(X_\Gamma)=0$$

be the filtration such that

$$F^r H^d(X_\Gamma)/F^{r+1}H^d(X_\Gamma) = E_\infty^{r,d-r} \quad .$$

Corollary 10:

Assume that $A = H^0(Sp(K))$ is a field of characteristic 0. If d is even we have

$$H^s(X_\Gamma) \cong \begin{cases} A & \text{if } 0 \le s \le 2d,\ s \ne d,\ s \text{ even}, \\ A^{(d+1)\mu(\Gamma)+1} & \text{if } s = d, \\ 0 & \text{otherwise} \end{cases}$$

and

$$F^r H^d(X_\Gamma)/F^{r+1}H^d(X_\Gamma) \cong \begin{cases} A^{\mu(\Gamma)} & \text{if } 0 \le r \le d,\ r \ne \frac{d}{2}, \\ A^{\mu(\Gamma)+1} & \text{if } r = \frac{d}{2}\ . \end{cases}$$

If d is odd we have

$$(d+1)\mu(\Gamma) - 2 \le \dim_A H^d(X_\Gamma) \le (d+1)\mu(\Gamma) \quad .$$

In $\mathbb{Q}_\ell$-adic cohomology the spectral sequence always degenerates for reasons of weight. In case $d = 1$ the next result was proved in [48] by a different method.

Theorem 4:

For $\ell \ne p$ we have

$$H^s_{et}(\overline{X}_\Gamma, \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell\left(-\frac{s}{2}\right) & \text{if } 0 \le s \le 2d,\ s \ne d,\ s \text{ even}, \\ 0 & \text{if } s \ne d \text{ odd or } s > 2d \end{cases}$$

and

$$F^r H^d_{et}(\overline{X}_\Gamma, \mathbb{Q}_\ell)/F^{r+1}H^d_{et}(\overline{X}_\Gamma, \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(r-d)^{\mu(\Gamma)} & \text{if } 0 \le r \le d,\ r \ne \frac{d}{2}, \\ \mathbb{Q}_\ell\left(-\frac{d}{2}\right)^{\mu(\Gamma)+1} & \text{if } r = \frac{d}{2}\ . \end{cases}$$

In de Rham cohomology the spectral sequence degenerates as well, and we can use it in order to strengthen a result of Mustafin ([39] 4.1.II).

Theorem 5:

If K is of characteristic 0 we have

$$H^d_{DR}(X_\Gamma) \cong \begin{cases} K^{(d+1)\mu(\Gamma)} & \text{if } d \text{ is odd}, \\ K^{(d+1)\mu(\Gamma)+1} & \text{if } d \text{ is even} \end{cases}$$

and

$$H^s(X_\Gamma, \Omega^j_{X_\Gamma}) \cong \begin{cases} K & \text{if } s = j \neq \frac{d}{2}\ , \\ 0 & \text{if } s \neq j,\ s + j \neq d\ . \end{cases}$$

Proof: By [39] 4.1.I the analytic variety X_Γ is algebraizable to a projective variety over K. The GAGA-principle ([28]) then implies that the coherent and the de Rham cohomology of X_Γ are equal to the corresponding algebraic cohomology groups. But since K has characteristic 0 we know in the algebraic context that the de Rham spectral sequence for X_Γ degenerates and furthermore that the strong Lefschetz theorem holds. From the first fact we deduce that

$$\dim_K H^s_{DR}(X_\Gamma) = \sum_{j=0}^{s} \dim_K H^{s-j}(X_\Gamma, \Omega^j)$$

and from the second that

$$\dim_K H^{2s}_{DR}(X_\Gamma) \geq \dim_K H^s(X_\Gamma, \Omega^s) \geq 1\ \ .$$

Because of this last inequality our above spectral sequence has to degenerate so that $\dim_K H^s_{DR}(X_\Gamma) = 0$ for $s \neq d$ odd, $= 1$ for $s \neq d$ even, and is as asserted for $s = d$.

For both, $\mathbb{Q}_\ell$-adic and de Rham cohomology we have a natural nilpotent monodromy operator on $H^d(X_\Gamma)$ ([15] 1.7.2 and [25]). We expect that our filtration $F^\cdot H^d(X_\Gamma)$ in both cases is the associated monodromy filtration ([15] 1.6) and that $F^\cdot H^d_{DR}(X_\Gamma)$ is opposite to the Hodge-de Rham filtration.

In the last part of this Paragraph we will deduce from Theorem 1 a proof of (a slightly weakened version of) the p-adic Shimura isomorphism stated in [44] p. 226. We assume from now on that K is of characteristic 0 and, for simplicity, we put

$$O := \mathcal{O}(\Omega^{(2)}) \ \text{ and } \ \Omega := \Omega^1(\Omega^{(2)})\ \ .$$

In [44] p. 225 there was constructed a $SL_2(K)$-equivariant homomorphism

$$I : \Omega/dO \longrightarrow C_{har}(\mathcal{BT}, K)$$

where on the right hand side we have the space of K-valued harmonic 1-cochains on the Bruhat-Tits tree $\mathcal{BT}$. It is an exercise to check that under the natural isomorphism

$$\begin{aligned} C_{har}(\mathcal{BT}, K) = \ker(\mathrm{Hom}_{\mathbb{Z}}(C^\infty(\mathbb{P}^1(K), \mathbb{Z}), K) &\longrightarrow K) \\ \mu &\longmapsto \mu(1) \end{aligned}$$

(compare [44] Lemma on p. 226) the map I corresponds to the isomorphism

$$H^1_{DR}(\Omega^{(2)}) = \ker(\mathrm{Hom}_{\mathbb{Z}}(C^\infty(\mathbb{P}^1(K), \mathbb{Z}), K) \longrightarrow K)$$

given in Theorem 1. We therefore have the exact sequence of $SL_2(K)$-modules

$$0 \longrightarrow K \xrightarrow{\subseteq} O \xrightarrow{d} \Omega \xrightarrow{I} C_{har}(\mathcal{BT}, K) \longrightarrow 0 \quad .$$

More generally, tensoring with the natural $SL_2(K)$-representation

$$\mathrm{Sym}^n := n\text{-th symmetric power of } K^2$$

we obtain the exact sequence of $SL_2(K)$-modules

$$0 \to \mathrm{Sym}^n \to O \underset{K}{\otimes} \mathrm{Sym}^n \to \Omega \underset{K}{\otimes} \mathrm{Sym}^n \to C_{har}(\mathcal{BT}, \mathrm{Sym}^n) \to 0$$

for any $n \geq 0$.

Let $x \in O$ denote the coordinate function

$$\begin{aligned} x : \Omega^{(2)}(\hat{\bar{K}}) &\longrightarrow \hat{\bar{K}} \\ [z_0 : z_1] &\longmapsto \frac{z_0}{z_1} \quad . \end{aligned}$$

The O-module Ω is free with generator dx. Also let $e_1 = (1,0)$, $e_2 = (0,1)$ be the standard basis of K^2. We consider the filtration

$$O \underset{K}{\otimes} \mathrm{Sym}^n = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^{n+1} = 0$$

which is defined by

$$\begin{aligned} F^q := &\text{ the } O\text{-submodule generated by} \\ &(xe_1 + e_2)^q e_1^{n-q-j} e_2^j \ \text{ for } \ 0 \leq j \leq n-q \quad ; \end{aligned}$$

it induces a corresponding filtration

$$\Omega \underset{K}{\otimes} \mathrm{Sym}^n = \Omega \underset{O}{\otimes} F^0 \supseteq \Omega \underset{O}{\otimes} F^1 \supseteq \ldots \supseteq \Omega \underset{O}{\otimes} F^{n+1} = 0 \quad .$$

(This is inspired by [1].) Because of

$$g(xe_1 + e_2) = (-\gamma x + \alpha)^{-1} \cdot (xe_1 + e_2) \ \text{ for } \ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(K)$$

both filtrations are $SL_2(K)$-invariant. $O \underset{K}{\otimes} \mathrm{Sym}^n$ is a free O-module with basis $e_1^{n-q} e_2^q$ for $0 \leq q \leq n$. On the other hand from $e_1^i e_2 = (xe_1 + e_2)e_1^i - xe_1^{i+1}$ we deduce inductively that

$$(xe_1 + e_2)^q e_1^{n-q-j} e_2^j \in O(xe_1 + e_2)^q e_1^{n-q} + F^{q+1} \ \text{ for } \ 1 \leq j \leq n-q \quad .$$

Therefore the elements $(xe_1+e_2)^q e_1^{n-q}$ for $0 \leq q \leq n$ also generate $O \underset{K}{\otimes} \mathrm{Sym}^n$ as an O-module and then have to be an O-basis, too. This shows that

$$\begin{aligned} O &\longrightarrow F^q/F^{q+1} \\ f &\longmapsto f(xe_1+e_2)^q e_1^{n-q} \bmod F^{q+1} \end{aligned}$$

is an O-module isomorphism for $0 \leq q \leq n$. Using the formula

$$\alpha e_1 + \gamma e_2 = (-\gamma x + \alpha) e_1 + \gamma (xe_1 + e_2)$$

we compute

$$\begin{aligned} & g[f(xe_1+e_2)^q e_1^{n-q}] \\ & \quad = g(f) \cdot (-\gamma x+\alpha)^{-q} \cdot (xe_1+e_2)^q \cdot (\alpha e_1 + \gamma e_2)^{n-q} \\ & \quad = g(f) \cdot (-\gamma x+\alpha)^{-q} \cdot (xe_1+e_2)^q \cdot [(-\gamma x+\alpha)e_1 + \gamma(xe_1+e_2)]^{n-q} \\ & \quad \equiv g(f) \cdot (-\gamma x+\alpha)^{n-2q} \cdot (xe_1+e_2)^q e_1^{n-q} \bmod F^{q+1} \end{aligned}$$

for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(K)$. This means that the above isomorphism becomes an O-module and a $SL_2(K)$-module isomorphism

$$\begin{aligned} \Theta_q : O[n-2q] &\longrightarrow F^q/F^{q+1} \\ f &\longmapsto f(xe_1+e_2)^q e_1^{n-q} \bmod F^{q+1} \end{aligned}$$

if, for any $m \in \mathbb{Z}$, $O[m]$ denotes O with the twisted $SL_2(K)$-action

$$\pi_m(g)f := g(f) \cdot (-\gamma x+\alpha)^m \quad \text{for} \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad .$$

Similarly we have an O-$SL_2(K)$-isomorphism

$$\begin{aligned} \Theta'_q : O[n-2q-2] &\longrightarrow \Omega \underset{O}{\otimes} F^q / \Omega \underset{O}{\otimes} F^{q+1} \\ f &\longmapsto dx \otimes f(xe_1+e_2)^q e_1^{n-q} \bmod \ldots \quad . \end{aligned}$$

The identity

$$(*) \qquad \begin{aligned} & (d \otimes id)(f(xe_1+e_2)^q e_1^{n-q}) \\ & \quad = dx \otimes qf(xe_1+e_2)^{q-1} e_1^{n-q+1} + df \otimes (xe_1+e_2)^q e_1^{n-q} \end{aligned}$$

for $1 \leq q \leq n$ implies

$$(d \otimes id)(f(xe_1+e_2)^q e_1^{n-q}) \equiv dx \otimes qf(xe_1+e_2)^{q-1} e_1^{n-q+1} \bmod \Omega \underset{O}{\otimes} F^q \quad .$$

This is the commutativity of the diagram

$$\begin{array}{ccc} O[n-2q] & \xrightarrow{q\cdot} & O[n-2q] \\ \Theta_q \big\downarrow & & \big\downarrow \Theta'_{q-1} \\ F^q/F^{q+1} & \xrightarrow{d\otimes id} & \Omega \underset{O}{\otimes} F^{q-1}/\Omega \underset{O}{\otimes} F^q \end{array}$$

for $1 \leq q \leq n$. As a consequence we obtain that $d \otimes id$ induces an isomorphism

$$F^1 \xrightarrow{\cong} \Omega \underset{K}{\otimes} \mathrm{Sym}^n / \Omega \underset{O}{\otimes} F^n$$

so that we have in particular the $SL_2(K)$-invariant decomposition

$$\Omega \underset{K}{\otimes} \mathrm{Sym}^n = (d \otimes id)(F^1) \oplus (\Omega \underset{O}{\otimes} F^n) \quad .$$

We see that our original exact sequence can be simplified to

$$0 \to \mathrm{Sym}^n \to O \underset{K}{\otimes} \mathrm{Sym}^n / F^1 \to \Omega \underset{O}{\otimes} F^n \to C_{har}(\mathcal{BT}, \mathrm{Sym}^n) \to 0 \quad .$$

In order to compute the map in the middle we observe that the above identity $(*)$ also implies

$$dx \otimes f(xe_1 + e_2)^q e_1^{n-q} \equiv -\frac{1}{q+1} df \otimes (xe_1 + e_2)^{q+1} e_1^{n-(q+1)} \operatorname{mod}(d \otimes id)(F^1)$$

for $0 \leq q < n$. Inductively we get

$$dx \otimes fe_1^n \equiv (-1)^n \frac{1}{n!} dx \otimes f^{(n)}(xe_1 + e_2)^n \operatorname{mod}(d \otimes id)(F^1) \quad .$$

Here, as usual, we put $f^{(n)} := (f^{(n-1)})'$ where $df = f'dx$. In particular, the diagram

$$\begin{array}{ccc} O \underset{K}{\otimes} \mathrm{Sym}^n / F^1 & \longrightarrow & \Omega \underset{O}{\otimes} F^n \\ \Theta_0 \big\uparrow & & \big\uparrow \Theta'_n \\ O[n] & \longrightarrow & O[-n-2] \\ & f \mapsto (-1)^n \frac{1}{n!} f^{(n+1)} & \end{array}$$

is commutative. In $O[n]$ we have the $SL_2(K)$-invariant subspace P_n of all polynomials in x of degree $\leq n$; obviously it is the kernel of the map $f \mapsto f^{(n+1)}$. Thus we finally arrive at the exact sequence of $SL_2(K)$-modules

$$\begin{array}{ccccccccc} 0 & \to & P_n & \xrightarrow{\subseteq} & O[n] & \to & O[-n-2] & \to & C_{har}(\mathcal{BT}, \mathrm{Sym}^n) & \to & 0 \\ & & & & & f \mapsto f^{(n+1)} & & f \mapsto (I \otimes id)(dx \otimes f(xe_1 + e_2)^n) & & & \end{array} \quad .$$

We now fix a cocompact discrete subgroup $\Gamma \subseteq SL_2(K)$. Let $\mathcal{M}_{n+2}(\Gamma)$ denote the space of all K-rational automorphic forms of weight $n+2$ for Γ, i.e., the space of all $f \in O$ such that

$$f \circ g = f \cdot (\gamma x + \delta)^{n+2} \quad \text{for all} \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma \quad .$$

Theorem: (p-adic Shimura isomorphism)

The map

$$\begin{array}{rcl} I_n : \mathcal{M}_{n+2}(\Gamma) & \longrightarrow & H^0(\Gamma, C_{har}(\mathcal{BT}, \mathrm{Sym}^n)) \\ f & \longmapsto & (I \otimes id)(dx \otimes f(xe_1 + e_2)^n) \end{array}$$

is an isomorphism.

Proof: Because of

$$\mathcal{M}_{n+2}(\Gamma) = O[-n-2]^\Gamma$$

the above exact sequence shows that the injectivity of the map I_n is equivalent to the vanishing of $(O[n]/P_n)^\Gamma$. This vanishing will be established later on as a consequence of more general considerations (see Corollary 12). For the moment we will take it as granted. It then remains to prove that the two K-vector spaces in the assertion have the same finite dimension. By passing to an appropriate normal subgroup of finite index we may assume that Γ is a free group of rank r. According to [44] we then have

$$\dim_K \mathcal{M}_{n+2}(\Gamma) = \begin{cases} r & \text{if } n = 0 \ , \\ (n+1)(r-1) & \text{if } n \geq 1 \ . \end{cases}$$

On the other hand the space of harmonic 1-cochains on $\mathcal{BT}$ is the linear dual of the Steinberg representation:

$$C_{har}(\mathcal{BT}, K) = \mathrm{Hom}_K(V_\phi(K), K) \quad .$$

Since Sym^n carries a natural $SL_2(K)$-invariant nondegenerate bilinear form we obtain

$$\begin{aligned} H^0(\Gamma, C_{har}(\mathcal{BT}, \mathrm{Sym}^n)) &= H^0(\Gamma, \mathrm{Hom}_K(V_\phi(K), \mathrm{Sym}^n)) \\ &= H^0(\Gamma, \mathrm{Hom}_K(V_\phi(K) \underset{K}{\otimes} \mathrm{Sym}^n, K)) \\ &= \mathrm{Hom}_K(H_0(\Gamma, V_\phi(K) \underset{K}{\otimes} \mathrm{Sym}^n), K) \\ &= \mathrm{Hom}_K(H^1(\Gamma, \mathrm{Sym}^n), K) \quad . \end{aligned}$$

Here the last identity is a special case of the Borel-Serre duality for discrete groups ([3] 6.2). Since Γ is free a standard Euler-Poincaré characteristic computation gives

$$\dim_K H^1(\Gamma, \mathrm{Sym}^n) - \dim_K H^0(\Gamma, \mathrm{Sym}^n) = (n+1) \cdot (r-1) \quad .$$

But we have $H^0(\Gamma, \mathrm{Sym}^n) = 0$ for $n \geq 1$.

In order to deal with the remaining problem of the vanishing of $(O[n]/P_n)^\Gamma$ we will analyze the p-adic principal series representations which were introduced by Morita. We fix a locally analytic character $\chi : K^\times \to K^\times$ and put

$$\begin{aligned}\mathcal{L}_\chi := \ & K\text{-vector space of all locally analytic} \\ & \text{functions } \Phi : K \times K \backslash (0,0) \to K \text{ such that} \\ & \Phi(z.,z.) = \chi(z)\Phi(.,.) \text{ for all } z \in K^\times \ .\end{aligned}$$

The group $SL_2(K)$ acts on $\mathcal{L}_\chi$ by

$$g(\Phi)(x,y) := \Phi(\delta x - \beta y, -\gamma x + \alpha y) \ \text{ for } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad .$$

Furthermore $\mathcal{L}_\chi$ in a natural way is a topological vector space. For the definition which is a little bit technical we refer to [35]. The above $SL_2(K)$-action is continuous. This is the p-adic principal series studied by Morita ([35]). In order to analyze the continuous linear forms on $\mathcal{L}_\chi$ we "cover" it in the following way by K-Banach spaces. Let $m \geq 1$ be minimal such that

$$\chi \text{ is analytic on } \{z \in K^\times : |z-1| \leq |\pi|^m\} \quad .$$

The congruence subgroup

$$B_m := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(o) : |\alpha - 1|, |\delta - 1| \leq |\pi|^m, |\beta| \leq |\pi|^{m-1}, |\gamma| \leq |\pi| \right\}$$

acts on the disk

$$\Delta := \{z \in K : |z| \leq |\pi|^{m-1}\}$$

by fractional linear transformations. It also acts isometrically on the Banach space $\mathcal{A}_\chi$ of K-analytic functions on Δ by

$$g(f)(z) := \chi(-\gamma z + \alpha) \cdot f\left(\frac{\delta z - \beta}{-\gamma z + \alpha}\right) \ \text{ for } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad .$$

(In case $K = \mathbb{Q}_p$ the B_m-representation $\mathcal{A}_\chi$ is studied in [43].) We then have the injective continuous B_m-equivariant map

$$\begin{aligned}&\mathcal{A}_\chi \longrightarrow \mathcal{L}_\chi \\ &\quad f \longmapsto \Phi_f(x,y) := \begin{cases} \chi(y) f\left(\dfrac{x}{y}\right) & \text{if } |x| \leq |\pi^{m-1} y| \ , \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

(compare [35] p. 198) which for simplicity we will view as an inclusion. It is not too difficult to see (loc. cit.) that

$$\sum_{g\in SL_2(K)} g\mathcal{A}_\chi = \mathcal{L}_\chi \tag{1}$$

holds true. Now let $\| \ \|$ denote the norm on the K-Banach space $\mathcal{A}_\chi$. For any continuous linear form $L : \mathcal{L}_\chi \to K$ we can define a kind of generalized operator norm

$$\|L\|(g) := \sup\left\{ \frac{|L(g(\Phi_f))|}{\|f\|} : f \in \mathcal{A}_\chi \backslash \{0\} \right\} \quad .$$

Since B_m acts isometrically on $\mathcal{A}_\chi$ we obtain in this way a norm function

$$\begin{aligned} \|L\| : SL_2(K)/B_m &\longrightarrow \mathbb{R}_{\geq 0} \\ gB_m &\longmapsto \|L\|(g) \quad . \end{aligned}$$

The property (1) implies that

$$L = 0 \ \text{ if and only if } \|L\| = 0 \quad . \tag{2}$$

It is also clear that the map $L \mapsto \|L\|$ is $SL_2(K)$-equivariant in the obvious sense.

Lemma 11:

If $L : \mathcal{L}_\chi \to K$ is a Γ-invariant continuous linear form then the norm function $\|L\|$ is bounded.

Proof: We only have to note that by the cocompactness of Γ the set $\Gamma\backslash SL_2(K)/B_m$ is finite.

In order to describe the growth of $\|L\|$ we let $R \subseteq o$ be a fixed set of representatives for the cosets in $\pi^{m-1}o/\pi^{m+1}o$. Put

$$h_\beta := \begin{pmatrix} \pi & \pi^{-1}\beta \\ 0 & \pi^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi & 0 \\ 0 & \pi^{-1} \end{pmatrix} \in SL_2(K)$$

and

$$\Delta_\beta := \{z \in K : |z - \beta| \leq |\pi|^{m+1}\} \ \text{ for } \beta \in R \quad .$$

We have

$$\Delta = \dot{\bigcup_{\beta\in R}} \Delta_\beta \ \text{ and } \ \Delta_\beta = h_\beta(\Delta)$$

(viewing h_β as a fractional linear transformation). For any $f \in \mathcal{A}_\chi$ we define

$$f_\beta := (f \mid \Delta_\beta) \circ h_\beta \in \mathcal{A}_\chi \quad .$$

It is clear that

$$\|f_\beta\| \le \|f\| \quad .$$

On the other hand a straightforward computation shows that

$$\chi(\pi) \cdot \Phi_f = \sum_{\beta \in R} h_\beta(\Phi_{f_\beta}) \quad .$$

For any given continuous linear form L on $\mathcal{L}_\chi$ we therefore obtain

$$|\chi(\pi)| \cdot \frac{|L(\Phi_f)|}{\|f\|} \le \max_\beta \frac{|L(h_\beta(\Phi_{f_\beta}))|}{\|f_\beta\|} \quad .$$

Since f was arbitrary this implies

$$|\chi(\pi)| \cdot \|L\|(1) \le \max_\beta \|L\|(h_\beta) \quad .$$

Replacing L by $L(g.)$ we even see that

(3) $$|\chi(\pi)| \cdot \|L\|(g) \le \max_\beta \|L\|(gh_\beta)$$

holds true for any $g \in SL_2(K)$.

Theorem 6:

If $|\chi(\pi)| > 1$ then there is no nonzero Γ-invariant continuous linear form on $\mathcal{L}_\chi$.

Proof: By Lemma 11 any Γ-invariant continuous linear form L on $\mathcal{L}_\chi$ has a bounded norm function $\|L\|$. But the assumption $|\chi(\pi)| > 1$ together with (3) then implies that $\|L\| = 0$. Because of (2) the linear form L itself has to be zero.

Corollary 12:

For $n \ge 0$ we have $(O[n]/P_n)^\Gamma = 0$.

Proof: The character $z \mapsto z^{-n-2}$ fulfills the assumption of Theorem 6. On the other hand it is shown in [34] p. 292 and [36] p. 394 that $O[n]/P_n$ can be identified $SL_2(K)$-equivariantly with the topological dual of $\mathcal{L}_{z^{-n-2}}$.

Corollary 13:

There are no nonzero automorphic forms of weight 1 for Γ.

Proof: In loc. cit. Morita also shows that $O[-1]$ is $SL_2(K)$-equivariantly isomorphic to the topological dual of $\mathcal{L}_{z^{-1}}$.

By working with the tree of Γ instead of $\mathcal{BT}$ the p-adic Shimura isomorphism can be obtained in an analogous way for any finitely generated discrete subgroup in $SL_2(K)$ with an infinite limit set. A much more direct proof is given in [46]. It was pointed out by de Shalit that contrary to what is claimed in [44] in general not every cohomology class in $H^1(\Gamma, \mathrm{Sym}^n)$ contains a harmonic cochain; nevertheless this holds true if $n = 0$ and as is shown in [46] for any $n \geq 0$ if Γ is arithmetic. Still another proof is to be found in [49].

Remark:

Assume that Γ acts without fixed points on $\Omega^{(2)}$. It follows from $(O/K)^\Gamma = (dO)^\Gamma = 0$ that the filtration $F^{\cdot}H^1_{DR}(X_\Gamma)$ which we introduced earlier is opposite to the de Rham filtration. This confirms in the case of $\Omega^{(2)}$ our general expectation which we explained after Theorem 5. In particular we have the natural decomposition

$$H^1_{DR}(X_\Gamma) = H^1(\Gamma, K) \oplus H^0(X_\Gamma, \Omega^1) \quad .$$

Using the isomorphism $I_0 : H^0(X_\Gamma, \Omega^1) = M_2(\Gamma) \xrightarrow{\cong} H^0(\Gamma, C_{har}(\mathcal{BT}, K)) = H^1(\Gamma, K)$ we can rewrite this as a natural isomorphism

$$H^1_{DR}(X_\Gamma) = H^1(\Gamma, K) \oplus I_0^{-1}(H^1(\Gamma, K)) \quad .$$

§6 Resolutions of $GL_{d+1}(K)$-modules

In a rather general context we study in the following the homology of certain coefficient systems on a simplicial complex. This is then applied to construct the resolutions of our $GL_{d+1}(K)$-modules $V_I(\mathbb{Z})$ which we needed in the last Paragraph.

Let $\mathcal{K}$ be a finite-dimensional simplicial complex, say, of dimension d. In order to make notations simpler we equip $\mathcal{K}$ with a fixed orientation and we let denote $[\tau : \sigma]$ the corresponding incidence numbers; nothing will really depend on this choice. A coefficient system $\underline{\underline{A}}$ on $\mathcal{K}$ consists of

— abelian groups A_σ for each simplex σ of $\mathcal{K}$, and

— homomorphisms $r_\sigma^{\sigma'} : A_{\sigma'} \to A_\sigma$ for each pair $\sigma \subseteq \sigma'$ of simplices of $\mathcal{K}$ such that $r_\sigma^\sigma = \mathrm{id}$ and $r_\sigma^{\sigma''} = r_\sigma^{\sigma'} \circ r_{\sigma'}^{\sigma''}$ whenever $\sigma \subseteq \sigma' \subseteq \sigma''$.

Any such coefficient system gives rise to a homological complex: Let $\mathcal{K}_q$ denote the set of all q-simplices of $\mathcal{K}$. We have the boundary map

$$\begin{aligned} \partial : \bigoplus_{\tau\in\mathcal{K}_{q+1}} A_\tau &\longrightarrow \bigoplus_{\sigma\in\mathcal{K}_q} A_\sigma \\ (a_\tau)_\tau &\longmapsto \Big(\sum_{\tau\supseteq\sigma} [\tau : \sigma] \cdot r_\sigma^\tau(a_\tau)\Big)_\sigma \quad . \end{aligned}$$

A standard computation then shows that

$$\bigoplus_{\tau\in\mathcal{K}_d} A_\tau \xrightarrow{\partial} \ldots \xrightarrow{\partial} \bigoplus_{\sigma\in\mathcal{K}_0} A_\sigma$$

is a complex. For example, if $\underline{\underline{A}}$ is the constant coefficient system given by $\mathbb{Z}$ we obtain in this way the usual chain complex of $\mathcal{K}$.

We are interested in coefficient systems of the following form: Let T be a fixed profinite set and suppose that for each 0-simplex $\sigma \in \mathcal{K}_0$ there is given a continuous surjection $T \to T_\sigma$ onto a finite set T_σ. For an arbitrary simplex $\tau = \{\sigma_0, \ldots, \sigma_q\} \in \mathcal{K}_q$ we put

$$T_\tau := T_{\sigma_0} \coprod_T \ldots \coprod_T T_{\sigma_q} \quad .$$

For any pair $\sigma \subseteq \tau$ of simplices of $\mathcal{K}$ we then have a commutative diagram of continuous surjections

$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & T_\sigma \arrow[dl] \\ & T_\tau & \end{tikzcd} \quad .$$

It is clear that the groups

$$C(T_\sigma, \mathbb{Z}) := \text{ all functions } T_\sigma \longrightarrow \mathbb{Z}$$

form a coefficient system on $\mathcal{K}$. In fact all the groups $C(T_\sigma, \mathbb{Z})$ can be viewed as subgroups of $C^\infty(T, \mathbb{Z})$ so that the transition maps of this coefficient system become inclusion maps. Furthermore the corresponding complex is augmented in a natural way

$$\bigoplus_{\tau \in \mathcal{K}_d} C(T_\tau, \mathbb{Z}) \xrightarrow{\partial} \ldots \xrightarrow{\partial} \bigoplus_{\sigma \in \mathcal{K}_0} C(T_\sigma, \mathbb{Z}) \longrightarrow C^\infty(T, \mathbb{Z})$$
$$(\varphi_\sigma)_\sigma \longmapsto \sum_\sigma \varphi_\sigma \quad .$$

We want to establish a sufficient condition under which this complex is an acyclic resolution of $C^\infty(T, \mathbb{Z})$.

First we observe that for any simplex σ we have the simplicial profinite set

$$T.^{(\sigma)}: \quad \ldots \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} T \underset{T_\sigma}{\times} T \underset{T_\sigma}{\times} T \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} T \underset{T_\sigma}{\times} T \rightrightarrows T$$

together with a continuous map

$$T.^{(\sigma)} \longrightarrow T_\sigma$$

to the constant simplicial (pro)finite set T_σ. We immediately note that the induced cohomological complex

$$0 \longrightarrow C(T_\sigma, \mathbb{Z}) \longrightarrow C^\infty(T_0^{(\sigma)}, \mathbb{Z}) \longrightarrow C^\infty(T_1^{(\sigma)}, \mathbb{Z}) \longrightarrow \ldots$$

is exact. This follows from §3 Lemma 4 since the fiber of $T.^{(\sigma)} \to T_\sigma$ in a fixed $t \in T_\sigma$ is the simplicial profinite set

$$\ldots \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} T_t \times T_t \times T_t \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} T_t \times T_t \rightrightarrows T_t$$

where T_t is the fiber of $T \to T_\sigma$ in t. A simplicial profinite set of this form namely is cohomologically trivial as we have recalled already in the proof of §3 Lemma 3.

Each surjection $T_\sigma \to T_\tau$ for $\sigma \subseteq \tau$ extends in an obvious way to a map of augmented simplicial profinite sets

$$\begin{array}{ccc} T.^{(\sigma)} & \longrightarrow & T_\sigma \\ \downarrow & & \downarrow \\ T.^{(\tau)} & \longrightarrow & T_\tau \quad ; \end{array}$$

similarly each surjection $T \to T_\sigma$ extends to a map $T. \to T.^{(\sigma)}$ of the constant simplicial profinite set $T.$ into $T.^{(\sigma)}$ such that

$$\begin{array}{ccccc} & & T.^{(\sigma)} & & \\ & \nearrow & & \searrow & \\ T. & & \longrightarrow & & T_\sigma \end{array}$$

is commutative. By passing to functions we have corresponding commutative diagrams between coefficient systems on $\mathcal{K}$. All this can be expressed by saying that we have the biaugmented double complex

$$
\begin{array}{ccccccccc}
 & 0 & & & 0 & & 0 & & \\
 & \downarrow & & & \downarrow & & \downarrow & & \\
0 \to & \bigoplus\limits_{\tau\in\mathcal{K}_d} C(T_\tau,\mathbb{Z}) & \to & \ldots \to & \bigoplus\limits_{\sigma\in\mathcal{K}_0} C(T_\sigma,\mathbb{Z}) & \dashrightarrow & C^\infty(T,\mathbb{Z}) & \to 0 & \\
 & \downarrow & & & \downarrow & & \| & & \\
0 \to & \bigoplus\limits_{\tau\in\mathcal{K}_d} C^\infty(T,\mathbb{Z}) & \to & \ldots \to & \bigoplus\limits_{\sigma\in\mathcal{K}_0} C^\infty(T,\mathbb{Z}) & \dashrightarrow & C^\infty(T,\mathbb{Z}) & \to 0 & \\
 & \downarrow & & & \downarrow & & \downarrow{\scriptstyle 0} & & \\
0 \to & \bigoplus\limits_{\tau\in\mathcal{K}_d} C^\infty(T\underset{T_\tau}{\times}T,\mathbb{Z}) & \to & \ldots \to & \bigoplus\limits_{\sigma\in\mathcal{K}_0} C^\infty(T\underset{T_\sigma}{\times}T,\mathbb{Z}) & \dashrightarrow & C^\infty(T\underset{T}{\times}T,\mathbb{Z}) & \to 0 & \\
 & \downarrow & & & \downarrow & & \| & & \\
0 \to & \bigoplus\limits_{\tau\in\mathcal{K}_d} C^\infty(T\underset{T_\tau}{\times}T\underset{T_\tau}{\times}T,\mathbb{Z}) & \to & \ldots \to & \bigoplus\limits_{\sigma\in\mathcal{K}_0} C^\infty(T\underset{T_\sigma}{\times}T\underset{T_\sigma}{\times}T,\mathbb{Z}) & \dashrightarrow & C^\infty(T\underset{T}{\times}T\underset{T}{\times}T,\mathbb{Z}) & \to 0 & \\
 & \downarrow & & & \downarrow & & \downarrow{\scriptstyle 0} & & \\
 & \vdots & & & \vdots & & \vdots & , &
\end{array}
$$

resp.

$$
\begin{array}{ccc}
\bigoplus\limits_{\sigma\in\mathcal{K}.} C(T_\sigma,\mathbb{Z}) & \longrightarrow & C^\infty(T,\mathbb{Z}) \\
\downarrow & & \downarrow \\
\bigoplus\limits_{\sigma\in\mathcal{K}.} C^\infty(T.^{(\sigma)},\mathbb{Z}) & \longrightarrow & C^\infty(T.,\mathbb{Z})
\end{array} \quad .
$$

As already noted the columns are exact, resp. the perpendicular arrows in the second diagram are quasi-isomorphisms. Therefore let us fix some $m \geq 0$ and consider the line

$$0 \to \bigoplus_{\tau\in\mathcal{K}_d} C^\infty(T_m^{(\tau)},\mathbb{Z}) \to \ldots \to \bigoplus_{\sigma\in\mathcal{K}_0} C^\infty(T_m^{(\sigma)},\mathbb{Z}) \to C^\infty(T,\mathbb{Z}) \to 0 \quad .$$

In order to sheafify this sequence we put $T^{m+1} := T \times \ldots \times T$ and let denote $j^{(\sigma)} : T_m^{(\sigma)} \to T^{m+1}$ the inclusion map and $\delta : T \to T^{m+1}$ the diagonal map. We then have the obvious complex of sheaves

$$(*) \qquad 0 \to \bigoplus_{\tau\in\mathcal{K}_d} j_*^{(\tau)}\mathbb{Z} \to \ldots \to \bigoplus_{\sigma\in\mathcal{K}_0} j_*^{(\sigma)}\mathbb{Z} \to \delta_*\mathbb{Z} \to 0$$

on T^{m+1} of which our line is the complex of global sections (the global section functor on a compact space commutes with arbitrary direct sums of sheaves!). Since the sheaves in $(*)$ are soft it is sufficient for the exactness of our line that $(*)$ is an exact complex of sheaves (compare the argument in the proof of §3 Lemma 4); the latter can be checked stalkwise. For $\underline{t} = (t_0, \ldots, t_m) \in T^{m+1}$ we have

$$(j_*^{(\sigma)}\mathbb{Z})_{\underline{t}} = \begin{cases} \mathbb{Z} & \text{if } \underline{t} \in T_m^{(\sigma)}, \\ 0 & \text{otherwise.} \end{cases}$$

This suggests the following definition:

$$\begin{aligned} \mathcal{K}^{(t_0,\ldots,t_m)} := {} & \text{the subcomplex of } \mathcal{K} \text{ consisting of all simplices } \sigma \\ & \text{such that } t_0, \ldots, t_m \text{ are not all identified under the} \\ & \text{map } T \to T_\sigma. \end{aligned}$$

Equivalently, σ is a simplex of $\mathcal{K}^{(t_0,\ldots,t_m)}$ if and only if $\underline{t} \notin T_m^{(\sigma)}$ if and only if $(j_*^{(\sigma)}\mathbb{Z})_{\underline{t}} = 0$. We make now the additional assumption that the natural map $T \to \prod\limits_{\sigma \in \mathcal{K}_0} T_\sigma$ is injective. Then $\mathcal{K}^{(t_0,\ldots,t_m)}$ is empty if and only if $t_0 = \ldots = t_m$ if and only if $(\delta_*\mathbb{Z})_{\underline{t}} \neq 0$. We obtain that the complex of stalks of $(*)$ in $\underline{t}$ is the relative augmented chain complex of the pair $\mathcal{K}^{(t_0,\ldots,t_m)} \subseteq \mathcal{K}$. Therefore if $\mathcal{K}$ and all its non-empty subcomplexes $\mathcal{K}^{(t_0,\ldots,t_m)}$ are contractible the complexes $(*)$ are exact. Going back to our biaugmented double complex we then see that all its lines apart possibly from the first one are exact. But since the columns are exact anyway the first line has to be exact, too. We sum up this discussion in the following criterion.

Proposition 1:
Assume that the map $T \to \prod\limits_{\sigma \in \mathcal{K}_0} T_\sigma$ is injective and that $\mathcal{K}$ and all its non-empty subcomplexes $\mathcal{K}^{(t_0,\ldots,t_m)}$ for $t_i \in T$ are contractible. Then the complex

$$0 \to \bigoplus_{\tau \in \mathcal{K}_d} C(T_\tau, \mathbb{Z}) \to \ldots \to \bigoplus_{\sigma \in \mathcal{K}_0} C(T_\sigma, \mathbb{Z}) \to C^\infty(T, \mathbb{Z}) \to 0$$

is exact.

We want to apply this result in the concrete situation where the underlying simplicial complex is the Bruhat-Tits building $\mathcal{BT}$ and where the given profinite set is $T := G/P_I$ for some fixed subset $I \subseteq \Delta = \{1, \ldots, d\}$. Let σ be a simplex in $\mathcal{BT}$. Then

$$B_\sigma := \{g \in G : g\sigma = \sigma \ \text{ and } \ \det g \in o^\times\}$$

is a compact open subgroup in G; it has a unique maximal normal pro-p-subgroup U_σ which itself is compact open. The set

$$T_\sigma := U_\sigma \backslash T$$

is finite. We have the continuous projection $T \to T_\sigma$ and

$$C(T_\sigma, \mathbb{Z}) = C^\infty(G/P_I, \mathbb{Z})^{U_\sigma} \quad .$$

It is also obvious that $U_\sigma \subseteq U_\tau$ if $\sigma \subseteq \tau$.

Lemma 2:

If $\sigma_0, \ldots, \sigma_q$ are the vertices of the simplex σ in $\mathcal{BT}$ then U_σ is generated by $U_{\sigma_0} \cup \ldots \cup U_{\sigma_q}$.

Proof: Let τ be that maximal simplex in $\mathcal{BT}$ for which $B_\tau = B$ is the standard Iwahori subgroup considered in §4. If $\tau_0, \ldots, \tau_d$ are the vertices of τ appropriately enumerated we have

$$B_{\tau_j} = y_j GL_{d+1}(o) y_j^{-1} \quad .$$

Using the notations introduced in §4 we observe that

$$(1) \qquad B_{\tau_0} \cap B_{\tau_j} = B_{\Delta\setminus\{j\}} \quad \text{and}$$

$$(2) \qquad B \cap U_{\Delta\setminus\{j\}} \subseteq y_j U_{\tau_0} y_j^{-1} = U_{\tau_j} \quad .$$

By conjugation we may assume that $\sigma \subseteq \tau$ and $\sigma_0 = \tau_0, \sigma_1 = \tau_{j_1}, \ldots, \sigma_q = \tau_{j_q}$. From (1) we then deduce that

$$B_\sigma = B_J \quad \text{with} \quad J := \Delta\setminus\{j_1, \ldots, j_q\} \quad .$$

Furthermore one easily checks that

$$U_\sigma = U_{\sigma_0} \cdot (B \cap U_J) \quad .$$

By (2) we therefore are reduced to prove that

$$B \cap U_J \subseteq U_{\sigma_0} \cdot (B \cap U_{\Delta\setminus\{j_1\}}) \cdot \ldots \cdot (B \cap U_{\Delta\setminus\{j_q\}}) \quad .$$

Reducing $\bmod\, \pi$ this amounts to the statement that the unipotent radical of a fixed parabolic subgroup is equal to the product of the unipotent radicals of the maximal proper parabolic subgroups containing the given one. This is a well-known fact (compare [4] 3.2) which for GL_{d+1} can be seen easily using elementary matrix transformations.

The Lemma says that

$$T_\sigma = T_{\sigma_0} \coprod_T \ldots \coprod_T T_{\sigma_q}$$

so that we are in the situation which was discussed above.

Lemma 3:

The map $T \to \prod\limits_{\sigma \in \mathcal{BT}_0} T_\sigma$ *is injective.*

Proof: Since points in the same fiber of the map in question cannot be separated by functions in the image of the homomorphism

$$\bigoplus_{\sigma \in \mathcal{BT}_0} C(T_\sigma, \mathbb{Z}) \longrightarrow C^\infty(T, \mathbb{Z})$$

it suffices to show that the latter is surjective. According to §4 Corollary 9' the G-module $C^\infty(T, \mathbb{Z})$ is generated by the characteristic function $\chi_{B_I P_I}$. If $\sigma_0 \in \mathcal{BT}_0$ is a vertex such that $U_{\sigma_0} \subseteq B_I$ then $\chi_{B_I P_I} \in C^\infty(T, \mathbb{Z})^{U_{\sigma_0}} = C(T_{\sigma_0}, \mathbb{Z})$. Therefore the above G-module homomorphism contains a generator of the right hand side in its image and consequently has to be surjective.

The Bruhat-Tits building is contractible. Therefore it remains to deal with the last assumption in Proposition 1 that all the non-empty subcomplexes $\mathcal{BT}^{(t_0,\ldots,t_m)}$ of $\mathcal{BT}$ are contractible. This requires a more elaborate argument. We will first describe $\mathcal{BT}^{(t_0,\ldots,t_m)}$ as the union of an explicitly given family of apartments in $\mathcal{BT}$. As a convenient technical tool we interprete the sets T_σ in terms of lattices. Recall that a simplex σ in $\mathcal{BT}$ is a family $\sigma = \{[L_0], \ldots, [L_q]\}$ of similarity classes of o-lattices in K^{d+1} such that

$$L_0 \underset{\neq}{\subseteq} L_1 \underset{\neq}{\subseteq} \ldots \underset{\neq}{\subseteq} L_q \underset{\neq}{\subseteq} \pi^{-1} L_0 =: L_{q+1} \quad .$$

It is clear that U_σ is the subgroup of all elements in G which fix each L_j and induce the identity on each L_{j+1}/L_j. We now introduce

$$\begin{aligned} \mathcal{T}(L_0, \ldots, L_q) := {} & \text{set of all sequences } (X_0, \ldots, X_q) \text{ of } o\text{-lattices} \\ & \text{such that } L_j \subseteq X_j \subseteq L_{j+1} \text{ for all } 0 \le j \le q; \end{aligned}$$

it is partially ordered by

$$(X_0, \ldots, X_q) \le (Y_0, \ldots, Y_q) \;\text{ if }\; X_j \subseteq Y_j \;\text{ for all }\; 0 \le j \le q \quad .$$

Furthermore we call

$$\dim(X_0, \ldots, X_q) := \sum_{j=0}^{q} \dim_{o/\pi o} X_j / L_j$$

the dimension of the element $(X_0, \ldots, X_q) \in \mathcal{T}(L_0, \ldots, L_q)$. Assume that $\Delta \backslash I = \{i_0 < i_1 < \ldots < i_r\}$. A flag $\xi_0 \le \ldots \le \xi_r$ in $\mathcal{T}(L_0, \ldots, L_q)$ is called of type I if $\dim \xi_\nu = i_\nu$ for all $0 \le \nu \le r$. It is rather obvious that the map

$$\begin{aligned} \mathcal{T}(L_0) & \longrightarrow \mathcal{T}(L_0, \ldots, L_q) \\ X & \longmapsto (L_0 + X \cap L_1, L_1 + X \cap L_2, \ldots, L_q + X \cap L_{q+1}) \end{aligned}$$

is order and dimension preserving. Also the group U_σ acts on $\mathcal{T}(L_0)$ in an order and dimension preserving way.

Proposition 4:

The above map induces a bijection

$$U_\sigma \backslash \{\textit{flags of type } I \textit{ in } \mathcal{T}(L_0)\} \xrightarrow{\sim} \{\textit{flags of type } I \textit{ in } \mathcal{T}(L_0, \ldots, L_q)\} \ .$$

Proof: To check surjectivity is an easy exercise. The injectivity amounts to the following statement in linear algebra: Let V be a finite dimensional vector space over some field equipped with a flag $0 = V_0 \underset{\neq}{\subseteq} V_1 \underset{\neq}{\subseteq} \ldots \underset{\neq}{\subseteq} V_q \underset{\neq}{\subseteq} V_{q+1} = V$ of subspaces; furthermore let $U_0 \subseteq \ldots \subseteq U_r \subseteq V$ and $U'_0 \subseteq \ldots \subseteq U'_r \subseteq V$ be two other flags of subspaces such that $V_j + U_\nu \cap V_{j+1} = V_j + U'_\nu \cap V_{j+1}$ for all $0 \leq j \leq q$ and $0 \leq \nu \leq r$. Then there exists a $g \in GL(V)$ which fixes each V_j, induces the identity on each V_{j+1}/V_j and fulfills $g(U_\nu) = U'_\nu$ for all $0 \leq \nu \leq r$. By induction with respect to q this is immediately reduced to another slightly more general statement: Let $V_1 \subseteq V$ be a subspace and let $U_0 \subseteq \ldots \subseteq U_r \subseteq V$ and $U'_0 \subseteq \ldots \subseteq U'_r \subseteq V$ be two flags of subspaces such that $U_\nu \cap V_1 = U'_\nu \cap V_1$ and $\overline{g}(V_1 + U_\nu/V_1) = V_1 + U'_\nu/V_1$ for all $0 \leq \nu \leq r$ and some fixed element $\overline{g} \in GL(V/V_1)$. Then there exists a $g \in GL(V)$ which induces the identity on V_1 and $\overline{g}$ on V/V_1 and fulfills $g(U_\nu) = U'_\nu$ for all $0 \leq \nu \leq r$. The proof of this is left to the reader.

Similarly as before a flag $W_0 \subseteq \ldots \subseteq W_r$ of K-subspaces in K^{d+1} is called of type I if $\dim_K W_\nu = i_\nu$ for all $0 \leq \nu \leq r$. We have the bijection

$$\begin{aligned} T = G/P_I &\xrightarrow{\sim} \text{set of all flags of type } I \text{ in } K^{d+1} \\ gP_I &\longmapsto g(\tau_I^*) \end{aligned}$$

where

$$\tau_I^* := \left(\sum_{i=1}^{i_0} K e_i \subseteq \sum_{i=1}^{i_1} K e_i \subseteq \ldots \subseteq \sum_{i=1}^{i_r} K e_i \right) \quad .$$

Corollary 5:

The map

$$\begin{array}{ccc} U_\sigma \backslash T & \xrightarrow{\sim} & \textit{set of all flags of type } I \textit{ in } \mathcal{T}(L_0, \ldots, L_q) \\ (W_0 \subseteq \ldots \subseteq W_r) \bmod U_\sigma & \longmapsto & (\xi_0 \leq \ldots \leq \xi_r) \end{array}$$

with $\xi_\nu := (L_0 + W_\nu \cap L_1, \ldots, L_q + W_\nu \cap L_{q+1})$ *is a bijection.*

Proof: We have

$$L_j + (L_0 + W_\nu \cap \pi^{-1} L_0) \cap L_{j+1} = L_j + W_\nu \cap L_{j+1} \quad .$$

Therefore by Proposition 4 we are reduced to treat the case $q = 0$. Surjectivity is easy to check again. For the injectivity let $t = (W_0 \subseteq \ldots \subseteq W_r)$ and $t' = (W'_0 \subseteq \ldots \subseteq W'_r)$ be two flags in T such that $L_0 + W_\nu \cap \pi^{-1}L_0 = L_0 + W'_\nu \cap \pi^{-1}L_0$ for all $0 \leq \nu \leq r$. Let $P_t \subseteq G$ be that parabolic subgroup which fixes t. Since B_σ is a maximal compact subgroup we find by the Iwasawa decomposition $G = B_\sigma P_t$ a $g \in B_\sigma$ such that $g(t) = t'$. On the other hand let $P_{\overline{t}} \subseteq GL(\pi^{-1}L_0/L_0) = B_\sigma/U_\sigma$ be that parabolic subgroup which fixes $\overline{t} := (\overline{W}_0 \subseteq \ldots \subseteq \overline{W}_r)$ with $\overline{W}_\nu := L_0 + W_\nu \cap \pi^{-1}L_0/L_0$. Our assumption implies $g \bmod U_\sigma \in P_{\overline{t}}$. Since any element in $P_{\overline{t}}$ can be lifted to $B_\sigma \cap P_t$ we find a $h \in B_\sigma \cap P_t$ such that $g \bmod U_\sigma = h \bmod U_\sigma$. We then have $gh^{-1} \in U_\sigma$ with $gh^{-1}(t) = t'$.

In the following we always will make the above identifications and view elements in T, resp. T_σ, as flags of subspaces, resp. lattices. As already mentioned our aim is to describe the subcomplexes $\mathcal{BT}^{(t_0,\ldots,t_m)}$ in terms of apartments in $\mathcal{BT}$. We therefore have to recall briefly this notion. Any basis $v_1, \ldots, v_{d+1}$ of the vector space K^{d+1} determines an apartment $\mathcal{A}$ in $\mathcal{BT}$ which is the full subcomplex generated by all vertices of the form

$$[\pi^{a_1} o v_1 + \ldots + \pi^{a_{d+1}} o v_{d+1}] \quad \text{with} \quad a_1, \ldots, a_{d+1} \in \mathbb{Z} \quad .$$

Its topological realization $|\mathcal{A}|$ is a d-dimensional affine space. The same basis also determines an apartment $\partial\mathcal{A}$ in the topological Tits building $\mathcal{T}.^{(d)}$ which is the full (and finite) subcomplex generated by all vertices of the form

$$Kv_{\alpha_1} + \ldots + Kv_{\alpha_t} \quad \text{with} \quad \phi \neq \{\alpha_1, \ldots, \alpha_t\} \underset{\neq}{\subset} \{1, \ldots, d+1\} \quad .$$

Its topological realization $|\partial\mathcal{A}|$ is a $(d-1)$-sphere. In the Borel-Serre compactification of $|\mathcal{BT}|$ (see [3]) $|\partial\mathcal{A}|$ is the boundary of $|\mathcal{A}|$. Since we don't need this fact in the moment we by definition call $\partial\mathcal{A}$ the boundary of $\mathcal{A}$.

Lemma 6:

Let $W', W'' \in T$ where $\#\Delta\backslash I = 1$. Any apartment $\mathcal{A}$ in $\mathcal{BT}$ such that W' and some 1-dimensional subspace $W \subseteq W''\backslash W'$ are vertices in the boundary $\partial\mathcal{A}$ is contained in $\mathcal{BT}^{(W',W'')}$.

Proof: Fix W and $\mathcal{A}$ as in the statement and let $\sigma = \{[L_0], \ldots, [L_q]\}$ with $L_0 \subsetneqq L_1 \subsetneqq \ldots \subsetneqq L_q \subsetneqq \pi^{-1}L_0 = L_{q+1}$ be a simplex in $\mathcal{A}$. We then find a basis $v_1, \ldots, v_{d+1}$ of K^{d+1} such that

$$W' = Kv_1 + \ldots + Kv_{k-1} \quad \text{for some} \quad 2 \leq k \leq d+1 \quad ,$$
$$W = Kv_k \ , \quad \text{and} \quad L_0 = ov_1 + \ldots + ov_{d+1} \quad .$$

In order to prove the assertion it suffices to find an $0 \le j_0 \le q$ such that

$$L_{j_0} + W \cap L_{j_0+1} \not\subseteq L_{j_0} + W' \cap L_{j_0+1} \quad .$$

Put

$$j_0 := \min\{0 \le j \le q : \pi^{-1} v_k \in L_{j+1}\} - 1 \quad .$$

Since $\pi^{-1} v_k \notin L_0$ and $\pi^{-1} v_k \in L_{q+1}$ we have $0 \le j_0 \le q$. By definition $\pi^{-1} v_k$ is contained in $W \cap L_{j_0+1}$. On the other hand it is easy to check that $\pi^{-1} v_k \notin L_{j_0} + W' \cap L_{j_0+1}$.

Lemma 7:

Let $t = (W_0 \subseteq \ldots \subseteq W_r) \in T$ and let $W' \subseteq K^{d+1}$ be a subspace with $\dim_K W' = \dim_K W_r$. Then $\mathcal{BT}^{(W_r, W')}$ is contained in the union of all apartments in $\mathcal{BT}$ which have t and some 1-dimensional subspace $W \subseteq W' \backslash W_r$ as simplices in their boundary.

Proof: Let $\sigma = \{[L_0], \ldots, [L_q]\}$ with $L_0 \subseteq L_1 \subseteq \ldots \subseteq L_q \subseteq \pi^{-1} L_0 = L_{q+1}$ be a simplex in $\mathcal{BT}^{(W_r, W')}$. We then have

$$(L_0 + W_r \cap L_1, \ldots, L_q + W_r \cap L_{q+1}) \ne (L_0 + W' \cap L_1, \ldots, L_q + W' \cap L_{q+1})$$

but

$$\begin{aligned} \dim(L_0 + W_r \cap L_1, \ldots, L_q + W_r \cap L_{q+1}) &= \dim_K W_r = \dim_K W' \\ &= \dim(L_0 + W' \cap L_1, \ldots, L_q + W' \cap L_{q+1}); \end{aligned}$$

therefore there has to be a $0 \le j_0 \le q$ such that

$$L_{j_0} + W' \cap L_{j_0+1} \not\subseteq L_{j_0} + W_r \cap L_{j_0+1} \quad .$$

We now consider the flag

$$\begin{array}{l} L_0 \subseteq L_0 + W_0 \cap L_1 \subseteq \ldots \subseteq L_0 + W_r \cap L_1 \subseteq \\ L_1 \subseteq \ldots \\ \quad \vdots \\ L_{j_0} \subseteq L_{j_0} + W_0 \cap L_{j_0+1} \subseteq \ldots \subseteq L_{j_0} + W_r \cap L_{j_0+1} \subsetneq L_{j_0} + W_r \cap L_{j_0+1} + W' \cap L_{j_0+1} \subseteq \\ L_{j_0+1} \subseteq \ldots \\ \quad \vdots \\ L_q \subseteq L_q + W_0 \cap L_{q+1} \subseteq \ldots \subseteq L_q + W_r \cap L_{q+1} \subseteq \\ L_{q+1} \end{array}$$

viewed as a flag of subspaces in $\pi^{-1}L_0/L_0$. We fix a $o/\pi o$-basis $\overline{v}_1, \ldots, \overline{v}_{d+1}$ of $\pi^{-1}L_0/L_0$ such that

— any subspace in the above flag is generated by some of the $\overline{v}_\alpha$'s, and

— there is an $1 \leq \alpha_0 \leq d+1$ with

$$\overline{v}_{\alpha_0} \in (L_{j_0} + W' \cap L_{j_0+1}/L_0) \backslash (L_{j_0} + W_r \cap L_{j_0+1}/L_0) \quad .$$

Although some of the inclusions in the above flag might be equalities for any fixed $\overline{v}_\alpha$ precisely one of the following cases holds true:

1) $\overline{v}_\alpha \in L_j + W_\nu \cap L_{j+1}/L_0$ and $\notin L_j + W_{\nu-1} \cap L_{j+1}/L_0$ for some $0 \leq j \leq q$ and $1 \leq \nu \leq r$, or

2) $\overline{v}_\alpha \in L_j + W_0 \cap L_{j+1}/L_0$ and $\notin L_j/L_0$ for some $0 \leq j \leq q$, or

3) $\overline{v}_\alpha \in L_{j+1}/L_0$ and $\notin L_j + W_r \cap L_{j+1}/L_0$ for some $0 \leq j \leq q$ and $\alpha \neq \alpha_0$, or

4) $\overline{v}_\alpha = \overline{v}_{\alpha_0}$.

In cases 1) and 2) we find

$$v_\alpha \in W_\nu \cap L_{j+1} \quad \text{such that} \quad v_\alpha \equiv \overline{v}_\alpha \bmod L_j \quad ;$$

in case 3) we find

$$v_\alpha \in L_{j+1} \quad \text{such that} \quad v_\alpha \equiv \overline{v}_\alpha \bmod L_0 \quad ;$$

finally in case 4) we find

$$v_{\alpha_0} \in W' \cap L_{j_0+1} \quad \text{such that} \quad v_{\alpha_0} \equiv \overline{v}_{\alpha_0} \bmod L_{j_0} \quad .$$

The $v_1, \ldots, v_{d+1}$ constructed in this way form a K-basis of K^{d+1} such that any of the subspaces $W_0, \ldots, W_r$ is generated by some of the v_α's. Put $W := Kv_{\alpha_0}$; we then have $W \subseteq W' \backslash W_r$. Therefore the apartment $\mathcal{A}$ in $\mathcal{BT}$ determined by this basis has t and W as simplices in its boundary. It also contains σ since $L_0 = \pi o v_1 + \ldots + \pi o v_{d+1}$ and any L_j/L_0 is generated by some of the $v_\alpha \bmod L_0$.

Proposition 8:

Let $t_0, \ldots, t_m \in T$ such that $t_\mu = (W_0^{(\mu)} \subseteq \ldots \subseteq W_r^{(\mu)})$. Then $\mathcal{BT}^{(t_0,\ldots,t_m)}$ is equal to the union of all apartments in $\mathcal{BT}$ which have, for some $0 \leq \nu \leq r$ and *some $1 \leq \mu \leq m$, the flag $(W_0^{(0)} \subseteq \ldots \subseteq W_\nu^{(0)})$ and some 1-dimensional subspace $W \subseteq W_\nu^{(\mu)} \backslash W_\nu^{(0)}$ as simplices in their boundary.*

Proof: This is an immediate consequence of the previous two Lemmata once we observe that

$$\begin{aligned}\mathcal{BT}^{(t_0,\ldots,t_m)} &= \bigcup_{1\leq\mu\leq m} \mathcal{BT}^{(t_0,t_\mu)} \\ &= \bigcup_{1\leq\mu\leq m}\bigcup_{0\leq\nu\leq r} \mathcal{BT}^{(W_\nu^{(0)},W_\nu^{(\mu)})}\end{aligned}$$

holds true.

All apartments which occur in the statement of Proposition 8 have the common vertex $W_0^{(0)}$ in their boundary. This enables us to explore the geodesic action of $W_0^{(0)}$ on $\mathcal{BT}$ for our purpose. Any proper subspace $0 \underset{\neq}{\subset} V \underset{\neq}{\subset} K^{d+1}$ defines a simplicial self-map

$$\begin{aligned}\gamma_V : \mathcal{BT} &\longrightarrow \mathcal{BT} \\ [L] &\longmapsto [\pi^{-1}L \cap V + L]\end{aligned}$$

called the geodesic action. In order to see its simplicial nature assume that $L_0 \subseteq L_1 \subseteq \pi^{-1}L_0$ are o-lattices such that $[\pi^{-1}L_0 \cap V + L_0] = [\pi^{-1}L_1 \cap V + L_1]$. Because of $\pi^{-1}L_0 \cap V + L_0 \subseteq \pi^{-1}L_1 \cap V + L_1 \subseteq \pi^{-1}(\pi^{-1}L_0 \cap V + L_0)$ this means that

$$\pi^{-1}L_1 \cap V + L_1 = \pi^{-1}L_0 \cap V + L_0 \quad \text{or} \quad = \pi^{-1}(\pi^{-1}L_0 \cap V + L_0) \quad .$$

In the first case we obtain that

$$\begin{aligned}\dim_K V &= \dim_{o/\pi o} \pi^{-1}L_0 \cap V + L_0/L_0 \\ &= \dim_{o/\pi o}(\pi^{-1}L_1 \cap V + L_1/L_1) + \dim_{o/\pi o} L_1/L_0 \\ &= \dim_K V + \dim_{o/\pi o} L_1/L_0\end{aligned}$$

and similarly in the second case that

$$\dim_K V = \dim_K V + \dim_{o/\pi o} \pi^{-1}L_0/L_1 \quad .$$

This implies $[L_0] = [L_1]$.

Remark:

i. $\{[L], \gamma_V[L]\}$ *is a 1-simplex in* $\mathcal{BT}$*;*
ii. $[L]$ *is a vertex in* $\mathcal{BT}^{(W',W'')}$ *if and only if* $\gamma_{W'}[L] \neq \gamma_{W''}[L]$.

Lemma 9:

If the subcomplex $\mathcal{K}$ of $\mathcal{BT}$ is a union of apartments which have V as a vertex in their boundary then we have:
i. $\gamma_V(\mathcal{K}) \subseteq \mathcal{K}$*;*
ii. the map $|\gamma_V|$ *induced by* γ_V *on the topological realization* $|\mathcal{K}|$ *is homotopic to the identity.*

Proof: i. We can assume that $\mathcal{K} = \mathcal{A}$ is a single apartment and it suffices to show that with any vertex $[L]$ in $\mathcal{A}$ also $\gamma_V[L]$ lies in $\mathcal{A}$. But we find a basis $v_1, \ldots, v_{d+1}$ of K^{d+1} such that

$$\begin{aligned} &V = Kv_1 + \ldots + Kv_k \quad \text{for some } 1 \leq k \leq d \\ &\text{and} \quad L = ov_1 + \ldots + ov_{d+1} \quad . \end{aligned}$$

Then $[\pi^{-1}L \cap V + L] = [\pi^{-1}ov_1 + \ldots + \pi^{-1}ov_k + ov_{k+1} + \ldots + ov_{d+1}]$ obviously lies in $\mathcal{A}$, too.

ii. On $|\mathcal{BT}|$ we have the metric d which restricted to an apartment is the Euclidean metric on the affine space. Furthermore any two points $x, y \in |\mathcal{BT}|$ are joined by a unique geodesic $[xy]$ which lies in every apartment containing x and y. For any real number $0 \leq t \leq 1$ let $tx + (1-t)y$ be the unique point $z \in [xy]$ such that $d(x,z) = (1-t)d(x,y)$. The argument in the proof of part i. shows that with any point $x \in |\mathcal{K}|$ the whole geodesic $[x\ |\gamma_V|(x)]$ is contained in $|\mathcal{K}|$. Therefore the map

$$\begin{aligned} |\mathcal{K}| \times [0,1] &\longrightarrow |\mathcal{K}| \\ (x,t) &\longmapsto tx + (1-t)|\gamma_V|(x) \end{aligned}$$

is well-defined and provides the required homotopy once it turns out to be continuous. But it is the restriction of the composed map

$$\begin{array}{ccccc} |\mathcal{BT}| \times [0,1] & \to & |\mathcal{BT}| \times |\mathcal{BT}| \times [0,1] & \to & |\mathcal{BT}| \\ (x,t) & \mapsto & (x, |\gamma_V|(x), t) & & \\ & & (x,y,t) & \mapsto & tx + (1-t)y \ ; \end{array}$$

here the first arrow is continuous since $|\gamma_V|$ is continuous and the second arrow is continuous according to [9] (2.5.15).

Lemma 10:

Let $\mathcal{K}$ be a subcomplex of $\mathcal{BT}$ with the following two properties:

— $\mathcal{K}$ is a union of apartments which have V as a vertex in their boundary, and

— for any simplex σ in $\mathcal{BT}$ there is a $m \in \mathbb{N}$ such that $\gamma_V^m(\sigma)$ lies in $\mathcal{K}$.

Then $\mathcal{K}$ is contractible.

Proof: By a theorem of Whitehead it suffices to show that $\pi_n(|\mathcal{K}|, x) = 0$ for all $n \geq 0$ and $x \in |\mathcal{K}|$. Let $f : (S^n, s_0) \to (|\mathcal{K}|, x)$ be a continuous map. Since $\mathcal{BT}$ is contractible we find a base point preserving homotopy

$$F : S^n \times [0,1] \to |\mathcal{BT}| \quad \text{with} \quad F(.,0) = x \quad \text{and} \quad F(.,1) = f \quad .$$

The image of F being compact is contained in the topological realization of a finite subcomplex of $\mathcal{BT}$. Our assumptions then guarantee that there is a $m \in \mathbb{N}$ such that the image of $|\gamma_V|^m \circ F$ is contained in $|\mathcal{K}|$. This means that the homotopy class of $|\gamma_V|^m \circ f$ in $\pi_n(|\mathcal{K}|, y)$ with $y := |\gamma_V|^m(x)$ is trivial. But according to our first assumption and the previous Lemma the map $|\gamma_V|^m : \pi_n(|\mathcal{K}|, x) \to \pi_n(|\mathcal{K}|, y)$ is a bijection. Therefore already the homotopy class of f was trivial.

Lemma 11:

Let $W \subseteq K^{d+1}$ be a 1-dimensional subspace such that $V \cap W = \{0\}$. For any simplex σ in $\mathcal{BT}$ there is a $m \in \mathbb{N}$ such that $\gamma_V^m(\sigma)$ is contained in an apartment of $\mathcal{BT}$ which has V and W as vertices in its boundary.

Proof: The simplex σ is of the form $\sigma = \{[L_0], \ldots, [L_q]\}$ with $L_0 \subseteq L_1 \subseteq \ldots \subseteq L_q \subseteq \pi^{-1}L_0 = L_{q+1}$. For any $m \in \mathbb{N}$ we put $L_j^{(m)} := \pi^{-m}L_j \cap V + L_j$ so that we have

$$\gamma_V^m(\sigma) = \{[L_0^{(m)}], \ldots, [L_q^{(m)}]\} \quad .$$

In order to prove the assertion it suffices to find a $m \in \mathbb{N}$ and a $0 \leq j \leq q$ such that

$$L_j^{(m)} + W \cap L_{j+1}^{(m)} \not\subseteq L_j^{(m)} + V \cap L_{j+1}^{(m)} \quad .$$

Then the required apartment can be constructed by exactly the same procedure as in the proof of Lemma 7. It is easiest to deduce a contradiction from the assumption that

$$L_j^{(m)} + W \cap L_{j+1}^{(m)} \subseteq L_j^{(m)} + V \cap L_{j+1}^{(m)} \quad \text{for all } m \in \mathbb{N} \quad \text{and } 0 \leq j \leq q \quad .$$

Since $\dim_K W = 1$ we know from Corollary 5 that given $m \in \mathbb{N}$ there is precisely one j such that $L_j^{(m)} \subsetneqq L_j^{(m)} + W \cap L_{j+1}^{(m)}$. By the pigeonhole principle there must be then one j which we fix from now on and some infinite subset $M \subseteq \mathbb{N}$ such that

$$L_j^{(m)} \subsetneqq L_j^{(m)} + W \cap L_{j+1}^{(m)} \subseteq L_j^{(m)} + V \cap L_{j+1}^{(m)} \quad \text{for all } m \in M \quad .$$

Inserting the definition of $L_j^{(m)}$ we obtain

$$\begin{aligned} \pi^{-m}L_j \cap V + L_j &\subsetneqq (\pi^{-m}L_j \cap V + L_j) + W \cap (\pi^{-m}L_{j+1} \cap V + L_{j+1}) \\ &\subseteq \pi^{-m}L_{j+1} \cap V + L_j \quad \text{for all } m \in M \quad . \end{aligned}$$

This means that for any $m \in M$ there is an element

$$w_m \in W \cap (\pi^{-m} L_{j+1} \cap V + L_{j+1})$$

which is of the form

$$w_m = \pi^{-m} v_m + \ell_m \quad \text{with } v_m \in L_{j+1} \cap V \quad \text{and } \ell_m \in L_j$$

and which is not contained in $\pi^{-m} L_j \cap V + L_j$. The latter implies that $v_m \notin L_j \cap V$. In the identity

$$\pi^m w_m = v_m + \pi^m \ell_m$$

all three terms viewed as sequences in m lie in compact subsets of K^{d+1} so that we may assume replacing M by a smaller subset that all three sequences are even convergent. Now the $\pi^m \ell_m \in \pi^m L_j$ obviously converge to 0; similarly the $\pi^m w_m \in W \cap (L_{j+1} \cap V + \pi^m L_{j+1})$ converge to an element in $W \cap V = \{0\}$, that is to 0. But the v_m lie in the closed set $(L_{j+1} \cap V) \backslash (L_j \cap V)$ so that they cannot converge to 0. This is the wanted contradiction.

Proposition 12:

The subcomplexes $\mathcal{BT}^{(t_0,\ldots,t_m)}$ in $\mathcal{BT}$ are contractible whenever the elements $t_0, \ldots, t_m$ $\in T$ are not all equal.

Proof: The flag t_μ is of the form $t_\mu = (W_0^{(\mu)} \subseteq \ldots \subseteq W_r^{(\mu)})$. Since not all the $t_0, \ldots, t_m$ are equal we find a smallest $0 \leq \nu \leq r$ such that

$$W_\nu^{(0)} \neq W_\nu^{(\mu)} \quad \text{for some } 1 \leq \mu \leq m \quad .$$

We fix a 1-dimensional subspace $W \subseteq W_\nu^{(\mu)} \backslash W_\nu^{(0)}$. From Lemma 6 and Proposition 8 we then deduce that the subcomplex $\mathcal{BT}^{(t_0,\ldots,t_m)}$

— is a union of apartments in $\mathcal{BT}$ which have $V := W_\nu^{(0)}$ as vertex in their boundary, and

— contains all apartments which have V and W as vertices in their boundary.

In this situation Lemma 11 says that $\mathcal{BT}^{(t_0,\ldots,t_m)}$ fulfills the assumptions of Lemma 10 and therefore is contractible.

We now have established all the assumptions of Proposition 1 for $\mathcal{BT}$ and the profinite set $T = G/P_I$ so that Proposition 1 gives the following result.

Theorem 7:

The complex

$$0 \longrightarrow \underset{\tau \in \mathcal{BT}_d}{\oplus} C^\infty(G/P_I, \mathbb{Z})^{U_\tau} \longrightarrow \ldots \longrightarrow \underset{\sigma \in \mathcal{BT}_0}{\oplus} C^\infty(G/P_I, \mathbb{Z})^{U_\sigma}$$
$$\longrightarrow C^\infty(G/P_I, \mathbb{Z}) \longrightarrow 0$$

is exact for any subset $I \subseteq \Delta$.

From this Theorem we want to deduce a similar result for the G-modules $V_I(\mathbb{Z})$. We start with the simple observation that any smooth G-module A (by which we mean an abelian group A with smooth G-action) gives rise to a coefficient system $\underline{\underline{A}}$ on the Bruhat-Tits building $\mathcal{BT}$ in which $A_\sigma := A^{U_\sigma}$ and the transition maps are the obvious inclusions. The corresponding augmented homological complex reads

$$0 \longrightarrow \underset{\tau \in \mathcal{BT}_d}{\oplus} A^{U_\tau} \longrightarrow \ldots \longrightarrow \underset{\sigma \in \mathcal{BT}_0}{\oplus} A^{U_\sigma} \longrightarrow A \longrightarrow 0 \quad .$$

For $A = C^\infty(G/P_I, \mathbb{Z})$ this is exactly the situation which we have discussed above. Clearly the coefficient system $\underline{\underline{A}}$ is functorial in the smooth G-module A. We therefore have, for any complex $\ldots \to A_0 \to \ldots \to A_m \to \ldots$ of smooth G-modules, the double complex

$$\begin{array}{ccccccccc}
 & & \vdots & & & & \vdots & & \vdots & & \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \to & \underset{\tau \in \mathcal{BT}_d}{\oplus} A_m^{U_\tau} & \to & \ldots & \to & \underset{\sigma \in \mathcal{BT}_0}{\oplus} A_m^{U_\sigma} & \to & A_m & \to & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
 & & \vdots & & & & \vdots & & \vdots & & \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \to & \underset{\tau \in \mathcal{BT}_d}{\oplus} A_0^{U_\tau} & \to & \ldots & \to & \underset{\sigma \in \mathcal{BT}_0}{\oplus} A_0^{U_\sigma} & \to & A_0 & \to & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
 & & \vdots & & & & \vdots & & \vdots & & .
\end{array}$$

Since the U_σ are pro-p-groups the above functor $A \mapsto \underline{\underline{A}}$ is exact on smooth $G - \mathbb{Z}\left[\frac{1}{p}\right]$-modules. Some of the complications which we have to overcome in the following are caused by our insisting in $\mathbb{Z}$-coefficients. By different techniques one can show that the above homological complex is exact for all irreducible smooth $G - \mathbb{Q}$-modules A for which $A^{U_\sigma} \neq \{0\}$ for some vertex $\sigma \in \mathcal{BT}_0$. We hope to come back to this in another paper.

The obvious idea to treat the G-modules $V_I(\mathbb{Z})$ is to resolve them by the G-modules $C^\infty(G/P_J, \mathbb{Z})$ for $J \supseteq I$. For some of the subsets $I \subseteq \Delta$ we have done this implicitly already in §3 Proposition 6 by investigating certain topological Tits buildings. This has to be generalized appropriately. Since $V_\Delta(\mathbb{Z}) = C^\infty(G/P_\Delta, \mathbb{Z}) = \mathbb{Z}$ we may assume in the following that $I \underset{\neq}{\subset} \Delta$. Define $\mathcal{T}.^I$ to be the simplicial profinite set of all flags $W_0 \subseteq \ldots \subseteq W_r$ of subspaces in K^{d+1} such that $\dim_K W_\nu \in \Delta\backslash I$ for all $0 \leq \nu \leq r$. Furthermore let $N\mathcal{T}_r^I \subseteq \mathcal{T}_r^I$ denote the open and closed subset of "nondegenerate" flags $W_0 \underset{\neq}{\subset} \ldots \underset{\neq}{\subset} W_r$; $N\mathcal{T}_r^I$ consists of all flags of type $J \supseteq I$ with $\#\Delta\backslash J = r+1$. (Neglecting that in §3 we worked in the dual space we have $\mathcal{T}.^{(k)} = \mathcal{T}.^{\{k+1,\ldots,d\}}$.)

Proposition 13:

The reduced cohomology $\tilde{H}^r(|\mathcal{T}.^I|, \mathbb{Z})$ vanishes for $r \neq d-1-\#I$.

Proof: Put $k := d - \#I$. Since $N\mathcal{T}_r^I = \phi$ for $r \geq k$ it is clear that $\tilde{H}^r(|\mathcal{T}.^I|, \mathbb{Z}) = 0$ for $r \geq k$ (compare the proof of §3 Proposition 6). We will prove the vanishing for $r < k-1$ by induction with respect to d. The case $d = 1$ is trivial. For general d we use the spectral sequence of a certain double complex. Set

$$\begin{aligned} Y.^I := {} & \text{simplicial profinite set of all sequences } (H_0, \ldots, H_r) \\ & \text{of 1-dim. subspaces } H_i \subseteq K^{d+1} \text{ such that} \\ & \dim_K \textstyle\sum_{i=0}^r H_i \leq j \text{ for some } j \in \Delta\backslash I \end{aligned}$$

and

$$\begin{aligned} Z..^I := {} & \text{bisimplicial profinite set of all } (W_0 \subseteq \ldots \subseteq W_r; H_0, \ldots, H_s) \in \mathcal{T}_r^I \times Y_s^I \\ & \text{such that } \sum_{i=0}^s H_i \subseteq W_0. \end{aligned}$$

From the obvious projection maps

$$\begin{array}{ccc} & & Y.^I \\ & & \uparrow \\ \mathcal{T}.^I & \longleftarrow & Z..^I \end{array}$$

we obtain the biaugmented double complex

$$\begin{array}{ccc} & & C^\infty(Y.^I, \mathbb{Z}) \\ & & \downarrow \\ C^\infty(\mathcal{T}.^I, \mathbb{Z}) & \longrightarrow & C^\infty(Z..^I, \mathbb{Z}) \end{array} \quad .$$

The same argument as in §3 Proposition 5 shows that the horizontal augmentation map is a quasi-isomorphism. Therefore the second spectral sequence of this double complex reads

$$E_1^{r,s} = H^s(C^\infty(Z._r^I, \mathbb{Z})) \Longrightarrow H^{r+s}(\mathrm{Tot}\, C^\infty(Z..^I, \mathbb{Z})) = H^{r+s}(|\mathcal{T}.^I|, \mathbb{Z}) \quad .$$

We claim the following partial exactness property of the perpendicular augmentation map: For $0 \leq r \leq k-1$ the sequence

$$0 \longrightarrow C^\infty(Y_r^I, \mathbb{Z}) \longrightarrow C^\infty(Z_{0r}^I, \mathbb{Z}) \longrightarrow \ldots \longrightarrow C^\infty(Z_{k-1-r,r}^I, \mathbb{Z})$$

is exact.

According to the argument in the proof of §3 Lemma 4 (i.e., [20] Chap. II) this can be checked on the fibers of the projection map

$$f. : Z_{.r}^I \longrightarrow Y_r^I \quad .$$

Fix $h = (H_0, \ldots, H_r) \in Y_r^I$ and put $W := \sum_{i=0}^{r} H_i$ and $m := \dim_K W$; we have $m \leq r+1$. The fiber $f.^{-1}(h)$ is the simplicial profinite set of all flags $W_0 \subsetneq \ldots \subsetneq W_r$ such that $\dim_K W_\nu \in \Delta \backslash I$ and $W \subseteq W_0$.

There are two cases to distinguish. First let us assume that $m \in \Delta \backslash I$. Then W is a vertex in $f.^{-1}(h)$ which therefore is cohomologically trivial by the argument in §3 Lemma 3. On the other hand if $m \in I$ then

$$\begin{aligned} f.^{-1}(h) &\longrightarrow \mathcal{T}(K^{d+1}/W).^{I'} \\ (W_0 \subsetneq \ldots \subsetneq W_r) &\longmapsto (W_0/W \subseteq \ldots \subsetneq W_r/W) \end{aligned}$$

is an isomorphism of simplicial profinite sets where the right hand side is the topological Tits building of all flags in K^{d+1}/W whose dimensions are contained in $\{1, \ldots, d-m\} \backslash I'$ with $I' := \{i - m : i \in I \text{ and } i > m\}$. By the induction hypothesis the reduced cohomology of $|\mathcal{T}(K^{d+1}/W).^{I'}|$ vanishes in degrees $< d-m-1-\#I'$. Because of $m \in I$ we have $\#I' < \#I$. We conclude that the reduced cohomology of $|f.^{-1}(h)|$ vanishes in degrees $< k-1-r \leq k-m = d-m-\#I \leq d-m-1-\#I'$. This establishes our claim. In terms of the above spectral sequence it means that

$$\begin{aligned} &E_1^{r,s} = 0 \;\text{ for } r+s < k-1 \;\text{ and } s \neq 0 \;\;, \\ &E_1^{r,0} = C^\infty(Y_r^I, \mathbb{Z}) \;\text{ for } r < k-1, \;\text{ and } C^\infty(Y_{k-1}^I, \mathbb{Z}) \subseteq E_1^{k-1,0} \quad . \end{aligned}$$

Since certainly $d - \#I \leq j$ for some $j \in \Delta \backslash I$ we have

$$Y_r^I = \underbrace{Y_0^I \times \ldots \times Y_0^I}_{r+1-\text{times}} \quad \text{for } r \leq k-1 \quad .$$

The contracting argument in the proof of §3 Lemma 3 then implies that

$$\begin{array}{ccccccccccc} 0 & \to & \mathbb{Z} & \to & C^\infty(Y_0^I, \mathbb{Z}) & \to & \ldots & \to & C^\infty(Y_{k-2}^I, \mathbb{Z}) & \to & C^\infty(Y_{k-1}^I, \mathbb{Z}) \\ & & \| & & \| & & & & \| & & \cap \\ 0 & \to & \mathbb{Z} & \to & E_1^{0,0} & \to & \ldots & \to & E_1^{k-2,0} & \to & E_1^{k-1,0} \end{array}$$

is exact. Altogether we obtain that $E_2^{r,s} = 0$ for $0 < r+s < k-1$ and $E_2^{0,0} = \mathbb{Z}$ which in the abutment gives the vanishing of $\tilde{H}^r(|\mathcal{T}.^I|, \mathbb{Z})$ for $0 \leq r < k-1$. q.e.d.

As already discussed several times we have

$$C^\infty(N\mathcal{T}_r^I, \mathbb{Z}) = \bigoplus_{\substack{I \subseteq J \subseteq \Delta \\ \#\Delta\setminus J = r+1}} C^\infty(G/P_J, \mathbb{Z}) \quad .$$

Therefore Proposition 13 can be reformulated by saying that

$$0 \to \mathbb{Z} \to C^\infty(N\mathcal{T}_0^I, \mathbb{Z}) \to \ldots \to C^\infty(N\mathcal{T}_{d-1-\#I}^I, \mathbb{Z}) \to V_I(\mathbb{Z}) \to 0$$

is an exact sequence of smooth G-modules. In the corresponding diagram

$$\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
\bigoplus_{\sigma\in\mathcal{BT}.} V_I(\mathbb{Z})^{U_\sigma} & \longrightarrow & V_I(\mathbb{Z}) \\
\uparrow & & \uparrow \\
\bigoplus_{\sigma\in\mathcal{BT}.} C^\infty(N\mathcal{T}_{d-1-\#I}^I, \mathbb{Z})^{U_\sigma} & \longrightarrow & C^\infty(N\mathcal{T}_{d-1-\#I}, \mathbb{Z}) \\
\uparrow & & \uparrow \\
\vdots & & \vdots \\
\uparrow & & \uparrow \\
\bigoplus_{\sigma\in\mathcal{BT}.} C^\infty(N\mathcal{T}_0^I, \mathbb{Z})^{U_\sigma} & \longrightarrow & C^\infty(N\mathcal{T}_0^I, \mathbb{Z}) \\
\uparrow & & \uparrow \\
\bigoplus_{\sigma\in\mathcal{BT}.} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

all horizontal arrows apart possibly from the first one are quasi-isomorphisms by Theorem 7. Our aim is to prove that the first arrow is a quasi-isomorphism, too. For this it is sufficient to check that all the left hand columns are exact. Let us fix a simplex $\sigma = \{[L_0], \ldots, [L_q]\}$ in $\mathcal{BT}$ with $L_0 \subsetneqq \ldots \subsetneqq L_q \subsetneqq \pi^{-1}L_0 = L_{q+1}$. Because of

$$C^\infty(N\mathcal{T}_r^I, \mathbb{Z})^{U_\sigma} = C(U_\sigma \backslash N\mathcal{T}_r^I, \mathbb{Z})$$

we deduce from Corollary 5 that

$$C^\infty(N\mathcal{T}_0^I, \mathbb{Z})^{U_\sigma} \longrightarrow \ldots \longrightarrow C^\infty(N\mathcal{T}_{d-1-\#I}^I, \mathbb{Z})^{U_\sigma}$$

is the cochain complex of the flag complex associated with the poset

$$\mathcal{T}^I(\sigma) := \{\xi \in \mathcal{T}(L_0, \ldots, L_q) : \dim \xi \in \Delta\backslash I\} \quad .$$

We only remark that $\mathcal{T}^I(\sigma)$ up to canonical isomorphism does not depend on the choice of $L_0, \ldots, L_q$. In the following we will not distinguish notationally between a poset and its associated flag complex.

Proposition 14:

The reduced cohomology $\tilde{H}^r(|\mathcal{T}^I(\sigma)|, \mathbb{Z})$ vanishes for $r \neq d-1-\#I$.

Proof: This can be proved in a completely analogous way as Proposition 13. Therefore we only explain how to translate the notions we have used in that argument into the present context. First of all it is convenient to work in a slightly more general setting. Suppose we are given finitely many $o/\pi o$-vector spaces $\overline{L}_0, \ldots, \overline{L}_q$ such that

$$\dim_{o/\pi o} \overline{L}_0 + \ldots + \dim_{o/\pi o} \overline{L}_q = d+1 \quad .$$

We then have the poset

$$\begin{aligned} \mathcal{T}(\overline{L}_0, \ldots, \overline{L}_q) :=& \text{set of all sequences } (\overline{X}_0, \ldots, \overline{X}_q) \\ & \qquad \text{where } \overline{X}_j \text{ is an } o/\pi o\text{-subspace of } \overline{L}_j \end{aligned}$$

partially ordered by

$$(\overline{X}_0, \ldots, \overline{X}_q) \leq (\overline{Y}_0, \ldots, \overline{Y}_q) \ \text{ if } \overline{X}_j \subseteq \overline{Y}_j \ \text{ for all } 0 \leq j \leq q \quad .$$

Any element $(\overline{X}_0, \ldots, \overline{X}_q) \in \mathcal{T}(\overline{L}_0, \ldots, \overline{L}_q)$ has the dimension

$$\dim(\overline{X}_0, \ldots, \overline{X}_q) := \sum_{j=0}^{q} \dim_{o/\pi o} \overline{X}_j \quad .$$

Furthermore we may define the sum of any two such elements as

$$(\overline{X}_0, \ldots, \overline{X}_q) + (\overline{Y}_0, \ldots, \overline{Y}_q) := (\overline{X}_0 + \overline{Y}_0, \ldots, \overline{X}_q + \overline{Y}_q) \quad .$$

For any proper subset $I \underset{\neq}{\subset} \Delta$ we consider the poset

$$\mathcal{T}^I(\overline{L}_0, \ldots, \overline{L}_q) := \{\xi \in \mathcal{T}(\overline{L}_0, \ldots, \overline{L}_q) : \dim \xi \in \Delta \backslash I\} \quad .$$

The assertion now is proved simultaneously for all these posets by induction with respect to d. The argument is exactly the same as in the proof of Proposition 13; actually it can be somewhat simplified since no profinite topology is involved.

Proposition 15:

$\tilde{H}^{d-1-\#I}(|\mathcal{T}^I(\sigma)|, \mathbb{Z}) = \tilde{H}^{d-1-\#I}(|\mathcal{T}_\cdot^I|, \mathbb{Z})^{U_\sigma} = V_I(\mathbb{Z})^{U_\sigma}$.

Proof: We will show that the sequence

$$\left(\bigoplus_{i \in \Delta \backslash I} C^\infty(G/P_{I \cup \{i\}}, \mathbb{Z}) \right)^{U_\sigma} \longrightarrow C^\infty(G/P_I, \mathbb{Z})^{U_\sigma} \longrightarrow V_I(\mathbb{Z})^{U_\sigma} \longrightarrow 0$$

is exact. Since U_σ is a pro-p-group taking U_σ-invariants is exact if we deal with $\mathbb{Z}\left[\frac{1}{p}\right]$-modules. Therefore our sequence certainly is exact after tensoring with $\mathbb{Z}\left[\frac{1}{p}\right]$. By conjugation we may assume that $B_\sigma = B_J$ for some $J \subseteq \Delta$ is one of the standard parahoric subgroups considered in §4. If σ_0 denotes the vertex $\sigma_0 = \left[o^{d+1}\right]$ we then have

$$U_{\sigma_0} \subseteq U_\sigma \subseteq B_\sigma = B_J \subseteq B_{\sigma_0} = B_\Delta = GL_{d+1}(o) \quad .$$

Let $\overline{P}_I$, resp. $\overline{U}_I$, be the standard parabolic subgroups (containing the upper triangular matrices), resp. their unipotent radicals, in the group $\overline{G} := GL_{d+1}(o)/U_{\sigma_0} = GL_{d+1}(o/\pi o)$; they are defined in the same way as the P_I and U_I in §4. It is clear that

$$\overline{P}_I = B_I/U_{\sigma_0} \ , \ \overline{P}_J = B_\sigma/U_{\sigma_0} \ , \text{ and } \overline{U}_J = U_\sigma/U_{\sigma_0} \quad .$$

Using the Iwasawa decomposition $G = B_\Delta P_\phi$ we can rewrite our sequence in the following way:

$$\left(\bigoplus_{i\in\Delta\setminus I} C(\overline{G}/\overline{P}_{I\cup\{i\}}, \mathbb{Z})\right)^{\overline{U}_J} \longrightarrow C(\overline{G}/\overline{P}_I, \mathbb{Z})^{\overline{U}_J} \longrightarrow \left(V_I(\mathbb{Z})^{U_{\sigma_0}}\right)^{\overline{U}_J} \longrightarrow 0 \quad .$$

A drastically simplified version of §4 Proposition 4 (no topology is involved!) tells us that the subgroup

$$C(\dot{\bigcup_{w\in V^I}} \overline{P}_\phi w\overline{P}_I/\overline{P}_I, \mathbb{Z})^{\overline{U}_J} = C(\dot{\bigcup_{w\in V^I}} \overline{U}_J\backslash\overline{P}_\phi w\overline{P}_I/\overline{P}_I, \mathbb{Z})$$

in

$$C(\overline{G}/\overline{P}_I, \mathbb{Z})^{\overline{U}_J} = C(\overline{U}_J\backslash G/\overline{P}_I, \mathbb{Z})$$

is a complement of the image of the left hand term in the above sequence, i.e., of the subgroup

$$\sum_{i\in\Delta\setminus I} C(\overline{U}_J\backslash\overline{G}/\overline{P}_{I\cup\{i\}}, \mathbb{Z}) \quad .$$

We recall that

$$V^I = W^I \setminus \bigcup_{i\in\Delta\setminus I} W^{I\cup\{i\}}$$

is the set of all permutations $w \in W$ which are of minimal length in their coset wW_I but not in any coset $wW_{I\cup\{i\}}$ for $i \in \Delta\setminus I$. In this way we are reduced to prove that

$$C(\dot{\bigcup_{w\in V^I}} \overline{P}_\phi w\overline{P}_I/\overline{P}_I, \mathbb{Z}) \longrightarrow V_I(\mathbb{Z})^{U_{\sigma_0}}$$

is bijective. Both sides are $\mathbb{Z}$-free, the left side obviously and the right side since $V_I(\mathbb{Z})$ is $\mathbb{Z}$-free by the argument in the proof of §4 Corollary 5. We already know that the map is bijective after tensoring with $\mathbb{Z}\left[\frac{1}{p}\right]$. Therefore it is at least injective with torsion cokernel. For its surjectivity it then suffices that the cokernel of the map

$$C(\dot{\bigcup_{w\in V^I}} \overline{P}_\phi w\overline{P}_I/\overline{P}_I, \mathbb{Z}) \longrightarrow V_I(\mathbb{Z})$$

is torsionfree. This in turn certainly is the case if the dual map

$$\mathrm{Hom}_{\mathbb{Z}}(V_I(\mathbb{Z}), \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C(\dot{\bigcup_{w\in V^I}} \overline{P}_\phi w\overline{P}_I/\overline{P}_I, \mathbb{Z}), \mathbb{Z})$$

is surjective. We observe that this still is a map of B_ϕ-modules. For any $h \in G$ we have the Dirac distribution

$$\begin{aligned} \delta_h : C^\infty(G/P_I, \mathbb{Z}) &\longrightarrow \mathbb{Z} \\ \psi &\longmapsto \psi(h) \quad . \end{aligned}$$

The action of G on the Dirac distributions is described by the formula

$${}^g(\delta_h) = \delta_{gh} \quad .$$

We now consider the linear form

$$\delta_I := \sum_{w\in W_{\Delta\setminus I}} (-1)^{\ell(w)}\delta_w$$

on $C^\infty(G/P_I, \mathbb{Z})$. Fix an $i \in \Delta\setminus I$ and let $\psi \in C^\infty(G/P_{I\cup\{i\}}, \mathbb{Z})$ be some function. From $\psi(.s_i) = \psi$ and from the decomposition

$$W_{\Delta\setminus I} = (W_{\Delta\setminus I} \cap W^{\{i\}}) \dot{\cup} (W_{\Delta\setminus I} \cap W^{\{i\}})s_i$$

we deduce

$$\begin{aligned} \delta_I(\psi) &= \sum_{w\in W_{\Delta\setminus I}} (-1)^{\ell(w)}\psi(w) \\ &= \sum_{w\in W_{\Delta\setminus I}\cap W^{\{i\}}} (-1)^{\ell(w)}\psi(w) + \sum_{w\in W_{\Delta\setminus I}\cap W^{\{i\}}} (-1)^{\ell(ws_i)}\psi(ws_i) \\ &= 0 \quad . \end{aligned}$$

Therefore δ_I lies in $\mathrm{Hom}_{\mathbb{Z}}(V_I(\mathbb{Z}), \mathbb{Z})$. For any $v \in V^I$ the restriction of the linear form ${}^v(\delta_I)$ to $C(\dot{\bigcup\limits_{w \in V^I}} \overline{P}_\phi w \overline{P}_I / \overline{P}_I, \mathbb{Z})$ is equal to

$$\sum_{\substack{w \in W_{\Delta \setminus I} \\ vw \in V^I W_I}} (-1)^{\ell(w)} \delta_{vw} = \sum_{v' \in V^I} m(v, v') \cdot \delta_{v'}$$

where

$$m(v, v') := \sum_{\substack{w \in W_{\Delta \setminus I} \\ vw \in v' W_I}} (-1)^{\ell(w)} \quad .$$

Because of $W_{\Delta \setminus I} \cap W_I = \{1\}$ the pair $(w, w') \in W_{\Delta \setminus I} \times W_I$ in the identity

$$vw = v'w'$$

is uniquely determined by the pair $(v, v') \in V^I \times V^I$. This implies that $m(v, v') \in \{0, \pm 1\}$ and that $m(v, v) = 1$. Since v' is the element of minimal length in its coset $v'W_I$ we have $v' \leq v'w'$ ($\leq$ denotes the Bruhat order). Also any $v \in V^I$ is the unique element of maximal length in its coset $vW_{\Delta \setminus I}$ so that $vw \leq v$. Together we obtain

$$v' \leq v \quad \text{if} \quad m(v, v') \neq 0 \quad .$$

We see that the matrix formed by the coefficients $m(v, v')$ is integrally invertible. Therefore an appropriate integral linear combination of the ${}^{v'}(\delta_I)$ with $v' \in V^I$ restricts to δ_v on $C(\dot{\bigcup\limits_{w \in V^I}} \overline{P}_\phi w \overline{P}_I / \overline{P}_I, \mathbb{Z})$. Since the δ_v for $v \in V^I$ generate

$$\mathrm{Hom}_{\mathbb{Z}}(C(\dot{\bigcup_{w \in V^I}} \overline{P}_\phi w \overline{P}_I / \overline{P}_I, \mathbb{Z}), \mathbb{Z})$$

as a B_ϕ-module we have established the wanted surjectivity. q.e.d.

We know now that the columns in our initial diagram are exact and as already explained this implies the result we were aiming at.

Theorem 8:

The complex

$$0 \longrightarrow \bigoplus_{\tau \in \mathcal{BT}_d} V_I(\mathbb{Z})^{U_\tau} \longrightarrow \ldots \longrightarrow \bigoplus_{\sigma \in \mathcal{BT}_0} V_I(\mathbb{Z})^{U_\sigma} \longrightarrow V_I(\mathbb{Z}) \longrightarrow 0$$

is exact for all subsets $I \subseteq \Delta$.

We want to complete this discussion by relating $\mathcal{T}^I(\sigma)$ to that simplicial complex which most naturally is associated with the simplex σ. This is the combinatorial Tits building $\mathcal{T}_\sigma$ of the finite reductive group B_σ/U_σ which also can be interpreted as the link of the simplex σ in $\mathcal{BT}$. It is convenient to put ourselves into the slightly more general framework of those simplicial complexes which were introduced in the proof of Proposition 14. We repeat that, given $o/\pi o$-vector spaces $\overline{L}_0, \ldots, \overline{L}_q$ such that

$$\dim_{o/\pi o} \overline{L}_0 + \ldots + \dim_{o/\pi o} \overline{L}_q = d+1 \quad ,$$

we have the poset

$$\begin{aligned} \mathcal{T}(\overline{L}_0, \ldots, \overline{L}_q) := &\text{ set of all sequences } (\overline{X}_0, \ldots, \overline{X}_q) \\ &\text{ where } \overline{X}_j \text{ is an } o/\pi o\text{-subspace of } \overline{L}_j \end{aligned}$$

with the dimension function

$$\dim(\overline{X}_0, \ldots, \overline{X}_q) := \sum_{j=0}^{q} \dim_{o/\pi o} \overline{X}_j \quad .$$

If we put

$$\overline{L} := \overline{L}_0 \oplus \ldots \oplus \overline{L}_q$$

then we may also consider the poset

$$\begin{aligned} \mathcal{T}[\overline{L}_0, \ldots, \overline{L}_q] := &\text{ set of all } o/\pi o\text{-subspaces } \overline{X} \subseteq \overline{L} \\ &\text{ such that } \overline{X} = (\overline{X} \cap \overline{L}_0) \oplus \ldots \oplus (\overline{X} \cap \overline{L}_q) \end{aligned}$$

ordered by inclusion. It is clear that

$$\begin{array}{ccc} \mathcal{T}[\overline{L}_0, \ldots, \overline{L}_q] & \xrightarrow{\sim} & \mathcal{T}(\overline{L}_0, \ldots, \overline{L}_q) \\ \overline{X} & \longmapsto & (\overline{X} \cap \overline{L}_0, \ldots, \overline{X} \cap \overline{L}_q) \end{array}$$

is a dimension preserving isomorphism of posets. On the other hand we fix a semisimple element $s \in GL(\overline{L})$ such that

$$GL(\overline{L}_0) \times \ldots \times GL(\overline{L}_q) = \text{ centralizer of } s \text{ in } GL(\overline{L}) \quad .$$

One then easily checks that $\mathcal{T}[\overline{L}_0, \ldots, \overline{L}_q]$ is the subposet of fixed points of s in $\mathcal{T}[\overline{L}]$:

$$\mathcal{T}[\overline{L}_0, \ldots, \overline{L}_q] = \mathcal{T}[\overline{L}]^s = \mathcal{T}(\overline{L})^s \quad .$$

For any proper subset $I \subsetneq \Delta$ we had considered the poset

$$\mathcal{T}^I(\overline{L}_0, \ldots, \overline{L}_q) := \{\xi \in \mathcal{T}(\overline{L}_0, \ldots, \overline{L}_q) : \dim \xi \in \Delta \backslash I\} \quad .$$

Obviously the $\mathcal{T}^I(\sigma)$ are of this form. Using the above isomorphism we obtain

$$\mathcal{T}^I(\overline{L}_0, \ldots, \overline{L}_q) \cong \{\overline{X} \in \mathcal{T}^\phi(\overline{L})^s : \dim_{o/\pi o} \overline{X} \in \Delta \backslash I\} \quad .$$

The combinatorial Tits building of the group $GL(\overline{L})$, resp. $GL(\overline{L}_0) \times \ldots \times GL(\overline{L}_q)$, by definition is $\mathcal{T}^\phi(\overline{L})$, resp. the join $\mathcal{T}^\phi(\overline{L}_0) * \ldots * \mathcal{T}^\phi(\overline{L}_q)$. In [14] (7.4) it is shown that there is a $GL(\overline{L}_0) \times \ldots \times GL(\overline{L}_q)$-equivariant homeomorphism

$$\mathrm{Susp}^q \;\; |\mathcal{T}^\phi(\overline{L}_0) * \ldots * \mathcal{T}^\phi(\overline{L}_q)| \cong |\mathcal{T}^\phi(\overline{L})^s|$$

where Susp^q denotes the q-fold suspension. We see in particular that for a q-simplex σ in $\mathcal{BT}$ the topological realizations $|\mathcal{T}^I(\sigma)|$ of our simplicial complexes $\mathcal{T}^I(\sigma)$ are B_σ/U_σ-equivariantly homeomorphic to closed subspaces in $\mathrm{Susp}^q\, |\mathcal{T}_\sigma|$. For general I it seems rather complicated to find a description of those subspaces which is intrinsic in terms of $\mathcal{T}_\sigma$. For $I = \phi$ we of course have $|\mathcal{T}^\phi(\sigma)| \cong \mathrm{Susp}^q\, |\mathcal{T}_\sigma|$. This latter remark shows for example that the term $V_\phi(\mathbb{Z})^{U_\sigma}$ in the resolution of Theorem 8 for the Steinberg module $V_\phi(\mathbb{Z})$ is the Steinberg module of the finite reductive group B_σ/U_σ (this can also be seen directly from the computations in the proof of Proposition 15).

The resolutions as stated in the Theorems 7 and 8 have the defect not to be G-equivariant. The reason is that in the above considerations we have fixed an orientation of $\mathcal{BT}$ and this cannot be done in a G-invariant way. But this is not a serious problem. Recall that an ordered q-simplex of $\mathcal{BT}$ is a sequence $(\sigma_0, \ldots, \sigma_q)$ of vertices such that $\{\sigma_0, \ldots, \sigma_q\}$ is a q-simplex in $\mathcal{BT}$. Two such ordered q-simplices are called equivalent if they differ by an even permutation of the vertices; the corresponding equivalence classes are called oriented q-simplices and are denoted by $< \sigma_0, \ldots, \sigma_q >$. Let $\mathcal{BT}_{(q)}$ be the set of all oriented q-simplices of $\mathcal{BT}$. If now A is a smooth G-module we put

$$\begin{aligned} C_c^{or}(\mathcal{BT}_{(q)}, A) := \;& \text{group of all maps } \omega : \mathcal{BT}_{(q)} \to A \\ & \text{such that} \\ & \text{— } \omega \text{ has finite support,} \\ & \text{— } \omega(< \sigma_0, \ldots, \sigma_q >) \in A^{U_{\{\sigma_0, \ldots, \sigma_q\}}}, \\ & \text{— } \omega(< \sigma_{\iota(0)}, \ldots, \sigma_{\iota(q)} >) = \mathrm{sgn}(\iota) \cdot \omega(< \sigma_0, \ldots, \sigma_q >) \\ & \text{for any permutation } \iota. \end{aligned}$$

The group G acts on $C_c^{or}(\mathcal{BT}_{(q)}, A)$ via

$$(g\omega)(< \sigma_0, \ldots, \sigma_q >) := g(\omega(< g^{-1}\sigma_0, \ldots, g^{-1}\sigma_q >)) \quad .$$

With respect to the boundary map

$$\begin{array}{rcl} \partial : C_c^{or}(\mathcal{BT}_{(q+1)}, A) & \longrightarrow & C_c^{or}(\mathcal{BT}_{(q)}, A) \\ \omega & \longmapsto & (<\sigma_0, \ldots, \sigma_q> \mapsto \sum\limits_{\substack{\{\sigma,\sigma_0,\ldots,\sigma_q\} \\ \in \mathcal{BT}_{q+1}}} \omega(<\sigma, \sigma_0, \ldots, \sigma_q>)) \end{array}$$

we then have the augmented complex of smooth G-modules

$$\begin{array}{ccccccc} C_c^{or}(\mathcal{BT}_{(d)}, A) & \xrightarrow{\partial} & \ldots & \xrightarrow{\partial} & C_c^{or}(\mathcal{BT}_{(0)}, A) & \longrightarrow & A \\ & & & & \omega & \longmapsto & \sum\limits_{\sigma \in \mathcal{BT}_{(0)}} \omega(\sigma) \end{array} .$$

It is rather clear that this complex is isomorphic to our original complex

$$\bigoplus_{\tau \in \mathcal{BT}_d} A^{U_\tau} \longrightarrow \ldots \longrightarrow \bigoplus_{\sigma \in \mathcal{BT}_0} A^{U_\sigma} \longrightarrow A \quad .$$

Proposition 16:

Let $\Gamma \subseteq PGL_{d+1}(K)$ be a cocompact discrete subgroup. Any of the G-modules $C^\infty(G/P_I, \mathbb{Z})$ and $V_I(\mathbb{Z})$ for $I \subseteq \Delta$ has a projective resolution by finitely generated free $\mathbb{Z}[\Gamma]$-modules; if Γ is torsionfree then there exists such a resolution of length $\leq d$.

Proof: Let A denote one of these G-modules. By the argument in the proof of §4 Corollary 5 we know that A is $\mathbb{Z}$-free so that the $\mathbb{Z}$-modules A^{U_σ} are finitely generated and free. Let us first assume that Γ is torsionfree. Then Γ acts freely on $\mathcal{BT}$ with a finite number of orbits in the q-simplices for each q. Therefore the $C_c^{or}(\mathcal{BT}_{(q)}, A)$ are finitely generated free $\mathbb{Z}[\Gamma]$-modules and the above constructed resolution has the required properties. In general, Γ contains a normal subgroup Γ' of finite index which is torsionfree ([19] 2.7). Let $B. := \overset{.+1}{\underset{\mathbb{Z}}{\otimes}} \mathbb{Z}[\Gamma/\Gamma']$ denote the (unnormalized) bar resolution of the trivial Γ/Γ'-module $\mathbb{Z}$. If now $C.$ is a resolution of the required type for A as a Γ/Γ'-module then $B. \underset{\mathbb{Z}}{\otimes} C.$ has the required properties for A as a Γ-module.

References

1. Báyer, P., Neukirch, J.: On Automorphic Forms and Hodge Theory. Math. Ann. 257, 137-155 (1981)

2. Bernstein, I.N., Zelevinskii, A.V.: Representations of the group $GL(n, F)$ where F is a non-archimedean local field. Russian Math. Surveys 31:3, 1-68 (1976)

3. Borel, A., Serre, J.-P.: Cohomologie d'immeubles et de groupes S-arithmétiques. Topology 15, 211-232 (1976)

4. Borel, A., Tits, J.: Compléments à l'article: "Groupes réductifs". Publ. Math. IHES 41, 253-276 (1972)

5. Borel, A., Wallach, N.: Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups. Ann. Math. Studies 94. Princeton University Press 1980

6. Bosch, S., Güntzer, U., Remmert, R.: Non-Archimedean Analysis. Berlin-Heidelberg-New York: Springer 1984

7. Bourbaki, N.: Groupes et algèbres de Lie, Chap. 4-6. Paris: Masson 1981

8. Brown, K.S.: Buildings. Berlin-Heidelberg-New York: Springer 1989

9. Bruhat, F., Tits, J.: Groupes réductifs sur un corps local.I. Données radicielles valuées. Publ. Math. IHES 41, 5-252 (1972)

10. Cartier, P.: Les arrangements d'hyperplane: Un chapitre de géométrie combinatoire. Sém. Bourbaki 1980/81, exp. 561, Lecture Notes in Math. 901. Berlin-Heidelberg-New York: Springer 1981

11. Casselman, W.: Introduction to the theory of admissible representations of $\wp$-adic reductive groups. Preprint

12. Casselman, W.: On a p-adic vanishing theorem of Garland. Bull. Soc. AMS 80, 1001-1004 (1974)

13. Casselman, W.: A new non-unitarity argument for p-adic representations. J. Fac. Sci. Univ. Tokyo 28, 907-928 (1981)

14. Curtis, C.W., Lehrer, G.I., Tits, J.: Spherical Buildings and the Character of the Steinberg Representation. Invent. math. 58, 201-210 (1980)

15. Deligne, P.: La conjecture de Weil II. Publ. Math. IHES 52, 137-252 (1980)

16. Drinfel'd, V.G.: Elliptic modules. Math. USSR Sbornik 23, 561-592 (1974)

17. Drinfel'd, V.G.: Coverings of p-adic symmetric regions. Funct. Anal. Appl. 10, 107-115 (1976)

18. Fresnel, J., van der Put, M.: Géométrie Analytique Rigide et Applications. Boston:Birkhäuser 1981

19. Garland, H.: p-adic curvature and the cohomology of discrete subgroups of p-adic groups. Ann. Math. 97, 375-423 (1973)

20. Godement, R.: Topologie algébrique et théorie des faisceaux. Paris: Hermann 1964

21. Goldmann, O., Iwahori, N.: The space of $\wp$-adic norms. Acta Math. 109, 137-177 (1963)

22. Hartshorne, R.: Residues and Duality. Lecture Notes in Math. 20. Berlin-Heidelberg-New York: Springer 1966

23. Hartshorne, R.: On the de Rham cohomology of algebraic varieties. Publ. Math. IHES 45, 5-99 (1975)

24. Hiller, H.: Geometry of Coxeter Groups. Boston: Pitman 1982

25. Hyodo, O.: On the de Rham-Witt complex attached to a semistable family. Preprint 1988

26. Iwahori, N.: Generalized Tits System (Bruhat Decomposition) on $\wp$-Adic Semisimple Groups. In: Algebraic Groups and Discontinuous Subgroups. Proc. Symp. Pure Math. 9, pp. 71-83. American Math. Soc. 1966

27. Jensen, C.U.: Les Foncteurs Dérivés de $\varprojlim$ et leurs Applications en Théorie des Modules. Lecture Notes in Math. 254. Berlin-Heidelberg-New York: Springer 1972

28. Kiehl, R.: Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedischen Funktionentheorie. Invent. math. 2, 191-214 (1967)

29. Kiehl, R.: Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie. Invent. math. 2, 256-273 (1967)

30. Kurihara, A.: Construction of p-adic unit balls and the Hirzebruch proportionality. Amer. J. Math. 102, 565-648 (1980)

31. MacLane, S.: Homology. Berlin-Heidelberg-New York: Springer 1975

32. Mehlmann, F.: Ein Beweis für einen Satz von Raynaud über flache Homomorphismen affinoider Algebren. Schriftenreihe Math. Institut Univ. Münster, 2. Serie, Heft 19. Münster: 1981

33. Milne, J.S.: Etale Cohomology. Princeton University Press 1980

34. Morita, Y.: Analytic Representations of SL_2 over a $\wp$-Adic Number Field, II. Automorphic Forms of Several Variables, Taniguchi Symp. 1983, Progress in Math. 46, pp. 282-297. Boston-Basel-Stuttgart: Birkhäuser 1984

35. Morita, Y.: Analytic Representations of SL_2 over a $\wp$-Adic Number Field, III. Automorphic Forms and Number Theory, Adv. Studies Pure Math. 7, pp. 185-222. Amsterdam: North-Holland 1985

36. Morita, Y., Schikhof, W.: Duality of projective limit spaces and inductive limit spaces over a nonspherically complete nonarchimedean field. Tôhoku Math. J. 38, 387-397 (1986)

37. Mumford, D.: An analytic construction of degenerating curves over complete local rings. Compositio Math. 24, 129-174 (1972)

38. Mumford, D.: Abelian Varieties. Oxford University Press 1974

39. Mustafin, G.A.: Nonarchimedean uniformization. Math. USSR Sbornik 34, 187-214 (1978)

40. van der Put, M.: A note on p-adic uniformization. Proc. Kon. Ned. Akad. Wet. A 90, 313-318 (1987)

41. Rapoport, M.: On the bad reduction of Shimura varieties. In Automorphic Forms, Shimura Varieties, and L-functions II (Eds. L.Clozel, J.S.Milne), Perspectives in Math. 11, pp. 253-321. Boston: Academic Press 1990

42. Raynaud, M.: Geometrie analytique rigide d'apres Tate, Kiehl,... Bull. Soc. math. France, Mémoire 39-40, 319-327 (1974)

43. Robert, A.: Représentations p-adiques irréductibles de sous-groupes ouverts de $SL_2(\mathbb{Z}_p)$. C.R. Acad. Sc. Paris 298, 237-240 (1984)

44. Schneider, P.: Rigid-analytic L-transforms. In: Number Theory, Noordwijkerhout 1983. Lecture Notes in Math. 1068, pp. 216-230. Berlin-Heidelberg-New York: Springer 1984

45. Segal, G.: Classifying spaces and spectral sequences. Publ. Math. IHES 34, 105-112 (1968)

46. de Shalit, E.: Eichler Cohomology and Periods of Modular Forms on p-adic Schottky Groups. J. reine angew. Math. 400, 3-31 (1989)

47. Spanier, E.H.: Algebraic Topology. New York: McGraw-Hill 1966

48. Stuhler, U.: Über die Kohomologie einiger arithmetischer Varietäten I. Math. Ann. 273, 685-699 (1986)

49. Teitelbaum, J.: Values of p-adic L-functions and a p-adic Poisson kernel. Invent. math. 101, 395-410 (1990)

SGA 1 Grothendieck, A.: Revêtements Etales et Groupe Fondamental. Lecture Notes in Math. 224. Berlin-Heidelberg-New York: Springer 1971

SGA 3 Demazure, M., Grothendieck, A.: Schémas en Groupes I. Lecture Notes in Math.151. Berlin-Heidelberg-New York: Springer 1970

SGA 4 Artin, M., Grothendieck, A., Verdier, J.L.: Théorie des Topos et Cohomologie Etale des Schémas. Lecture Notes in Math. 270. Berlin-Heidelberg-New York: Springer 1972