p-adic points of motives

by Peter Schneider

Classically one has two tools to study the group of rational points of an abelian variety over a local number field: One is the logarithm map into the tangent space. The other one is the connecting homomorphism arising from the Kummer sequence into the Galois cohomology of the Tate module. In their seminal paper [BK] Bloch/Kato define the group of points of a motive over a local number field directly in terms of the Galois cohomology of its p-adic realization. Moreover using the Faltings-Fontaine-Messing theory of p-adic Galois representations they construct an exponential map from the de Rham realization into the group of points.

In this talk we observe that the conjectural formalism of motivic cohomology is perfectly suited to extend the classical Kummer theoretic approach to the setting of motives. We will compare this to the Bloch/Kato construction. In this way we will be naturally led to a conjectural expression of the Bloch/Kato groups of points as the cohomology of certain complexes of sheaves. These latter complexes really exist independently of the formalism of motivic cohomology; the conjecture only concerns the bijectivity of the map between the two sides in the expression. Under additional assumptions we will prove that the two sides at least have the same dimension. We will finish with some speculations on a formal deformation theory of motivic cohomology.

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1. Review of some of the local results in [BK]

Throughout the paper $K/\mathbb{Q}_p$ is a fixed finite extension, $G_K := \mathrm{Gal}(\overline{K}/K)$ is the Galois group of an algebraic closure $\overline{K}$ of K, and $K_0 \subseteq K$ is the maximal unramified subextension. For any finite dimensional $\mathbb{Q}_p$-vector space V with a continuous G_K-action we define

$$\begin{aligned} Crys(V) &:= H^0(K, B_{crys} \otimes V) \quad \text{and} \\ DR(V) &:= H^0(K, B_{DR} \otimes V) \quad . \end{aligned}$$

The former is a K_0-vector space with a Frobenius f, the latter is a K-vector space with a decreasing de Rham filtration $DR(V)^{\cdot}$. For a summary of the properties of the rings B_{crys} and B_{DR} we refer to [BK]. One has

$$\dim_{K_0} Crys(V) \subseteq \dim_K DR(V) \subseteq \dim_{\mathbb{Q}_p} V$$

together with a natural injection

$$K \underset{K_0}{\otimes} Crys(V) \hookrightarrow DR(V) \quad .$$

The G_K-representation V is called

$$\begin{array}{llllll} \text{de Rham} & \text{if} & \dim_K DR(V) & = & \dim_{\mathbb{Q}_p} V & , \quad \text{resp.} \\ \text{crystalline} & \text{if} & \dim_{K_0} Crys(V) & = & \dim_{\mathbb{Q}_p} V & . \end{array}$$

We put

$$\begin{aligned} H^1_e(K,V) &:= \ker(H^1(K,V) \longrightarrow H^1(K, B^{f=1}_{crys} \otimes V)) \quad \text{and} \\ H^1_f(K,V) &:= \ker(H^1(K,V) \longrightarrow H^1(K, B_{crys} \otimes V)) \quad ; \end{aligned}$$

here $H^*(K,.)$ denotes as usual the continuous Galois cohomology of G_K. The fundamental exact diagram in [BK] is

$$\begin{array}{ccccccc} 0 \longrightarrow \mathbb{Q}_p & \stackrel{diag}{\longrightarrow} & B^{f=1}_{crys} \oplus B^+_{DR} & \stackrel{\beta}{\longrightarrow} & B_{DR} & \longrightarrow 0 \\ \| & & \downarrow\subseteq & & \downarrow(0,\mathrm{id}) & \\ 0 \longrightarrow \mathbb{Q}_p & \stackrel{diag}{\longrightarrow} & B_{crys} \oplus B^+_{DR} & \stackrel{\gamma}{\longrightarrow} & B_{crys} \oplus B_{DR} & \longrightarrow 0 \end{array}$$

where

$$\beta(x,y) := x - y \quad \text{and} \quad \gamma(x,y) := (x - f(x), x - y) \quad .$$

Tensoring with V and passing to cohomology gives in case V is de Rham (then $H^1(K, B^+_{DR} \otimes V) \to H^1(K, B_{DR} \otimes V)$ is injective!) the exact diagram

$$\begin{array}{ccccc} 0 \to H^0(K,V) & \stackrel{diag}{\to} & Crys(V)^{f=1} \oplus DR(V)^0 & \stackrel{\beta}{\to} & \\ \| & & \downarrow\subseteq & & \\ 0 \to H^0(K,V) & \stackrel{diag}{\to} & Crys(V) \oplus DR(V)^0 & \stackrel{\gamma}{\to} & \\ & \to & DR(V) & \to H^1_e(K,V) \to 0 & \\ & & \downarrow(0,id) & \downarrow\subseteq & \\ & \to & Crys(V) \oplus DR(V) & \to H^1_f(K,V) \to 0. & \end{array}$$

Assuming from now on that V is de Rham we obtain in particular:

— $\dim_{\mathbb{Q}_p} H^1_f(K,V) = \dim_{\mathbb{Q}_p} DR(V)/DR(V)^0 + \dim_{\mathbb{Q}_p} H^0(K,V)$;

— $H^1_f(K,V)/H^1_e(K,V) \cong Crys(V)/(1-f)Crys(V)$;

— the connecting homomorphism induces a surjective map

$$\exp : DR(V)/DR(V)^0 \twoheadrightarrow H^1_e(K,V)$$

with kernel $Crys(V)^{f=1}/H^0(K,V)$.

Defining

$$P(V;u) := \det_{K_0}(1 - f^{[K_0:\mathbb{Q}_p]}u;\, Crys(V))$$

we then have the following

Fact:

If V is de Rham with $P(V;1) \neq 0$ then

$$\exp : DR(V)/DR(V)^0 \xrightarrow{\cong} H^1_e(K,V) = H^1_f(K,V)$$

is an isomorphism.

At least philosophocally this theory applies to the following geometric situation: Let M be a motive over K which is integral in the sense that we can speak about its $\mathbb{Z}_p$-adic realization T. Faltings' proof of the de Rham conjecture of Fontaine suggests that the $\mathbb{Q}_p$-adic realization $V = T \underset{\mathbb{Z}_p}{\otimes} \mathbb{Q}_p$ of M always should be de Rham. Also we should have $P(V;1) \neq 0$ if M has weight ≤ -1. In this situation Bloch/Kato propose

$$A(K) := H^1_f(K,T) := \text{ inverse image of } H^1_f(K,V) \text{ in } H^1(K,T)$$

as a candidate for the (pro-p-part of the) "group of K-rational points" of M. (Actually they take T to be a Galois lattice in V.) And the isomorphism

$$\exp : DR(V)/DR(V)^0 \xrightarrow{\cong} A(K) \otimes \mathbb{Q}$$

they consider as an "exponential map". One basic problem in the local arithmetic of M then is to understand this map on an "integral level". Since, e.g., Iwasawa theory is concerned with ramified $\mathbb{Z}_p$-extensions it is absolutely crucial not to restrict the ramification type of the field K when dealing with this problem.

2. The classical case

Let o_K, resp. $o_{\overline{K}}$, denote the ring of integers in K, resp. $\overline{K}$. Consider a connected p-divisible group $\mathcal{G}$ over o_K. Its Tate module is

$$T := T(\mathcal{G}) := \varprojlim \mathcal{G}_{p^\mu}(o_{\overline{K}}) \quad .$$

Fontaine has shown ([Fon] §6) that the G_K-representation $V := T \underset{\mathbb{Z}_p}{\otimes} \mathbb{Q}_p$ is de Rham. The assumption that $\mathcal{G}$ is connected implies $P(V;1) \neq 0$. We therefore have the isomorphism

$$\exp : DR(V)/DR(V)^0 \xrightarrow{\cong} H^1_e(K,V) \quad .$$

Classically one starts with the exact ([Tat] §2.4 Cor. 1) <u>Kummer sequence</u>

$$0 \longrightarrow T \longrightarrow \varprojlim \mathcal{G}(o_{\overline{K}}) \longrightarrow \mathcal{G}(o_{\overline{K}}) \longrightarrow 0$$

which induces the injective connecting homomorphism

$$\mathcal{G}(o_K) \overset{\delta}{\hookrightarrow} H^1(K,T) \quad .$$

Its relation to the exponential map exp is as follows.
Step 1: In case of the multiplicative group we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & \varprojlim \hat{\mathbb{G}}_m(o_{\overline{K}}) & \longrightarrow & \hat{\mathbb{G}}_m(o_{\overline{K}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \log[\ ]} & & \downarrow{\scriptstyle \log} & & \\
0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & B_{crys}^{f=p} \cap B_{DR}^{+} & \longrightarrow & \mathbb{C}_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle =} & & \downarrow \quad \begin{array}{c} x \\ \downarrow \\ xt^{-1} \otimes t \end{array} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & B_{crys}^{f=1} \otimes \mathbb{Z}_p(1) & \longrightarrow & (B_{DR}/B_{DR}^{+})(1) & \longrightarrow & 0
\end{array}$$

(see [BK] p. 360). Tensoring with $T(-1)$ and passing to cohomology leads to the commutative diagram

$$\begin{array}{ccc}
H^0(K, \hat{\mathbb{G}}_m(o_{\overline{K}}) \otimes T(-1)) & \xrightarrow{\delta} & H^1(K,T) \\
\downarrow & & \\
H^0(K, \mathbb{C}_p \otimes T(-1)) & & \Bigg\downarrow \\
\downarrow & & \\
DR(V)/DR(V)^0 & \xrightarrow{\exp} & H^1(K,V)
\end{array} \quad .$$

Step 2: If $\mathcal{G}'$ denotes the dual p-divisible group then Cartier duality says that

$$\mathcal{G}'_{p^\mu}(o_{\overline{K}}) = \mathrm{Hom}_{o_{\overline{K}}}(\mathcal{G}_{p^\mu}, \hat{\mathbb{G}}_m)$$

and therefore that

$$\mathrm{Hom}_{\mathbb{Z}_p}(T, T(\hat{\mathbb{G}}_m)) = T(\mathcal{G}') = \varprojlim \mathcal{G}'_{p^\mu}(o_{\overline{K}}) = \mathrm{Hom}_{o_{\overline{K}}}(\mathcal{G}, \hat{\mathbb{G}}_m) \quad .$$

This implies the existence of a natural G_K-equivariant map

$$\mathcal{G}(o_{\overline{K}}) \longrightarrow \hat{\mathbb{G}}_m(o_{\overline{K}}) \otimes T(-1)$$

which makes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & \varprojlim \mathcal{G}(o_{\overline{K}}) & \longrightarrow & \mathcal{G}(o_{\overline{K}}) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & \varprojlim \hat{\mathbb{G}}_m(o_{\overline{K}}) \otimes T(-1) & \longrightarrow & \hat{\mathbb{G}}_m(o_{\overline{K}}) \otimes T(-1) & \longrightarrow & 0
\end{array}$$

commutative.
Combining the two steps we obtain the commutative diagram

$$\begin{array}{ccc}
\mathcal{G}(o_K) & \overset{\delta}{\hookrightarrow} & H^1_e(K,T) \\
\downarrow & & \\
H^0(K,\mathbb{C}_p\otimes T(-1)) & & \downarrow \\
\downarrow\cong & & \\
DR(V)/DR(V)^0 & \underset{\cong}{\xrightarrow{\exp}} & H^1_e(K,V)
\end{array} \quad .$$

Via Hodge-Tate theory the space $H^0(K,\mathbb{C}_p\otimes T(-1))$ can be naturally identified with the tangent space of $\mathcal{G}$ ([Tat] §4 Thm. 3). The map $\mathcal{G}(o_K)\to H^0(K,\mathbb{C}_p\otimes T(-1))$ then becomes the usual logarithm map for $\mathcal{G}$ which is a local isomorphism ([Tat] §2.4).

Résumé:

From the Kummer sequence for $\mathcal{G}$ we obtain a local inverse

$$\log:\mathcal{G}(o_K)\longrightarrow DR(V)/DR(V)^0$$

of the exponential map.

3. A digression on weight arguments

From now on we fix a smooth projective variety X over K and integers $i\geq 0$ and $n\in\mathbb{Z}$ and we consider the motive $M:=H^i(X)(n)$. This means that we consider the G_K-representation $V:=H^i(\overline{X},\mathbb{Q}_p(n))$ together with the Galois lattice

$$T:=H^i(\overline{X},\mathbb{Z}_p(n))/tor\hookrightarrow V=H^i(\overline{X},\mathbb{Q}_p(n))$$

in it; here $\overline{X}:=X\underset{K}{\times}\overline{K}$ and cohomology if not indicated otherwise always is etale cohomology. Faltings ([Fal]) has proved that V is de Rham. We have weight(M) $=i-2n$.

Since the group G_K has cohomological dimension 2 the groups $H^*(K,V)$ can be nonzero at most for $0\leq *\leq 2$. Using Poincaré duality together with a polarization we deduce a G_K-isomorphism

$$\mathrm{Hom}_{\mathbb{Q}_p}(V,\mathbb{Q}_p)\cong V(i-2n)\quad .$$

From local Tate duality we therefore obtain that

$$H^2(K,V)\text{ is dual to }H^0(K,V(i+1-2n))\quad .$$

Throughout the paper we assume that X has good reduction. We fix a smooth projective model $\mathcal{X}$ of X over $\mathrm{Spec}(o_K)$ and we let $Y := \mathcal{X} \underset{o_K}{\times} k$ denote the reduction of $\mathcal{X}$ over the residue class field k of o_K. By Faltings ([Fal]) V is crystalline and

$$Crys(V) = [H^i_{crys}(Y/W(k)) \underset{W(k)}{\otimes} K_0](n)$$

identifies with the (twisted) crystalline cohomology of the reduction Y. The latter implies by an argument of Katz/Messing ([KM]) based on Deligne's proof of the Weil conjectures that the reciprocal roots of $P(V;u)$ are algebraic integers of complex absolute value $q^{(i-2n)/2} = q^{weight(M)/2}$. We see that if $\mathrm{weight}(M) \neq 0$ then we have $P(V;1) \neq 0$ as well as $H^0(K,V) = 0$.

Lemma:

i. If $i \neq 2n$ then $P(V;1) \neq 0$ and $H^0(K,V) = 0$;
ii. if $i \neq 2n-2$ then $H^2(K,V) = 0$;
iii. if $i \neq 2n-1$ then $H^1(K,V) = H^{i+1}(X, \mathbb{Q}_p(n))$.

Proof: i. This restates the result of our discussion above. ii. By duality as explained above this reduces to the assertion i. for the motive $H^i(X)(i+1-n)$. iii. This then follows from the Hochschild-Serre spectral sequence for G_K.

4. Kummer sequences through motivic cohomology

From now on assume $n \geq 0$ and let $\mathbb{Z}(n)_X$ or simply $\mathbb{Z}(n)$ denote the n-th Beilinson-Lichtenbaum complex on the etale site X_{et} of X. One hopes that such objects fulfilling a certain list of nice properties exist and form the ultimate "motivic" cohomology theory. For the background we refer to [Lic1]. The properties in that list which will be of use for us will be introduced one after the other when they are needed. We will number them by "Mot?" followed by a reference to where this property is stated in the literature possibly with some explanation. All our assertions which depend on those hypothetical properties will be called "Claims".

To begin with the most important property for our purposes is:

Mot 1: ([Lic1])
There are compatible distinguished triangles of complexes of sheaves on X_{et}

$$\begin{tikzcd} & \mathbb{Z}/p^\mu\mathbb{Z}(n) \arrow[dl, "+1"'] & \\ \mathbb{Z}(n) \arrow[rr, "p^\mu"] & & \mathbb{Z}(n) \arrow[ul] \end{tikzcd} \qquad \text{for } \mu \geq 1 \quad .$$

The associated cohomology sequences are

$$0 \to H^i(\overline{X}, \mathbb{Z}(n))/p^\mu \to H^i(\overline{X}, \mathbb{Z}/p^\mu\mathbb{Z}(n)) \to H^{i+1}(\overline{X}, \mathbb{Z}(n))^{p^\mu=0} \to 0.$$

We define

$$H^*(\overline{X}, \mathbb{Z}(n))^0 := \text{maximal } p\text{-divisible subgroup in } H^*(\overline{X}, \mathbb{Z}(n)) \quad .$$

Our first aim is to show that the above exact sequences in the projective limit lead, for $i \neq 2n$, to the exact Kummer sequence

$$(*) \qquad 0 \to T \to \varprojlim H^{i+1}(\overline{X}, \mathbb{Z}(n))^0 \to H^{i+1}(\overline{X}, \mathbb{Z}(n))^0 \to 0$$

where the projective limit in the middle is formed with respect to multiplication by p as transition maps. For trivial reasons we obtain the exact sequence

$$0 \to \varprojlim H^i(\overline{X}, \mathbb{Z}(n))/p^\mu \to H^i(\overline{X}, \mathbb{Z}_p(n)) \to$$
$$\to \varprojlim H^{i+1}(\overline{X}, \mathbb{Z}(n)) \to H^{i+1}(\overline{X}, \mathbb{Z}(n))^0 \to 0 \quad .$$

Claim 1:

$$\begin{aligned} H^*(\overline{X}, \mathbb{Z}(n))^0 &= \ker(H^*(\overline{X}, \mathbb{Z}(n)) \to \varprojlim H^*(\overline{X}, \mathbb{Z}(n))/p^\mu) \\ &= \ker(H^*(\overline{X}, \mathbb{Z}(n)) \to H^*(\overline{X}, \mathbb{Z}_p(n))) \quad . \end{aligned}$$

Proof: The second equality follows from the injectivity of the map

$$\varprojlim H^*(\overline{X}, \mathbb{Z}(n))/p^\mu \longrightarrow H^*(\overline{X}, \mathbb{Z}_p(n)) \quad .$$

Concerning the first equality the inclusion "$\subseteq$" is clear since $H^*(\overline{X}, \mathbb{Z}_p(n))$ is a finitely generated $\mathbb{Z}_p$-module. On the other hand if p^s is the order of the torsion subgroup in $H^*(\overline{X}, \mathbb{Z}_p(n))$ and if x is in the kernel under consideration then, for any $r \geq 0$, we find a $y_r \in H^*(\overline{X}, \mathbb{Z}(n))$ such that $x = p^r y_r$; we get $x = p(p^s y_{s+1})$ where $p^s y_{s+1}$ also lies in the kernel. This shows that the kernel is p-divisible.

Again because of $H^{i+1}(\overline{X}, \mathbb{Z}_p(n))$ being a finitely generated $\mathbb{Z}_p$-module Claim 1 implies that

$$\varprojlim H^{i+1}(\overline{X}, \mathbb{Z}(n))^0 = \varprojlim H^{i+1}(\overline{X}, \mathbb{Z}(n)) \quad .$$

Since this group obviously is p-torsion free it remains to show that

$$\varprojlim H^i(\overline{X}, \mathbb{Z}(n))/p^\mu \quad \text{is finite for } i \neq 2n \quad .$$

This follows from the next Claim.

Claim 2:

If $i \neq 2n$ then $H^i(\overline{X}, \mathbb{Z}(n))^0$ is of finite index in $H^i(\overline{X}, \mathbb{Z}(n))$.

Proof: By Claim 1 it suffices to show that

$$\mathrm{im}(H^i(\overline{X}, \mathbb{Z}(n)) \longrightarrow H^i(\overline{X}, \mathbb{Z}_p(n))) \text{ is finite } .$$

We use

<u>Mot 2</u>: (implicitly in [Lic1])
The formation of $\mathbb{Z}(n)$ commutes with proetale base change.

This implies that

$$H^*(\overline{X}, \mathbb{Z}(n)) = \varinjlim_{\substack{K \subseteq L \subseteq \overline{K} \\ L/K \ finite}} H^*(X_{/L}, \mathbb{Z}(n)) \quad .$$

Therefore the above image is contained in that part of $H^i(\overline{X}, \mathbb{Z}_p(n))$ on which G_K acts discretely. But as explained in Section 3 the discrete part of $H^i(\overline{X}, \mathbb{Q}_p(n))$ is zero if $i \neq 2n$.

Comment:

The above considerations actually show that

$$H^i(\overline{X}, \mathbb{Z}(n))/H^i(\overline{X}, \mathbb{Z}(n))^0 = \mathrm{Tor}\, H^i(\overline{X}, \mathbb{Z}_p(n)) \text{ for } i \neq 2n \quad .$$

Is the cokernel of

$$H^{2n}(\overline{X}, \mathbb{Z}(n))/H^{2n}(\overline{X}, \mathbb{Z}(n))^0 \hookrightarrow H^{2n}(\overline{X}, \mathbb{Z}_p(n))^{discrete}$$

finite?

The above Kummer sequence $(*)$ provides us, for $i \neq 2n$, with the connecting homomorphism

$$H^0(K, H^{i+1}(\overline{X}, \mathbb{Z}(n))^0) \xrightarrow{\delta} H^1(K, T) \quad .$$

Claim 3:

If $i \neq 2n$ then $\ker(\delta)$ *is uniquely p-divisible.*

Proof: By our weight assumption we have $H^0(K,T)=0$ and therefore

$$\ker(\delta) = \varprojlim H^0(K, H^{i+1}(\overline{X}, \mathbb{Z}(n))^0)$$

which obviously is uniquely p-divisible.

Example: For $i=n=1$ we have $\mathrm{im}(\delta) = \mathrm{Pic}(X)^0(K)$. This follows from

<u>Mot 3</u>: ([Lic1])
$\mathbb{Z}(1) = \mathbb{G}_m[-1]$.

Assuming $i \neq 2n$ we would like to have a commutative diagram

$$(**) \qquad \begin{array}{ccccc} ? & \subseteq & H^0(K, H^{i+1}(\overline{X}, \mathbb{Z}(n))^0) & \xrightarrow{\delta} & H^1(K,T) \\ \downarrow & & & & \downarrow \\ DR(V)/DR(V)^0 & & \underset{\cong}{\xrightarrow{\exp}} & & H^1_f(K,V) \subseteq H^1(K,V). \end{array}$$

(E.g., for $i=0$ and $n=1$ we have $? = o_K^\times \subseteq K^\times$.) This would lead to a local inverse

$$\begin{array}{ccccc} & & \delta(?) & \subseteq & A(K) \\ & \swarrow^{\log} & & & \downarrow \\ DR(V)/DR(V)^0 & & \underset{\cong}{\xrightarrow{\exp}} & & A(K)\otimes\mathbb{Q} \end{array}$$

of the exponential map. Of course the question remained whether $\delta(?)$ is of finite index in $A(K)$, even whether $\delta(?)$ is a $\mathbb{Z}_p$-submodule of $A(K)$ (and also whether $\delta(?)$ only depends on M and not on the particular representation $M = H^i(X)(n))$.

5. The comparing diagram

A first simplification of the wanted diagram $(**)$ is achieved by using Galois descent. We will exclude the cases $i = 2n$, $2n \pm 1$ which are more complicated.

Claim 4: (Galois descent)

For $i \neq 2n$, $2n \pm 1$ we have the exact diagram

$$
\begin{array}{ccccc}
 & & H^0(K, H^{i+1}(\overline{X}, \mathbb{Z}(n))^0) & & \\
 & & \downarrow\subseteq & & \\
H^{i+1}(X, \mathbb{Z}(n)) & \longrightarrow & H^0(K, H^{i+1}(\overline{X}, \mathbb{Z}(n))) & \longrightarrow & \textit{finite} \\
 & & \downarrow & & \\
 & & \textit{finite} & & .
\end{array}
$$

Proof: The column follows from Claim 2. For the row we look at the Hochschild-Serre spectral sequence (whose existence follows from Mot 2)

$$H^r(K, H^s(\overline{X}, \mathbb{Z}(n))) \Longrightarrow H^{r+s}(X, \mathbb{Z}(n)) \quad .$$

Since K has strict cohomological dimension 2 it suffices to prove that

$$H^2(K, H^i(\overline{X}, \mathbb{Z}(n))) \ \text{ is finite} \quad .$$

Passing in the cohomology sequences associated with the triangles in Mot 1 to the direct limit we obtain the exact sequence

$$0 \to H^{i-1}(\overline{X}, \mathbb{Z}(n)) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^{i-1}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^i(\overline{X}, \mathbb{Z}(n))(p) \to 0 \ .$$

By Claim 2 the first group vanishes. We therefore have

$$H^i(\overline{X}, \mathbb{Z}(n))^0(p) = H^{i-1}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))^0$$

where also on the right hand side the superscript ^{0} stands for the maximal divisible subgroup. This implies

$$H^2(K, H^i(\overline{X}, \mathbb{Z}(n))^0) = H^2(K, H^{i-1}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))^0) \quad .$$

By the arguments in Section 3 the right hand side is dual to the Galois invariants of a Galois lattice in $H^{i-1}(\overline{X}, \mathbb{Q}_p(i-n))$ and consequently is zero because of $i - 1 \neq 2(i-n)$. Again by Claim 2 the left hand group differs from $H^2(K, H^i(\overline{X}, \mathbb{Z}(n)))$ only by a finite group.

We have seen in Section 3 that for $i \neq 2n-1$ the Hochschild-Serre spectral sequence induces the isomorphism

$$H^1(K, V) \cong H^{i+1}(X, \mathbb{Q}_p(n)) \quad .$$

In case $i \neq 2n,\ 2n \pm 1$ the diagram

$$\begin{array}{ccccc}
H^0(K, H^{i+1}(\overline{X}, \mathbb{Z}(n))^0) & \xrightarrow{\delta} & H^1(K,T) & \longrightarrow & H^1(K,V) \\
\downarrow & & & & \\
H^0(K, H^{i+1}(\overline{X}, \mathbb{Z}(n))) & & & & \updownarrow \cong \\
\uparrow & & & & \\
H^{i+1}(X, \mathbb{Z}(n)) & \xrightarrow{c_{et}} & H^{i+1}(X, \mathbb{Z}_p(n)) & \longrightarrow & H^{i+1}(X, \mathbb{Q}_p(n))
\end{array}$$

then for a completely formal reason is commutative after tensoring with $\mathbb{Q}$ (so that by Claim 4 the upper, resp. lower, perpendicular arrow on the left hand side becomes an isomorphism, resp. surjective). The natural map c_{et} is induced by the distinguished triangles in Mot 1. What we are looking for therefore is (in case $i \neq 2n,\ 2n \pm 1$) a commutative diagram

$$(**_K) \qquad \begin{array}{ccccc}
? & \subseteq H^{i+1}(X, \mathbb{Z}(n)) & \xrightarrow{c_{et}} & & H^{i+1}(X, \mathbb{Q}_p(n)) \\
\Big\downarrow & & & & \updownarrow \cong \\
DR(V)/DR(V)^0 & & \underset{\cong}{\xrightarrow{\exp}} & H^1_f(K,V) \subseteq & H^1(K,V)
\end{array} \quad .$$

Remarks:

1) By Mot 1 the map c_{et} factorizes through an injective map

$$\varprojlim H^{i+1}(X, \mathbb{Z}(n))/p^\mu \overset{c_{et}}{\hookrightarrow} H^{i+1}(X, \mathbb{Z}_p(n)) \quad .$$

2) We have

$$\begin{aligned}
\dim_{\mathbb{Q}_p} H^1(K,V) &= \sum_{r \geq 0} (-1)^{r+1} \dim_{\mathbb{Q}_p} H^r(K,V) \\
&= [K : \mathbb{Q}_p] \cdot \dim_{\mathbb{Q}_p} V
\end{aligned}$$

provided $i \neq 2n$ and $2n-2$; the first equality is a consequence of the Lemma in Section 3 and the second one follows from the theorem about local Euler-Poincaré characteristics. On the other hand Faltings' result gives a natural identification

$$DR(V) = H^i_{DR}(X)(n)$$

with the (twisted) algebraic de Rham cohomology of X over K. For $i \neq 2n$ we obtain

$$\begin{aligned}
\dim_{\mathbb{Q}_p} H^1_f(K,V) &= [K : \mathbb{Q}_p] \cdot \dim_K H^i_{DR}(X)/F^n \\
&= [K : \mathbb{Q}_p] \cdot (\dim_{\mathbb{Q}_p} V - \dim_K F^n H^i_{DR}(X)) \quad .
\end{aligned}$$

This shows that

$$H^1_f(K,V) = H^1(K,V)$$

if $i < n$ with the exception of the case $i = 0$, $n = 1$. In this situation the existence of a diagram $(**_K)$ of course is trivial.

6. The vanishing cycle spectral sequence

One may view the existence of a diagram $(**_K)$ as a statement about the image of c_{et}. From this point of view it seems important to find a characterization of that subspace in $H^{i+1}(X, \mathbb{Q}_p(n))$ which corresponds to $H^1_f(K,V)$:

$$\begin{array}{ccc} H^1_f(K,V) & \subseteq & H^1(K,V) \\ \updownarrow\cong & & \updownarrow\cong \\ ? & \subseteq & H^{i+1}(X,\mathbb{Q}_p(n)) \end{array} \qquad (i \neq 2n-1) \quad .$$

We will approach this question by using the p-adic vanishing cycle spectral sequence. Let

$$Y \xrightarrow{\sigma} \mathcal{X} \xleftarrow{\tau} X$$

denote the immersions of the fibers into the model $\mathcal{X}$. It seems that the canonical filtration $t_{\leq .}R\tau_*\mathcal{F}^\cdot$ of the total direct image complex $R\tau_*\mathcal{F}^\cdot$ behaves rather differently in the two cases $\mathcal{F}^\cdot = \mathbb{Z}(n)$ and $\mathcal{F}^\cdot = \mathbb{Z}/p^\mu\mathbb{Z}(n)$, respectively. This is what we want to explore.

In complete generality the decomposition theorem in etale topology states that for any bounded below complex of sheaves $\mathcal{C}^\cdot$ on the small etale site of $\mathcal{X}$ one has a natural distinguished triangle

$$\begin{array}{ccccc} & & R\tau_*\tau^*\mathcal{C}^\cdot & & \\ & {\scriptstyle +1}\swarrow & & \nwarrow & \\ \sigma_*R\sigma^!\mathcal{C}^\cdot & & \longrightarrow & & \mathcal{C}^\cdot \end{array} \quad .$$

If $\mathcal{C}^\cdot$ is acyclic in degrees $> n$ then

$$\begin{array}{ccccc} & & t_{\leq n}R\tau_*\tau^*\mathcal{C}^\cdot & & \\ & {\scriptstyle +1}\swarrow & & \nwarrow & \\ \sigma_*t_{\leq n+1}R\sigma^!\mathcal{C}^\cdot & & \longrightarrow & & \mathcal{C}^\cdot \end{array}$$

is a distinguished triangle, too, and

$$R^{n+j}\tau_*\tau^*\mathcal{C}^\cdot = \sigma_*R^{n+j+1}\sigma^!\mathcal{C}^\cdot \quad \text{for } j \geq 1 \quad .$$

Because of

Mot 4: ([Lic1])
$\mathbb{Z}(0) = \mathbb{Z}$, and $\mathbb{Z}(n)$, for $n \geq 1$, is acyclic outside $[1, n]$.

we may apply this to the complex $\mathcal{C}^{\cdot} = \mathbb{Z}(n)_{\mathcal{X}}$. Using

Mot 5: (Purity, [Lic2] and [Mil2])
$t_{\leq n+1} R\sigma^! \mathbb{Z}(n)_{\mathcal{X}} = \mathbb{Z}(n-1)_Y[-2]$

we obtain the distinguished triangle

$$\begin{array}{ccc} & t_{\leq n} R\tau_* \mathbb{Z}(n)_X & \\ {}^{+1}\swarrow & & \nwarrow \\ \sigma_* \mathbb{Z}(n-1)_Y[-2] & \longrightarrow & \mathbb{Z}(n)_{\mathcal{X}} \end{array}$$

and the isomorphisms

$$R^{n+j} \tau_* \mathbb{Z}(n)_X = \sigma_* R^{n+j+1} \sigma^! \mathbb{Z}(n)_{\mathcal{X}} \quad \text{for } j \geq 1 \quad .$$

The associated cohomology sequence reads

$$\begin{aligned} \ldots \longrightarrow H^{i-1}(Y, \mathbb{Z}(n-1)) \longrightarrow H^{i+1}(\mathcal{X}, \mathbb{Z}(n)) \longrightarrow H^{i+1}(\mathcal{X}, t_{\leq n} R\tau_* \mathbb{Z}(n)_X) \\ \longrightarrow H^i(Y, \mathbb{Z}(n-1)) \longrightarrow \ldots \end{aligned}$$

We now invoke

Mot 6: ([Lic1])
$H^*(Y, \mathbb{Z}(n-1))$ is finite for $* \neq 2n-2, 2n$ and is finitely generated for $* = 2n-2$.

Excluding the cases $i = 2n$, $2n \pm 1$, or $2n-2$ (which involve the codimension $n-1$ cycles on Y through the term $H^{2n-2}(Y, \mathbb{Z}(n-1)))$ we see that the map

$$H^{i+1}(\mathcal{X}, \mathbb{Z}(n)) \longrightarrow H^{i+1}(\mathcal{X}, t_{\leq n} R\tau_* \mathbb{Z}(n)_X)$$

should have finite kernel and cokernel. Next we have to look at the map

$$H^{i+1}(\mathcal{X}, t_{\leq n} R\tau_* \mathbb{Z}(n)_X) \longrightarrow H^{i+1}(\mathcal{X}, R\tau_* \mathbb{Z}(n)_X) = H^{i+1}(X, \mathbb{Z}(n)) \quad .$$

We hope that its kernel and cokernel are torsion and therefore could be neglected in looking for our logarithm map since the latter will take values in a $\mathbb{Q}_p$-vector space. For trivial reasons, of course, the map is an isomorphism if $i < n$. What are the reasons for our hope?

A) (Compare [Mil2] p. 74 last paragraph) It might be possible that the purity axiom Mot 5 can be complemented by the requirement that the sheaves

$$R^j \sigma^! \mathbb{Z}(n)_{\mathcal{X}} \quad \text{for } j \geq n+2 \geq 3 \quad \text{are } p\text{-primary torsion} \quad .$$

Then the sheaves

$$R^j \tau_* \mathbb{Z}(n)_X \quad \text{for } j > n \geq 1 \quad \text{are } p\text{-primary torsion, too} \quad ,$$

which would imply that the kernels and cokernels in question are p-primary torsion if $n \geq 1$.

Claim 5:

In case $X = \mathrm{Spec}(K)$ we have $R^j\tau_*\mathbb{Z}(n)_X = 0$ for $j > n \geq 1$.

Proof: If I denotes the inertia subgroup in G_K we have

$$R^j\tau_*\mathbb{Z}(n)_X = H^j(I, \mathbb{Z}(n)_X) \quad .$$

Since I has strict cohomological dimension 2 and $\mathbb{Z}(n)$ is acyclic in degrees $> n$ by Mot 4 it is clear that $H^j(I, \mathbb{Z}(n)_X) = 0$ for $j > n+2$. The same argument shows that for $j = n+2$ it suffices to prove the vanishing of $H^2(I, h^n(\mathbb{Z}(n)_X))$. In case $n = 1$ this is, by Mot 3, the well-known vanishing of $H^2(I, \overline{K}^\times)$. For $n \geq 2$ it follows from Mot 1 that $h^n(\mathbb{Z}(n)_X)$ is uniquely divisible and therefore cohomologically trivial. The vanishing of $H^{n+1}(I, \mathbb{Z}(n)_X)$ is a consequence of

<u>Mot 7</u>: (Hilbert 90, [Lic1])
$R^{n+1}\alpha_*\mathbb{Z}(n) = 0$ where $\alpha : X_{et} \to X_{Zar}$ is the natural morphism of sites.

B) In the presence of other expected properties of the complexes $\mathbb{Z}(n)$ the purity axiom Mot 5 actually can be sharpened:

Claim 6:

$R^{n+1}\tau_*\mathbb{Z}(n)_X = \sigma_* R^{n+2}\sigma^!\mathbb{Z}(n)_{\mathcal{X}} = 0.$

Proof: Let $\overline{x}$ be a geometric point of $\mathcal{X}$. We have to show that

$$(R^{n+1}\tau_*\mathbb{Z}(n)_X)_{\overline{x}} = H^{n+1}(\mathcal{O}_{\mathcal{X},\overline{x}}[\frac{1}{p}], \mathbb{Z}(n))$$

(use Mot 2) vanishes. The axiom about the K-theoretic Beilinson complexes

<u>Mot 8</u>: ([Lic1])
$\mathbb{Z}_B(n) = t_{\leq n} R\alpha_*\mathbb{Z}(n)$

together with Mot 7 implies that

$$H^{n+1}(\mathcal{O}_{\mathcal{X},\overline{x}}[\frac{1}{p}], \mathbb{Z}(n)) = H^{n+1}_{Zar}(\mathcal{O}_{\mathcal{X},\overline{x}}[\frac{1}{p}], \mathbb{Z}_B(n)) \quad .$$

But in the K-theoretic localization sequence

$$\begin{aligned} \ldots \longrightarrow H^{n+1}_{Zar}(\mathcal{O}_{\mathcal{X},\overline{x}}, \mathbb{Z}_B(n)) \longrightarrow H^{n+1}_{Zar}(\mathcal{O}_{\mathcal{X},\overline{x}}[\frac{1}{p}], \mathbb{Z}_B(n)) \longrightarrow \\ \longrightarrow H^{n}_{Zar}(\mathcal{O}_{\mathcal{X},\overline{x}}/p, \mathbb{Z}_B(n-1)) \longrightarrow \ldots \end{aligned}$$

the outer terms vanish being the Zariski cohomology over a local ring in a degree in which the coefficient complex is acyclic.

As a consequence the map in question would be an isomorphism even for $i \leq n$.

C) A similar argument as in the proof of Claim 6 shows that

$$R\tau_*^{Zar}\mathbb{Z}_B(n)_X \ \text{ is acyclic in degrees } > n \quad .$$

Because of

$$\begin{aligned} t_{\leq n}R\tau_*^{Zar}\mathbb{Z}_B(n)_X &= t_{\leq n}R\tau_*^{Zar}(t_{\leq n}R\alpha_*\mathbb{Z}(n)_X) \\ &= t_{\leq n}R\alpha_*(t_{\leq n}R\tau_*\mathbb{Z}(n)_X) \end{aligned}$$

this leads to the factorization

$$\begin{array}{ccc} H^*_{Zar}(X, \mathbb{Z}_B(n)) & \longrightarrow & H^*(X, \mathbb{Z}(n)) \\ & \searrow \qquad \nearrow & \\ & H^*(\mathcal{X}, t_{\leq n}R\tau_*\mathbb{Z}(n)_X) & \end{array}$$

showing that we loose nothing coming from K-theory. Also it seems reasonable to believe that the horizontal arrow always has torsion kernel and cokernel. For $\mathbb{Z}(1) = \mathbb{G}_m[-1]$ this is shown to be the case in [Gro] Prop. 1.4.

We have the obvious commutative diagram

$$\begin{array}{ccc} H^{i+1}(\mathcal{X}, \mathbb{Z}(n)) & \longrightarrow & H^{i+1}(X, \mathbb{Z}(n)) \\ \downarrow & & \downarrow c_{et} \\ H^{i+1}(\mathcal{X}, \mathbb{Z}(n)_{\mathcal{X}} \overset{\mathbb{L}}{\otimes} \mathbb{Z}/p^\mu\mathbb{Z}) & \longrightarrow & H^{i+1}(X, \mathbb{Z}(n)_X \overset{\mathbb{L}}{\otimes} \mathbb{Z}/p^\mu\mathbb{Z}) \\ & & \| \\ & & H^{i+1}(X, \mathbb{Z}/p^\mu\mathbb{Z}(n)) \end{array}$$

where the identity on the right hand side comes from Mot 1. So far we have discussed the upper horizontal arrow. We now turn to the computation of the complexes

$$S_\mu(n) := \mathbb{Z}(n)_{\mathcal{X}} \overset{\mathbb{L}}{\otimes} \mathbb{Z}/p^\mu\mathbb{Z} \quad .$$

With $\mathbb{Z}(n)_{\mathcal{X}}$ (by Mot 4) also $S_\mu(n)$ is acyclic in degrees $> n$. The decomposition theorem then gives us the distinguished triangle

$$\begin{array}{ccc} & t_{\leq n}R\tau_*\tau^*S_\mu(n) & \\ {}^{+1}\swarrow & & \nwarrow \\ \sigma_*t_{\leq n+1}R\sigma^!S_\mu(n) & \longrightarrow & S_\mu(n) \end{array} \quad .$$

For the further arguments observe first that $.\overset{\mathbb{L}}{\otimes}\mathbb{Z}/p^\mu\mathbb{Z}$ commutes with any other triangulated functor.

$*$ From Mot 1 we deduce

$$\tau^*S_\mu(n) = \tau^*\mathbb{Z}(n)_{\mathcal{X}} \overset{\mathbb{L}}{\otimes} \mathbb{Z}/p^\mu\mathbb{Z} = \mathbb{Z}(n)_X \overset{\mathbb{L}}{\otimes} \mathbb{Z}/p^\mu\mathbb{Z} = \mathbb{Z}/p^\mu\mathbb{Z}(n) \quad .$$

$*$ From Claim 6 we deduce that the distinguished triangle

$$\begin{array}{ccc} & R\sigma^! S_\mu(n) & \\ {}^{+1}\swarrow & & \nwarrow \\ R\sigma^! \mathbb{Z}(n)_{\mathcal{X}} & \xrightarrow{p^\mu} & R\sigma^! \mathbb{Z}(n)_{\mathcal{X}} \end{array}$$

remains a distinguished triangle after $t_{\leq n+1}$-truncation. Using in addition the purity axiom Mot 5 we therefore obtain

$$\begin{aligned} t_{\leq n+1} R\sigma^! S_\mu(n) &= (t_{\leq n+1} R\sigma^! \mathbb{Z}(n)_{\mathcal{X}}) \overset{\mathbb{L}}{\otimes} \mathbb{Z}/p^\mu\mathbb{Z} \\ &= (\mathbb{Z}(n-1)_Y \overset{\mathbb{L}}{\otimes} \mathbb{Z}/p^\mu\mathbb{Z})[-2] \quad . \end{aligned}$$

About the p-torsion in characteristic p we have

<u>Mot 9</u>: ([Mil2])

$$\mathbb{Z}(n)_Y \overset{\mathbb{L}}{\otimes} \mathbb{Z}/p^\mu\mathbb{Z} = \nu_\mu(n)[-n].$$

We recall the definition of the sheaves $\nu_\mu(n)$ on the etale site Y_{et}: For $n=0$ put

$$\nu_\mu(0) := \mathbb{Z}/p^\mu\mathbb{Z} \subseteq W_\mu\mathcal{O}_Y \quad .$$

Using the homomorphism

$$\begin{aligned} d\log : \mathcal{O}_Y^\times &\longrightarrow W_\mu\Omega_Y^1 \\ f &\longmapsto \frac{d\underline{f}}{\underline{f}} \ , \ \underline{f} := \text{ Teichmüller representative of } f \quad , \end{aligned}$$

we define for $n \geq 1$

$$\begin{aligned} \nu_\mu(n) := & \text{additive subsheaf in } W_\mu\Omega_Y^n \text{ generated etale} \\ & \text{locally by the sections } d\log f_1 \wedge \ldots \wedge d\log f_n \quad . \end{aligned}$$

Equivalently we have

$$\begin{aligned} \nu_\mu(n) = \text{ image } (\underline{\underline{K}}^{Milnor}_{n,Y_{et}} & \overset{symbol\ map}{\longrightarrow} W_\mu\Omega_Y^n) \\ \{f_1, \ldots, f_n\} & \longmapsto d\log f_1 \wedge \ldots \wedge d\log f_n. \end{aligned}$$

Putting everything together we have the distinguished triangle

$$\begin{array}{ccc} & t_{\leq n} R\tau_* \mathbb{Z}/p^\mu\mathbb{Z}(n) & \\ {}^{+1}\swarrow & & \nwarrow \\ \sigma_*\nu_\mu(n-1)[-n-1] & \longrightarrow & S_\mu(n) \end{array} \quad ,$$

resp. by translation

$$S_\mu(n) = \text{ cone } (t_{\leq n} R\tau_* \mathbb{Z}/p^\mu \mathbb{Z}(n) \longrightarrow \sigma_* \nu_\mu(n-1)[-n])[-1] \quad .$$

This cone is determined by a sheaf homomorphism

$$s : R^n \tau_* \mathbb{Z}/p^\mu \mathbb{Z}(n) \longrightarrow \sigma_* \nu_\mu(n-1)$$

at which we will take a closer look:
As a consequence of Claim 6 the natural homomorphism

$$R^n \tau_* \mathbb{Z}(n)_X \twoheadrightarrow R^n \tau_* \mathbb{Z}/p^\mu \mathbb{Z}(n)$$

is surjective so that s is determined by the composite homomorphism

$$R^n \tau_* \mathbb{Z}(n)_X \longrightarrow \sigma_* \nu_\mu(n-1)$$

or, equivalently by adjointness, by a homomorphism

$$\sigma^* R^n \tau_* \mathbb{Z}(n)_X \longrightarrow \nu_\mu(n-1) \quad .$$

The decomposition theorem together with Mot 4 gives us the exact sequence

$$h^n(\mathbb{Z}(n)_{\mathcal{X}}) \longrightarrow R^n \tau_* \mathbb{Z}(n)_X \longrightarrow \sigma_* R^{n+1} \sigma^! \mathbb{Z}(n)_{\mathcal{X}} \longrightarrow 0 \quad ,$$

resp.

$$\sigma^* h^n(\mathbb{Z}(n)_{\mathcal{X}}) \longrightarrow \sigma^* R^n \tau_* \mathbb{Z}(n)_X \longrightarrow R^{n+1} \sigma^! \mathbb{Z}(n)_{\mathcal{X}} \longrightarrow 0 \quad .$$

By the purity axiom Mot 5 the right hand side is equal to $h^{n-1}(\mathbb{Z}(n-1)_Y)$. We now use

<u>Mot 10</u>: ([Lic2] and [Mil2])
$R^n \alpha_* \mathbb{Z}(n) = \underline{\underline{K}}^{Milnor}_{n,Zar}$.

It implies

$$h^n(\mathbb{Z}(n)) = \underline{\underline{K}}^{Milnor}_{n,et}$$

so that the above exact sequence finally becomes

$$\sigma^* \underline{\underline{K}}^{Milnor}_{n,\mathcal{X}_{et}} \longrightarrow \sigma^* R^n \tau_* \mathbb{Z}(n)_X \longrightarrow \underline{\underline{K}}^{Milnor}_{n-1,Y_{et}} \longrightarrow 0 \quad .$$

The expected compatibility of the various axioms with multiplicative structures (see [Mil2]) then shows that the exact diagram

$$\begin{array}{ccc}
\sigma^* \underline{\underline{K}}^{Milnor}_{n,\mathcal{X}_{et}}/p^\mu & & \\
\downarrow & & \\
\sigma^* R^n \tau_* \mathbb{Z}(n)_X/p^\mu & \xrightarrow{\cong} & \sigma^* R^n \tau_* \mathbb{Z}/p^\mu \mathbb{Z}(n) \\
\downarrow & & \downarrow \sigma^* s \\
\underline{\underline{K}}^{Milnor}_{n-1,Y_{et}}/p^\mu & \underset{symbol\ map}{\xrightarrow{\cong}} & \nu_\mu(n-1) \\
\downarrow & & \\
0 & &
\end{array}$$

is commutative (the lower isomorphism comes from Mot 9 and Mot 10). Since by Mot 10 and Mot 4 we have

$$\underline{\underline{K}}^{Milnor}_{n,\mathcal{X}_{et}}/p^\mu = h^n(\mathbb{Z}(n)_{\mathcal{X}})/p^\mu = h^n(S_\mu(n))$$

we arrive at an exact sequence

$$0 \longrightarrow \underline{\underline{K}}^{Milnor}_{n,\mathcal{X}_{et}}/p^\mu \longrightarrow R^n \tau_* \mathbb{Z}/p^\mu \mathbb{Z}(n) \xrightarrow{s} \sigma_* \nu_\mu(n-1) \longrightarrow 0 \quad .$$

7. A conjecture

Bloch/Kato show in [BK0] (6.1.1) that the composed homomorphism

$$(\sigma^* \tau_* \mathbb{G}_m)^{\otimes n} \longrightarrow (\sigma^* R^1 \tau_* \mathbb{Z}/p^\mu \mathbb{Z}(1))^{\otimes n} \xrightarrow[cupproduct]{} \sigma^* R^n \tau_* \mathbb{Z}/p^\mu \mathbb{Z}(n)$$

is surjective. This enables them by using the above symbol map to actually define the homomorphism $\sigma^* s$ (and then also s) independently of any conjecture about motivic cohomology (loc. cit. (6.6)). We therefore may take

$$S_\mu(n) = \text{ cone } (t_{\leq n} R\tau_* \mathbb{Z}/p^\mu \mathbb{Z}(n) \longrightarrow \sigma_* \nu_\mu(n-1)[-n])[-1]$$

as the definition of the complexes $S_\mu(n)$. Put

$$\begin{aligned}
H^*(\mathcal{X}, S_{\mathbb{Z}_p}(n)) &:= \varprojlim H^*(\mathcal{X}, S_\mu(n)) \qquad \text{and} \\
H^*(\mathcal{X}, S_{\mathbb{Q}_p}(n)) &:= H^*(\mathcal{X}, S_{\mathbb{Z}_p}(n)) \otimes \mathbb{Q}_p \quad .
\end{aligned}$$

Conjecture:

For $i \neq 2n,\ 2n-1$ the natural map $H^{i+1}(\mathcal{X}, S_{\mathbb{Q}_p}(n)) \to H^{i+1}(X, \mathbb{Q}_p(n))$ is injective and its image corresponds to $H^1_f(K,V)$ under the isomorphism $H^{i+1}(X, \mathbb{Q}_p(n)) \cong H^1(K,V)$, i.e.:

$$\begin{array}{ccc} H^{i+1}(\mathcal{X}, S_{\mathbb{Q}_p}(n)) & \hookrightarrow & H^{i+1}(X, \mathbb{Q}_p(n)) \\ \updownarrow \cong & & \updownarrow \cong \\ H^1_f(K,V) & \subseteq & H^1(K,V) \end{array} \quad .$$

As explained above this Conjecture makes sense independently of the existence of motivic cohomology.
Concerning the evidence we begin by looking at the case $i < n$ and not $i = 0$, $n = 1$; then $H^1_f(K,V) = H^1(K,V)$. It is known that

$$\varprojlim H^*(Y, \nu_\mu(n)) \otimes \mathbb{Q}_p = 0 \quad \text{for } * \neq n, n+1$$

(see [Mil2] p. 80 first paragraph). It follows that the map

$$H^{i+1}(\mathcal{X}, S_{\mathbb{Q}_p}(n)) \longrightarrow [\varprojlim H^{i+1}(\mathcal{X}, t_{\leq n} R\tau_* \mathbb{Z}/p^\mu \mathbb{Z}(n))] \otimes \mathbb{Q}_p$$

is injective for $i \neq 2n, 2n-1$ and is bijective for $i \neq 2n, 2n-1, 2n-2$. This shows that

$$H^{i+1}(\mathcal{X}, S_{\mathbb{Q}_p}(n)) \longrightarrow H^{i+1}(X, \mathbb{Q}_p(n))$$

is injective for $i < n$ and is an isomorphism for $i < n$ and not $i = 0$, $n = 1$ so that the Conjecture holds in the latter case.
If $X = \mathrm{Spec}(K)$ and $i = 0$ only the case $n = 1$ is not covered by that consideration. But we know that

$$\begin{array}{ccc} o_K^\times \otimes \mathbb{Q} & \subseteq & (\varprojlim K^\times / K^{\times p^\mu}) \otimes \mathbb{Q} \\ \updownarrow \cong & & \updownarrow \cong \\ H^1_f(K,V) & \subseteq & H^1(K,V) \end{array} \quad ;$$

on the other hand we have

$$\begin{aligned} H^1(\mathrm{Spec}(o_K), S_{\mathbb{Q}_p}(1)) &= \ker(H^1(K, \mathbb{Q}_p(1)) \xrightarrow{s} H^0(k, \mathbb{Z}_p)) \\ &= \ker((\varprojlim K^\times / K^{\times p^\mu}) \otimes \mathbb{Q}_p \xrightarrow{valuation} \mathbb{Q}_p) \\ &= o_K^\times \otimes \mathbb{Q} \quad . \end{aligned}$$

Let us now assume that $K/\mathbb{Q}_p$ is unramified and that $n < p$. Then more evidence can be obtained from considering the syntomic sheaves of Fontaine/Messing and Kato. On the syntomic site $\mathcal{X}_{syn}$ we have the sheaves

$$\begin{aligned} \mathcal{O}_\mu(V) &:= H^0(V_\mu, \mathcal{O}) \quad , \\ \mathcal{O}_\mu^{crys}(V) &:= H^0((V_1 \to \operatorname{Spec}(o_K/p^\mu))_{crys}, \mathcal{O}^{crys}) \quad , \\ J_\mu &:= \ker(\mathcal{O}_\mu^{crys} \to \mathcal{O}_\mu) \quad , \quad \text{and} \\ J_\mu^{[n]} &:= n\text{-th divided power of } J_\mu \quad ; \end{aligned}$$

here we have put $V_\mu := V \underset{o_K}{\times} (o_K/p^\mu)$ for any syntomic scheme V over o_K.
It is shown in [FM] II.2.3 and III.1.1 that (for $n < p$) the homomorphism

$$J_\mu^{[n]} \xrightarrow{1-p^{-n}f} \mathcal{O}_\mu^{crys}$$

is well-defined and surjective. Denoting its kernel by $s_\mu(n)$ we therefore have the exact sequence

$$0 \longrightarrow s_\mu(n) \longrightarrow J_\mu^{[n]} \xrightarrow{1-p^{-n}f} \mathcal{O}_\mu^{crys} \longrightarrow 0 \quad .$$

Put

$$H^*_{syn}(\mathcal{X}, s_{\mathbb{Q}_p}(n)) := [\varprojlim H^*_{syn}(\mathcal{X}, s_\mu(n))] \otimes \mathbb{Q}_p \quad .$$

Passing to cohomology then gives, again by [FM], the exact sequence

$$\begin{aligned} 0 \to H^i_{DR}(X)/(1-p^{-n}f)F^n \to H^{i+1}_{syn}(\mathcal{X}, s_{\mathbb{Q}_p}(n)) \\ \to \{v \in F^n H^{i+1}_{DR}(X) : fv = p^n v\} \to 0 \quad . \end{aligned}$$

From the crystalline Weyl conjecture ([KM]) we know that $1 - p^{-n}f$ is an automorphism of $H^*_{DR}(X) = H^*_{crys}(Y/W(k)) \otimes \mathbb{Q}_p$ if $* \neq 2n$. Hence we obtain natural $\mathbb{Q}_p$-linear isomorphisms

$$H^i_{DR}(X)/F^n \underset{1-p^{-n}f}{\overset{\cong}{\longrightarrow}} H^i_{DR}(X)/(1-p^{-n}f)F^n \overset{\cong}{\longrightarrow} H^{i+1}_{syn}(\mathcal{X}, s_{\mathbb{Q}_p}(n))$$

if $i \neq 2n, 2n-1$. Moreover Faltings' result gives an isomorphism

$$DR(V)/DR(V)^0 \cong H^i_{DR}(X)/F^n \quad .$$

We see that

$$\dim_{\mathbb{Q}_p} H^{i+1}_{syn}(\mathcal{X}, s_{\mathbb{Q}_p}(n)) = \dim_{\mathbb{Q}_p} DR(V)/DR(V)^0$$

holds for $n < p$ and $i \neq 2n, 2n-1$. On the other hand if $\beta : \mathcal{X}_{syn} \to \mathcal{X}_{et}$ denotes the natural morphism of sites then Kurihara has established in [Kur] the existence of distinguished triangles

$$\begin{array}{ccc} & \nu_\mu(n-1)[-n] & \\ {}^{+1}\swarrow & & \nwarrow \\ \sigma^* R\beta_* s_\mu(n) & \longrightarrow & \sigma^* t_{\leq n} R\tau_* \mathbb{Z}/p^\mu\mathbb{Z}(n) \end{array}$$

provided $n < p-1$. This means that we have (noncanonical) isomorphisms in the derived category

$$\sigma^* R\beta_* s_\mu(n) \sim \sigma^* S_\mu(n)$$

and therefore by the proper base change theorem isomorphisms on cohomology groups

$$H^*_{syn}(\mathcal{X}, s_\mu(n)) \cong H^*(\mathcal{X}, S_\mu(n))$$

provided $n < p-1$. In this way we get the following result in the direction of our Conjecture.

Proposition:

If $K/\mathbb{Q}_p$ is unramified and if $n < p-1$ then we have

$$\dim_{\mathbb{Q}_p} H^{i+1}(\mathcal{X}, S_{\mathbb{Q}_p}(n)) = \dim_{\mathbb{Q}_p} H^1_f(K, V) \quad \textit{for } i \neq 2n, 2n-1 \quad .$$

Of course one may hope that the various isomorphisms in the above considerations can be made compatible in such a way that one has the commutative diagram

$$\begin{array}{ccccc} H^{i+1}_{syn}(\mathcal{X}, s_{\mathbb{Q}_p}(n)) & \xrightarrow{\cong} & H^{i+1}(\mathcal{X}, S_{\mathbb{Q}_p}(n)) & \hookrightarrow & H^{i+1}(X, \mathbb{Q}_p(n)) \\ \updownarrow \cong & & & & \\ H^i_{DR}(X)/(1-p^{-n}f)F^n & & & & \updownarrow \cong \\ \updownarrow \cong & & & & \\ DR(V)/DR(V)^0 & \xrightarrow[\exp]{\cong} & H^1_f(K, V) & \xrightarrow{\subseteq} & H^1(K, V) \end{array}$$

thereby fully proving the Conjecture in this case.

8. Motivic Tate modules

The next step would be to understand the map

$$H^{i+1}(\mathcal{X}, \mathbb{Z}(n)) \longrightarrow H^{i+1}(\mathcal{X}, S_{\mathbb{Z}_p}(n)) \quad .$$

This seems not to be possible at present. Nevertheless we want to propagate the point of view that

$$\mathcal{G}_n^{i+1}(L) := \ker(H^{i+1}(\mathcal{X} \times o_L, \mathbb{Z}(n)) \longrightarrow H^{i+1}(\mathcal{X} \times k_L, \mathbb{Z}(n)))$$

for L varying through the finite extensions of K is a kind of generalization of the formal group of an abelian variety; here o_L, resp. k_L, denotes the ring of integers in L, resp. the residue class field of o_L. (Possibly $H^{i+1}(\mathcal{X} \times o_L, \mathbb{Z}(n))$ contains a large uniquely divisible subgroup which one may want to divide out.) In this light our discussion so far has dealt with the "tangent space"

$$L \longmapsto H^i_{DR}(X_{/L})/F^n$$

and, if $i \neq 2n, 2n-1$, the conjectural "logarithm map"

$$\mathcal{G}_n^{i+1}(L) \longrightarrow H^{i+1}(\mathcal{X} \times o_L, S_{\mathbb{Q}_p}(n)) \cong H^1_f(L, V) \cong H^i_{DR}(X_{/L})/F^n$$

into the tangent space. A third natural object to consider is the Tate module

$$\mathrm{Tate}(\mathcal{G}_n^{i+1}) := \left[\varprojlim_{\mu} \varinjlim_{L} \mathcal{G}_n^{i+1}(L)_{p^\mu} \right] \otimes \mathbb{Q}_p \quad .$$

We can say something about it under the assumption that the reduction Y is Hodge-Witt (i.e., the slope spectral sequence for Y degenerates at E_1). Put $\overline{\mathcal{X}} := \mathcal{X} \times o_{\overline{K}}$ and let $\overline{\tau} : \overline{X} \hookrightarrow \overline{\mathcal{X}}$ denote the corresponding open immersion.

Lemma:

$\varinjlim_{L} H^*(\mathcal{X} \times o_L, S_\mu(n)) = H^*(\overline{\mathcal{X}}, t_{\leq n} R\overline{\tau}_* \mathbb{Z}/p^\mu \mathbb{Z}(n)).$

Proof: For any finite extension L/K let $\tau_L : X_{/L} \hookrightarrow \mathcal{X} \times o_L$ be the natural open immersion. Now fix finite extensions $K \subseteq L \subseteq L'$ and let e denote the ramification index of the extension L'/L. It follows from the explicit description in [BK0] (6.6.1) of the homomorphism s which defines the cone $S_\mu(n)$ that the following diagram is commutative (and exact):

$$\begin{array}{ccc}
\cdots & & \cdots \\
\downarrow & & \downarrow \\
H^{*-n-1}(Y\times k_L,\nu_\mu(n-1)) & \xrightarrow{e\cdot can} & H^{*-n-1}(Y\times k_{L'},\nu_\mu(n-1)) \\
\downarrow & & \downarrow \\
H^*(\mathcal{X}\times o_L,S_\mu(n)) & \xrightarrow{can} & H^*(\mathcal{X}\times o_{L'},S_\mu(n)) \\
\downarrow & & \downarrow \\
H^*(\mathcal{X}\times o_L,t_{\leq n}R\tau_{L*}\mathbb{Z}/p^\mu\mathbb{Z}(n)) & \xrightarrow{can} & H^*(\mathcal{X}\times o_{L'},t_{\leq n}R\tau_{L'*}\mathbb{Z}/p^\mu\mathbb{Z}(n)) \\
\downarrow & & \downarrow \\
H^{*-n}(Y\times k_L,\nu_\mu(n-1)) & \xrightarrow{e\cdot can} & H^{*-n}(Y\times k_{L'},\nu_\mu(n-1)) \\
\downarrow & & \downarrow \\
\cdots & & \cdots
\end{array}$$

This implies that in the limit we have

$$\varinjlim_L H^*(\mathcal{X}\times o_L,S_\mu(n)) = \varinjlim_L H^*(\mathcal{X}\times o_L,t_{\leq n}R\tau_{L*}\mathbb{Z}/p^\mu\mathbb{Z}(n))\ .$$

For formal reasons the right hand side is equal to $H^*(\overline{\mathcal{X}},t_{\leq n}R\overline{\tau}_*\mathbb{Z}/p^\mu\mathbb{Z}(n))$.

Consider now the exact sequences

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
\varinjlim_L H^i(\mathcal{X}\times o_L,\mathbb{Z}(n))/p^\mu & & \\
\downarrow & & \\
\varinjlim_L H^i(\mathcal{X}\times o_L,S_\mu(n)) & = & H^i(\overline{\mathcal{X}},t_{\leq n}R\overline{\tau}_*\mathbb{Z}/p^\mu\mathbb{Z}(n)) \\
\downarrow & & \\
\varinjlim_L H^{i+1}(\mathcal{X}\times o_L,\mathbb{Z}(n))_{p^\mu} & & \\
\downarrow & & \\
0 & & .
\end{array}$$

Viewed as a short exact sequence of projective systems with respect to μ we see that the upper term has surjective transition maps. In the projective limit we therefore still have a surjection

$$\varprojlim_\mu H^i(\overline{\mathcal{X}},t_{\leq n}R\overline{\tau}_*\mathbb{Z}/p^\mu\mathbb{Z}(n)) \to \varprojlim_\mu \varinjlim_L H^{i+1}(\mathcal{X}\times o_L,\mathbb{Z}(n))_{p^\mu} \to 0\ .$$

In order to be able to say more we now assume the following conjecture of Bloch in [Blo] §3 to hold:

$$(B1)\quad \begin{array}{l}\text{If } Y \text{ is Hodge-Witt then the spectral sequences}\\ E_2^{a,b} = H^a(\overline{\mathcal{X}}, R^b\overline{\tau}_*\mathbb{Z}/p^\mu\mathbb{Z}(n)) \Longrightarrow H^{a+b}(\overline{X}, \mathbb{Z}/p^\mu\mathbb{Z}(n))\\ \text{degenerate at least up to torsion which is bounded independently of } \mu.\end{array}$$

This conjecture was proved by Kato ([Kat]) under the additional assumptions that $K/\mathbb{Q}_p$ is unramified and that $\dim X < p-1$. Let $F^{\cdot}H^*(\overline{X}, \mathbb{Z}/p^\mu\mathbb{Z}(n))$ denote the filtration which is induced by the above spectral sequence, i.e.,

$$F^{*-b}H^*(\ldots) = \mathrm{im}(H^*(\overline{\mathcal{X}}, t_{\leq b}R\overline{\tau}_*\mathbb{Z}/p^\mu\mathbb{Z}(n)) \to H^*(\overline{X}, \mathbb{Z}/p^\mu\mathbb{Z}(n)))$$
$$\text{and } F^{\cdot}H^*(\ldots)/F^{\cdot+1}H^*(\ldots) = E_\infty^{\cdot,*-\cdot}\ .$$

It induces correspondings filtrations on $H^*(\overline{X}, \mathbb{Z}_p(n))$ and $H^*(\overline{X}, \mathbb{Q}_p(n))$. Assuming from now on that Y is Hodge-Witt (B1) implies that

$$\varprojlim_\mu H^i(\overline{\mathcal{X}}, t_{\leq n}R\overline{\tau}_*\mathbb{Z}/p^\mu\mathbb{Z}(n)) \otimes \mathbb{Q}_p = F^{i-n}H^i(\overline{X}, \mathbb{Q}_p(n)) = F^{i-n}V\ .$$

Consider the commutative diagram

$$\begin{array}{ccc}
 & & 0\\
 & & \downarrow\\
 & & H^i(\overline{X}, \mathbb{Z}(n))/p^\mu\\
 & & \downarrow\\
\varinjlim_L H^i(\mathcal{X}\times o_L, S_\mu(n)) & \longrightarrow & H^i(\overline{X}, \mathbb{Z}/p^\mu\mathbb{Z}(n))\\
\downarrow & & \downarrow\\
\varinjlim_L H^{i+1}(\mathcal{X}\times o_L, \mathbb{Z}(n))_{p^\mu} & \longrightarrow & H^{i+1}(\overline{X}, \mathbb{Z}(n))_{p^\mu}\\
\downarrow & & \downarrow\\
0 & & 0
\end{array}\ .$$

By (B1) and the Lemma the kernels of the horizontal maps are bounded independently of μ. The same is true in case $i \neq 2n$ for the first term in the right column by Claim 2. Therefore passing to the projective limit with respect to μ and tensoring with $\mathbb{Q}_p$ leads, for $i \neq 2n$, to the commutative diagram

$$\begin{array}{ccc}
[\varprojlim_\mu \varinjlim_L H^i(\mathcal{X}\times o_L, S_\mu(n))]\otimes\mathbb{Q}_p & \overset{\cong}{\to} & [\varprojlim_\mu \varinjlim_L H^{i+1}(\mathcal{X}\times o_L, \mathbb{Z}(n))_{p^\mu}]\otimes\mathbb{Q}_p\\
\downarrow\cong & & \\
F^{i-n}H^i(\overline{X}, \mathbb{Q}_p(n)) & & \downarrow\\
\downarrow\subseteq & & \\
H^i(\overline{X}, \mathbb{Q}_p(n)) & \overset{\cong}{\to} & [\varprojlim_\mu H^{i+1}(\overline{X}, \mathbb{Z}(n))_{p^\mu}]\otimes\mathbb{Q}_p
\end{array}\ .$$

On the other hand from Mot 9 we have the exact sequences

$$\begin{aligned} 0 \to H^i(\mathcal{X} \times k_L, \mathbb{Z}(n))/p^\mu &\to H^{i-n}(\mathcal{X} \times k_L, \nu_\mu(n)) \\ &\to H^{i+1}(\mathcal{X} \times k_L, \mathbb{Z}(n))_{p^\mu} \to 0 \quad . \end{aligned}$$

In the limit with respect to L they give (by Mot 2) the exact sequences

$$0 \to H^i(\overline{Y}, \mathbb{Z}(n))/p^\mu \to H^{i-n}(\overline{Y}, \nu_\mu(n)) \to H^{i+1}(\overline{Y}, \mathbb{Z}(n))_{p^\mu} \to 0$$

where $\overline{Y} := \mathcal{X} \times \overline{k} = Y \times \overline{k}$ with $\overline{k}$ the residue class field of $o_{\overline{K}}$.

Claim 7:

For $i \neq 2n$ the group $H^i(\overline{Y}, \mathbb{Z}(n))$ contains a p-divisible subgroup such that the quotient has finite exponent.

Proof: This follows by the same arguments as in the proofs of Claims 1 and 2 from the subsequent facts which can be found in [Mil1] Prop. 3.1 and Cor. 6.4:
— The torsion subgroup in $H^{i-n}(\overline{Y}, \nu_{\mathbb{Z}_p}(n)) := \varprojlim_\mu H^{i-n}(\overline{Y}, \nu_\mu(n))$ has a finite exponent.
— That part of $H^{i-n}(\overline{Y}, \nu_{\mathbb{Z}_p}(n))$ on which $\mathrm{Gal}(\overline{k}/k)$ acts discretely is torsion provided $i \neq 2n$.

We therefore obtain, for $i \neq 2n$,

$$[\varprojlim_\mu H^{i+1}(\overline{Y}, \mathbb{Z}(n))_{p^\mu}] \otimes \mathbb{Q}_p = \varprojlim_\mu H^{i-n}(\overline{Y}, \nu_\mu(n)) \otimes \mathbb{Q}_p =: H^{i-n}(\overline{Y}, \nu_{\mathbb{Q}_p}(n)) \ .$$

Putting these results together we arrive at the following computation of the Tate module.

Claim 8:

If Y is Hodge-Witt and (B1) holds then we have, for $i \neq 2n$,

$$\mathrm{Tate}(\mathcal{G}_n^{i+1}) = \ker(F^{i-n}H^i(\overline{X}, \mathbb{Q}_p(n)) \longrightarrow H^{i-n}(\overline{Y}, \nu_{\mathbb{Q}_p}(n))) \quad .$$

Of course the map

$$F^{i-n}H^i(\overline{X}, \mathbb{Q}_p(n)) \longrightarrow H^{i-n}(\overline{Y}, \nu_{\mathbb{Q}_p}(n))$$

is induced by the natural sheaf homomorphisms

$$\begin{array}{ccc} \mathbb{Z}(n)_{\mathcal{X}\times o_L} \overset{\mathbb{L}}{\otimes} \mathbb{Z}/p^\mu\mathbb{Z} & \longrightarrow & \sigma_{L*}\mathbb{Z}(n)_{Y\times k_L} \overset{\mathbb{L}}{\otimes} \mathbb{Z}/p^\mu\mathbb{Z} \\ \| & & \| \\ S_\mu(n)_{\mathcal{X}\times o_L} & \longrightarrow & \sigma_{L*}\nu_\mu(n)_{Y\times k_L}[-n] \end{array}$$

where $\sigma_L : Y \times k_L \hookrightarrow \mathcal{X} \times o_L$ denotes the obvious closed immersion (the vertical identification on the right hand side comes from Mot 9). A similar analysis as in Section 7 based again on [BK0] (6.6) shows that the lower homomorphism can be defined independently of the existence of motivic cohomology. Following [Blo] we set

$$U^{i+1-n}H^i(\overline{X}, \mathbb{Q}_p(n)) := \ker(F^{i-n}H^i(\overline{X}, \mathbb{Q}_p(n)) \longrightarrow H^{i-n}(\overline{Y}, \nu_{\mathbb{Q}_p}(n))) \quad .$$

Always assuming (B1) to hold it is easy to see that

$$F^{i+1-n}H^i(\overline{X}, \mathbb{Q}_p(n)) \subseteq U^{i+1-n}H^i(\overline{X}, \mathbb{Q}_p(n)) \quad .$$

Since over $\overline{K}$ the Tate twist commutes with forming cohomology this in particular means that

$$[U^{i+1-(n-1)}H^i(\overline{X}, \mathbb{Q}_p(n-1))](1) \subseteq U^{i+1-n}H^i(\overline{X}, \mathbb{Q}_p(n)) \quad .$$

If $i \neq 2n, 2n-2$ the latter can be reformulated by Claim 8 as

$$\mathrm{Tate}(\mathcal{G}^{i+1}_{n-1})(1) \subseteq \mathrm{Tate}(\mathcal{G}^{i+1}_n) \quad .$$

Bloch in [Blo] §3 also conjectures the following:

$$(B2) \quad \begin{array}{l} \text{If } Y \text{ is Hodge-Witt then the functor} \\ L \mapsto \ker(H^{i+1-n}_{Zar}(\mathcal{X} \times o_L, \underline{\underline{K}}^{Milnor}_n) \to H^{i+1-n}_{Zar}(Y \times k_L, \underline{\underline{K}}^{Milnor}_n)) \\ \text{has a "piece" } L \mapsto \mathcal{B}^{i+1}_n(L) \text{ which is a } p\text{-divisible formal} \\ \text{group over } o_K \text{ with Tate module} \\ \mathrm{Tate}(\mathcal{B}^{i+1}_n) = U^{i+1-n}H^i(\overline{X}, \mathbb{Q}_p(n))/U^{i+1-(n-1)}H^i(\overline{X}, \mathbb{Q}_p(n-1))(1) \; . \end{array}$$

In case $i \neq 2n$ we then would have the surjection

$$\mathrm{Tate}(\mathcal{G}^{i+1}_n) \twoheadrightarrow \mathrm{Tate}(\mathcal{B}^{i+1}_n) \quad ;$$

does it come from a homomorphism $\mathcal{G}^{i+1}_n \to \mathcal{B}^{i+1}_n$?
From Mot 10 we obtain

$$H^{i+1-n}_{Zar}(\mathcal{X} \times o_L, \underline{\underline{K}}^{Milnor}_n) = H^{i+1-n}_{Zar}(\mathcal{X} \times o_L, R^n\alpha_*\mathbb{Z}(n)) \quad .$$

Since the right hand side by the Leray spectral sequence for α_* is closely related to $H^{i+1}(\mathcal{X} \times o_L, \mathbb{Z}(n))$ this conjecture (B2) seems to fit well into our general point of view.

If $\mathcal{G}_n^{i+1}$ is some kind of generalized formal group then it should have a "Dieudonné module". This means understanding the functor

$$A \longmapsto \mathcal{G}_n^{i+1}(A) := \ker(H^{i+1}(\mathcal{X} \times A, \mathbb{Z}(n)) \longrightarrow H^{i+1}(\mathcal{X} \times A_{red}, \mathbb{Z}(n)))$$

on artinian o_K-algebras A. Writing down this functor requires the existence of the complexes $\mathbb{Z}(n)_U$ at least for any scheme U such that the associated scheme U_{red} is noetherian and regular. But so far all proposals for the complexes $\mathbb{Z}(n)$ are insensitive to nilpotent elements.

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