The cohomology of local systems on p-adically uniformized varieties

by Peter Schneider

The first part of this paper is a sequel to [SS]. We will extend the computations there to a certain class of non-constant local systems. Let K be a finite extension of the field $\mathbb{Q}_p$ and let d be a natural number. Then

$$\Omega^{(d+1)} := \mathbb{P}^d_{/K} \setminus \text{union of all } K\text{-rational hyperplanes}$$

is an open rigid analytic subvariety of $\mathbb{P}^d_{/K}$ on which $PGL_{d+1}(K)$ acts in a natural way ([Dri]). We fix a cocompact discrete subgroup $\Gamma \subseteq PGL_{d+1}(K)$ which has no fixed points on $\Omega^{(d+1)}$. It is known ([Dri]) that

— $X_\Gamma := \Gamma\backslash\Omega^{(d+1)}$ is a proper and smooth (rigid) analytic variety over K (according to [Mus] it is even projective algebraic), and that

— the projection map $\pi : \Omega^{(d+1)} \to X_\Gamma$ is an etale covering.

We also fix a finite dimensional $K[\Gamma]$-module M. It gives rise to a local system $\mathcal{M}$, i.e., a locally constant sheaf $\mathcal{M}$ on the etale site $X_{\Gamma,et}$ on X_Γ by

$$\mathcal{M}(U) := \underline{M}(\Omega^{(d+1)} \underset{X_\Gamma}{\times} U)^\Gamma$$

where $\underline{M}$ is the constant sheaf on $\Omega^{(d+1)}_{et}$ corresponding to M. The de Rham cohomology of X_Γ with coefficients in the local system $\mathcal{M}$ is defined to be the etale hypercohomology

$$H^*_{DR}(X_\Gamma, \mathcal{M}) := H^*(X_\Gamma, \Omega^{\cdot}_{X_\Gamma/K} \underset{K}{\otimes} \mathcal{M})$$

where $\Omega^{\cdot}_{X_\Gamma/K}$ is the complex of holomorphic differential forms on $X_{\Gamma,et}$.

Remarks:

1) $H^*_{DR}(X_\Gamma) := H^*_{DR}(X_\Gamma, K)$ is the usual rigid analytic de Rham cohomology computed in the analytic topology (see [SS] proof of §5 Prop. 2).

2) If Γ is torsionfree then π is an analytic covering and $\mathcal{M}$ already is locally constant in the analytic topology. Therefore in this case $H^*_{DR}(X_\Gamma, \mathcal{M})$ can be computed as the hypercohomology of the complex of locally free $\mathcal{O}_{X_\Gamma}$-modules $\Omega^{\cdot}_{X_\Gamma/K} \underset{K}{\otimes} \mathcal{M}$ in the analytic topology. By the GAGA theorems of Kiehl $H^*_{DR}(X_\Gamma, \mathcal{M})$ even is defined algebraically (compare [Mal] p. 152/153).

Our aim in this paper is to compute the groups $H^*_{DR}(X_\Gamma, \mathcal{M})$. In the first Paragraph we will determine their dimensions as K-vector spaces. In particular we obtain that interesting cohomology only exists in middle dimension d. In the second Paragraph we will discuss the Hodge-de Rham spectral sequence for $H^*_{DR}(X_\Gamma, \mathcal{M})$. We conjecture that the Hodge-de Rham filtration on $H^d_{DR}(X_\Gamma, \mathcal{M})$ is opposite to the filtration induced by the covering spectral sequence in Paragraph 1. A consequence would be the existence of a Hodge type decomposition of $H^d_{DR}(X_\Gamma, \mathcal{M})$. Using a result of Mustafin we prove this Conjecture for $d \leq 2$ and any M which contains a Γ-invariant lattice.

In the last Paragraph we analyze the case where M comes from a K-rational representation of $SL_{d+1}(K)$. In this situation the Hodge-de Rham spectral sequence will usually not degenerate. Our aim is to understand the complex $\Omega^{\cdot}(\Omega^{(d+1)}) \underset{K}{\otimes} M$ of global sections as a complex of $SL_{d+1}(K)$-representations. For this purpose we introduce a class of representations (in K-vector spaces) called the holomorphic discrete series of $SL_{d+1}(K)$ and begin its systematic study. This will enable us to show that the complex $\Omega^{\cdot}(\Omega^{(d+1)}) \underset{K}{\otimes} M$ is naturally and $SL_{d+1}(K)$-equivariantly quasi-isomorphic to a much simpler complex of discrete series representations. In this way we obtain a "reduced" Hodge-de Rham spectral sequence which has a chance to degenerate. Our theory can be viewed as a p-adic analog of the Bernstein-Gelfand-Gelfand resolutions in the theory of Verma modules; it was very much inspired by reading [Fal], [FC] Chap. VI §5 and [Zuc]. Even our proofs are similar in spirit although the arguments have to be arranged in a different way; after all the analogy is not complete: The Bernstein-Gelfand-Gelfand resolution is exact whereas our complex has explicitly computed non-trivial homology. One of the reasons for this difference is that over a p-adic field the Lie algebra is too weak an object to study the representations of the group. In this paper the holomorphic discrete series of $SL_{d+1}(K)$ is treated in a purely algebraic way. Eventually one must introduce locally convex topologies on these representations; in particular this will be necessary for the question of irreducibility which is not touched here. For $SL_2(K)$ this class of representations was introduced and studied in a rather complete way by Morita and Murase ([Mor]).

It should be clear to the reader that not every local system on X_Γ is of the form considered here. The reason for this is that $\Omega^{(d+1)}$ is not the universal etale covering of X_Γ. If not stated otherwise all cohomology in this paper is (rigid) etale cohomology.

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§1 The covering spectral sequence

We first recall the following result of Garland, Casselman, Prasad, and Borel/Wallach.

Proposition 1:

$H^r(\Gamma, M)$ is finite dimensional for all $r \geq 0$ and vanishes for $r \neq 0, d$.

Proof: In case $d = 1$ this follows from the fact that by Ihara's theorem Γ contains a normal subgroup of finite index which is finitely generated and free ([GP] I §3 or [Ser2] II §1.5). In case $d \geq 2$ this is proved in [BW] XIII 3.7.

The basic invariant of the Γ-module M which will enter into our computation is

$$\mu(\Gamma, M) := \dim_K H^d(\Gamma, M) \quad .$$

Also let $M^* := \mathrm{Hom}_K(M, K)$ denote the dual Γ-module.

Remark:

If Γ is torsionfree then

$$\mu(\Gamma, M) = \dim_K M \cdot \mu(\Gamma, K) + (-1)^d \cdot \dim_K M/M^\Gamma \quad .$$

Proof: Proposition 1 and [Ser1] Remark on p. 85.

In this Paragraph we want to establish the subsequent Theorem. It should be said right away that we will rely very much on the ideas developed by Casselman and Borel/Wallach in their proofs of Proposition 1.

Theorem 1:

If the trivial Γ-module K is not a Jordan-Hölder factor of M then we have

i. $H^r_{DR}(X_\Gamma, \mathcal{M}) = 0$ *for* $r \neq d$;

ii. $\dim_K H^d_{DR}(X_\Gamma, \mathcal{M}) = \begin{cases} \mu(\Gamma, M) + \mu(\Gamma, M^*) & \text{if } d = 1 \ , \\ (d+1) \cdot \mu(\Gamma, M) & \text{if } d \geq 2 \ . \end{cases}$

Remark:

In case $d \geq 2$ the assumption in Theorem 1 can be simplified. By Proposition 1 we then have $H^1(\Gamma, M') = 0$ for all Γ-submodules $M' \subseteq M$. Therefore the trivial Γ-module K is not a Jordan-Hölder factor of M if and only if $M^\Gamma = 0$.

Our starting point is the covering spectral sequence

$$H^r(\Gamma, H^s(\Omega^{(d+1)}, \Omega^{\cdot}_{X_\Gamma/K} \underset{K}{\otimes} \mathcal{M})) \Longrightarrow H^{r+s}_{DR}(X_\Gamma, \mathcal{M})$$

whose existence was established in [SS] §5 Prop. 2. The restriction of $\Omega^{\cdot}_{X_\Gamma/K}$ to $\Omega^{(d+1)}_{et}$ by definition is equal to $\Omega^{\cdot}_{\Omega^{(d+1)}/K}$. On the other hand because of the isomorphism

$$\begin{aligned} \coprod_{g\in\Gamma} \Omega^{(d+1)} &\xrightarrow{\sim} \Omega^{(d+1)} \underset{X_\Gamma}{\times} \Omega^{(d+1)} \\ z \text{ in component } g &\longmapsto (gz, z) \end{aligned}$$

the restriction of $\mathcal{M}$ to $\Omega^{(d+1)}_{et}$ is equal to $\underline{M}$. We obtain

$$H^*(\Omega^{(d+1)}, \Omega^{\cdot}_{X_\Gamma/K} \underset{K}{\otimes} \mathcal{M}) = H^*_{DR}(\Omega^{(d+1)}) \underset{K}{\otimes} M$$

so that our spectral sequence becomes

$$H^r(\Gamma, H^s_{DR}(\Omega^{(d+1)}) \underset{K}{\otimes} M) \Longrightarrow H^{r+s}_{DR}(X_\Gamma, \mathcal{M}) \quad .$$

The de Rham cohomology of $\Omega^{(d+1)}$ was computed in [SS] Thm. 1 and §4 Lemma 1. In order to recall the result we need some more notations. For any subset $I \subseteq \Delta := \{1, \ldots, d\}$ let $P_I \subseteq G := PGL_{d+1}(K)$ be that parabolic subgroup which stabilizes the flag

$$(\sum_{i=1}^{i_0} Ke_i \underset{\neq}{\subseteq} \sum_{i=1}^{i_1} Ke_i \underset{\neq}{\subseteq} \ldots \underset{\neq}{\subseteq} \sum_{i=1}^{i_r} Ke_i)$$

in K^{d+1} where $\{i_0 < i_1 < \ldots < i_r\} = \Delta \backslash I$ and $e_1, \ldots, e_{d+1}$ is the standard basis. Furthermore if A is any abelian group let $C^\infty(G/P_I, A)$ be the group of all A-valued locally constant functions on the compact and totally disconnected space G/P_I. We put

$$\begin{aligned} V_I(A) &:= C^\infty(G/P_I, A) / \sum_{i\in\Delta\backslash I} C^\infty(G/P_{I\cup\{i\}}, A) \\ &= V_I(\mathbb{Z}) \otimes A \end{aligned}$$

which in an obvious way is a G-module. It was proved in loc.cit. that there is a natural G-equivariant isomorphism

$$H^s_{DR}(\Omega^{(d+1)}) = \mathrm{Hom}_K(V_{\{1,\ldots,d-s\}}(K), K) \qquad \text{if} \quad 0 \le s \le d$$

and that

$$H^s_{DR}(\Omega^{(d+1)}) = 0 \qquad \text{if} \quad s > d\ .$$

This implies

$$\begin{aligned} H^r(\Gamma, H^s_{DR}(\Omega^{(d+1)}) \underset{K}{\otimes} M) &= H^r(\Gamma, \mathrm{Hom}_K(V_{\{1,\ldots,d-s\}}(K), K) \underset{K}{\otimes} M) \\ &= H^r(\Gamma, \mathrm{Hom}_K(V_{\{1,\ldots,d-s\}}(K), M)) \\ &= \mathrm{Ext}^r_{K[\Gamma]}(V_{\{1,\ldots,d-s\}}(K), M) \end{aligned}$$

for $0 \leq s \leq d$ and we can rewrite our spectral sequence as

$$E_2^{r,s} = \left\{ \begin{array}{ll} \mathrm{Ext}^r_{K[\Gamma]}(V_{\{1,\ldots,d-s\}}(K), M) & \text{if } 0 \leq s \leq d \\ 0 & \text{otherwise} \end{array} \right\} \Longrightarrow H^{r+s}_{DR}(X_\Gamma, \mathcal{M}) \quad .$$

Our task therefore is to determine the dimensions of the K-vector spaces $\mathrm{Ext}^r_{K[\Gamma]}(V_I(K), M)$.

Proposition 2:

If the trivial Γ-module K is not a Jordan-Hölder factor of M then we have:

i. $\dim_K \mathrm{Ext}^r_{K[\Gamma]}(V_I(K), M) = \begin{cases} \mu(\Gamma, M) & \textit{if } r = \#I \neq 0 \ , \\ \mu(\Gamma, M^*) & \textit{if } r = \#I = 0 \ , \\ 0 & \textit{otherwise} \ ; \end{cases}$

ii. $\mu(\Gamma, M) = \mu(\Gamma, M^*)$ *if* $d \geq 2$.

Proof: Because of $V_\Delta(K) = K$ the assertion i. in case $I = \Delta$ is Proposition 1.

Next we settle the other extreme case $I = \phi$ by using Borel-Serre duality.

Lemma 3:

$\mathrm{Ext}^*_{K[\Gamma]}(V_\phi(K), M) \cong \mathrm{Hom}_K(H^{d-*}(\Gamma, M^*), K)$.

Proof: This results from combining the following three identifications. Firstly we have for almost trivial reasons that

$$\mathrm{Hom}_K(H_*(\Gamma, .), K) = \mathrm{Ext}^*_{K[\Gamma]}(., K) \quad .$$

Secondly because of the finite dimensionality of M^* the functor $\mathrm{Hom}_K(M^*, .)$ respects injective $K[\Gamma]$-modules which implies

$$\mathrm{Ext}^*_{K[\Gamma]}(V_\phi(K) \underset{K}{\otimes} M^*, K) = \mathrm{Ext}^*_{K[\Gamma]}(V_\phi(K), M) \quad .$$

Thirdly the Borel-Serre duality for Γ ([BS] §6) gives a natural isomorphism

$$H_*(\Gamma, V_\phi(K) \underset{K}{\otimes} M) \cong H^{d-*}(\Gamma, M) \quad .$$

The assumption in [BS] that Γ is torsionfree is superfluous in our context: Since our Γ-modules are vector spaces the above isomorphism holds for Γ if it holds for a normal subgroup of finite index in Γ (observe [BE] Prop. 1.8). But Γ always contains a normal subgroup of finite index which is torsionfree ([Gar] 2.7).

Going back to the proof of Proposition 2 we see that Lemma 3 reduces our assertion i. in the case $I = \phi$ to Proposition 1. In particular the whole assertion is proved for $d = 1$. We therefore will always assume in the following that $d \geq 2$.

Lemma 4:

For any field extension L/K we have

$$\mathrm{Ext}^*_{K[\Gamma]}(V_I(K), M) \underset{K}{\otimes} L = \mathrm{Ext}^*_{L[\Gamma]}(V_I(L), M \underset{K}{\otimes} L) \quad .$$

Proof: This follows by the universal coefficient theorem from the fact that $V_I(K)$ has a projective resolution by finitely generated free $K[\Gamma]$-modules ([SS] §6 Prop. 16, compare also the proof of §5 Prop. 4).

In the rest of the proof of Proposition 2 we therefore may and will work over the complex numbers $\mathbb{C}$ instead of over K. Note that our assumption on the Jordan-Hölder factors of M is preserved under scalar extension. This is a consequence of the fact that any irreducible $K[\Gamma]$-module is absolutely semisimple ([Bou] §7.5 Prop. 5 and 7). By induction with respect to a Jordan-Hölder series of M we also may assume that M is an irreducible Γ-module. But then a deep theorem of Margulis tells us that M carries a Γ-invariant positive definite hermitian form, i.e., that M is a unitary Γ-module ([BW] p. 373 or [Mar] IX.5.12(i)). This immediately implies the assertion ii. The further argument for proving the assertion i. from this point on is completely analogous to the reasoning in [SS] §5 for the case of the constant local system. We therefore give only a brief sketch. Consider the induced representation

$$\mathrm{Ind}_\Gamma(M) := \{f \in C^\infty(G, M) : f(g\gamma) = \gamma^{-1} f(g) \text{ for all } g \in G, \gamma \in \Gamma\}$$

with G acting by left translations. Since Γ is cocompact and M is unitary $\mathrm{Ind}_\Gamma(M)$ is an admissible (pre-)unitary representation of G. Shapiro's lemma ([Cas2] A.8) implies

$$\mathrm{Ext}^*_{\mathbb{C}[\Gamma]}(V_I(\mathbb{C}), M) = \mathrm{Ext}^*_G(V_I(\mathbb{C}), \mathrm{Ind}_\Gamma(M))$$

where Ext^*_G is the Ext-functor on the category of smooth G-representations.

Proposition 5:

If V is any admissible (pre-)unitary representation of G then we have

$$\mathrm{Ext}^r_G(V_I(\mathbb{C}), V) \cong \begin{cases} V^G & \text{if } r = \#\Delta\backslash I \neq \#I \ , \\ \mathrm{Hom}_G(V_\phi(\mathbb{C}), V) & \text{if } r = \#I \neq \#\Delta\backslash I \ , \\ V^G \oplus \mathrm{Hom}_G(V_\phi(\mathbb{C}), V) & \text{if } r = \#I = \#\Delta\backslash I \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Proof: This is the content of [SS] §5 from the paragraph before Prop. 5 til Prop. 9.

We apply this to $V = \mathrm{Ind}_\Gamma(M)$. Since $\mathrm{Ind}_\Gamma(M)^G = M^\Gamma = 0$ by assumption we obtain

$$\mathrm{Ext}^r_{\mathbb{C}[\Gamma]}(V_I(\mathbb{C}), M) \cong \begin{cases} \mathrm{Hom}_G(V_\phi(\mathbb{C}), \mathrm{Ind}_\Gamma(M)) & \text{if } r = \#I \ , \\ 0 & \text{otherwise .} \end{cases}$$

But Lemma 3 says that

$$\begin{aligned} \mathrm{Hom}_G(V_\phi(\mathbb{C}), \mathrm{Ind}_\Gamma(M)) &= \mathrm{Hom}_{\mathbb{C}[\Gamma]}(V_\phi(\mathbb{C}), M) \\ &= \mathrm{Hom}_{\mathbb{C}}(H^d(\Gamma, M^*), \mathbb{C}) \quad . \end{aligned}$$

This completes the proof of Proposition 2.

Remark:

The above proof reproduces the well-known result that $\mu(\Gamma, M)$ for a unitary $\mathbb{C}[\Gamma]$-module M is the multiplicity of the Steinberg representation $V_\phi(\mathbb{C})$ in $\mathrm{Ind}_\Gamma(M)$ (compare [Bor] Thm. 7.2 and [Cas1] Thm. 3).

Theorem 1 now is immediate from inserting the result in Proposition 2 into our covering spectral sequence.

Corollary 6:

For $d \geq 2$ we have

$$\dim_K H^r_{DR}(X_\Gamma, \mathcal{M}) = \begin{cases} \dim_K M^\Gamma & \textit{if } 0 \leq r \leq 2d, r \neq d, \\ & \qquad r \textit{ even}, \\ \dim_K M^\Gamma + (d+1) \cdot \mu(\Gamma, M) & \textit{if } r = d, d \textit{ even}, \\ (d+1) \cdot \mu(\Gamma, M) & \textit{if } r = d, d \textit{ odd}, \\ 0 & \textit{otherwise.} \end{cases}$$

For $d = 1$ we have

$$\dim_K H^r_{DR}(X_\Gamma, \mathcal{M}) = \begin{cases} \dim_K M^\Gamma & \textit{if } r = 0 \ , \\ \mu(\Gamma, M) + \mu(\Gamma, M^*) & \textit{if } r = 1 \ , \\ \dim_K (M^*)^\Gamma & \textit{if } r = 2 \ , \\ 0 & \textit{otherwise .} \end{cases}$$

Proof: Considering first the case $d \geq 2$ we note that as a Γ-module M^Γ is a direct summand of M. Because of

$$\mathrm{Ext}^1_{K[\Gamma]}(M/M^\Gamma, M^\Gamma) = \mathrm{Ext}^1_{K[\Gamma]}(M/M^\Gamma, K) \underset{K}{\otimes} M^\Gamma = H^1(\Gamma, (M/M^\Gamma)^*) \underset{K}{\otimes} M^\Gamma$$

this follows from Proposition 1. On the other hand we had already remarked that M/M^Γ does not have the trivial Γ-module K as a Jordan-Hölder factor. Therefore the assertion is a consequence of Theorem 1 together with [SS] §5 Thm. 5.
Now assume $d=1$. Then the covering spectral sequence only involves $V_\Delta(K) = K$ and $V_\phi(K)$ and becomes

$$E_2^{r,s} = \left\{ \begin{array}{ll} \mathrm{Ext}^r_{K[\Gamma]}(K,M) & \text{if } s=0 \\ \mathrm{Ext}^r_{K[\Gamma]}(V_\phi(K),M) & \text{if } s=1 \\ 0 & \text{otherwise} \end{array} \right\} \Longrightarrow H^{r+s}_{DR}(X_\Gamma,\mathcal{M}) \quad .$$

Taking Proposition 1 and Lemma 3 into account it comes down to the isomorphisms

$$H^0_{DR}(X_\Gamma,\mathcal{M}) \cong M^\Gamma \quad \text{and} \quad H^2_{DR}(X_\Gamma,\mathcal{M}) \cong \mathrm{Hom}_K((M^*)^\Gamma,K)$$

and the exact sequence

$$0 \longrightarrow H^1(\Gamma,M) \longrightarrow H^1_{DR}(X_\Gamma,\mathcal{M}) \longrightarrow \mathrm{Hom}_K(H^1(\Gamma,M^*),K) \longrightarrow 0 \quad .$$

Actually we have proved a slightly stronger result. Let

$$H^d_{DR}(X_\Gamma,\mathcal{M}) = F^0_\Gamma \supseteq F^1_\Gamma \supseteq \ldots \supseteq F^{d+1}_\Gamma = 0$$

be the filtration induced by the covering spectral sequence (i.e., $F^i_\Gamma/F^{i+1}_\Gamma = E^{i,d-i}_\infty$). We have computed the dimension of each step in this filtration.

Theorem 2:

The covering spectral sequence degenerates. For $d \geq 2$ and $0 \leq i \leq d$ we have

$$\dim_K F^i_\Gamma/F^{i+1}_\Gamma = \left\{ \begin{array}{ll} \mu(\Gamma,M) & \textit{if } 2i \neq d\ , \\ \mu(\Gamma,M) + \dim_K M^\Gamma & \textit{if } 2i = d\ ; \end{array} \right.$$

for $d=1$ we have

$$\dim_K F^0_\Gamma/F^1_\Gamma = \mu(\Gamma,M^*) \quad \textit{and} \quad \dim_K F^1_\Gamma = \mu(\Gamma,M) \quad .$$

Proof: In case $d=1$ see the proof of Corollary 6. In case $d \geq 2$ again by an argument in the proof of Corollary 6 it suffices to consider either the trivial Γ-module K which is dealt with in [SS] §5 Thm. 3 and Thm. 5 or a Γ-module M not having K as a Jordan-Hölder factor which we treated in Proposition 2.

Corollary 7:

For $d \geq 2$ we have $H^1(X_\Gamma, \mathcal{M}) = 0$.

Proof: Since $\mathcal{M}$ is the 0-th cohomology sheaf of the complex $\Omega^{\cdot}_{X_\Gamma/K} \underset{K}{\otimes} \mathcal{M}$ we have the injective edge homomorphism $H^1(X_\Gamma, \mathcal{M}) \hookrightarrow H^1_{DR}(X_\Gamma, \mathcal{M})$.

§2 The de Rham filtration

The Hodge- de Rham spectral sequence is the standard hypercohomology spectral sequence

$$E_1^{r,s} = H^s(X_\Gamma, \Omega^r_{X_\Gamma/K} \underset{K}{\otimes} \mathcal{M}) \Longrightarrow H^{r+s}_{DR}(X_\Gamma, \mathcal{M}) \quad .$$

It induces the de Rham filtration

$$H^d_{DR}(X_\Gamma, \mathcal{M}) = F^0_{DR} \supseteq F^1_{DR} \supseteq \ldots \supseteq F^{d+1}_{DR} = 0$$

on the abutment (i.e., $F^i_{DR}/F^{i+1}_{DR} = E^{i,d-i}_\infty$). If M is a trivial Γ-module then as a consequence of the GAGA-principle ([Kie]) the Hodge-de Rham spectral sequence degenerates. Also the corresponding Hodge numbers can be computed; for $d = 2$ this is contained in [Mum].

Proposition 1:

If Γ is a torsionfree subgroup in $SL_{d+1}(K)$ then we have

$$\dim_K H^s(X_\Gamma, \Omega^r_{X_\Gamma/K}) = \begin{cases} \mu(\Gamma, K) + 1 & \text{if } r + s = d, r = s \quad , \\ \mu(\Gamma, K) & \text{if } r + s = d, r \neq s \quad , \\ 1 & \text{if } r = s \neq \frac{d}{2} \quad , \\ 0 & \text{if } r \neq s, r + s \neq d \quad . \end{cases}$$

Proof: The case $r + s \neq d$ was settled (for any Γ) in [SS] §5 Thm. 5. I am grateful to A. Kurihara for pointing out to me the following argument for the case $r + s = d$. Fixing r we so far know that

$$(-1)^d \cdot \dim_K H^{d-r}(X_\Gamma, \Omega^r_{X_\Gamma/K}) = (-1)^r \cdot \chi(\Omega^r_{X_\Gamma/K}) - 1 + \delta_{r\frac{d}{2}}$$

where

$$\chi(.) := \sum_{s=0}^{d} (-1)^s \dim_K H^s(X_\Gamma, .)$$

denotes the Euler-Poincaré characteristic and $\delta_{r\frac{d}{2}}$ is the Kronecker symbol. On the other hand it is a consequence of the proportionality theorem in [Kur] that

$$\begin{aligned}\chi(\Omega^r_{X_\Gamma/K}) &= \mu(\Gamma\backslash SL_{d+1}(K)) \cdot \chi(\Omega^r_{\mathbb{P}^d/K}) \\ &= (-1)^r \cdot \mu(\Gamma\backslash SL_{d+1}(K))\end{aligned}$$

where μ denotes the Euler-Poincaré measure on $SL_{d+1}(K)$. We obtain

$$\dim_K H^{d-r}(X_\Gamma, \Omega^r_{X_\Gamma/K}) = (-1)^d \cdot (\mu(\Gamma\backslash SL_{d+1}(K)) - 1 + \delta_{r\frac{d}{2}}) \quad .$$

By definition of μ we have

$$\mu(\Gamma\backslash SL_{d+1}(K)) = \sum_s (-1)^s \cdot \dim_K H^s(\Gamma, K)$$

which by §1 Proposition 1 means that

$$\mu(\Gamma\backslash SL_{d+1}(K)) = 1 + (-1)^d \cdot \mu(\Gamma, K) \quad .$$

We will see later on that even for very natural Γ-modules M the Hodge-de Rham spectral sequence does not degenerate. Nevertheless there is evidence that the de Rham filtration has the following property.

Conjecture:

The filtrations $F^\cdot_\Gamma$ and $F^\cdot_{DR}$ are opposite to each other, i.e.,

$$H^d_{DR}(X_\Gamma, \mathcal{M}) = F^i_\Gamma \oplus F^{d+1-i}_{DR} \quad \textit{for any } 0 \leq i \leq d+1 \quad .$$

An immediate consequence of this Conjecture would be the Hodge type decomposition

$$H^d_{DR}(X_\Gamma, \mathcal{M}) = \bigoplus_{i=0}^{d} (F^i_\Gamma \cap F^{d-i}_{DR})$$

with (by Theorem 2)

$$\dim_K(F^i_\Gamma \cap F^{d-i}_{DR}) = \begin{cases} \mu(\Gamma, M) & \text{if } d \geq 2, 2i \neq d \text{ or } d = i = 1 \quad , \\ \mu(\Gamma, M^*) & \text{if } d = 1, i = 0 \quad , \\ \mu(\Gamma, M) + \dim_K M^\Gamma & \text{if } 2i = d \quad . \end{cases}$$

Since $X := \Omega^{(d+1)}$ is a Stein-space ([SS] §1 Prop. 4) its de Rham cohomology can be computed from the complex $\Omega^{\cdot}(X)$ of global holomorphic differential forms on X. Hence we have

$$\begin{aligned} H^*(\Gamma, H^s_{DR}(X) \underset{K}{\otimes} M) &= H^*(\Gamma, h^s(\Omega^{\cdot}(X) \underset{K}{\otimes} M)) \quad , \\ H^*(X_\Gamma, \Omega^r_{X_\Gamma/K} \underset{K}{\otimes} \mathcal{M}) &= H^*(\Gamma, \Omega^r(X) \underset{K}{\otimes} M) \ , \quad \text{and} \\ H^*_{DR}(X_\Gamma, \mathcal{M}) &= H^*(\Gamma, \Omega^{\cdot}(X) \underset{K}{\otimes} M) \quad . \end{aligned}$$

The two filtrations $F^{\cdot}_\Gamma$ and $F^{\cdot}_{DR}$ are induced by the two hypercohomology spectral sequences for the complex $\Omega^{\cdot}(X) \underset{K}{\otimes} M$:

$$\begin{aligned} F^i_\Gamma &= \mathrm{im}(H^d(\Gamma, t_{\leq d-i}\Omega^{\cdot}(X) \underset{K}{\otimes} M) \to H^d(\Gamma, \Omega^{\cdot}(X) \underset{K}{\otimes} M)) \\ &= \ker(H^d(\Gamma, \Omega^{\cdot}(X) \underset{K}{\otimes} M) \to H^d(\Gamma, t_{\geq d+1-i}\Omega^{\cdot}(X) \underset{K}{\otimes} M)) \text{ and} \\ F^i_{DR} &= \mathrm{im}(H^d(\Gamma, \Omega^{\cdot}_{\geq i}(X) \underset{K}{\otimes} M) \to H^d(\Gamma, \Omega^{\cdot}(X) \underset{K}{\otimes} M)) \\ &= \ker(H^d(\Gamma, \Omega^{\cdot}(X) \underset{K}{\otimes} M) \to H^d(\Gamma, \Omega^{\cdot}_{\leq i-1}(X) \underset{K}{\otimes} M)) \quad . \end{aligned}$$

Here we have used the following notations for the various truncations of a complex: If

$$C^{\cdot} = (C^0 \xrightarrow{d} C^1 \xrightarrow{d} \ldots)$$

is any complex we put

$$\begin{array}{rcl} t_{\leq i}C^{\cdot} &:=& (C^0 \to \ldots \to C^{i-1} \to \ \ker(d) \ \to \ 0 \ \to \ldots) \ , \\ C^{\cdot}_{\leq i} &:=& (C^0 \to \ldots \to C^{i-1} \to \ C^i \ \to \ 0 \ \to \ldots) \ , \\ t_{\geq i}C^{\cdot} &:=& (0 \to \ldots \to 0 \to C^i/\mathrm{im}(d) \to C^{i+1} \to \ldots) \ , \text{and} \\ C^{\cdot}_{\geq i} &:=& (0 \to \ldots \to 0 \to C^i \to C^{i+1} \to \ldots) \ . \end{array}$$

Also let

$$B^{\cdot}(X) \subseteq \Omega^{\cdot}(X)$$

denote the subgroup of exact forms in $\Omega^{\cdot}(X)$.

Lemma 2:

i. $F^{d-i}_\Gamma \cap F^{i+1}_{DR} = 0$ *if and only if the natural map*

$$H^d(\Gamma, t_{\leq i}\Omega^{\cdot}(X) \underset{K}{\otimes} M) \to H^d(\Gamma, \Omega^{\cdot}_{\leq i}(X) \underset{K}{\otimes} M) \quad \textit{is injective} \ ;$$

ii. $F_\Gamma^{d+1-i} + F_{DR}^i = H_{DR}^d(X_\Gamma, \mathcal{M})$ *if and only if the natural map*

$$H^d(\Gamma, \Omega^{\cdot}_{\geq i}(X) \underset{K}{\otimes} M) \to H^d(\Gamma, t_{\geq i}\Omega^{\cdot}(X) \underset{K}{\otimes} M) \text{ is surjective ;}$$

iii. the Hodge-de Rham spectral sequence degenerates and the Conjecture holds if

$$H^*(\Gamma, B^i(X) \underset{K}{\otimes} M) = 0 \quad \text{for } 1 \leq i \leq d \quad .$$

Proof: i. From the commutative diagram of complexes

$$\begin{tikzcd} & \Omega^{\cdot}(X) \arrow[dr] & \\ t_{\leq i}\Omega^{\cdot}(X) \arrow[ur] \arrow[rr] & & \Omega^{\cdot}_{\leq i}(X) \end{tikzcd}$$

we obtain the commutative diagram of cohomology groups

$$\begin{tikzcd} & & H_{DR}^d(X_\Gamma, \mathcal{M}) \arrow[dr] & & \\ & F_\Gamma^{d-i} \arrow[ur, "\subseteq"] & & F_{DR}^0/F_{DR}^{i+1} \arrow[d] & \\ H^d(\Gamma, t_{\leq i}\Omega^{\cdot}(X) \underset{K}{\otimes} M) \arrow[ur, "\cong"] \arrow[rrr] & & & H^d(\Gamma, \Omega^{\cdot}_{\leq i}(X) \underset{K}{\otimes} M). \end{tikzcd}$$

The left oblique arrow is an isomorphism since the covering spectral sequence degenerates.

ii. Similarly we have commutative diagrams

$$\begin{tikzcd} & \Omega^{\cdot}(X) \arrow[dr] & \\ \Omega^{\cdot}_{\geq i}(X) \arrow[ur] \arrow[rr] & & t_{\geq i}\Omega^{\cdot}(X) \end{tikzcd}$$

and

$$\begin{tikzcd} & & H_{DR}^d(X_\Gamma, \mathcal{M}) \arrow[dr] & & \\ & F_{DR}^i \arrow[ur, "\subseteq"] & & F_\Gamma^0/F_\Gamma^{d+1-i} \arrow[d, "\cong"] & \\ H^d(\Gamma, \Omega^{\cdot}_{\geq i}(X) \underset{K}{\otimes} M) \arrow[ur] \arrow[rrr] & & & H^d(\Gamma, t_{\geq i}\Omega^{\cdot}(X) \underset{K}{\otimes} M) \end{tikzcd}$$

where now the right oblique arrow is an isomorphism.

iii. Our assumption implies that the maps considered in i. and ii. even are bijective so that the Conjecture holds. It also implies that in the commutative diagram

$$\begin{tikzcd} & H_{DR}^s(X_\Gamma, \mathcal{M}) \arrow[dr] & \\ H^s(\Gamma, t_{\leq i}\Omega^{\cdot}(X) \underset{K}{\otimes} M) \arrow[ur] \arrow[rr] & & H^s(\Gamma, \Omega^{\cdot}_{\leq i}(X) \underset{K}{\otimes} M) \end{tikzcd}$$

for any $s \geq 0$ the horizontal arrow is an isomorphism. Therefore the right oblique arrow is surjective for all $s \geq 0$ which amounts to the degeneration of the Hodge-de Rham spectral sequence.

There are two important classes of Γ-modules.

Definition:

i. M is called integral if it contains a Γ-invariant o_K-lattice where o_K denotes the ring of integers in K.

ii. M is called algebraic if there is a cocompact discrete subgroup $\tilde{\Gamma} \subseteq SL_{d+1}(K)$ which maps into Γ and such that M as a $\tilde{\Gamma}$-module comes from a K-rational representation of SL_{d+1}.

Remark:

Assume $d \geq 2$. Margulis proves that there always is a cocompact discrete subgroup $\tilde{\Gamma} \subseteq SL_{d+1}(K)$ which maps into Γ and such that M as a $\tilde{\Gamma}$-module comes from a continuous representation of $SL_{d+1}(K)$ over K ([Mar] IX.1.11, 5.13, and 5.14 (iii)). This result together with [BT] leads to a rather precise description of arbitrary Γ-modules in terms of algebraic ones involving Galois twists and tensor products. In particular, if $K = \mathbb{Q}_p$ then every Γ-module is algebraic ([Mar] I.2.6.1 (ii)).

For integral Γ-modules we have the following partial result of Mustafin.

Proposition 3:

If M is integral then $H^*(\Gamma, B^1(X) \underset{K}{\otimes} M) = 0$.

Proof: [Mus] Thm. 3.1.II.b.

Corollary 4:

If M is integral then we have:

i. $H^d_{DR}(X_\Gamma, \mathcal{M}) = F^d_\Gamma \oplus F^1_{DR}$;

ii. $F^d_\Gamma \cong F^0_{DR}/F^1_{DR} \cong H^d(X_\Gamma, \mathcal{O}_{X_\Gamma} \underset{K}{\otimes} \mathcal{M})$ *has dimension* $\mu(\Gamma, M)$;

iii. $H^s(X_\Gamma, \mathcal{O}_{X_\Gamma} \underset{K}{\otimes} \mathcal{M}) = 0$ *for* $0 < s < d$.

Proof: The assertion i. and the isomorphisms in ii. are deduced by the technique in the proof of Lemma 2. The other assertions follow from §1 Proposition 1 and the isomorphisms

$$H^s(X_\Gamma, \mathcal{O}_{X_\Gamma} \underset{K}{\otimes} \mathcal{M}) = H^s(\Gamma, \mathcal{O}(X) \underset{K}{\otimes} M) \cong H^s(\Gamma, M) \quad .$$

Corollary 5:

If M is integral then we have:

i. $H^*(\Gamma, B^d(X) \underset{K}{\otimes} M) = 0$;

ii. $H^d_{DR}(X_\Gamma, \mathcal{M}) = F^1_\Gamma \oplus F^d_{DR}$;

iii. $F^0_\Gamma / F^1_\Gamma \cong F^d_{DR} \cong H^0(X_\Gamma, \Omega^d_{X_\Gamma/K} \underset{K}{\otimes} \mathcal{M})$ *has dimension* $\mu(\Gamma, M^*)$;

iv. $H^s(X_\Gamma, \Omega^d_{X_\Gamma/K} \underset{K}{\otimes} \mathcal{M}) = 0$ *for* $0 < s < d$.

Proof: We have the commutative diagram of cupproduct pairings

$$\begin{array}{ccccc}
H^{d-*}(\Gamma, M^*) & \times\, H^*(\Gamma, h^d(\Omega^\cdot(X)) \underset{K}{\otimes} M) & \longrightarrow & H^d(\Gamma, h^d(\Omega^\cdot(X))) \cong K \\
\downarrow & \uparrow & & \uparrow\cong \\
H^{d-*}(\Gamma, \mathcal{O}(X) \underset{K}{\otimes} M^*) \times & H^*(\Gamma, \Omega^d(X) \underset{K}{\otimes} M) & \longrightarrow & H^d(\Gamma, \Omega^d(X)) \cong K & .
\end{array}$$

Because of $h^d(\Omega^\cdot(X)) = \mathrm{Hom}_K(V_\phi(K), K)$ the pairing in the upper row is non-degenerate by Borel-Serre duality for Γ (compare Lemma 3 and the discussion preceding Proposition 2 in §1). The pairing in the lower row is nondegenerate by Serre duality for the projective smooth variety X_Γ (we may pass to a torsion-free normal subgroup of finite index in Γ so that the cohomology groups involved are defined algebraically). The left hand arrow is an isomorphism by Proposition 3 since with M also M^* is integral. Therefore the arrow in the middle is an isomorphism, too, which is equivalent to the assertion i.
Similarly as before the assertion ii. and the isomorphisms in iii. now follow by the arguments in the proof of Lemma 2. Moreover we obtain

$$\begin{aligned}
H^s(X_\Gamma, \Omega^d_{X_\Gamma/K} \underset{K}{\otimes} \mathcal{M}) &= H^s(\Gamma, \Omega^d(X) \underset{K}{\otimes} M) \cong H^s(\Gamma, h^d(\Omega^\cdot(X)) \underset{K}{\otimes} M) \\
&= H^s(\Gamma, \mathrm{Hom}_K(V_\phi(K), M)) = \mathrm{Ext}^s_{K[\Gamma]}(V_\phi(K), M) \\
&\cong \mathrm{Hom}_K(H^{d-s}(\Gamma, M^*), K)
\end{aligned}$$

where the last isomorphism by §1 Lemma 3 comes from Borel-Serre duality. The assertion iv. therefore again is a consequence of §1 Proposition 1.

Corollary 6:

For $d = 1$ or 2 and integral M the Hodge-de Rham spectral sequence degenerates and the Conjecture holds.

In view of these results one might expect that the vanishing criterion of Lemma 2 iii is fulfilled for integral Γ-modules so that in this case not only the Conjecture would hold but also the Hodge-de Rham spectral sequence would degenerate.

§3 The holomorphic discrete series of $SL_{d+1}(K)$

We now want to study more closely the case of an algebraic Γ-module. Since our Conjecture holds for Γ if it holds for a normal subgroup of finite index in Γ we change our notations slightly and assume from now on that Γ is a cocompact discrete subgroup in $G := SL_{d+1}(K)$ and that M is the underlying K-vector space of an irreducible K-rational representation of $SL_{d+1}(K)$. First we introduce that class of G-representations which will turn up in the analysis of the complex $\Omega^{\cdot}(X) \underset{K}{\otimes} M$. The projection morphism

$$\begin{aligned} SL_{d+1} &\longrightarrow \mathbb{P}^d \\ g &\longmapsto g([1:0:\ldots:0]) \end{aligned}$$

where $[z_0 : \ldots : z_d]$ are the homogeneous coordinates on $\mathbb{P}^d$ induces an isomorphism

$$SL_{d+1}/\mathbf{P} \xrightarrow{\sim} \mathbb{P}^d \quad ;$$

here $\mathbf{P} \subseteq SL_{d+1}$ is the parabolic subgroup of all matrices of the form

$$\begin{pmatrix} * & * & \ldots & * \\ 0 & * & \ldots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \ldots & * \end{pmatrix} \quad .$$

Let $x_1, \ldots, x_d \in \mathcal{O}(X)$ denote the coordinate functions

$$\begin{array}{cccc} x_i : & X & \longrightarrow & \mathbf{A}^1 \\ & [z_0 : \ldots : z_d] & \longmapsto & \frac{z_i}{z_0} \end{array} \quad .$$

Over X the above projection morphism has the natural section

$$\begin{aligned} X &\longrightarrow SL_{d+1} \\ z &\longmapsto u(z) := \begin{pmatrix} 1 & 0 & . & . & . & 0 \\ x_1(z) & 1 & & & & . \\ . & 0 & . & & & . \\ . & & & . & & . \\ . & & & & . & 0 \\ x_d(z) & 0 & . & . & 0 & 1 \end{pmatrix} \end{aligned} \quad .$$

In this way we obtain the map

$$\begin{aligned} \mu : G &\longrightarrow \mathbf{P}(K(x_1, \ldots, x_d)) \subseteq \mathbf{P}(\mathcal{O}(X)) \\ g &\longmapsto u(g(z))^{-1} \cdot g \cdot u(z) \quad . \end{aligned}$$

A straightforward computation shows that viewing $\mu(g)(z)$ as a matrix-valued function on X the "automorphy factor" relation

$$\mu(gh)(z) = \mu(g)(hz) \cdot \mu(h)(z)$$

holds true for $g, h \in G$. Now let

$$\mathbf{L} := \text{ matrices of the form } \begin{pmatrix} * & 0 & \ldots & 0 \\ 0 & * & \ldots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \ldots & * \end{pmatrix} \quad , \text{ resp.}$$

$$\mathbf{U} := \text{ matrices of the form } \begin{pmatrix} 1 & * & . & . & . & * \\ 0 & 1 & 0 & . & . & 0 \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & 0 \\ 0 & . & . & . & 0 & 1 \end{pmatrix} \quad ,$$

be the standard Levi component, resp. the unipotent radical, of $\mathbf{P}$ so that $\mathbf{U}$ is normal in $\mathbf{P}$ and $\mathbf{P} = \mathbf{L} \cdot \mathbf{U}$. Furthermore let V be the underlying K-vector space of a K-rational representation of $\mathbf{L}$. By letting act $\mathbf{U}$ trivially we view V as a representation of $\mathbf{P}$. Then $\mathbf{P}(\mathcal{O}(X))$ acts $\mathcal{O}(X)$-linearly on

$$D^V := \mathcal{O}(X) \underset{K}{\otimes} V \quad .$$

It is an immediate consequence of the "automorphy factor" relation that

$$g(f \otimes v) := f(g^{-1}.) \cdot \mu(g^{-1})^{-1}(1 \otimes v) \quad \text{for } f \otimes v \in \mathcal{O}(X) \underset{K}{\otimes} V$$

defines a K-linear action of G on D^V.

Since $\mathbf{L}$ is naturally isomorphic to GL_d we see that the irreducible K-rational representations V_λ of $\mathbf{L}$ are parametrized by the set of dominant integral weights

$$\Lambda_{\mathbf{L}} := \{(\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d : \lambda_1 \geq \ldots \geq \lambda_d\} \quad .$$

Any $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d$ defines a character

$$\chi_\lambda : \begin{pmatrix} a_0 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix} \longmapsto \prod_{i=1}^{d} a_i^{\lambda_i}$$

on the torus of diagonal matrices in $\mathbf{L}$. The representation V_λ then is characterized by the existence of an (up to scalar multiples) unique highest weight vector $v_\lambda \in V_\lambda \backslash \{0\}$ such that

$$av_\lambda = \chi_\lambda(a) v_\lambda \quad \text{for any } a = \begin{pmatrix} a_0 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix} \quad .$$

Examples: a. For

$$\lambda = (\underbrace{1, \ldots, 1}_{j \ times}, 0, \ldots, 0)$$

V_λ is the j-th exterior power of the d-dimensional standard representation of $\mathbf{L} \cong GL_d$ ([Jan] II.2.15).

b. $V_{(-\lambda_d, \ldots, -\lambda_1)}$ is the contragredient representation of $V_{(\lambda_1, \ldots, \lambda_d)}$ ([Jan] II.2.5).

Definition:

The family of G-representations $D_\lambda := D^{V_\lambda}$ for $\lambda \in \Lambda_{\mathbf{L}}$ is called the holomorphic discrete series of G.

Our interest in this notion comes from the fact that the spaces $\Omega^j(X)$ as G-representations belong to the holomorphic discrete series of G. In order to see this we observe that the coordinate functions $x_1, \ldots, x_d$ define an open embedding of X into the affine d-space over K. Therefore

$$\Omega^j(X) = \bigoplus_{1 \leq i_1 < \ldots < i_j \leq d} \mathcal{O}(X) dx_{i_1} \wedge \ldots \wedge dx_{i_j}$$

is a free $\mathcal{O}(X)$-module. In particular, as a K-vector space we have

$$\Omega^j(X) = \mathcal{O}(X) \underset{K}{\otimes} \Omega^j_\circ$$

with

$$\Omega^j_\circ := \bigoplus_{1\le i_1<\ldots<i_j\le d} K dx_{i_1}\wedge\ldots\wedge dx_{i_j} \subseteq \Omega^j(X) \quad .$$

For

$$g = \begin{pmatrix} a_{00} & \ldots & a_{0d} \\ \vdots & & \vdots \\ a_{d0} & \ldots & a_{dd} \end{pmatrix} \in G$$

we compute

$$g^{-1}(dx_i) = \sum_{h=1}^{d} \frac{a_{ih}\cdot A_0 - a_{0h}\cdot A_i}{A_0^2}\cdot dx_h$$

with

$$A_\ell := a_{\ell 0} + a_{\ell 1}x_1 + \ldots + a_{\ell d}x_d \;\text{ for } \ell = 0,\ldots,d \quad .$$

In case $g\in \mathbf{L}(K)$ this simplifies to

$$g^{-1}(dx_i) = a_{00}^{-1}\cdot(a_{i1}dx_1 + \ldots + a_{id}dx_d) \quad ;$$

in other words g stabilizes $\Omega^1_\circ$ and with respect to the K-basis $dx_1,\ldots,dx_d$ is given by the matrix

$$\det(\varphi(g)^{-1})\cdot\varphi(g)^{-1t} \quad .$$

Here

$$\varphi : \mathbf{P} \overset{pr}{\twoheadrightarrow} \mathbf{L} \cong GL_d$$

is the obvious homomorphism. Hence $\mathbf{L}(K)$ stabilizes the subspace $\Omega^j_\circ$ in $\Omega^j(X)$ and as $\mathbf{L}(K)$-modules we have

$$\Omega^j_\circ \cong \bigwedge^j(V^* \underset{K}{\otimes} \bigwedge^d V^*) \cong \bigwedge^j V^* \underset{K}{\otimes} (\bigwedge^d V^*)^{\otimes j}$$

where V^* denotes the contragredient of the d-dimensional standard representation of $\mathbf{L}\cong GL_d$. In view of the above Examples we obtain

$$\Omega^j_\circ \cong V_{0(j)} \quad \text{with } 0(j) := (-j,\ldots,-j,\underbrace{-j-1,\ldots,-j-1}_{j\ times}) \quad .$$

($0(j)$ is the special case, for $\lambda = 0$, of a $\lambda(j)$ which will be defined later on.)

For general $g\in G$ again, another straightforward computation shows that

$$\mu(g) = \begin{pmatrix} A_0 & a_{01} & \ldots & a_{0d} \\ 0 & a_{11} - a_{01}A_1A_0^{-1} & \ldots & a_{1d} - a_{0d}A_1A_0^{-1} \\ \vdots & \vdots & & \vdots \\ 0 & a_{d1} - a_{01}A_dA_0^{-1} & \ldots & a_{dd} - a_{0d}A_dA_0^{-1} \end{pmatrix} \quad .$$

This means that

$$g(dx_i) = R_{1i}dx_1 + \ldots + R_{di}dx_d$$

where

$$\begin{pmatrix} R_{11} & \ldots & R_{1d} \\ \vdots & & \vdots \\ R_{d1} & \ldots & R_{dd} \end{pmatrix} = \det(\varphi(\mu(g^{-1}))) \cdot \varphi(\mu(g^{-1}))^t$$
$$= \det(\varphi(\mu(g^{-1}))) \cdot [\varphi(\mu(g^{-1})^{-1})]^{-1t} \quad .$$

In this way we have proved the following result.

Proposition 1:

As a G-representation we have $\Omega^j(X) \cong D_{0(j)}$ for $0 \leq j \leq d$.

Sometimes it is convenient to work with a different realization of the holomorphic discrete series representations D_λ. We define

$$\begin{aligned} \tilde{X} :=& \text{preimage of } X \text{ under the above} \\ & \text{projection morphism } SL_{d+1} \to \mathbb{P}^d \quad . \end{aligned}$$

This is an open rigid analytic subvariety of $SL_{d+1/K}$ upon which $\mathbf{P}$ acts rigid K-analytically from the right. Furthermore let

$$\mathbf{B} := \text{ Borel subgroup in } SL_{d+1} \text{ of upper triangular matrices} \quad .$$

Any χ_λ can be viewed in the obvious way as a K-algebraic homomorphism

$$\chi_\lambda : \mathbf{B} \longrightarrow \mathbb{G}_m \quad .$$

For any $\lambda \in \mathbb{Z}^d$ we put

$$\tilde{D}_\lambda := \{f \in \mathcal{O}(\tilde{X}) : f(gb) = \chi_\lambda(b^{-1})f(g) \ \text{ for all } g \in \tilde{X}, b \in \mathbf{B}\} \quad ;$$

this is a G-representation with respect to G acting by left translations. On $\mathbb{Z}^d$ we have the permutation $w_d(\lambda_1, \ldots, \lambda_d) := (\lambda_d, \ldots, \lambda_1)$.

Lemma 2:

For $\lambda \in \Lambda_{\mathbf{L}}$ we have $D_\lambda \cong \tilde{D}_{w_d(\lambda)}$ as G-representations; if $\lambda \notin w_d(\Lambda_{\mathbf{L}})$ then $\tilde{D}_\lambda = 0$.

Before we give the proof we have to recall the description of V_λ as an induced representation: By [Jan] II.2.6 and 5.6 we have

$$\mathrm{ind}^{\mathbf{L}}_{w_d\mathbf{L}\cap\mathbf{B}w_d}(\chi_\lambda) = \begin{cases} V_\lambda & \text{if } \lambda \in \Lambda_{\mathbf{L}} \ , \\ 0 & \text{if } \lambda \notin \Lambda_{\mathbf{L}} \ . \end{cases}$$

Here w_d is viewed as the permutation matrix

$$w_d = \begin{pmatrix} \varepsilon & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & ⋰ & \vdots \\ 0 & 1 & \dots & 0 \end{pmatrix} \qquad \text{with } \varepsilon := (-1)^{\frac{d(d-1)}{2}}$$

so that $w_d\mathbf{L}\cap\mathbf{B}w_d$ is the transpose of $\mathbf{L}\cap\mathbf{B}$; the homomorphism χ_λ extends to both of these latter groups. Next we note that

$$\begin{aligned} \mathrm{ind}^{\mathbf{L}}_{w_d\mathbf{L}\cap\mathbf{B}w_d}(\chi_\lambda) &\xrightarrow{\sim} \mathrm{ind}^{\mathbf{L}}_{\mathbf{L}\cap\mathbf{B}}(\chi_{w_d(\lambda)}) \\ s &\longmapsto (g \mapsto s(gw_d)) \end{aligned}$$

is a $\mathbf{L}$-equivariant isomorphism. We therefore obtain

$$\begin{aligned} V_\lambda = &\text{space of all } K\text{-algebraic morphisms } s : \mathbf{L} \to \mathbf{A}^1 \text{ such} \\ &\text{that } s(gb) = \chi_{w_d(\lambda)}(b^{-1})s(g) \text{ for all } g \in \mathbf{L},\ b \in \mathbf{L}\cap\mathbf{B} \end{aligned}$$

in case $\lambda \in \Lambda_{\mathbf{L}}$; otherwise the right hand side vanishes. These morphisms s also can be viewed as the global algebraic sections of a certain line bundle $\mathcal{L}(w_d(\lambda))$ on $\mathbf{L}/\mathbf{L}\cap\mathbf{B}$ ([Jan] I.5.12 and 5.16). Since the latter scheme is projective the GAGA-principle implies that any global rigid K-analytic section of $\mathcal{L}(w_d(\lambda))$ already is algebraic. We finally arrive at the following description:

$$\begin{aligned} V_\lambda = &\text{space of all rigid } K\text{-analytic morphisms } s : \mathbf{L} \to \mathbf{A}^1 \text{ such} \\ &\text{that } s(gb) = \chi_{w_d(\lambda)}(b^{-1})s(g) \text{ for all } g \in \mathbf{L},\ b \in \mathbf{L}\cap\mathbf{B} \end{aligned}$$

for $\lambda \in \Lambda_{\mathbf{L}}$; if $\lambda \notin \Lambda_{\mathbf{L}}$ then the right hand side vanishes.

Proof of Lemma 2: First assume that $\lambda \in \Lambda_{\mathbf{L}}$. We consider the K-linear map

$$\begin{aligned} \Psi : D_\lambda = \mathcal{O}(X) \otimes V_\lambda &\longrightarrow \tilde{D}_{w_d(\lambda)} \\ f \otimes s &\longmapsto (g \mapsto f(z_g) \cdot s(u(z_g)^{-1} g)) \quad . \end{aligned}$$

Here $z_g \in X$ denotes the image of $g \in \tilde{X}$ and s is viewed as the composed morphism $\mathbf{P} \twoheadrightarrow \mathbf{L} \xrightarrow{s} \mathbf{A}^1$. This map is G-equivariant: Fixing a K-basis $s_1, \ldots, s_m$ of V_λ we have, for $h \in G$,

$$h(f \otimes s) = f(h^{-1}.) \cdot s(\mu(h^{-1}) \ . \ ) = f(h^{-1}.) \cdot \sum_{n=1}^{m} f_n^{(h)} \otimes s_n$$

with appropriate functions $f_n^{(h)} \in \mathcal{O}(X)$. Applying Ψ gives

$$\Psi(h(f \otimes s))(g) = f(h^{-1} z_g) \cdot \sum_{n=1}^{m} f_n^{(h)}(z_g) \cdot s_n(u(z_g)^{-1} g) \quad .$$

Since the $\mathbf{P}$-action on V_λ is rational we have

$$\sum_{n=1}^{m} f_n^{(h)}(z_g) \cdot s_n = s(\mu(h^{-1})(z_g) \ . \ ) \quad .$$

We therefore obtain

$$\begin{aligned} \Psi(h(f \otimes s))(g) &= f(h^{-1} z_g) \cdot s(\mu(h^{-1})(z_g) \cdot u(z_g)^{-1} g) \\ &= f(h^{-1} z_g) \cdot s(u(h^{-1} z_g)^{-1} h^{-1} g) \\ &= f(z_{h^{-1}g}) \cdot s(u(z_{h^{-1}g})^{-1} h^{-1} g) \\ &= \Psi(f \otimes s)(h^{-1} g) = h(\Psi(f \otimes s))(g) \quad . \end{aligned}$$

In order to construct an inverse map we observe that for $f \in \tilde{D}_{w_d(\lambda)}$ and any $z \in X$ the morphism

$$\begin{aligned} s_z : \mathbf{L} &\longrightarrow \mathbf{A}^1 \\ g &\longmapsto f(u(z) g) \end{aligned}$$

is contained in V_λ and therefore can be written as

$$s_z = f_1(z) \cdot s_1 + \ldots + f_m(z) \cdot s_m$$

for appropriate functions $f_1, \ldots, f_m \in \mathcal{O}(X)$. It is easily checked that the map

$$\begin{aligned} \tilde{D}_{w_d(\lambda)} &\longrightarrow \mathcal{O}(X) \otimes V_\lambda \\ f &\longmapsto f_1 \otimes s_1 + \ldots + f_m \otimes s_m \end{aligned}$$

is inverse to the previous map Ψ.

The definition of the morphisms s_z associated to some $f \in \tilde{D}_{w_d(\lambda)}$ makes sense for any $\lambda \in \mathbb{Z}^d$. But in case $\lambda \notin \Lambda_{\mathbf{L}}$ we must have $s_z = 0$; since $\tilde{X} = u(X)\mathbf{L}\mathbf{U}$ and f is $\mathbf{U}$-invariant this implies $f = 0$ and consequently $\tilde{D}_{w_d(\lambda)} = 0$.

Coming back to our irreducible K-rational representation M of SL_{d+1} we similarly have that $M = M_\lambda$ is characterized by its highest weight χ_λ for some

$$\lambda \in \Lambda := \{(\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d : 0 \geq \lambda_1 \geq \ldots \geq \lambda_d\} \quad .$$

Remark 3:

For $\lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda$ we have:
i. M_λ is an irreducible G-representation;
ii. M_λ is contained as a G-subrepresentation in $D_{\lambda'}$ for $\lambda' := (-\lambda_d, \lambda_1 - \lambda_d, \ldots, \lambda_{d-1} - \lambda_d)$.

Proof: i. This follows from the fact that G is Zariski dense in SL_{d+1}.
ii. What we have recalled above about induced representations applies mutatis mutandis to M_λ. Of course here we have to work with the permutation matrix

$$w_{d+1} := \begin{pmatrix} 0 & \ldots & \pm 1 \\ \vdots & \iddots & \vdots \\ 1 & \ldots & 0 \end{pmatrix}$$

the sign being chosen in such a way that w_{d+1} has determinant 1. Observing that $\chi_\lambda(w_{d+1} \,.\, w_{d+1}) = \chi_{w_d(\lambda')}$ we obtain

$$M_\lambda = \mathrm{ind}_{\mathbf{B}}^{SL_{d+1}}(\chi_{w_d(\lambda')}) \quad .$$

Restriction of morphisms then defines a G-equivariant map

$$M_\lambda \longrightarrow \tilde{D}_{w_d(\lambda')} \cong D_{\lambda'}$$

which is injective since $\tilde{X}$ is Zariski dense in SL_{d+1}.

The symmetric group $W := S_{d+1}$ on the letters $\{0, \ldots, d\}$ acts on $\mathbb{Z}^d$ by

$$w(\lambda_1, \ldots, \lambda_d) := (\lambda_{w^{-1}(1)} - \lambda_{w^{-1}(0)}, \ldots, \lambda_{w^{-1}(d)} - \lambda_{w^{-1}(0)})$$

for $w \in W$ with the convention that $\lambda_0 := 0$. For the subgroup $W_{\mathbf{L}} := \{w \in W : w(0) = 0\} \cong S_d$ this is the obvious permutation action. The conceptual meaning of this action is explained by the fact that viewing $w \in W$ as a (signed) permutation matrix in SL_{d+1} we have

$$\chi_{w(\lambda)} = \chi_\lambda(w^{-1} \,.\, w) \quad \text{for all } \lambda \in \mathbb{Z}^d \quad .$$

The already considered permutations w_d and w_{d+1} are the elements of greatest length in $W_{\mathbf{L}}$ and W, respectively. The cycles $c_j := (01\ldots j) \in W$ for $0 \le j \le d$ form a set of representatives for the right cosets of $W_{\mathbf{L}}$ in W, i.e.,

$$W = W_{\mathbf{L}} c_0 \dot\cup \ldots \dot\cup W_{\mathbf{L}} c_d \quad .$$

Moreover we have:

— c_j is the unique element of minimal length in its coset $W_{\mathbf{L}} c_j$, and

— $w_d c_j w_{d+1} = c_{d-j}$.

For any $\lambda \in \Lambda$ and any $0 \le j \le d$ we now define

$$\lambda(j) := c_{d-j}(\lambda) + 0(j) \in \Lambda_{\mathbf{L}} \quad .$$

Remark 4:

We have $\lambda(j) = -w_d(c_j(-w_{d+1}(\lambda) + \rho) - \rho)$ where $\rho := (-1, -2, \ldots, -d)$ corresponds to half the sum of the $\mathbf{B}$-positive roots of SL_{d+1}.

Our aim in the following is to construct a G-equivariant complex

$$M_\lambda \longrightarrow D_{\lambda(0)} \longrightarrow \ldots \longrightarrow D_{\lambda(d)}$$

for any $\lambda \in \Lambda$ which is G-equivariantly quasi-isomorphic to the complex

$$M_\lambda \xrightarrow{\subseteq} \Omega^0(X) \underset{K}{\otimes} M_\lambda \xrightarrow{d\otimes id} \ldots \xrightarrow{d\otimes id} \Omega^d(X) \underset{K}{\otimes} M_\lambda \quad .$$

In case $\lambda = 0$ Proposition 1 says that the latter complex already is of the wanted form.

Example: For $d = 1$ the wanted complex is of the form

$$M_\lambda \longrightarrow D_{-\lambda} \longrightarrow D_{\lambda-2}$$

and was constructed in [SS] pp. 95-97.

The Remark 3ii provides us, for general $\lambda \in \Lambda$, with a G-equivariant injection

$$M_\lambda \hookrightarrow D_{\lambda(0)} = D_{c_d(\lambda)} \quad .$$

How is this map related to the inclusion $M_\lambda \hookrightarrow \Omega^0(X) \underset{K}{\otimes} M_\lambda$? We will see that $\Omega^0(X) \underset{K}{\otimes} M_\lambda$ carries a natural filtration of which $D_{\lambda(0)}$ is the top quotient. Consider quite generally arbitrary K-rational representations V of $\mathbf{L}$ and N of SL_{d+1}. Also note that the G-representation $D^{V'}$ is defined for any K-rational representation V' of $\mathbf{P}$ (i.e., $\mathbf{U}$ does not have to act trivially on V'). If not explicitly stated otherwise we always view $V \underset{K}{\otimes} N$ as a $\mathbf{P}$-representation where $\mathbf{P}$ acts on the first, resp. second, factor through the projection $\mathbf{P} \twoheadrightarrow \mathbf{L}$, resp. the inclusion $\mathbf{P} \hookrightarrow SL_{d+1}$.

Lemma 5:

The map

$$D^V \underset{K}{\otimes} N \xrightarrow{\cong} D^{V \underset{K}{\otimes} N}$$
$$(f \otimes v) \otimes e \longmapsto f \cdot (v \otimes u(z)^{-1}(e))$$

is an $\mathcal{O}(X)$-linear and G-equivariant isomorphism (where G acts diagonally on the left hand side).

Proof: Straightforward.

In particular any $\mathbf{P}$-invariant filtration $F^{\cdot}N$ on N induces a G-equivariant filtration

$$F^{\cdot}(D^V \underset{K}{\otimes} N) := \mathcal{O}(X) \cdot (V \otimes u(z)(F^{\cdot}N))$$

on $D^V \underset{K}{\otimes} N$ such that

$$gr_F^{\cdot}(D^V \underset{K}{\otimes} N) \cong D^{V \underset{K}{\otimes} gr_F^{\cdot} N} \quad .$$

There is a distinguished such filtration $F^{\cdot}N$. Its definition is based on the homomorphism

$$\begin{array}{rccc} I: & \mathbb{Z}^d & \longrightarrow & \mathbb{Q} \\ & (\lambda_1, \ldots, \lambda_d) & \longmapsto & -\frac{1}{d+1}(\lambda_1 + \ldots + \lambda_d) \quad . \end{array}$$

We have:

— $I(0(j)) = j$;
— $I(w\lambda) = I(\lambda) + \lambda_{w^{-1}(0)}$ for $w \in W$; in particular $I(w\lambda) = I(\lambda)$ for $w \in W_{\mathbf{L}}$.

The $\mathbf{B}$-simple roots of SL_{d+1} are $\chi_{\varepsilon_1}, \ldots, \chi_{\varepsilon_d}$ with

$$\begin{aligned} \varepsilon_1 &:= (-2, -1, \ldots, -1) \quad , \\ \varepsilon_2 &:= (1, -1, 0, \ldots, 0) \quad , \\ &\vdots \\ \varepsilon_d &:= (0, \ldots, 0, 1, -1) \quad . \end{aligned}$$

The map I is the unique homomorphism such that

$$I(\varepsilon_1) = 1 \quad \text{and} \quad I(\varepsilon_2) = \ldots = I(\varepsilon_d) = 0 \quad .$$

On $\mathbb{Z}^d$ we have the usual partial order

$$\nu \leq \lambda \quad \text{if} \quad \lambda - \nu \in \mathbb{N}_0\varepsilon_1 + \ldots + \mathbb{N}_0\varepsilon_d$$

(where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). Clearly

$$I(\lambda) - I(\nu) \in \mathbb{N}_0 \quad \text{if} \quad \nu \leq \lambda \quad .$$

For any $\nu \in \mathbb{Z}^d$ let

$$N^{(\nu)} := \left\{ e \in N : ae = \chi_\nu(a)e \quad \text{for any } a = \begin{pmatrix} a_0 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix} \right\}$$

denote the corresponding weight subspace in N. For any rational number $r \in \mathbb{Q}$ we now define

$$\Phi^r N := \bigoplus_{\substack{\nu \in \mathbb{Z}^d \\ I(\nu) \geq r}} N^{(\nu)} \quad .$$

Lemma 6:

i. $\Phi^r N$ *is a* $\mathbf{P}$*-submodule of* N*;*

ii. $\mathbf{U}$ *acts trivially on* $\Phi^r N / \Phi^{r+1} N$*.*

Proof: i. Because of $wN^{(\nu)} = N^{(w\nu)}$ for $w \in W$ we obviously have that $\Phi^r N$ is $W_{\mathbf{L}}$-invariant. Since $W_{\mathbf{L}}$ and $\mathbf{B}$ together generate $\mathbf{P}$ it therefore suffices to show that $\Phi^r N$ is a $\mathbf{B}$-submodule. But from the general theory we know ([Jan] II.1.19) that

$$\mathbf{B} \cdot N^{(\nu)} \subseteq \bigoplus_{\nu' \geq \nu} N^{(\nu')} \quad .$$

ii. Again by [Jan] II.1.19 we have

$$\mathbf{U}e \in e + \sum_{1 \leq i \leq d} \sum_{m_i \in \mathbb{N}} N^{(\nu + m_i(\varepsilon_1 + \ldots + \varepsilon_i))} \quad \text{for } e \in N^{(\nu)} \quad .$$

In case $N = M_\lambda$ the $\nu \in \mathbb{Z}^d$ such that $M_\lambda^{(\nu)} \neq 0$ fulfill

$$w_{d+1}\lambda \leq \nu \leq \lambda$$

([Jan] II.2.2); for those ν therefore $I(\lambda) - I(\nu)$ is an integer such that

$$0 \leq I(\lambda) - I(\nu) \leq -\lambda_d \quad .$$

We define the renormalized filtration

$$F^r M_\lambda := \Phi^{I(\lambda)+r} M_\lambda \quad \text{for } r \in \mathbb{Z} \quad ;$$

then

$$M_\lambda = F^{\lambda_d} M_\lambda \supseteq F^{\lambda_d + 1} M_\lambda \supseteq \ldots \supseteq F^0 M_\lambda \supseteq F^1 M_\lambda = 0 \quad .$$

Lemma 7:

i. $F^0 M_\lambda \cong V_\lambda$*;*

ii. $M_\lambda / F^{\lambda_d+1} M_\lambda \cong V_{c_d(\lambda)}$*;*

iii. $gr_F^{\lambda_j} M_\lambda$*, for any* $0 \leq j \leq d$*, contains* $V_{c_j(\lambda)}$ *as a direct factor with multiplicity one.*

Proof: We will denote the unipotent radical of some algebraic group by rad(.). i. From

$$\mathrm{rad}(\mathbf{B}) = \mathrm{rad}(\mathbf{B} \cap \mathbf{L}) \cdot \mathbf{U}$$

we obtain by Lemma 6 ii that

$$(F^0 M_\lambda)^{\mathrm{rad}(\mathbf{B}\cap\mathbf{L})} = (F^0 M_\lambda)^{\mathrm{rad}(\mathbf{B})} = M_\lambda^{(\lambda)} \quad .$$

This implies our assertion. ii. It is not difficult to check that as a $\mathbf{L}$-representation $M_\lambda / F^{\lambda_d+1} M_\lambda$ is the contragredient of $F^0 M_{-w_{d+1}(\lambda)}$ and therefore by i. is isomorphic to $V_{w_d w_{d+1}(\lambda)} = V_{c_d(\lambda)}$. iii. In order that M_λ as a $\mathbf{L}$-representation contains V_ν it is necessary and sufficient that

$$M_\lambda^{(\nu)} \cap M_\lambda^{\mathrm{rad}(\mathbf{B}\cap\mathbf{L})} \neq 0 \quad .$$

An explicit computation shows that

$$\mathrm{rad}(\mathbf{B} \cap \mathbf{L}) \subseteq c_j \, \mathrm{rad}(\mathbf{B}) c_j^{-1}$$

for any $0 \leq j \leq d$. We see that

$$\begin{aligned} M_\lambda^{(c_j(\lambda))} \cap M_\lambda^{\mathrm{rad}(\mathbf{B}\cap\mathbf{L})} &\supseteq c_j M_\lambda^{(\lambda)} \cap M_\lambda^{c_j \, \mathrm{rad}(\mathbf{B}) c_j^{-1}} \\ &= c_j (M_\lambda^{(\lambda)} \cap M_\lambda^{\mathrm{rad}(\mathbf{B})}) \\ &= c_j M_\lambda^{(\lambda)} \neq 0 \quad . \end{aligned}$$

Therefore $V_{c_j(\lambda)}$ occurs in M_λ and its multiplicity is one since $M_\lambda^{(c_j(\lambda))}$ is 1-dimensional. Moreover it has to occur in $gr_F^{\lambda_j} M_\lambda$ because of $I(c_j(\lambda)) = I(\lambda) + \lambda_j$.

Viewing M_λ and $V_{c_d(\lambda)}$ as induced representations the restriction of functions defines a $\mathbf{P}$-equivariant homomorphism

$$\begin{array}{ccc} M_\lambda & = & \mathrm{ind}_{\mathbf{B}}^{SL_{d+1}}(\chi_{w_{d+1}(\lambda)}) \\ & & \downarrow \Pi \\ V_{c_d(\lambda)} & = & \mathrm{ind}_{\mathbf{L}\cap\mathbf{B}}^{\mathbf{L}}(\chi_{w_d c_d(\lambda)}) \end{array} \quad .$$

As explained in Lemma 5 this Π then induces a G-equivariant homomorphism

$$\begin{aligned}\tilde{\Pi} : \mathcal{O}(X) \underset{K}{\otimes} M_\lambda &\longrightarrow \mathcal{O}(X) \underset{K}{\otimes} V_{c_d(\lambda)} = D_{c_d(\lambda)} \\ f \otimes e &\longmapsto (\mathrm{id} \otimes \Pi)(f \cdot u(z)^{-1}(e)) \quad .\end{aligned}$$

We leave it as an exercise to the reader to check that the diagram

$$\begin{array}{ccc} & & \tilde{D}_{w_{d+1}(\lambda)} \\ & & \uparrow \Psi \\ & \nearrow & D_{c_d(\lambda)} \\ & & \uparrow \tilde{\Pi} \\ M_\lambda & \xrightarrow{\subseteq} & \mathcal{O}(X) \underset{K}{\otimes} M_\lambda \end{array}$$

is commutative; here Ψ is the isomorphism in the proof of Lemma 2 and the oblique arrow is the restriction map in the proof of Remark 3 ii. (Note that if M_λ and $V_{c_d(\lambda)}$ are considered as abstract irreducible representations then the diagram still commutes up to a constant — use [Jan] II.2.8.) In particular we have $\Pi \neq 0$. Since we know from the proof of Lemma 7 iii that $V_{c_d(\lambda)}$ occurs with multiplicity one in M_λ it follows that Π induces an explicit isomorphism

$$M_\lambda / F^{\lambda_d + 1} M_\lambda \xrightarrow{\cong} V_{c_d(\lambda)} \quad .$$

As we have seen after Lemma 5 the filtration $F^{\cdot} M_\lambda$ induces, for any $0 \leq j \leq d$, a G-equivariant filtration

$$F^{\cdot}(\Omega^j(X) \underset{K}{\otimes} M_\lambda) := \Omega^j(X) \underset{\mathcal{O}(X)}{\otimes} \mathcal{O}(X) \cdot u(z)(F^{\cdot} M_\lambda)$$

on $\Omega^j(X) \underset{K}{\otimes} M_\lambda$.

Lemma 8:

For all $0 \leq j \leq d$ and $r \in \mathbb{Z}$ we have

$$(d \otimes \mathrm{id})[F^r(\Omega^j(X) \underset{K}{\otimes} M_\lambda)] \subseteq F^{r-1}(\Omega^{j+1}(X) \underset{K}{\otimes} M_\lambda) \quad .$$

Proof: The root subgroups

$$\begin{aligned}\rho_i : \mathbb{G}_a &\longrightarrow SL_{d+1} \\ b &\longmapsto \begin{pmatrix} 1 & . & . & . & 0 \\ \vdots & . & & & . \\ b & & . & & . \\ \vdots & & & . & . \\ 0 & . & . & . & 1 \end{pmatrix}\end{aligned}$$

for $1 \leq i \leq d$ give rise to the Lie algebra elements

$$X_i := (d\rho_i)(1) \in Lie(SL_{d+1}) \quad .$$

Define

$$X_{i,m} := X_i^m / m! \ \text{ for } m \geq 0$$

as elements in the universal enveloping algebra $U(Lie(SL_{d+1}))$. They commute with each other. In terms of the natural $U(Lie(SL_{d+1}))$-action on M_λ we have the formula

$$\rho_i(b)(e) = \sum_{m \geq 0} b^m \cdot X_{i,m}(e) \quad \text{for any } b \in \mathbb{G}_a \text{ and } e \in M_\lambda$$

([Jan] II.1.19 (6)). We deduce

$$\begin{aligned} u(z)(e) &= \rho_1(x_1(z)) \circ \ldots \circ \rho_d(x_d(z))(e) \\ &= \sum_{m_1,\ldots,m_d \geq 0} x_1(z)^{m_1} \cdot \ldots \cdot x_d(z)^{m_d} \cdot X_{1,m_1} \circ \ldots \circ X_{d,m_d}(e) \end{aligned}$$

and consequently

$$\begin{aligned} &(d \otimes \mathrm{id})(u(z)(e)) \\ &= \sum_{m_1,\ldots,m_d \geq 0} \sum_{1 \leq i \leq d} m_i x_1(z)^{m_1} \cdot \ldots \cdot x_i(z)^{m_i - 1} \cdot \ldots \cdot x_d(z)^{m_d} dx_i \cdot X_{1,m_1} \circ \ldots \circ X_{d,m_d}(e) \\ &= \sum_{1 \leq i \leq d} (\sum_{m_1,\ldots,m_d \geq 0} x_1(z)^{m_1} \cdot \ldots \cdot x_d(z)^{m_d} \cdot X_{1,m_1} \circ \ldots \circ X_{d,m_d}(X_i e)) dx_i \\ &= \sum_{1 \leq i \leq d} u(z)(X_i e) dx_i \end{aligned}$$

for any $e \in M_\lambda$. This implies

$$(d \otimes \mathrm{id})(\omega \otimes u(z)(e)) = d\omega \otimes u(z)(e) + \sum_{1 \leq i \leq d} (dx_i \wedge \omega) \otimes u(z)(X_i e)$$

for any $\omega \in \Omega^j(X)$ and $e \in M_\lambda$. Our assertion follows now since

$$X_i(M_\lambda^{(\nu)}) \subseteq M_\lambda^{(\nu - \varepsilon_1 - \ldots - \varepsilon_i)}$$

([Jan] II.1.19 (5)).

This result allows us to filter the complex $\Omega^{\cdot}(X) \underset{K}{\otimes} M_\lambda$ by the G-equivariant subcomplexes

$$\mathcal{F}^{r,\cdot} : F^r(\Omega^0(X) \underset{K}{\otimes} M_\lambda) \stackrel{d\otimes \mathrm{id}}{\longrightarrow} \ldots \stackrel{d\otimes \mathrm{id}}{\longrightarrow} F^{r-d}(\Omega^d(X) \underset{K}{\otimes} M_\lambda)$$

for $r \in \mathbb{Z}$. The G-equivariant spectral sequence corresponding to this filtration is

$$E_1^{r,s} = h^{r+s}(\mathcal{F}^{r,\cdot}/\mathcal{F}^{r+1,\cdot}) \Longrightarrow h^{r+s}(\Omega^{\cdot}(X) \underset{K}{\otimes} M_\lambda) \quad .$$

By the previous proof we have for the differential d of the complex $\mathcal{F}^{r,\cdot}/\mathcal{F}^{r+1,\cdot}$ the formula

$$(1) \qquad d(\omega \otimes u(z)e \bmod \ldots) = \sum_{1\le i\le d} (dx_i \wedge \omega) \otimes u(z)(X_i e) \bmod \ldots \quad .$$

In particular this differential is $\mathcal{O}(X)$-linear. On the other hand consider the complex of K-vector spaces

$$C^{\cdot}(M_\lambda) := \Omega^{\cdot}_{\circ} \underset{K}{\otimes} M_\lambda$$

with the differential

$$\begin{aligned} d : C^j(M_\lambda) &\longrightarrow C^{j+1}(M_\lambda) \\ \omega \otimes e &\longmapsto \sum_{i=1}^{d} (dx_i \wedge \omega) \otimes X_i e \quad . \end{aligned}$$

Recall that for

$$g = \begin{pmatrix} a_{00} & 0 & \ldots & 0 \\ 0 & a_{11} & \ldots & a_{1d} \\ \vdots & \vdots & & \vdots \\ 0 & a_{d1} & \ldots & a_{dd} \end{pmatrix} \in \mathbf{L}$$

we have

$$(2) \qquad g^{-1}(dx_i) = a_{00}^{-1} \cdot (a_{i1} dx_1 + \ldots + a_{id} dx_d) \quad .$$

Similarly we compute

$$(3) \qquad g X_i g^{-1} = a_{00}^{-1} \cdot (a_{1i} X_1 + \ldots + a_{di} X_d) \quad .$$

It follows from (2) and (3) that with $\mathbf{L}$ acting diagonally on $\Omega^j_\circ \underset{K}{\otimes} M_\lambda$ the complex $C^{\cdot}(M_\lambda)$ is $\mathbf{L}$-equivariant. By tensoring with $\mathcal{O}(X)$ we obtain the G-equivariant complex of $\mathcal{O}(X)$-modules

$$(D^{C^{\cdot}(M_\lambda)} = \mathcal{O}(X) \underset{K}{\otimes} C^{\cdot}(M_\lambda), \mathrm{id} \otimes d) \quad .$$

From the formula (1) it is clear that

$$D^{C^{\cdot}(M_\lambda)} = \mathcal{O}(X) \underset{K}{\otimes} C^{\cdot}(M_\lambda) \xrightarrow{\cong} \underset{r\in\mathbb{Z}}{\oplus} \mathcal{F}^{r,\cdot}/\mathcal{F}^{r+1,\cdot}$$
$$f \otimes \omega \otimes e \longmapsto f\omega \otimes u(z)(e)$$

is an $\mathcal{O}(X)$-linear and G-equivariant complex isomorphism. (We emphasize that in forming $D^{C^{\cdot}(M_\lambda)}$ we have to consider $C^{\cdot}(M_\lambda)$ as a $\mathbf{P}$-representation via the projection $\mathbf{P} \twoheadrightarrow \mathbf{L}$; the reason for this is Lemma 6 ii.) In particular we obtain an $\mathcal{O}(X)$-linear and G-equivariant isomorphism

$$D^{h^s(C^{\cdot}(M_\lambda))} \xrightarrow{\cong} \underset{r\in\mathbb{Z}}{\oplus} h^s(\mathcal{F}^{r,\cdot}/\mathcal{F}^{r+1,\cdot}) \quad .$$

We want to identify $h^*(C^{\cdot}(M_\lambda))$ as Lie algebra cohomology. Let $\mathbf{u}^-$ denote the Lie algebra of the transpose $\mathbf{U}^t$ of $\mathbf{U}$. The elements $X_1, \ldots, X_d$ considered in the proof of Lemma 8 form a K-basis of $\mathbf{u}^-$. Viewing M_λ as a $\mathbf{u}^-$-module in the usual way the complex of alternating cochains $C^{\cdot}(\mathbf{u}^-, M_\lambda)$ of $\mathbf{u}^-$ with values in M_λ is given by

$$C^{\cdot}(\mathbf{u}^-, M_\lambda) := \mathrm{Hom}_K(\bigwedge^{\cdot} \mathbf{u}^-, M_\lambda)$$

with differential (note that $\mathbf{u}^-$ is a commutative Lie algebra)

$$d\Theta(Y_1 \wedge \ldots \wedge Y_{j+1}) := \sum_{m=1}^{j+1} (-1)^{m+1} Y_m(\Theta(Y_1 \wedge \ldots \wedge \hat{Y}_m \wedge \ldots \wedge Y_{j+1}))$$

where $\hat{Y}_m$ as usual indicates the omission of Y_m. It computes the Lie algebra cohomology $H^*(\mathbf{u}^-, M_\lambda)$ of $\mathbf{u}^-$ with coefficients in M_λ. We have the obvious K-linear isomorphism

$$\bigwedge^{\cdot} \mathrm{Hom}_K(\mathbf{u}^-, K) \underset{K}{\otimes} M_\lambda = \mathrm{Hom}_K(\bigwedge^{\cdot} \mathbf{u}^-, K) \underset{K}{\otimes} M_\lambda \xrightarrow{\cong} C^{\cdot}(\mathbf{u}^-, M_\lambda) \quad .$$

If $X_1^*, \ldots, X_d^*$ is the K-basis of $\mathrm{Hom}_K(\mathbf{u}^-, K)$ which is dual to $X_1, \ldots, X_d$ then a straightforward computation shows that on the left hand side the differential becomes

$$d(\Xi \otimes e) = \sum_{i=1}^{d} (X_i^* \wedge \Xi) \otimes X_i e \quad .$$

Therefore associating X_i^* with dx_i induces a K-linear complex isomorphism

$$C^{\cdot}(M_\lambda) \xrightarrow{\cong} \bigwedge^{\cdot} \mathrm{Hom}_K(\mathbf{u}^-, K) \underset{K}{\otimes} M_\lambda \xrightarrow{\cong} C^{\cdot}(\mathbf{u}^-, M_\lambda)$$

which by the formulas (2) and (3) is $\mathbf{L}$-equivariant. In this way we obtain an $\mathcal{O}(X)$-linear and G-equivariant isomorphism

$$D^{H^s(\mathbf{u}^-,M_\lambda)} \xrightarrow{\cong} \bigoplus_{r\in\mathbb{Z}} h^s(\mathcal{F}^{r,\cdot}/\mathcal{F}^{r+1,\cdot})$$

for any $s \geq 0$. Due to a theorem of Kostant the $\mathbf{L}$-representations $H^s(\mathbf{u}^-, M_\lambda)$ are known: We have

$$H^s(\mathbf{u}^-, M_\lambda) \cong \begin{cases} V_{\lambda(s)} & \text{if } 0 \leq s \leq d\ , \\ 0 & \text{otherwise} \end{cases}$$

([Kos] Thm. 5.14 or [Lep] p. 504; also use the above Remark 4); note that by [Jan] II.2.8 this isomorphism is unique up to multiplication by a constant in $K^\times$.

Lemma 9:

We have a (up to multiplication by a constant in $K^\times$) natural $\mathcal{O}(X)$-linear and G-equivariant isomorphism

$$h^j(\mathcal{F}^{\lambda_{d-j}+j,\cdot}/\mathcal{F}^{\lambda_{d-j}+j+1,\cdot}) \cong D_{\lambda(j)}$$

for any $0 \leq j \leq d$; moreover $h^s(\mathcal{F}^{r,\cdot}/\mathcal{F}^{r+1,\cdot}) = 0$ for all other pairs (r, s).

Proof: In the proof of Lemma 7 we have seen that $V_{c_{d-j}(\lambda)}$ occurs with multiplicity one in M_λ and that it occurs in $gr_F^{\lambda_{d-j}} M_\lambda$. It follows that $V_{\lambda(j)}$ occurs with multiplicity one in $\Omega^j_\circ \underset{K}{\otimes} M_\lambda$ and that it occurs in $\Omega^j_\circ \underset{K}{\otimes} gr_F^{\lambda_{d-j}} M_\lambda$. This implies that under the map

$$\mathcal{O}(X) \underset{K}{\otimes} C^j(M_\lambda) \xrightarrow{\cong} \bigoplus_{r\in\mathbb{Z}} \mathcal{F}^{r,j}/\mathcal{F}^{r+1,j}$$

considered above $V_{\lambda(j)}$ is mapped into the summand $\mathcal{F}^{\lambda_{d-j}+j,j}/\mathcal{F}^{\lambda_{d-j}+j+1,j}$. Since this map is $\mathcal{O}(X)$-linear all of $D^{V_{\lambda(j)}} = \mathcal{O}(X) \underset{K}{\otimes} V_{\lambda(j)}$ then is mapped into that summand. On the other hand by Kostant's theorem the Lie algebra cohomology $H^j(\mathbf{u}^-, M_\lambda) = h^j(C^\cdot(M_\lambda))$ is represented by $V_{\lambda(j)} \subseteq C^j(M_\lambda)$. It therefore follows that our isomorphism

$$D^{V_{\lambda(j)}} \cong D^{H^j(\mathbf{u}^-,M_\lambda)} \xrightarrow{\cong} \bigoplus_{r\in\mathbb{Z}} h^j(\mathcal{F}^{r,\cdot}/\mathcal{F}^{r+1,\cdot})$$

in fact induces an isomorphism

$$D^{V_{\lambda(j)}} \xrightarrow{\cong} h^j(\mathcal{F}^{\lambda_{d-j}+j,\cdot}/\mathcal{F}^{\lambda_{d-j}+j+1,\cdot}) \quad .$$

For the E_1-terms of our spectral sequence we obtain

$$E_1^{r,s} \cong \begin{cases} D_{\lambda(j)} & \text{if } (r,s) = (\lambda_{d-j}+j, -\lambda_{d-j}) \text{ for some } 0 \leq j \leq d \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

For formal reasons the only possibly nonzero higher differentials between those terms are

$$d_{m(j)}^{\cdot,\cdot} : D_{\lambda(j)} \longrightarrow D_{\lambda(j+1)}$$

with $m(j) := \lambda_{d-j-1} - \lambda_{d-j} + 1$. This means that

$$D_{\lambda(.)} : D_{\lambda(0)} \xrightarrow{d_{m(0)}} D_{\lambda(1)} \xrightarrow{d_{m(1)}} \ldots \xrightarrow{d_{m(d-1)}} D_{\lambda(d)}$$

is a G-equivariant complex whose homology is the same as that of the complex $\Omega^{\cdot}(X) \underset{K}{\otimes} M_\lambda$. (Of course the complex $D_{\lambda(.)}$ is well-defined only up to multiplying each differential $d_{m(j)}$ by a constant in $K^\times$.) As a by-product we have the following result.

Proposition 10:

For any $0 \leq j \leq d$ the G-representation $\mathrm{Hom}_K(V_{\{1,\ldots,d-j\}}(K), M_\lambda)$ is isomorphic to a G-invariant subquotient of the holomorphic discrete series representation $D_{\lambda(j)}$.

Proof: We know that $\mathrm{Hom}_K(V_{\{1,\ldots,d-j\}}(K), M_\lambda) = h^j(\Omega^{\cdot}(X) \underset{K}{\otimes} M_\lambda)$.

Theorem 3:

We have an (explicit) G-equivariant quasi-isomorphism between the complexes $D_{\lambda(.)}$ and $\Omega^{\cdot}(X) \underset{K}{\otimes} M_\lambda$.

Proof: We define G-invariant subspaces $B^j \subseteq Z^j \subseteq \Omega^j(X) \underset{K}{\otimes} M_\lambda$ in such a way that

$$B^j / F^{\lambda_{d-j}+1}(\Omega^j(X) \underset{K}{\otimes} M_\lambda) \quad , \text{ resp. } Z^j / F^{\lambda_{d-j}+1}(\Omega^j(X) \underset{K}{\otimes} M_\lambda) \quad ,$$

is the image, resp. the kernel, of the differential in the complex $\mathcal{F}^{\lambda_{d-j}+j,\cdot}/\mathcal{F}^{\lambda_{d-j}+j+1,\cdot}$ in degree j. In particular we have $D_{\lambda(j)} \cong Z^j/B^j$. Also, for simplicity, we write d for the differential $d \otimes \mathrm{id}$ in the complex $\Omega^{\cdot}(X) \underset{K}{\otimes} M_\lambda$. The following properties of these subspaces are shown by diagram-chasing based on Lemma 9:

a. $dZ^{j-1} \subseteq Z^j + dB^{j-1}$,

b. $Z^j \cap \operatorname{im} d \subseteq dZ^{j-1} \subseteq \ker d \subseteq Z^j + \operatorname{im} d$,

c. $Z^j \cap dB^{j-1} \subseteq B^j$,

d. $B^j \cap \ker d \subseteq dB^{j-1}$.

The property a. means that

$$Z^0 \xrightarrow{d} Z^1 + dB^0 \xrightarrow{d} Z^2 + dB^1 \xrightarrow{d} \ldots \xrightarrow{d} Z^d + dB^{d-1}$$

is a subcomplex in $\Omega^{\cdot}(X) \underset{K}{\otimes} M_\lambda$. It follows from b. that the corresponding inclusion map is a quasi-isomorphism. By the property c. the maps

$$\begin{aligned} Z^j + dB^{j-1} &\longrightarrow Z^j/B^j \\ z + db &\longmapsto z \bmod B^j \end{aligned}$$

are well-defined and combine to a surjective homomorphism of complexes

$$\begin{array}{ccccccccc} Z^0 & \to & Z^1 + dB^0 & \to & Z^2 + dB^1 & \to & \ldots & \to & Z^d + dB^{d-1} \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ Z^0/B^0 & \stackrel{d_{m(0)}}{\longrightarrow} & Z^1/B^1 & \stackrel{d_{m(1)}}{\longrightarrow} & Z^2/B^2 & \stackrel{d_{m(2)}}{\longrightarrow} & \ldots & \stackrel{d_{m(d-1)}}{\longrightarrow} & Z^d/B^d \end{array} .$$

Finally as a consequence of d. this map is a quasi-isomorphism, too.

We now come back to our cocompact discrete subgroup Γ in $G = SL_{d+1}(K)$; for simplicity we assume that Γ has no fixed points on $X = \Omega^{(d+1)}$. We write $\mathcal{M}_\lambda := \mathcal{M}$ for the local system on X_Γ associated to $M = M_\lambda$. The above Theorem says that we have the spectral sequence

$$E_1^{r,s} = H^s(\Gamma, D_{\lambda(r)}) \Longrightarrow H_{DR}^{r+s}(X_\Gamma, \mathcal{M}_\lambda)$$

which we call the <u>reduced</u> Hodge-de Rham spectral sequence.

Conjecture:

I) The reduced Hodge-de Rham spectral sequence degenerates.
II) The filtration on $H^d_{DR}(X_\Gamma, \mathcal{M}_\lambda)$ induced by the reduced Hodge-de Rham spectral sequence is the de Rham filtration $F^{\cdot}_{DR}$.

It is obvious that an analog of §2 Lemma 2 can be proved for the reduced Hodge-de Rham spectral sequence. We finish this paper by looking at the case $d = 1$. Then λ is a nonpositive integer. Since the case $\lambda = 0$ of constant coefficients was dealt with in §2 we assume $\lambda < 0$ so that the only nonvanishing group is $H^1_{DR}(X_\Gamma, \mathcal{M}_\lambda)$. The two complexes under consideration are

$$\mathcal{O}(X) \underset{K}{\otimes} M_\lambda \xrightarrow{d\otimes \mathrm{id}} \Omega^1(X) \underset{K}{\otimes} M_\lambda$$

and

$$D_{-\lambda} \xrightarrow{d_{1-\lambda}} D_{\lambda-2} \quad ;$$

moreover

$$D_{-\lambda} \cong \mathcal{O}(X) \underset{K}{\otimes} M_\lambda / F^{\lambda+1}(\mathcal{O}(X) \underset{K}{\otimes} M_\lambda) \quad \text{and}$$
$$D_{\lambda-2} \cong F^1(\Omega^1(X) \underset{K}{\otimes} M_\lambda) \quad .$$

Lemma 9 implies that we have the G-invariant decomposition

$$\Omega^1(X) \underset{K}{\otimes} M_\lambda = F^1(\Omega^1(X) \underset{K}{\otimes} M_\lambda) \oplus (d \otimes \mathrm{id})(F^{\lambda+1}(\mathcal{O}(X) \underset{K}{\otimes} M_\lambda)) \quad .$$

It follows that the part II) of the above Conjecture holds true. On the other hand it was shown in [SS] §5 Corollary 12 and Theorem on p. 98 that

$$(D_{-\lambda}/M_\lambda)^\Gamma = (\mathrm{im}\, d_{1-\lambda})^\Gamma = 0$$

and that

$$(D_{\lambda-2})^\Gamma \xrightarrow{\cong} (\mathrm{coker}\, d_{1-\lambda})^\Gamma \quad \text{is bijective} \quad .$$

The first fact implies that

$$H^1(\Gamma, M_\lambda) \longrightarrow H^1(\Gamma, D_{-\lambda})$$

is injective and consequently that $F^1_\Gamma \cap F^1_{DR} = \{0\}$ (compare §2 Lemma 2 i). The second fact implies that $F^1_\Gamma + F^1_{DR} = H^1_{DR}(X_\Gamma, \mathcal{M}_\lambda)$ (compare §2 Lemma 2 ii). We obtain the Hodge type decomposition

$$H^1_{DR}(X_\Gamma, \mathcal{M}_\lambda) = F^1_\Gamma \oplus F^1_{DR}$$

thereby confirming our Conjecture in §2. Such a decomposition was established for the first time (by different methods) in [Sha]. The part I) of the above Conjecture amounts to the vanishing of $H^1(\Gamma, \mathrm{im}\, d_{1-\lambda})$ and seems not to be known at present. The (nonreduced) Hodge-de Rham spectral sequence does not degenerate in general: We have the exact sequence

$$\begin{aligned} 0 = H^0_{DR}(X_\Gamma, \mathcal{M}_\lambda) &\longrightarrow H^0(\Gamma, \mathcal{O}(X) \underset{K}{\otimes} M_\lambda) \\ &\longrightarrow H^0(\Gamma, \Omega^1(X) \underset{K}{\otimes} M_\lambda) \longrightarrow F^1_{DR} \longrightarrow 0 \quad . \end{aligned}$$

The second term contains $H^0(\Gamma, F^0(\mathcal{O}(X) \underset{K}{\otimes} M_\lambda)) \cong H^0(\Gamma, D_\lambda)$ which does not vanish in general. Namely $H^0(\Gamma, D_\lambda)$ is the space of all K-analytic automorphic forms of weight $-\lambda$ for Γ; if Γ is a free group on g generators and if $\lambda \leq -2$ then the dimension of this latter space is equal to $(g-1)(-\lambda-1)$ if $\lambda \leq -3$, resp. to g if $\lambda = -2$ (see [SS] p. 97/98).

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