Verdier duality on the building

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Dedicated to Martin Kneser

Let K be a nonarchimedean locally compact field and let G be the group of K-rational points of a connected reductive group over K. In [SS] a certain duality functor on smooth finite length representations of G is introduced and studied. The main tool is a construction which realizes such a G-representation as the homology, resp. the cohomology with compact support, of an equivariant coefficient system (or cosheaf), resp. an equivariant sheaf, on the building X of G. Since X is a locally compact space of finite dimension we have the Verdier duality functor on sheaves on X. It is therefore a natural question how those two duality functors on G-representations and on sheaves on X are related to each other. The answer which will be given in this paper is that they are intertwined by the cohomology functor with compact support. The technical problem which has to be solved is how to reinterprete coefficient systems on X in sheaf theoretic terms. Indeed, we will show that the bounded derived categories of the coefficient systems on X and of the (weakly constructible) sheaves on X are canonically equivalent. In other words that derived category has two different natural hearts. We will see that Verdier duality exchanges those two hearts and is given by the naive functor which applies linear duality to the stalks of a constructible sheaf. In some sense coefficient systems bear a certain resemblance to perverse sheaves. The reader will easily realize that this part of the paper works literally the same way for any locally finite simplicial complex. Using the formalism developed in [Sch] we extend these results in the last section to the equivariant setting and deduce the theorem that the cohomology with compact support transforms Verdier duality into representation theoretic duality.

1. Constructible sheaves on the building

Let X denote the Bruhat-Tits building of G. Its main structural features are the following (see [SS]I.1 for a brief overview):
– X carries a natural metric $d(\,.\,,\,.\,)$.
– X in a natural way is a locally finite d-dimensional polysimplicial complex where d is the semisimple K-rank of G. The corresponding (open) polysimplices are called facets.
– X carries a natural G-action which is isometric and respects the partition into facets.

We first review the contents of [KS]8.1 leaving to the reader the exercise to extend the proofs in loc. cit. from the simplicial to the polysimplicial setting. Let $\mathrm{Sh}(\mathbb{C}_X)$ denote the abelian category of sheaves of $\mathbb{C}$-vector spaces on X and let $D^b(X)$ be the corresponding bounded derived category.

Definition: ([KS]8.1.3)
A sheaf S in $\mathrm{Sh}(\mathbb{C}_X)$ is called weakly constructible, resp. constructible, if the restriction of S to any facet in X is constant, resp. is constant with finite dimensional stalks.

For any facet F in X the subset

$$\mathrm{St}(F) := \text{ union of all facets } F' \subseteq X \text{ such that } F \subseteq \overline{F'}$$

is called the star of F. These stars form a locally finite open covering of X. For any point $x \in X$ let $F(x)$ be the unique facet containing x and put

$$\mathrm{St}(x) := \mathrm{St}(F(x)) \quad .$$

Lemma 1:
For any weakly constructible sheaf S on X and any point $x \in X$ we have

$$H^*(\mathrm{St}(x), S|\mathrm{St}(x)) = H^*(F(x), S|F(x)) = \begin{cases} S_x & \text{if } * = 0 \ , \\ 0 & \text{if } * > 0 \ . \end{cases}$$

Proof: [KS]8.1.4.

Let w-$\mathrm{Cons}(X)$, resp. $\mathrm{Cons}(X)$, denote the full subcategory of weakly constructible, resp. constructible, sheaves in $\mathrm{Sh}(\mathbb{C}_X)$. These are thick abelian subcategories in the sense of [Har]p.38. Let $D^b_{w-c}(X)$, resp. $D^b_c(X)$, be the full triangulated subcategory of $D^b(X)$ consisting of those complexes $S^\cdot$ whose cohomology sheaves $h^\cdot(S^\cdot)$ are all weakly constructible, resp. constructible.

Proposition 2:
The natural functors

$$D^b(w\text{-}\mathrm{Cons}(X)) \xrightarrow{\sim} D^b_{w-c}(X) \quad \textit{and} \quad D^b(\mathrm{Cons}(X)) \xrightarrow{\sim} D^b_c(X)$$

are equivalences of categories.
Proof: [KS]8.1.10 and 8.1.11.

We will describe a quasi-inverse of the first functor. Note that to give a (weakly) constructible sheaf S on X is the same as to give
– $\mathbb{C}$-vector spaces S_F for any facet $F \subseteq X$, and
– linear maps $r_F^{F'} : S_{F'} \longrightarrow S_F$ for each pair of facets $F' \subseteq \overline{F}$ such that $r_F^F = id$ and $r_F^{F'} \circ r_{F'}^{F''} = r_F^{F''}$ whenever $F'' \subseteq \overline{F'}$ and $F' \subseteq \overline{F}$.
Namely, given any sheaf S put $S_F := S(\mathrm{St}(F))$ and let $r_F^{F'}$ be the restriction maps. Vice versa define a weakly constructible sheaf S by setting, for any open subset $\Omega \subseteq X$,

$$S(\Omega) := \mathbb{C}\text{-vector space of all maps } s : \Omega \longrightarrow \dot{\bigcup_{x\in\Omega}} S_{F(x)}$$

such that
– $s(x) \in S_{F(x)}$ for any $x \in \Omega$, and
– there is an open covering $\Omega = \bigcup_{i\in I} \Omega_i$ with
$r_{F(y)}^{F(x)}(s(x)) = s(y)$ for any $x \in \Omega_i, y \in \mathrm{St}(x) \cap \Omega_i$, and $i \in I$.

Composing these two constructions we obtain a functor

$$\beta : \mathrm{Sh}(\mathbb{C}_X) \longrightarrow w\text{-}\mathrm{Cons}(X)$$

which has the following properties ([KS]8.1.7 and 8.1.8):
– β is right adjoint to the inclusion functor $w\text{-}\mathrm{Cons}(X) \xrightarrow{\subseteq} \mathrm{Sh}(\mathbb{C}_X)$ and is, through adjunction, a left quasi-inverse for the latter;
– β is left exact;
– $R^*\beta(S)(\mathrm{St}(F)) = H^*(\mathrm{St}(F), S)$ for any facet F and any S in $\mathrm{Sh}(\mathbb{C}_X)$; in particular, weakly constructible sheaves are β-acyclic.
It is a formal consequence of these properties that the restriction of the derived functor

$$R\beta \ : \ D^b(X) \longrightarrow D^b(w\text{-}\mathrm{Cons}(X))$$

to $D^b_{w-c}(X)$ is the quasi-inverse we were looking for.

2. Coefficient systems

We recall from [SS] that a coefficient system $\mathcal{V} = (V_F)_F$ of complex vector spaces on X consists of

– $\mathbb{C}$-vector spaces V_F for each facet $F \subseteq X$, and
– linear maps $r^F_{F'} : V_F \longrightarrow V_{F'}$ for each pair of facets $F' \subseteq \overline{F}$ such that $r^F_F = id$ and $r^F_{F''} = r^{F'}_{F''} \circ r^F_{F'}$ whenever $F'' \subseteq \overline{F'}$ and $F' \subseteq \overline{F}$.
Let Coeff(X) denote the abelian category of those coefficient systems and let $D^b(\text{Coeff}(X))$ be the corresponding bounded derived category. With any coefficient system $\mathcal{V}$ we may associate the complex of oriented chains

$$C_c^{or}(X_{(d)}, \mathcal{V}) \xrightarrow{\partial} \ldots \xrightarrow{\partial} C_c^{or}(X_{(0)}, \mathcal{V}) \quad .$$

Here $X_{(q)}$ denotes the set of all q-dimensional oriented facets (F, c) and

$$C_c^{or}(X_{(q)}, \mathcal{V}) := \mathbb{C}\text{-vector space of all maps } \omega : X_{(q)} \longrightarrow \dot{\bigcup_{F \in X_q}} V_F$$

such that
– ω has finite support,
– $\omega((F, c)) \in V_F$, and, if $q \geq 1$,
– $\omega((F, -c)) = -\omega((F, c))$ for any $(F, c) \in X_{(q)}$.

The boundary map ∂ is defined by

$$\begin{aligned} \partial : C_c^{or}(X_{(q+1)}, \mathcal{V}) &\longrightarrow C_c^{or}(X_{(q)}, \mathcal{V}) \\ \omega &\longmapsto \Big((F', c') \longmapsto \sum_{\substack{(F,c) \in X_{(q+1)} \\ F' \subseteq \overline{F},\ \partial^F_{F'}(c) = c'}} r^F_{F'}(\omega((F, c))) \Big) . \end{aligned}$$

We call

$$H_*(X, \mathcal{V}) := h_*(C_c^{or}(X_{(\cdot)}, \mathcal{V}))$$

the homology of the coefficient system $\mathcal{V}$.

Next we will construct a natural functor

$$\sigma : D^b(\text{Coeff}(X)) \longrightarrow D^b(w\text{-Cons}(X)) \quad .$$

Fo any facet F let $j_F : F \xrightarrow{\subseteq} X$ be the corresponding locally closed immersion. Consider a fixed coefficient system $\mathcal{V} = (V_F)_F$ on X. We will use the same symbol V_F to also denote the constant sheaf with value V_F on F.

Remark 1:
i. If $j : F \xrightarrow{\subseteq} \overline{F}$ denotes the open immersion of a facet F into its closure then we have

$$j_* V_F = \textit{ constant sheaf with value } V_F \textit{ on } \overline{F} \quad ;$$

ii. for any pair of facets $F' \subseteq \overline{F}$ we have

$$(j_{F'})^* j_{F*} V_F = \text{ constant sheaf with value } V_F \text{ on } F' \ .$$

Proof: i. Obvious. ii. This follows from the first assertion.

For any $0 \leq q \leq d$ we define a sheaf $\mathcal{V}_q$ on X by setting, for any open subset $\Omega \subseteq X$,

$$\mathcal{V}_q(\Omega) := \mathbb{C}\text{-vector space of all maps } \omega : X_{(q)} \longrightarrow \dot{\bigcup_{F \in X_q}} (j_{F*} V_F)(\Omega)$$

such that

– $\omega((F,c)) \in (j_{F*} V_F)(\Omega)$ and, if $q \geq 1$,

– $\omega((F,-c)) = -\omega((F,c))$ for any $(F,c) \in X_{(q)}$.

Since X is locally finite we have noncanonically

$$\mathcal{V}_q \cong \prod_{F \in X_q} j_{F*} V_F \ = \bigoplus_{F \in X_q} j_{F*} V_F$$

where X_q denotes the set of all q-dimensional facets. From this we easily see that

$$(j_{F'})^* \mathcal{V}_q \cong \bigoplus_{\substack{F \subseteq \mathrm{St}(F') \\ \dim(F) = q}} (j_{F'})^* j_{F*} V_F$$

for any facet F'. It follows that $\mathcal{V}_q$ is weakly constructible. Clearly the boundary map ∂ extends to the sheaf level so that we obtain a complex of sheaves on X

$$\sigma(\mathcal{V}) := [\mathcal{V}_d \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{V}_0]$$

put in degrees $-d$ through 0. This construction is obviously functorial and by Remark 1 also exact in $\mathcal{V}$. Hence it gives rise to a functor σ between the respective derived categories.

Proposition 2:
The functor $\sigma : D^b(\mathrm{Coeff}(X)) \xrightarrow{\sim} D^b(w\text{-}\mathrm{Cons}(X))$ is an equivalence of categories.
Proof: In order to simplify the notations we fix an orientation of X. For any weakly constructible sheaf $S = (S_F)_F$ on X we construct a complex

$$\tau(S) = [S_0 \longrightarrow \cdots \longrightarrow S_d]$$

of coefficient systems on X put in degrees 0 through d in the following way. The coefficient system S_q, for $0 \leq q \leq d$, is defined by

$$S_q := \Big(\bigoplus_{\substack{F' \subseteq \mathrm{St}(F) \\ \dim F' = q}} S_{F'} \Big)_F$$

with the obvious inclusions as the transition maps. The differential in the complex on the level F is the relative cohomological differential for the pair $(X, X \setminus \mathrm{St}(F))$. This construction clearly is functorial and exact and hence induces a functor

$$\tau : D^b(w\text{-Cons}(X)) \longrightarrow D^b(\mathrm{Coeff}(X)) \quad .$$

We claim that τ is a quasi-inverse for σ. Recall that, for a complex $\mathcal{V}^{\cdot} = (V_F^{\cdot})_F$ of coefficient systems, we have

$$\sigma(\mathcal{V}^{\cdot}) = [\mathcal{V}_d^{\cdot} \longrightarrow \cdots \longrightarrow \mathcal{V}_0^{\cdot}]$$

with

$$\mathcal{V}_q^{\cdot} = \Big(\bigoplus_{\substack{F \subseteq \mathrm{St}(F') \\ \dim F = q}} V_F^{\cdot} \Big)_{F'} \quad .$$

We see that $\tau\sigma(\mathcal{V}^{\cdot})$ is given by the triple complex

$$\begin{array}{ccccc}
(\mathcal{V}_d^{\cdot})_d & \longrightarrow & \dots & \longrightarrow & (\mathcal{V}_0^{\cdot})_d \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
(\mathcal{V}_d^{\cdot})_0 & \longrightarrow & \dots & \longrightarrow & (\mathcal{V}_0^{\cdot})_0 \ .
\end{array}$$

We compute

$$(\mathcal{V}_i^{\cdot})_j = \Big(\bigoplus_{\substack{F' \subseteq \mathrm{St}(F) \\ \dim F' = j}} \ \bigoplus_{\substack{F'' \subseteq \mathrm{St}(F') \\ \dim F'' = i}} V_{F''}^{\cdot} \Big)_F \quad .$$

This shows in particular that $(\mathcal{V}_i^{\cdot})_j = 0$ for $i < j$ and that

$$(\mathcal{V}_i^{\cdot})_i = \Big(\bigoplus_{\substack{F' \subseteq \mathrm{St}(F) \\ \dim F' = i}} V_{F'}^{\cdot} \Big)_F \quad .$$

We claim that the natural map

$$\bigoplus_i (\mathcal{V}_i^{\cdot})_i = \Big(\bigoplus_{F' \subseteq \mathrm{St}(F)} V_{F'}^{\cdot} \Big)_F \xrightarrow{(\sum r_F^{F'})_F} \mathcal{V}^{\cdot} = (V_F^{\cdot})_F$$

induces a quasi-isomorphism $(\mathcal{V}^{\cdot}_{*})_{*} \longrightarrow \mathcal{V}^{\cdot}$. By a filtration argument we may assume that $\mathcal{V}^{\cdot}$ is a single coefficient system $\mathcal{V} = (V_F)_F$ which is supported on a single facet F_0 of dimension m, i.e., $V_F = 0$ for $F \neq F_0$. Put $V := V_{F_0}$. We then have

$$\mathcal{V}_i = 0 \text{ for } i \neq m, \quad \mathcal{V}_m = \left(\begin{cases} V & \text{if } F' \subseteq \overline{F_0} \\ 0 & \text{otherwise} \end{cases} \right)_{F'} \quad \text{and}$$
$$(\mathcal{V}_m)_j = \Big(\bigoplus_{\substack{F \subseteq \overline{F'}, F' \subseteq \overline{F_0} \\ \dim F' = j}} V \Big)_F \quad .$$

What we have to check therefore is the exactness, for each F, of the complex

$$0 \longrightarrow \bigoplus_{\substack{F \subseteq \overline{F'}, F' \subseteq \overline{F_0} \\ \dim F' = \dim F}} V \longrightarrow \ldots \longrightarrow \bigoplus_{\substack{F \subseteq \overline{F'}, F' \subseteq \overline{F_0} \\ \dim F' = \dim F_0}} \longrightarrow \begin{cases} V & \text{if } F = F_0 \\ 0 & \text{otherwise} \end{cases} \longrightarrow 0 \quad .$$

This is clear. In this way we have shown that $\tau\sigma \cong id$. The argument for the other identity $\sigma\tau \cong id$ is entirely analogous.

Lemma 3:
For any coefficient system $\mathcal{V} = (V_F)_F$ on X we have

$$H^*_c(X, \mathcal{V}_q) = \begin{cases} C^{or}_c(X_{(q)}, \mathcal{V}) & \text{if } * = 0, \\ 0 & \text{otherwise} . \end{cases}$$

Proof: Straightforward.

Corollary 4:
For any coefficient system $\mathcal{V}$ on X we have

$$H_*(X, \mathcal{V}) = H^{-*}_c(X, \sigma(\mathcal{V})) \quad .$$

In the following the complex

$$\sigma_X := \sigma((\mathbb{C})_F)$$

in $D^b(\mathrm{Cons}(X))$ belonging to the constant coefficient system $(\mathbb{C})_F$ with value $\mathbb{C}$ will play a distinguished role. There is an obvious exact functor

$$\begin{aligned} {}^* : w\text{-}\mathrm{Cons}(X) &\longrightarrow \mathrm{Coeff}(X) \\ S &\longmapsto S^* = (S^*_F)_F \end{aligned}$$

defined by $S_F^* := \mathrm{Hom}_{\mathbb{C}}(S(\mathrm{St}(F)), \mathbb{C})$. It follows from Lemma 1.1 that

$$j_{F*} S_F^* = j_{F*} \underline{\mathrm{Hom}}_{\mathbb{C}_F}(S|F, \mathbb{C}_F) = \underline{\mathrm{Hom}}_{\mathbb{C}_X}(S, j_{F*}\mathbb{C}_F)$$

and hence that

$$\sigma(S^*)^{\cdot} = \underline{\mathrm{Hom}}_{\mathbb{C}_X}(S, \sigma_X^{\cdot}) \quad .$$

Remark 5:
For any constant sheaf V on a facet F we have $R^i j_{F*} V = 0$ for $i \geq 1$.
Proof: Obvious.

Using this and [Bor]V.7.9 it follows that

$$\begin{aligned} \underline{\mathrm{Ext}}^i_{\mathbb{C}_X}(S, j_{F*}\mathbb{C}_F) &= \underline{\mathrm{Ext}}^i_{\mathbb{C}_X}(S, Rj_{F*}\mathbb{C}_F) = R^i j_{F*} R\,\underline{\mathrm{Hom}}_{\mathbb{C}_F}(S|F, \mathbb{C}_F) \\ &= R^i j_{F*} S_F^* = 0 \end{aligned}$$

for $i \geq 1$. As a consequence we obtain the equality

$$\sigma(S^*) = R\,\underline{\mathrm{Hom}}_{\mathbb{C}_X}(S, \sigma_X)$$

in $D^b(w\text{-Cons}(X))$. In other words we have the commutative diagram

$$\begin{array}{ccc} D^b(w\text{-Cons}(X)) & \xrightarrow{R\,\underline{\mathrm{Hom}}_{\mathbb{C}_X}(\,.\,,\sigma_X)} & D^b(w\text{-Cons}(X)) \\ * \searrow & & \nearrow \sigma \\ & D^b(\mathrm{Coeff}(X)) & \end{array} \quad .$$

We see in particular that the functor $R\,\underline{\mathrm{Hom}}_{\mathbb{C}_X}(\,.\,,\sigma_X)$ respects the subcategories $D^b_{w-c}(X)$ and $D^b_c(X)$ of $D^b(X)$.
For any point $x \in X$ the stalk in x of the complex σ_X is

$$(\sigma_X^q)_x \cong \bigoplus_{\substack{F \subseteq \mathrm{St}(F(x)) \\ \dim(F) = -q}} \mathbb{C} \quad .$$

This means that the stalks of the homology sheaves are the relative homology groups

$$h^q(\sigma_X)_x = H_{-q}(X, X \setminus \{x\}; \mathbb{C}) \quad .$$

But according to [BT]II.5.1.32 and [Tit]3.5.4 the link of a facet in X is isomorphic to the spherical building of a reductive group over a finite field. By the theorem of Solomon-Tits the latter is a bouquet of spheres. It follows that

$$H_*(X, X \setminus \{x\}; \mathbb{C}) \;=\; 0 \quad \text{for } * \neq d \quad .$$

In this way we have proved the following result.

Lemma 6:
σ_X coincides in $D^b(\mathrm{Cons}(X))$ with a single sheaf placed in degree $-d$.

3. Verdier duality

The formalism of Verdier duality applies since X is locally compact of cohomological dimension d. It ensures the existence of the dualizing complex ω_X in $D^b(X)$ (actually $h^j(\omega_X) \neq 0$ at most for $-d \leq j \leq 0$) together with a natural isomorphism

$$\mathrm{Ext}^*_{\mathbb{C}_X}(S, \omega_X) = \mathrm{Hom}_{\mathbb{C}}(H_c^{-*}(X, S), \mathbb{C})$$

for any S in $D^b(X)$ (compare [KS]3.1.10). The duality functor by definition is

$$\begin{array}{rcl} D_X : D^b(X) & \longrightarrow & D^b(X) \\ S & \longmapsto & R\,\underline{\mathrm{Hom}}_{\mathbb{C}_X}(S, \omega_X) \quad ; \end{array}$$

it comes equipped with the natural transformation of biduality

$$S \longrightarrow D_X D_X S \quad .$$

According to [Bor]V.7.1 we have

$$h^*(\omega_X)_x = \mathrm{Hom}_{\mathbb{C}}(H^{-*}(X, X \setminus \{x\}; \mathbb{C}), \mathbb{C}) = H_{-*}(X, X \setminus \{x\}; \mathbb{C})$$

for any $x \in X$. (The constant sheaf $\mathbb{C}_X$ is cohomologically constructible.) By the same argument as before it follows that $h^*(\omega_X) = 0$ for $* \neq -d$. Since, by the construction of ω_X, we have

$$h^{-d}(\omega_X)(\Omega) = \mathrm{Hom}_{\mathbb{C}}(H_c^d(\Omega, \mathbb{C}), \mathbb{C}) \quad \text{for } \Omega \subseteq X \text{ open}$$

this proves the following.

Lemma 1:
ω_X coincides in $D^b(X)$ with the sheaf

$$\Omega \longmapsto \mathrm{Hom}_{\mathbb{C}}(H_c^d(\Omega, \mathbb{C}), \mathbb{C})$$

placed in degree $-d$.

Proposition 2:
$\omega_X = \sigma_X$ in $D^b_c(X)$.
Proof: The filtration of X by its skeletons gives rise to the spectral sequence

$$E_1^{i,j} = \prod_{\dim F = d-i} \mathrm{Hom}_{\mathbb{C}}(H_c^{-(i+j)}(\Omega \cap F, \mathbb{C}), \mathbb{C}) \Rightarrow \mathrm{Hom}_{\mathbb{C}}(H_c^{-(i+j)}(\Omega, \mathbb{C}), \mathbb{C})$$

of presheaves on X. By [KS]3.3.6 we have

$$\text{sheafification of } \; E_1^{i,j} = \begin{cases} \prod\limits_{\dim F = d-i} j_{F^*} or_F & \text{if } j = -d\,, \\ 0 & \text{otherwise} \end{cases}$$

where or_F denotes the orientation sheaf of F. It follows that

$$\mathrm{Hom}_{\mathbb{C}}(H_c^d(\,.\,, \mathbb{C}), \mathbb{C}) = h^{-d}(\sigma_X) \quad .$$

Because of the above result it follows from section 2 that D_X respects the subcategories $D^b_{w-c}(X)$ and $D^b_c(X)$ and that

$$D_X = \sigma \circ (^*) \circ R\beta \text{ on } D^b_{w-c}(X) \quad .$$

Proposition 3:
Assume that S is in $D^b_c(X)$; we then have:

i. The natural transformation of biduality $S \xrightarrow{\sim} D_X D_X S$ is an isomorphism;

ii. $(D_X S)_x = \mathrm{Hom}_{\mathbb{C}}(R\Gamma_{\{x\}}(X, S), \mathbb{C})$ for any $x \in X$.
Proof: By an induction argument (compare [Bor] p.140) we may assume that S is a single constructible sheaf on X.
i. Consider the exact functor

$$\begin{array}{cccc} ^* : & \mathrm{Coeff}(X) & \longrightarrow & w\text{-}\mathrm{Cons}(X) \\ & \mathcal{V} = (V_F)_F & \longmapsto & \mathcal{V}^* = (V_F^*)_F \end{array}$$

defined by $V_F^* := \mathrm{Hom}_{\mathbb{C}}(V_F, \mathbb{C})$. We claim that

$$\mathcal{V}^* = \underline{\mathrm{Hom}}_{\mathbb{C}_X}(\sigma(\mathcal{V})^{\cdot}, \sigma_X^{\cdot}) \;\; \text{in } D^b(w\text{-}\mathrm{Cons}(X)) \quad .$$

Our assertion then follows immediately from the obvious identity

$$S^{**} = S \text{ for any constructible sheaf } S \quad .$$

The computation before the Remark 2.5 shows that the total complex $\underline{\mathrm{Hom}}_{\mathbb{C}_X}(\sigma(\mathcal{V})^{\cdot}, \sigma_X^{\cdot})$ is a complex in degrees 0 up to d with the term in degree q equal to

$$\prod_{\substack{F' \subseteq \overline{F} \\ \dim F' + q = \dim F}} j_{F'*}(V_F^*|F') \quad .$$

We more precisely claim that the obvious natural map

$$\mathcal{V}^* \longrightarrow \prod_F j_{F*} V_F^*$$

induces the wanted quasi-isomorphism. By an induction as well as a direct product argument we may assume that $\mathcal{V}$ is supported on a single facet F of dimension m, i.e., $\mathcal{V}^* = j_{F!} V_F^*$. In this case our claim amounts to the exactness of the complex

$$\begin{aligned} 0 \longrightarrow j_{F!} V_F^* \longrightarrow j_{F*} V_F^* \longrightarrow \bigoplus_{\substack{F' \subseteq \overline{F} \\ \dim F' = m-1}} j_{F'*}(V_F^*|F') \longrightarrow \\ \ldots \longrightarrow \bigoplus_{\substack{F' \subseteq \overline{F} \\ \dim F' = 0}} j_{F'*}(V_F^*|F') \longrightarrow 0 \quad . \end{aligned}$$

That exactness is a consequence of the fact that the corresponding complex of stalks in a point $x \in X$ computes the relative homology groups

$$H_*(\overline{F}, \overline{F} \setminus \{x\}; V_F^*) \cong \begin{cases} V_F^* & \text{if } x \in F \text{ and } * = \dim F, \\ 0 & \text{otherwise} . \end{cases}$$

ii. Because of i. it suffices to show that

$$H^*_{\{x\}}(X, D_X S) = \begin{cases} \mathrm{Hom}_{\mathbb{C}}(S_x, \mathbb{C}) & \text{if } * = 0, \\ 0 & \text{otherwise} . \end{cases}$$

Applying the first hypercohomology spectral sequence to the complex $\sigma(S^*)^{\cdot}$ which represents $D_X S$ this follows from

$$H^*_{\{x\}}(X, j_{F*} S_F^*) = \begin{cases} H^*_{\{x\}}(F, S_F^*) \cong S_F^* & \text{if } x \in F \text{ and } * = \dim F, \\ 0 & \text{otherwise} \end{cases}$$

(the isomorphism depends on the choice of an orientation of $F = F(x)$).

Let us put

$$\Omega_X := h^{-d}(\omega_X) = h^{-d}(\sigma_X)$$

so that up to quasi-isomorphism $\Omega_X[d]$ is the dualizing complex of sheaves on X. Moreover let $\pi : X \longrightarrow pt$ denote the natural map to the point.

Proposition 4:
$\pi^! = \Omega_X[d] \underset{\mathbb{C}_X}{\otimes} \pi^*$.
Proof: Let V be a vector space. The same argument as for the proof of Lemma 1 shows that $\pi^! V[-d]$ is quasi-isomorphic to the sheaf $\Omega \longmapsto \mathrm{Hom}_{\mathbb{C}}(H_c^d(\Omega, \mathbb{C}), V)$. Since $H_c^d(\Omega, \mathbb{C})$ is finite dimensional for any relatively compact Ω the latter sheaf is isomorphic to $\Omega_X \underset{\mathbb{C}_X}{\otimes} V_X$ where $V_X = \pi^* V$ is the constant sheaf with value V on X.

4. The equivariant duality functor

In [Sch] we discuss the homological algebra of the category $\mathrm{Sh}_G(Y)$ of G-equivariant sheaves on a locally compact G-space Y. In particular we introduce the G-equivariant dualizing complex of sheaves $\omega_{Y,G} \in D^b(\mathrm{Sh}_G(Y))$. The building X is a special G-space in the terminology of loc. cit. A weakly constructible sheaf $S = (S_F)_F$ on X is G-equivariant if, for any facet $F \subseteq X$, there is given a linear map

(0) $$g_F : S_F \longrightarrow S_{gF}$$

in such a way that

(1) $g_{hF} \circ h_F = (gh)_F$ for any $g, h \in G$ and any F,

(2) $1_F = id_{S_F}$ for any F,

(3) the diagram

$$\begin{array}{ccc} S_{F'} & \xrightarrow{g_{F'}} & S_{gF'} \\ r_F^{F'} \downarrow & & \downarrow r_{gF}^{gF'} \\ S_F & \xrightarrow{g_F} & S_{gF} \end{array}$$

is commutative for any $g \in G$ and any pair of facets $F' \subseteq \overline{F}$, and

(4) the induced action of the pointwise stabilizer P_F of F on S_F is smooth for any F.

Without the “continuity” condition (4), i.e., if only (0)-(3) are required S is called G^{dis}-equivariant. Any G^{dis}-equivariant sheaf S has a largest G-equivariant subsheaf S^{smooth} which in the weakly constructible case is given by $S^{\mathrm{smooth}} = (S_F^{\mathrm{smooth}})_F$ with

$$S_F^{\mathrm{smooth}} := \{s \in S_F : s \text{ is fixed by some open subgroup of } P_F\} \quad .$$

It is clear that Ω_X in a natural way is G^{dis}-equivariant. For a facet F let $P_{\mathrm{St}(F)} \subseteq G$ be the pointwise stabilizer of $\mathrm{St}(F)$; it is an open subgroup of G. Since Ω_X as a weakly constructible sheaf is given by

$$\Omega_X = (\mathrm{Hom}_{\mathbb{C}}(H_c^d(\mathrm{St}(F), \mathbb{C}), \mathbb{C}))_F$$

and since $P_{\mathrm{St}(F)}$ acts trivially on $H_c^d(\mathrm{St}(F), \mathbb{C})$ we see that Ω_X actually is G-equivariant.

Proposition 1:
$\omega_{X,G}$ is naturally quasi-isomorphic to $\Omega_X[d]$ with its canonical G-equivariant structure.
Proof: [Sch] 3.10 and 3.12.

In the following we freely use the various functors introduced in [Sch]. In particular let

$$D_{X,G} := R\,\underline{\mathrm{Hom}}_{X,\infty}(\,.\,, \omega_{X,G})$$

be the equivariant duality functor on $D^b(\mathrm{Sh}_G(X))$. On $D^b(\mathrm{Alg}(G))$ with $\mathrm{Alg}(G) := \mathrm{Sh}_G(pt)$ the equivariant duality functor simply is

$$D_{pt,G} := \mathrm{Hom}_{\mathbb{C}}^{\infty}(\,.\,, \mathbb{C}) \quad .$$

As before let $\pi : X \longrightarrow pt$ be the natural map to the point. The smooth Verdier duality then says ([Sch]) that

$$D_{pt,G} \circ R\pi_! = R\pi_{*,\infty} \circ D_{X,G}$$

holds true on $D^b(\mathrm{Sh}_G(X))$. The remarkable fact which we are now going to explain is that there is a second formula of this type.

Let $\mathrm{Cons}_G(X)$ denote the full subcategory of constructible sheaves in $\mathrm{Sh}_G(X)$. Also let $D^b_c(X,G)$ be the full triangulated subcategory of $D^b(\mathrm{Sh}_G(X))$ consisting of those complexes all of whose cohomology sheaves are constructible. Similarly as before the functors

$$D^b(\mathrm{Cons}_G(X)) \underset{R\beta}{\overset{\text{inclusion}}{\rightleftarrows}} D^b_c(X,G)$$

are quasi-inverse equivalences of categories. It should be noted that $R\beta$ is compatible with the forgetful functor ([Sch]1.3 together with the fact that c-soft sheaves on X are $H^0(\mathrm{St}(F), .)$-acyclic).

Proposition 2:
The diagram of functors

$$\begin{array}{ccc} D^b_c(X,G) & \overset{D_{X,G}}{\longrightarrow} & D^b_c(X,G) \\ \mathrm{For}\downarrow & & \downarrow \mathrm{For} \\ D^b_c(X) & \overset{D_X}{\longrightarrow} & D^b_c(X) \end{array}$$

is commutative.
Proof: Since constructible G-equivariant sheaves are special this follows from [Sch] 3.13.

Analogous to the above description of G-equivariant weakly constructible sheaves we have the notion of G-equivariant coefficient systems on X. They form an abelian category which we denote by $\mathrm{Coeff}_G(X)$. A coefficient system $\mathcal{V} = (V_F)_F$ is called constructible if each V_F is finite dimensional. Let $D^b_c(\mathrm{Coeff}_G(X))$ denote the full triangulated subcategory in $D^b(\mathrm{Coeff}_G(X))$ of those complexes whose cohomology coefficient systems are constructible. It is now straightforward to check that our earlier identity $D_X = \sigma \circ {}^*$ lifts to a commutative diagram of functors

$$(+) \qquad \begin{array}{ccccc} D^b_c(X,G) & & \overset{D_{X,G}}{\longrightarrow} & & D^b_c(X,G) \\ & * \searrow & & \nearrow \sigma & \\ & & D^b_c(\mathrm{Coeff}_G(X)) & & . \end{array}$$

In [SS] we have studied another duality functor for the category $\mathrm{Alg}(G)$. Let $\mathrm{Alg}_{fg}(G)$ be the full subcategory in $\mathrm{Alg}(G)$ of finitely generated smooth G-modules. This is a thick abelian subcategory ([Ber]3.12) so that we can consider the full triangulated subcategory $D^b_{fg}(\mathrm{Alg}(G))$ in $D^b(\mathrm{Alg}(G))$ of complexes whose cohomology G-modules are finitely generated. Put

$$\begin{aligned}\mathcal{H} := &\text{ space of all } \mathbb{C}\text{-valued locally constant} \\ &\text{functions with compact support on } G.\end{aligned}$$

It carries two commuting smooth G-actions by left and right translations, respectively. In forming $\mathrm{Hom}_G(V, \mathcal{H})$ for a $V \in \mathrm{Alg}(G)$ we always use the left translations action on $\mathcal{H}$ so that this still is a G-module through the right translation action on $\mathcal{H}$.

Lemma 3:
For any $V \in \mathrm{Alg}_{fg}(G)$ we have:

i. The G-module $\mathrm{Hom}_G(V, \mathcal{H})$ is smooth and finitely generated;

ii. V has a projective resolution in $\mathrm{Alg}(G)$ of finite length which lies in $\mathrm{Alg}_{fg}(G)$.

Proof: Since these are well known facts we only sketch the proofs.
i. All we have to do is to embed $\mathrm{Hom}_G(V, \mathcal{H})$ into a finitely generated smooth G-module. Let $v_1, \ldots, v_m$ be generators for V and let $H \subseteq G$ be a compact open subgroup which fixes each v_i. Then

$$\begin{array}{rcl}\mathrm{Hom}_G(V, \mathcal{H}) & \hookrightarrow & {}^H\mathcal{H} \oplus \ldots \oplus {}^H\mathcal{H} \\ f & \longmapsto & (f(v_1), \ldots, f(v_m))\end{array}$$

where ${}^H\mathcal{H}$ denotes the subspace of H-left invariant functions in $\mathcal{H}$ is such an embedding.
ii. Using [Ber]3.9 and 3.12 one can see that V has a projective resolution of possibly infinite length lying in $\mathrm{Alg}_{fg}(G)$. But on the other hand the theory of the extended Bruhat-Tits building implies that the projective dimension of the category $\mathrm{Alg}(G)$ is bounded by the K-rank of G.

It follows that the functor

$$D_{\mathcal{H}} := R\,\mathrm{Hom}_G(\,.\,, \mathcal{H}) : D^b_{fg}(\mathrm{Alg}(G)) \longrightarrow D^b_{fg}(\mathrm{Alg}(G))$$

is well defined. Although the assertions in [SS] II.2.1, II.2.2, III.1.1, and IV.1.3 apparently are more special the arguments in the proofs show that the diagram of functors

$$(++) \qquad \begin{array}{ccc} D^b_c(X, G) & \xrightarrow{R\pi_!} & D^b_{fg}(\mathrm{Alg}(G)) \\ * \downarrow & & \downarrow D_{\mathcal{H}} \\ D^b_c(\mathrm{Coeff}_G(X)) & \xrightarrow{LH_0(X,\,.\,)} & D^b_{fg}(\mathrm{Alg}(G)) \end{array}$$

is commutative. In order to simplify the situation a little bit we assume here and in the following that the centre of G is compact. The general case works the same way but one has to fix a central character in everything.

Proposition 4:
The diagram of functors

$$\begin{array}{ccc} D_c^b(X,G) & \stackrel{R\pi_!}{\longrightarrow} & D_{fg}^b(\mathrm{Alg}(G)) \\ D_{X,G} \downarrow & & \downarrow D_{\mathcal{H}} \\ D_c^b(X,G) & \stackrel{R\pi_!}{\longrightarrow} & D_{fg}^b(\mathrm{Alg}(G)) \end{array}$$

commutes.
Proof: Combine the diagrams (+) and (++) and Lemma 2.3.

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