Points and topologies in rigid geometry

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1. Introduction

This paper has its origin in notes of the second author and remarks in [P]. Its purpose is to clarify the various concepts of points of an affinoid space and topologies on sets of points which can be found in [B, D, H, P, S].

The set of ordinary points of an affinoid space X over a complete non-archimedean valued field k is too small for the collection of abelian sheaves on X. The family of stalks in the ordinary points does not detect the vanishing of a sheaf. One introduces the notion of a prime filter p on X to remedy this. The collection of all prime filters is called $\mathcal{P}(X)$. For every $p \in \mathcal{P}(X)$ and every abelian sheaf F on X one can form a stalk F_p which is an abelian group. The functor $F \mapsto F_p$ is exact and $F = 0$ if all F_p are zero.

The set $\mathcal{P}(X)$ has a natural topology on it. For this topology $\mathcal{P}(X)$ is quasi-compact but not Hausdorff. The topological space $\mathcal{P}(X)$ can be identified with the projective limit of a certain family of schemes of finite type over the residue field of k. For a field k with a discrete valuation this family consists of the special fibers of all formal schemes $\mathcal{X}$ of finite type and flat over the valuation ring k^0 of k such that the "generic fiber" $\mathcal{X} \otimes k$ is isomorphic to X.

The set $\mathcal{P}(X)$ is shown to be isomorphic to a collection of valuations $\mathrm{Val}(X)$ on the affinoid algebra $\mathcal{O}(X)$ of X. The latter space is the building block in Huber's approach to analytic spaces. The valuations of (real) rank 1 correspond to a certain subset $\mathcal{M}(X)$ of $\mathcal{P}(X)$. This subset consists of the maximal filters on X. There is a natural retraction map $r : \mathcal{P}(X) \to \mathcal{M}(X)$. Let $\mathcal{M}(X)_{qt}$ denote the set $\mathcal{M}(X)$ provided with the quotient topology. It turns out to be a compact Hausdorff space. The set $\mathcal{M}(X)$ plays a central role in Berkovich theory of analytic spaces. He provides $\mathcal{M}(X)$ with a certain topology which we will show to coincide with the quotient topology. This answers a question in [D].

The category of abelian sheaves on X turns out to be equivalent to the category of abelian sheaves on the topological space $\mathcal{P}(X)$. Using the retraction map $r : \mathcal{P}(X) \to \mathcal{M}(X)$ and the nice topological structure of $\mathcal{M}(X)_{qt}$ one finds that the category of overconvergent sheaves on X coincides with the category of abelian sheaves on $\mathcal{M}(X)_{qt}$. This fact was already proven in [S] and led to an easy proof of a base change theorem for rigid analytic spaces.

For part of our results about the spaces $\mathcal{P}(X)$ and $\mathrm{Val}(X)$ one can find brief indications already in [H].

In the last section we extend our results to general rigid spaces.

2. Points and valuations

Let X be an affinoid space over a complete non-archimedean valued field k. We will use the notations $k^0 := \{\lambda \in k : |\lambda| \leq 1\}$ and $\overline{k}$ for the residue field of k. Further π denotes some element in k with $0 < |\pi| < 1$. The algebra of functions on X will be denoted by $\mathcal{O}(X)$. This algebra has a presentation $\alpha : k\langle T_1, \ldots, T_m\rangle \twoheadrightarrow \mathcal{O}(X)$ which induces a norm $|\ |_\alpha$ on $\mathcal{O}(X)$ (compare [BGR] 6.1.1). The subring of elements with norm ≤ 1 is denoted by $\mathcal{O}(X)^0_\alpha$. The subring of the elements with spectral (semi-)norm ≤ 1 ([BGR] 6.2.1.2) is denoted by $\mathcal{O}(X)^0$. One defines $\mathcal{O}(X)^{00} \subseteq \mathcal{O}(X)^0$ to consist of the elements f such that $\lim_{n\to\infty} |f^n|_\alpha = 0$. An $f \in \mathcal{O}(X)$ belongs to $\mathcal{O}(X)^{00}$ if and only if its spectral (semi-)norm is < 1 ([BGR] 6.2.3.2). The ring $\mathcal{O}(X)^0/\mathcal{O}(X)^{00}$ is a finitely generated reduced $\overline{k}$-algebra and is called the *canonical reduction* of $\mathcal{O}(X)$. The corresponding affine scheme over $\overline{k}$ is denoted by $\overline{X}$ and its set of closed points by $\overline{X}_{cl}$. The dimension of $\overline{X}$ is equal to the dimension of X.

The space X as a set consists of the maximal ideals in $\mathcal{O}(X)$. The finite unions of affinoid subdomains in X are called special subsets. We always give X the following Grothendieck topology:
(1) The admissible subsets of X are the special subsets.
(2) For a special subset U the family $Cov(U)$ consists of the coverings by special subsets which have a finite subcovering.
A rational set $R(f_0, \ldots, f_n)$ in X is given by elements $f_0, \ldots, f_n \in \mathcal{O}(X)$ generating the unit ideal and is defined to be

$$R(f_0, \ldots, f_n) := R_X(f_0, \ldots, f_n) := \{x \in X : |f_0(x)| \geq |f_i(x)| \text{ for all } 1 \leq i \leq n\}\,.$$

For any rational set $U \subseteq X$ the coverings of the form

$$U = \bigcup_{0 \leq i \leq n} R_U(f_i, f_0, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n)$$

are called *rational coverings*. The rational sets together with their rational coverings generate the Grothendieck topology of X ([BGR] 8.2.2.2). This has two simple consequences for abelian sheaves on X. Firstly a sheaf is determined by its sections in rational sets. Secondly let P be a presheaf on X and let F be its sheafification; if P satisfies the sheaf axiom for rational coverings then we have $P(U) = F(U)$ for any rational set U.

A *filter* f on X is a collection of special subsets of X satisfying:

(p1) $X \in f$ and $\emptyset \notin f$;

(p2) if $U_1, U_2 \in f$ then $U_1 \cap U_2 \in f$;

(p3) if $U \in f$ and the special subset V contains U then $V \in f$.

A *prime filter* p is a filter which in addition fulfills:

(p4) If $U \in p$ and $U = U_1 \cup U_2$with special subsets U_i then U_1 or $U_2 \in p$.

This is, of course, equivalent to the condition:

(p4)' If $U = \bigcup_{i\in I} U_i$ is an admissible covering of $U \in p$ then $U_{i_0} \in p$ for some $i_0 \in I$.

Let $\mathcal{P}(X)$ denote the set of all prime filters on X. The filters on X are ordered with respect to inclusion. Let $\mathcal{M}(X)$ be the set of maximal filters. We have $\mathcal{M}(X) \subseteq \mathcal{P}(X)$. This is a special case of the following basic argument which later on will be used several times.

Remark 1:

Let s be any family of special subsets of X which is closed with respect to finite unions. Assume that there is a filter f on X such that $f \cap s = \emptyset$. Then there is a filter p on X containing f which is maximal with respect to $p \cap s = \emptyset$ and any such filter is a prime filter.

Proof: The existence of p follows by Zorn's lemma. It remains to verify (p4) for p. Let U, U_1, U_2 be special subsets in X such that $U \in p$ and $U = U_1 \cup U_2$. If $U_1 \notin p$ then the collection p_1 of special subsets of X defined by $V \in p_1$ if V contains some $W \cap U_1$ with $W \in p$ still satisfies $X \in p_1$, (p2), and (p3). Since p_1 is larger than p we must have, by the maximality of p, that p_1 contains a member of $s \cup \{\emptyset\}$. Assume now that neither U_1 nor U_2 belong to p. Then there are $V_i \in s \cup \{\emptyset\}$ and $W_i \in p$ such that $W_i \cap U_i \subseteq V_i$. Hence $W := W_1 \cap W_2 \in p$ satisfies $W \cap U \subseteq V_1 \cup V_2 \in s \cup \{\emptyset\}$. This is a contradiction.

A *valuation* $(\mathfrak{p}, A)$ on X is a pair consisting of a prime ideal $\mathfrak{p} \subseteq \mathcal{O}(X)$ and a valuation ring A in the field of fractions of $\mathcal{O}(X)/\mathfrak{p}$ such that:

(v1) $\mathcal{O}(X)^0 + \mathfrak{p}/\mathfrak{p} \subseteq A$;

(v2) the intersection of all $\pi^n A$ is 0.

Usually we will denote by ϕ the residue class map from $\mathcal{O}(X)$ into the field of fractions K of $\mathcal{O}(X)/\mathfrak{p}$. Let $\mathrm{Val}(X)$ be the set of all valuations on X.
We note that one can replace (v1) by the weaker condition
(v1)' $\phi(\mathcal{O}(X)^0_\alpha) \subseteq A$.

Indeed, by Noether normalization we have a finite monomorphism $k\langle T_1, \ldots, T_d\rangle \hookrightarrow \mathcal{O}(X)$ such that the T_i are mapped into $\mathcal{O}(X)^0_\alpha$. An element $f \in \mathcal{O}(X)$ with spectral (semi-)norm ≤ 1 is integral over $k^0\langle T_1, \ldots, T_d\rangle$ ([BGR] 6.3.4.1). This last ring is mapped by ϕ into A. Hence $\phi(f)$ is integral over A and therefore lies in A.
Also note that (v2) means that the valuation topology on A is the π-adic topology. It is sometimes convenient to replace A by its completion $\hat{A}$, the projective limit of the $A/\pi^n A$. It follows from (v2) that A is a subring of $\hat{A}$. One easily verifies that $\hat{A}$ has no zero divisors and is in fact a valuation ring. The field

of fractions of $\hat{A}$ will be denoted by $\hat{K}$. Let $\mathfrak{m}_A$ be the maximal ideal in A. The map $A \to A/\mathfrak{m}_A$ extends to a map $\hat{A} \to A/\mathfrak{m}_A$ and therefore the maximal ideal $\mathfrak{m}_{\hat{A}}$ contains $\mathfrak{m}_A$. Suppose now that $a \in K \backslash A$. Then $a^{-1} \in \mathfrak{m}_A$. Hence $a^{-1} \in \mathfrak{m}_{\hat{A}}$ and $a \notin \hat{A}$. This shows that $\hat{A} \cap K = A$. More general, for any $a \in K$ one has $a\hat{A} \cap K = aA$.

The aim of this section is to find a natural bijection $\mathrm{Val}(X) \to \mathcal{P}(X)$. Let a valuation $(\mathfrak{p}, A)$ be given. Define a family $p = p(\mathfrak{p}, A)$ of special subsets of X as follows:

> U belongs to p if it contains a rational set $R(f_0, \ldots, f_n)$ such that $\phi(f_i) \in \phi(f_0)A$ for all $1 \leq i \leq n$.

Proposition 2:

i. If $R(f_0, \ldots, f_n) \subseteq R(g_0, \ldots, g_m)$ and $\phi(f_i) \in \phi(f_0)A$ for all i then $\phi(g_j) \in \phi(g_0)A$ for all j;
ii. p is a prime filter on X.

Proof: i. The homomorphism $\phi : \mathcal{O}(X)^0_\alpha \to A$ extends to the homomorphism

$$\begin{aligned} \phi_1 : \mathcal{O}(X)^0_\alpha[T_1, \ldots, T_n] &\longrightarrow A \\ T_i &\longmapsto \frac{\phi(f_i)}{\phi(f_0)} \quad . \end{aligned}$$

By taking limits ϕ_1 extends uniquely to a homomorphism $\phi_2 : \mathcal{O}(X)^0_\alpha\langle T_1, \ldots, T_n\rangle \to \hat{A}$. The corresponding homomorphism $\mathcal{O}(X)\langle T_1, \ldots, T_n\rangle \to \hat{K}$ factors over $\phi_3 : R := \mathcal{O}(X)\langle T_1, \ldots, T_n\rangle/(f_1 - f_0T_0, \ldots, f_n - f_0T_n) \to \hat{K}$. We have $R = \mathcal{O}(R(f_0, \ldots, f_n))$. Let us note here that since the image of ϕ_3 is not $\{0\}$ it follows that $R(f_0, \ldots, f_n)$ is not empty. By construction $\phi_3(R^0_\beta) \subseteq \hat{A}$ where $|\ |_\beta$ is the norm on R induced by the given presentations of R and $\mathcal{O}(X)$. As noted before, this implies that also $\phi_3(R^0) \subseteq \hat{A}$.
We consider now the restriction map $\mathcal{O}(R(g_0, \ldots, g_m)) \to \mathcal{O}(R(f_0, \ldots, f_n))$. The images of the elements $\frac{g_j}{g_0}$ have spectral (semi-)norm ≤ 1 in R. Hence

$$\frac{\phi(g_j)}{\phi(g_0)} = \phi_3\left(\frac{g_j}{g_0}\right) \in \hat{A} \cap K = A \quad .$$

ii. We have already seen that the empty set does not belong to p. Further $X = R(1, 0)$ obviously belongs to p. This proves (p1). The formula $R(f_0, \ldots, f_n) \cap R(g_0, \ldots, g_m) = R(\ldots, f_ig_j, \ldots)$ proves (p2). The condition (p3) holds by construction. For (p4) we first consider the case of a rational covering of X given by $g_0, \ldots, g_m \in \mathcal{O}(X)$ generating the unit ideal. The fractional ideal $\phi(g_0)A + \ldots + \phi(g_m)A$ in K is generated by some $\phi(g_{i_0})$. Hence $R(g_{i_0}, g_0, \ldots, g_m) \in p$.

In order to prove (p4) in general it suffices, by the definition of p, to consider a rational set $U = R(f_0, \ldots, f_m)$. Because p is already known to be a filter (p4) for U is a consequence of (p4) for X by the following fact: For any admissible covering of U there exists a rational covering of X whose restriction to U is finer than the given covering. This is well-known but for the convenience of the reader we include the argument.
First we refine the given covering into a rational covering

$$U = \bigcup_{0 \leq j \leq m} U_j \quad \text{where} \quad U_j = R_U(g_j, g_0, \ldots, g_{j-1}, g_{j+1}, \ldots g_m)$$

with $g_0, \ldots, g_m \in \mathcal{O}(U)$ generating the unit ideal. Since g_j is invertible on U_j we find an $\varepsilon > 0$ such that

$$|g_j(x)| > \varepsilon \quad \text{for any } x \in U_j \text{ and any } 0 \leq j \leq m \quad .$$

Now we choose elements $g'_0, \ldots, g'_m \in \mathcal{O}(X)[f_0^{-1}]$ such that

$$|g_j(x) - g'_j(x)| < \varepsilon \quad \text{for any } x \in U \text{ and any } 0 \leq j \leq m \quad .$$

Then the g'_j cannot have a common zero in U and therefore generate the unit ideal in $\mathcal{O}(U)$; also by construction we have

$$U_j = R_U(g'_j, g'_0, \ldots, g'_{j-1}, g'_{j+1}, \ldots, g'_m) \quad \text{for any } 0 \leq j \leq m \quad .$$

Moreover we may multiply all the g'_j by an appropriate power of f_0 without changing the U_j. This shows that from the start we may assume without changing the given rational covering that the $g_0, \ldots, g_m$ extend to functions on X. In this situation we choose a $\lambda \in k^\times$ such that $|\lambda| < \varepsilon$. The rational covering of X given by the functions $\lambda, g_0, \ldots, g_m$ then has the wanted property.

We next construct a map $\mathcal{P}(X) \to \mathrm{Val}(X)$. Let p be a prime filter on X. We put

$$\mathcal{O}_p := \varinjlim_{U \in p} \mathcal{O}(U) = \varinjlim \{\mathcal{O}(U) : U \in p \text{ rational}\} \quad \text{and}$$
$$\mathcal{O}^0_p := \varinjlim_{U \in p} \mathcal{O}^0(U) = \varinjlim \{\mathcal{O}^0(U) : U \in p \text{ rational}\} \quad .$$

Define

$$\|f\|_p := \inf \|f\|_U \quad \text{for } f \in \mathcal{O}_p \quad ;$$

here $\| \ \|_U$ denotes the spectral (semi-)norm on U and the infimum is taken over all $U \in p$ such that f is defined on U. Let

$$\mathfrak{m}_p := \{f \in \mathcal{O}_p : \|f\|_p = 0\} \quad .$$

Clearly $\mathfrak{m}_p$ is an ideal in both rings $\mathcal{O}_p$ and $\mathcal{O}^0_p$. Put

$$k_p := \mathcal{O}_p/\mathfrak{m}_p \quad \text{and} \quad k^0_p := \mathcal{O}^0_p/\mathfrak{m}_p \quad .$$

Proposition 3:

i. $\mathfrak{m}_p$ *is the unique maximal ideal of* $\mathcal{O}_p$*;*
ii. k_p^0 *is a valuation ring with field of fractions* k_p*;*
iii. let $\mathfrak{p}$ *be the kernel of the homomorphism* $\mathcal{O}(X) \to k_p$ *and let* A *be the preimage of* k_p^0 *in the field of fractions of* $\mathcal{O}(X)/\mathfrak{p}$*; then* $\mathrm{val}(p) := (\mathfrak{p}, A)$ *is a valuation on* X*.*

Proof: i. Let $f \in \mathcal{O}_p$ be defined on some $U \in p$. If for some $\varepsilon \in |k|$, $\varepsilon > 0$, the set $\{x \in U : |f(x)| \geq \varepsilon\}$ belongs to p then f is an invertible element of $\mathcal{O}_p$. If not then $\{x \in U : |f(x)| \leq \varepsilon\} \in p$ for all ε and so $f \in \mathfrak{m}_p$.
ii. We will not distinguish in notation between $f \in \mathcal{O}_p$ and its image in k_p. Suppose that $f \in k_p$ does not lie in k_p^0. Let f be defined on some $U \in p$. Then $\{x \in U : |f(x)| \leq 1\}$ does not belong to p. Hence the set $\{x \in U : |f(x)| \geq 1\} \in p$ and so $f^{-1} \in k_p^0$.
iii. The only non-trivial item to verify is (v2). But $\pi^n A \subseteq \pi^n k_p^0$. Let $f \in \mathcal{O}_p$ represent an element in the intersection of all $\pi^n k_p^0$. Then clearly $\|f\|_p = 0$ and the image of f in k_p^0 is 0.

Theorem 4:

The maps $p(.)$ *and* $\mathrm{val}(.)$ *between* $\mathrm{Val}(X)$ *and* $\mathcal{P}(X)$ *are each others inverses.*

Proof: Let $(\mathfrak{p}, A)$ be given and let $p := p(\mathfrak{p}, A)$. In the proof of Proposition 2 we have seen that, for any $U = R(f_0, \ldots, f_n) \in p$, there is a unique continuous extension $\phi_U : \mathcal{O}(U) \to \hat{K}$ of ϕ satisfying $\phi_U(\mathcal{O}(U)^0) \subseteq \hat{A}$. This induces a homomorphism $\phi_p : \mathcal{O}_p \to \hat{K}$ such that $\phi_p(\mathcal{O}_p^0) \subseteq \hat{A}$. For $f \in \mathfrak{m}_p$ one has

$$f \in \bigcap_{n \in \mathbb{N}} \pi^n \mathcal{O}_p^0 \quad \text{and so} \quad \phi_p(f) \in \bigcap_{n \in \mathbb{N}} \pi^n \hat{A} = \{0\} \quad .$$

Hence ϕ_p induces an injection $k_p \subseteq \hat{K}$ such that $k_p^0 \subseteq \hat{A}$. Let $f \in k_p \cap \hat{A}$. One can represent f by some element $g \in \mathcal{O}(U)$ with $U \in p$. The condition $\phi_U(g) \in \hat{A}$ implies that $\{x \in U : |g(x)| \leq 1\} \in p$ and therefore $f \in k_p^0$. Hence $k_p^0 = k_p \cap \hat{A}$. This implies $k_p^0 \cap K = A$ and we have proved that $\mathrm{val}(p) = (\mathfrak{p}, A)$.
Let a prime filter p be given and let $(\mathfrak{p}, A) := \mathrm{val}(p)$. The filter induces a homomorphism $\psi : \mathcal{O}(X) \to k_p$. It is easily seen that a rational subset $U = R(f_0, \ldots, f_n)$ is contained in p if and only if $\psi(f_i) \in \psi(f_0) k_p^0$ for all $1 \leq i \leq n$. Since $A = K \cap k_p^0$ it follows that $p(\mathfrak{p}, A) = p$.

Corollary 5:

Let p *be a prime filter on* X *and let* $(\mathfrak{p}, A)$ *be the corresponding valuation; we then have:*

i. $\mathcal{O}_p$ *is a Henselian local ring;*

ii. k_p^0 *is Henselian with respect to its prime ideal* $\sqrt{\pi k_p^0}$*;*

iii. $\hat{A} = \varprojlim k_p^0/\pi^n k_p^0$*; in general* $\hat{A} \neq k_p^0$*;*

iv. there is a chain of subfields $\hat{K} \supseteq M \supseteq L$ *such that*

— $\hat{K}$ *is finite over* M*,*

— $L = k(T_1, \ldots, T_d)$ *is purely transcendental over* k *with* $d \leq \dim(X)$*, and*

— L *is* π*-dense in* M*, i.e., for any* $m \in M$ *there is a* $\ell \in L$ *with* $\ell - m \in \pi\hat{A}$*;*

v. Krull $\dim(A) \leq \dim(X) + 1$.

Proof: i. First of all we observe that by Hensel's lemma $\mathcal{O}_p$ is Henselian if $(\mathcal{O}_p)_{red}$ is Henselian. Hence we may assume that X is reduced. According to one of the equivalent conditions for being Henselian we have to show that any finite free $\mathcal{O}_p$-algebra R is a product of local rings. In other words we have to show that any idempotent $\overline{e}$ in $R \underset{\mathcal{O}_p}{\otimes} k_p$ lifts to an idempotent in R. We clearly find an $U \in p$ and a finite free $\mathcal{O}(U)$-algebra R' such that $R = R' \underset{\mathcal{O}(U)}{\otimes} \mathcal{O}_p$. Since X is reduced $\| \ \|_U$ is a complete norm on $\mathcal{O}(U)$ ([BGR] 6.2.4.1). According to [BGR] 6.1.1.6 and 6.1.3.3 there is a Banach algebra norm $\| \ \|$ on R' such that $(R', \| \ \|)$ is a normed $(\mathcal{O}(U), \| \ \|_U)$-algebra. After replacing U by a smaller set in p we may assume that $\overline{e}$ lifts to an element $e_0 \in R'$ such that $\|e_0^2 - e_0\| < 1$. Define a sequence $(e_n)_{n \geq 0}$ of elements in R' by

$$e_{n+1} := 3e_n^2 - 2e_n^3 \quad .$$

By construction one has

$$\begin{aligned} e_{n+1} - e_n &= (e_n^2 - e_n)(1 - 2e_n) \quad \text{and} \\ e_{n+1}^2 - e_{n+1} &= 4(e_n^2 - e_n)^3 - 3(e_n^2 - e_n)^2 \quad . \end{aligned}$$

It follows that the sequence $(e_n)_{n \geq 0}$ converges to an idempotent $e \in R'$ which lifts $\overline{e}$.

ii. This is proved in a similar way as the first assertion.

iii. The first statement is easily verified. An example for the inequality is provided by $X := Sp(k\langle T \rangle)$ and the valuation ring $A := \{\frac{f}{g} : f, g \in k\langle T \rangle, g \neq 0,$ and $\|f\|_X \leq \|g\|_X\}$.

iv. After dividing $\mathcal{O}(X)$ by a prime ideal we may suppose that ϕ is injective. The field of fractions K of $\mathcal{O}(X)$ is π-dense in $\hat{K}$. The algebra $\mathcal{O}(X)$ is finite over some $R := k\langle T_1, \ldots, T_d \rangle$ with d equal to the dimension of X. Let $M \subseteq \hat{K}$ denote the π-completion (in the obvious sense) of the field of fractions of R.

Then $\hat{K}$ is finite over M and M has $k(T_1, \ldots, T_d)$ as π-dense subfield.
v. In the valuation ring A the smallest non-zero prime ideal is $I := \sqrt{\pi A}$. Indeed, let $J \subset I$ be a prime ideal which is smaller than I. Choose $a \in J$. Let $n \in \mathbb{N}$ and suppose that $aA + \pi^n A = aA$. Then $\pi^n \in aA$ and $\pi \in J$ and one finds the contradiction $I \subseteq J$. Hence $aA + \pi^n A = \pi^n A$ for all $n \in \mathbb{N}$, hence $a \in \bigcap\limits_{n \in \mathbb{N}} \pi^n A = \{0\}$, and hence $J = 0$.
We will show that Krull $\dim(A/I) \leq \dim(X)$. The map $\phi : \mathcal{O}(X) \to K$ has the properties $\phi(\mathcal{O}(X)^0) \subseteq A$ and $\phi(\mathcal{O}(X)^{00}) \subseteq I$. The kernel of the induced homomorphism $\mathcal{O}(\overline{X}) \to A_1 := A/I$ is a prime ideal corresponding to a closed irreducible (and reduced) subset $Y \subseteq \overline{X}$. It can be seen that the induced injective map $\mathcal{O}(Y) \to A_1$ gives a bijection between the fields of fractions of the two rings. In other terms: A_1 is a valuation ring of the field of fractions of $\mathcal{O}(Y)$ containing $\mathcal{O}(Y)$. It is well known that the Krull dimension of such a valuation ring is $\leq$ the dimension of Y.

An ordinary point $x \in X$ can be identified with the prime filter $\{U : x \in U\}$. The valuation corresponding to x is given by the homomorphism $\mathcal{O}(X) \to \mathcal{O}(X)/\mathfrak{m}_x$ where $\mathfrak{m}_x$ denotes the maximal ideal corresponding to x. In this way X can be embedded in $\mathcal{P}(X)$ and $\mathrm{Val}(X)$, respectively.

A valuation $(\mathfrak{p}, A)$ on X is called an *analytic point* of X if the valuation ring A has rank 1. Recall that a valuation ring has rank 1 if and only if its Krull dimension is 1 ([M] 10.7). Obviously any ordinary point gives rise to an analytic point. Let p_1 and p_2 be two prime filters on X with corresponding valuations $(\mathfrak{p}_i, A_i)$. Then $p_1 \subseteq p_2$ if and only if $\mathfrak{p}_1 = \mathfrak{p}_2$ and A_2 is a localization of A_1 with respect to some prime ideal of A_1. This follows easily from Theorem 4. In particular the prime filter p is maximal if and only if the corresponding valuation ring A has rank 1. In other words: The subset of analytic points corresponds to the subset $\mathcal{M}(X)$ of maximal filters.
The analytic points of X can also be described in the following way. An analytic point a of X is a semi-norm $|\ |_a : \mathcal{O}(X) \to \mathbb{R}_{\geq 0}$ on the affinoid algebra $\mathcal{O}(X)$ of X satisfying:

(a1) $|f + g|_a \leq \max(|f|_a, |g|_a)$ for all $f, g \in \mathcal{O}(X)$;

(a2) $|fg|_a = |f|_a |g|_a$ for all $f, g \in \mathcal{O}(X)$;

(a3) $|\lambda|_a = |\lambda|$ for any $\lambda \in k$;

(a4) $|\ |_a : \mathcal{O}(X) \to \mathbb{R}_{\geq 0}$ is continuous with respect to the norm topology on $\mathcal{O}(X)$.

Still another characterization of analytic points will be given in Lemma 6.

As an example we give an explicit description of the prime filters on the disk $D = Sp(k\langle T \rangle)$. For convenience we suppose that k is algebraically closed. For any $d \in D$ and any $\rho \in |k^\times|$ with $0 < \rho \leq 1$ we write $D(d, \rho) := \{x \in D :$

$|x-d|\leq\rho\}$. This is a closed disk. We will write $D(d,\rho)^*$ for any subset of D of the form

$$\{x\in D(d,\rho):|x-a_i|=\rho \text{ for } i=1,\ldots,m\} \text{ with } a_1,\ldots,a_m\in D(d,\rho)\ .$$

This is a formal open subset of $D(d,\rho)$. We start by describing the analytic points of D (compare also [B] 1.4.4).
Let $|\ |_a$ be an analytic point. For every $d\in D$ one defines the real number $\rho(d):=|T-d|_a$. Then $D(d,\rho)$ belongs to a if and only if $\rho(d)\leq\rho$. This follows from the definition of the filter attached to $|\ |_a$.
For convenience we extend the notation, namely $D(d,\rho):=\{x\in D:|x-d|\leq\rho\}$ for any $\rho\in\mathbb{R}$, $0\leq\rho\leq 1$.
For two points $d_1,d_2\in D$ one has

$$D(d_1,\rho(d_1))\subseteq D(d_2,\rho(d_2)) \text{ or } D(d_2,\rho(d_2))\subseteq D(d_1,\rho(d_1))$$

because $|d_1-d_2|=|(T-d_1)-(T-d_2)|_a\leq\max(|T-d_1|_a,|T-d_2|_a)$. Now there are several possibilities: If $\cap\{D(d,\rho(d)):d\in D\}$ is
(1) an (ordinary) point $x\in D$ then a is the filter of all special subsets containing x;
(2) equal to $D(d_0,\rho_0)$ with $\rho_0\in|k^\times|$ then a consists of the special subsets containing some $D(d_0,\rho_0)^*$; further $|\ |_a$ is the spectral norm (or supremum norm) on $D(d_0,\rho_0)$;
(3) equal to $D(d_0,\rho_0)$ with $\rho_0\notin|k^\times|$ then $|\ |_a$ still is the supremum norm on $D(d_0,\rho_0)$; further $D(d,\rho)^*$ belongs to a if and only if $D(d,\rho)^*\supseteq\{x\in D:\rho_1\leq|x-d_0|\leq\rho_2\}$ with $\rho_1<\rho_0<\rho_2$ and $\rho_1,\rho_2\in|k^\times|$;
(4) empty then $|\ |_a$ is the infimum of the supremum norms taken over the sets $D(d,\rho(d))$; a special subset belongs to a if and only if it contains some $D(d,\rho(d))$; this case can only occur if the field k is not maximally complete.
The description of the prime filters p which are not maximal is more complicated. We will use the results and notations of the subsequent paragraph for this. Let $a:=r(p)$. For an a of the type (1), (3), or (4) above one sees that $\tilde{a}=a$; hence $p=a$. For a of type (2), we take for notational convenience $D=(d_0,\rho_0)$. For any $\overline{\lambda}\in\overline{k}$ one chooses a $\lambda\in k^0$ with residue $\overline{\lambda}$. Define $p_{\overline{\lambda}}$ to be the family of special subsets of D containing for some $\rho\in|k^\times|$, $\rho<1$, the ring domain $\{x\in D:\rho<|x-\lambda|<1\}$. This is in fact a prime filter and $r(p_{\overline{\lambda}})=a$. Using Lemma 6 one can show that any $p\neq a$ with $r(p)=a$ is equal to $p_{\overline{\lambda}}$ for a unique $\overline{\lambda}\in\overline{k}$. This finishes the description of all prime filters on D.

Coming back to the general situation we finally describe a natural retraction map

$$r:\mathcal{P}(X)\longrightarrow\mathcal{M}(X)\ .$$

Let p be any prime filter on X. The map $f\mapsto\|f\|_p$ satisfies (a1) — (a4), as is easily seen. We denote by $r(p)$ the analytic point of X (resp. the maximal filter) which corresponds to $\|\ \|_p$. One has that $R(f_1,\ldots,f_n)$ belongs to $r(p)$

if and only if $\|f_0\|_p \geq \|f_i\|_p$ for all $1 \leq i \leq n$. Moreover, this is equivalent to $R(\rho f_0, f_1, \ldots, f_n) \in p$ for every $\rho \in \sqrt{|k^\times|}, \rho > 1$. The notation which we use here and in the sequel is

$$R(\rho f_0, f_1, \ldots, f_n) := \{x \in X : \rho|f_0(x)| \geq |f_i(x)| \text{ for all } 1 \leq i \leq n\};$$

this is a rational subset ([BGR] 7.2.3).

Lemma 6:

i. A prime filter p is maximal if and only if it has the property that $R(f_0, \ldots, f_n) \in p$ if $R(\rho f_0, f_1, \ldots, f_n) \in p$ for every $\rho \in \sqrt{|k^\times|}$, $\rho > 1$;
ii. for a prime filter p the filter $r(p)$ is the unique maximal filter containing p;
iii. if A is the valuation ring corresponding to the prime filter p then the valuation ring corresponding to $r(p)$ is the localization of A at the prime ideal $\sqrt{\pi A}$.

Proof: Obvious.

We will need some further properties of this retraction map.

Lemma 7:

Let a be an analytic point of X with corresponding valuation $(\mathfrak{p}, A)$; we have:
i. define $\tilde{a}$ to be the family of all special subsets of X which contain $R(\rho f_0, f_1, \ldots, f_n)$ for some $R(f_0, \ldots, f_n) \in a$ and some $\rho \in \sqrt{|k^\times|}$, $\rho > 1$; then $\tilde{a}$ is equal to the intersection of all prime filters p with $r(p) = a$;
ii. the set $r^{-1}(\{a\})$ can be identified with the set of all valuation rings in the residue field $A/\mathfrak{m}_A$ of A which contain the image of the map $\mathcal{O}(X)^0 \to \mathcal{O}(\overline{X}) \to A/\mathfrak{m}_A$.

Proof: i. We know already that the prime filters p containing $\tilde{a}$ are precisely those with $r(p) = a$. In particular $\tilde{a}$ is contained in the intersection of all those prime filters. In order to see the reverse inclusion observe first that $\tilde{a}$ is a filter. If $U \notin \tilde{a}$ is any special subset then applying Remark 1 to the family $s := \{U\}$ we obtain a prime filter containing $\tilde{a}$ but not U. ii. This follows from Lemma 6.iii and [M] 10.1.

3. Topologies on $\mathcal{P}(X)$ and $\mathcal{M}(X)$

Let Z be a scheme of finite type over a field F. We denote by Z_{cl} the variety over F consisting of the closed points of Z. A prime filter on Z_{cl} is a collection of Zariski open subsets of Z_{cl} having the properties (p1) — (p4) as in section

2. Let $\mathcal{P}(Z_{cl})$ be the set of all those prime filters. Every point $z \in Z$ defines an irreducible closed subset $\overline{\{z\}}$ and a prime filter p_z consisting of the open subsets $U \subseteq Z_{cl}$ such that $U \cap \overline{\{z\}} \neq \emptyset$. We want to prove the following amusing fact.

Lemma 1:

The map $Z \to \mathcal{P}(Z_{cl})$ given by $z \mapsto p_z$ is a bijection.

Proof: The injectivity of the map is clear. For the surjectivity it suffices to consider an affine scheme Z. Let p be a prime filter on Z_{cl} and define

$$\mathcal{O}_p := \varinjlim_{U \in p} \mathcal{O}(U) \quad \text{and} \quad \mathfrak{m}_p := \varinjlim_{U \in p} \{f \in \mathcal{O}(U) : \{y \in U : f(y) \neq 0\} \notin p\} \quad .$$

Here $\mathcal{O}$ denotes the structure sheaf of Z as well as its inverse image on Z_{cl}. Clearly $\mathfrak{m}_p$ is an ideal in $\mathcal{O}_p$. If $f \in \mathcal{O}_p \backslash \mathfrak{m}_p$ is defined on $U \in p$ then the set $\{y \in U : f(y) \neq 0\}$ belongs to p and so f is invertible in $\mathcal{O}_p$. Hence $\mathcal{O}_p$ is a local ring. The kernel of $\phi : \mathcal{O}(Z) \to \mathcal{O}_p \to \mathcal{O}_p/\mathfrak{m}_p$ is some prime ideal I of $\mathcal{O}(Z)$. For $f \in \mathcal{O}(Z)$ put $Z_f := \{y \in Z : f(y) \neq 0\}$ which is Zariski open in Z; then $Z_f \cap Z_{cl} \in p$ if and only if $f \notin I$. Since the Z_f form a basis of the Zariski topology it follows from (p4) that $\mathcal{O}_p$ is the direct limit of all $\mathcal{O}(Z_f)$ with $f \notin I$. Therefore $\mathcal{O}_p$ is the localization of $\mathcal{O}(Z)$ with respect to the prime ideal I. The prime ideal I is a point z of Z and $p = p_z$.

On $\mathcal{P}(Z_{cl})$ we put the topology induced by its bijection with Z. One can verify that the open sets in $\mathcal{P}(Z_{cl})$ are the sets $\{p \in \mathcal{P}(Z_{cl}) : U \in p\}$ where $U \subseteq Z_{cl}$ is a Zariski open set.

We return now to the affinoid space X over k and its set of prime filters $\mathcal{P}(X)$. On this set we define a topology by taking as a basis for the open sets the sets $\tilde{U} := \{p \in \mathcal{P}(X) : U \in p\}$ where $U \subseteq X$ is a special subset. If U is an affinoid subdomain of X then the obvious bijection $\mathcal{P}(U) \xrightarrow{\sim} \tilde{U}$ is a homeomorphism. The next fact is a central result in [H].

Lemma 2:

i. The space $\mathcal{P}(X)$ is quasi-compact;
ii. for any analytic point $a \in \mathcal{M}(X) \subseteq \mathcal{P}(X)$ the closure of $\{a\}$ is the set $r^{-1}(\{a\})$, where r is the retraction map.

Proof: i. Let $\{\tilde{U}_j\}_{j \in J}$ be a covering of $\mathcal{P}(X)$. Assume that this covering has no finite subcovering. By (p4) this means that X does not belong to the family s of all finite unions of sets U_j. Applying Remark 2.1 to the filter $\{X\}$ we obtain a prime filter p such that $p \cap s = \emptyset$. On the other hand we have $p \in \tilde{U}_{j_0}$ for some $j_0 \in J$ and hence $U_{j_0} \in p \cap s$. This is a contradiction. ii. Let a prime filter p be given. By the definition of the topology we have

$$\mathcal{P}(X) \backslash \overline{\{p\}} = \{p' : p' \text{ contains some special subset } U \notin p\}$$

and hence $\overline{\{p\}} = \{p' : p' \subseteq p\}$. The assertion follows now from Lemma 2.6.ii.

The maximal ideal corresponding to a point in X is the kernel of a surjective k-algebra homomorphism $\psi : \mathcal{O}(X) \to \ell$ where ℓ is a finite extension of k. There is an induced $\overline{k}$-algebra homomorphism $\overline{\psi} : \mathcal{O}(\overline{X}) \to \overline{\ell}$. The kernel of $\overline{\psi}$ is a maximal ideal of $\mathcal{O}(\overline{X})$. This defines a map

$$red : X \longrightarrow \overline{X}_{cl}$$

which is called the *canonical reduction map.* It is surjective and for every Zariski open $V \subseteq \overline{X}_{cl}$ the preimage $red^{-1}V$ is a special subset in X ([BGR] 7.1.5.2 and 4).

Lemma 3:

There is a natural surjective continuous map $red : \mathcal{P}(X) \to \mathcal{P}(\overline{X}_{cl}) \cong \overline{X}$.

Proof: For $p \in \mathcal{P}(X)$ define

$$red(p) := \{V \subseteq \overline{X}_{cl} \text{ open } : red^{-1}V \in p\} \quad .$$

It is clear that $red(p)$ is a prime filter on $\overline{X}_{cl}$. The continuity of red follows from the definition of the topologies. Let a prime filter q on $\overline{X}_{cl}$ be given. The family

$$f := \text{ all special subsets } U \subseteq X \text{ such that } U \supseteq red^{-1}V \text{ for some } V \in q$$

is a filter on X. The family

$$\begin{aligned} s := \text{all special subsets } U \subseteq X \text{ such that } U \subseteq red^{-1}W \\ \text{for some open } W \subseteq \overline{X}_{cl} \text{ with } W \notin q \end{aligned}$$

is closed with respect to finite unions and fulfills $f \cap s = \emptyset$. Applying the Remark 2.1 we obtain a prime filter p on X such that $p \supseteq f$ and $p \cap s = \emptyset$. The former property implies that $q \subseteq red(p)$ and the latter that $red(p) \subseteq q$.

These considerations can be generalized in the following way. Consider the rational covering $\{U_0, \ldots, U_n\}$ of X given by elements $f_0, \ldots, f_n \in \mathcal{O}(X)$ generating the unit ideal. For every i, j one has that $\overline{U_i \cap U_j}$ is an open affine subscheme of $\overline{U_i}$ ([BGR] 7.2.6.3). The affine schemes $\overline{U_i}$ are glued together over these open subschemes. The result is a reduced scheme of finite type over $\overline{k}$ which we denote by $\overline{(X, f.)}$. The canonical reduction maps $U_i \to (\overline{U_i})_{cl}$ glue to a map

$$red(f.) : X \longrightarrow \overline{(X, f.)}_{cl}$$

which is called the reduction map of the rational covering given by $f. = \{f_0, \ldots, f_n\}$. Again it is surjective and for every open $V \subseteq \overline{(X, f.)}_{cl}$ the preimage $red(f.)^{-1}V$ is a special subset in X. As in Lemma 3 one can prove that this map extends naturally to a surjective continuous map

$$red(f.) : \mathcal{P}(X) \longrightarrow \mathcal{P}(\overline{(X, f.)}_{cl}) \cong \overline{(X, f.)} \quad .$$

This leads to a continuous map

$$\mathcal{P}(X) \longrightarrow \varprojlim \overline{(X, f.)}$$

in which the last space is provided with the topology of the projective limit.

Theorem 4:

$\mathcal{P}(X) \xrightarrow{\sim} \varprojlim \overline{(X, f.)}$ *is a (bijective) homeomorphism.*

Proof: First note that for any rational set $U = R(g_0, \ldots, g_m)$ in X the subset $\overline{U}_{cl}$ in $\overline{(X, g.)}_{cl}$ is open and $U = red(g.)^{-1}\overline{U}_{cl}$. This immediately implies that the map in question is injective. But it also shows that for any $\{p(f.)\}_{f.}$ in the projective limit on the right hand side the filter

$$\begin{aligned} p := &\text{ all special subsets } U \subseteq X \text{ such that } U \supseteq red(f.)^{-1}V \\ &\text{ for some } f. \text{ and some } V \in p(f.) \end{aligned}$$

on X is a prime filter. One easily checks that p is a preimage of $\{p(f.)\}_{f.}$. Hence our map is bijective. In order to see that the map is open let $U = R(g_0, \ldots, g_m)$ be again a rational set in X. The image of $\tilde{U} \subseteq \mathcal{P}(X)$ consists of the elements $\{p(f.)\}_{f.}$ in the projective limit such that $p(g.)$ contains the open set $\overline{U}_{cl} \subseteq \overline{(X, g.)}_{cl}$.

Assume the valuation of k to be discrete and let π generate the maximal ideal of k^0. The Raynaud functor associates to every formal scheme $\mathcal{X}$ of finite type and flat over k^0 a rigid analytic space $X := \mathcal{X} \otimes k$ called the "generic fiber" of $\mathcal{X}$. If the formal scheme is affine, i.e., $\mathcal{X}$ is the formal spectrum of a flat k^0-algebra of the type $R = k^0\langle T_1, \ldots, T_n\rangle/I$ then X is the affinoid space with algebra $\mathcal{O}(X) = R \underset{k^0}{\otimes} k$. One finds a reduction map $red : X \to \overline{\mathcal{X}}_{cl}$ where $\overline{\mathcal{X}} := \mathrm{Spec}(R \underset{k^0}{\otimes} \overline{k})$. In the general case one has a similar reduction map where $\overline{\mathcal{X}} := \mathcal{X} \underset{k^0}{\otimes} \overline{k}$ is the special fiber of $\mathcal{X}$. Blowing ups of $\mathcal{X}$ with respect to an ideal with support in the special fiber do not change X but change the reduction map.

Let X be a reduced affinoid space over k. The ring $\mathcal{O}(X)^0$ is noetherian, π-adically complete and $\mathcal{O}(X)^0/(\pi)$ is of finite type over $\overline{k}$ ([BGR] 6.4.1.6). This makes $\mathcal{X} := Spf(\mathcal{O}(X)^0)$ into a formal scheme of finite type and flat over k^0 with generic fiber X. The reduction map corresponding to $\mathcal{X}$ is what we have called the canonical reduction map. For any rational covering $\{U_1, \ldots, U_n\}$ of X given by elements $f_0, \ldots, f_n$ the glueing of the $Spf(\mathcal{O}(U_i)^0)$ gives a formal scheme $\mathcal{X}(f.)$ of finite type and flat over k^0 with generic fiber X and with reduction as explained before Theorem 4. This new formal scheme can be obtained from $\mathcal{X}$ by blowing up some ideal supported in the special fiber. One can show that the family $\mathcal{X}(f.)$ is cofinal in the collection of all formal schemes $\mathcal{Y}$ of the type above with X as generic fiber. This is the interpretation of Theorem 4 in terms of formal schemes.

For more general fields this interpretation remains valid but the formulation is more complicated due to the fact that the rings $k^0\langle T_1, \ldots, T_n\rangle$ are not noetherian.

Definition:

The Berkovich topology on $\mathcal{M}(X)$ is the coarsest topology such that, for every $f \in \mathcal{O}(X)$, the function $a \mapsto |f|_a$ on $\mathcal{M}(X)$ is continuous.

The description of analytic points in terms of semi-norms on $\mathcal{O}(X)$ shows that the functions in the above Definition separate points in $\mathcal{M}(X)$. Hence the Berkovich topology is Hausdorff. In the following $\mathcal{M}(X)$ always is equipped with the Berkovich topology.

Theorem 5:

The Berkovich topology on $\mathcal{M}(X)$ coincides with the quotient topology derived from the topology on $\mathcal{P}(X)$ and the retraction map $r : \mathcal{P}(X) \to \mathcal{M}(X)$; $\mathcal{M}(X)$ is compact.

Proof: We temporarily write $\mathcal{M}(X)_B$ for $\mathcal{M}(X)$ equipped with the Berkovich topology. It suffices to show that the map $r : \mathcal{P}(X) \to \mathcal{M}(X)_B$ is continuous. Since the first space is quasi-compact and hence the second one is compact the map then must be a quotient map. It is easy to see that the open subsets

$$M(f_0, \ldots, f_n) := \{a \in \mathcal{M}(X) : |f_0|_a > |f_i|_a \text{ for all } 1 \leq i \leq n\}$$

for any $f_0, \ldots, f_n \in \mathcal{O}(X)$ generating the unit ideal form a basis of the topology of $\mathcal{M}(X)_B$. Let p be prime filter on X. We have $r(p) \in M(f_0, \ldots, f_n)$ if and only if $R(\rho f_0, f_1, \ldots, f_n) \in r(p)$ for some $\rho \in \sqrt{|k^\times|}$, $\rho < 1$, which by Lemma 2.7.i is equivalent to $R(\rho' f_0, f_1, \ldots, f_n) \in p$ for all $\rho' > \rho$. It follows that

$$(*) \qquad r^{-1}M(f_0, \ldots, f_n) = \cup\{R(\rho f_0, f_1, \ldots, f_n)^{\sim} : \rho \in \sqrt{|k^\times|} \text{ and } \rho < 1\} \quad .$$

The right hand side is open in $\mathcal{P}(X)$.

The second statement in the above Theorem is due to Berkovich ([B]). But our proof is completely different.

4. Sheaves on X, $\mathcal{M}(X)$, and $\mathcal{P}(X)$

The inclusion $X \subseteq \mathcal{P}(X)$ is not very useful for comparing sheaves on the two spaces. But viewing X with its Grothendieck topology as well as the topological space $\mathcal{P}(X)$ as sites the functor $U \mapsto \tilde{U}$ defines a morphism of sites

$$\sigma : \mathcal{P}(X) \longrightarrow X \quad .$$

The following result was proved in [H].

Theorem 1:

The functors σ_* and σ^* are quasi-inverse equivalences between the categories of abelian sheaves on $\mathcal{P}(X)$ and on X.

Proof: Let F be an abelian sheaf on X. Then $\sigma^* F$ is the sheaf associated with the presheaf P given by

$$P(N) = \varinjlim_{N \subseteq \tilde{U}} F(U) \quad \text{for any open } N \subseteq \mathcal{P}(X) \quad .$$

Consider the case where N is of the form $N = \tilde{V}$ for some special subset $V \subseteq X$. A special subset U such that $\tilde{V} \subseteq \tilde{U}$ satisfies $V \subseteq U$. Indeed, suppose there is a point $x \in V \backslash U$. Its neighbourhood filter lies in $\tilde{V}$ but does not contain U which is a contradiction. It follows that $P(\tilde{V}) = F(V)$. In particular P already satisfies the sheaf axiom for coverings consisting of open sets of the form $\tilde{V}$. Since the latter form a basis of the topology of $\mathcal{P}(X)$ sheafification leaves the sections in such a set $\tilde{V}$ unchanged: We have

$$(\sigma_* \sigma^* F)(V) = (\sigma^* F)(\tilde{V}) = P(\tilde{V}) = F(V) \ \text{ and hence } \sigma_* \sigma^* F = F \quad .$$

Also for an abelian sheaf S on $\mathcal{P}(X)$ we obtain

$$(\sigma^* \sigma_* S)(\tilde{V}) = (\sigma_* S)(V) = S(\tilde{V}) \ \text{ and hence } \sigma^* \sigma_* S = S \quad .$$

This proof in particular shows that for any abelian sheaf F on X and any special subset $U \subseteq X$ we have

$$(\sigma^* F)(\tilde{U}) = F(U) \quad .$$

It follows that

$$F_p := \varinjlim_{U \in p} F(U) = (\sigma^* F)_p \ \text{ for any } p \in \mathcal{P}(X)$$

where the right hand side is the stalk in the usual sense of the sheaf $\sigma^* F$ in the point p. As a consequence we obtain that the functors $F \mapsto F_p$ are exact and that $F = 0$ if all $F_p = 0$.

An abelian (pre)sheaf F on X is called *overconvergent* if, for all $f_0, \ldots, f_n \in \mathcal{O}(X)$ generating the unit ideal, we have

$$F(R(f_0, \ldots, f_n)) = \varinjlim F(R(\rho f_0, f_1, \ldots, f_n))$$

where the limit is taken over all $\rho \in \sqrt{|k^\times|}$, $\rho > 1$. One can verify that this is the same notion as that of a conservative sheaf in [S] as well as that of a constructible sheaf in [P]. We note that the sheaf associated to an overconvergent presheaf is also overconvergent. A sheaf on $\mathcal{P}(X)$ will be called overconvergent if the corresponding sheaf on X is overconvergent. The following theorem is one of the main results in [S]. The proof given here is however more direct.

Theorem 2:

The retraction map $r : \mathcal{P}(X) \to \mathcal{M}(X)$ gives rise to quasi-inverse equivalences r_* and r^* between the category of overconvergent sheaves on $\mathcal{P}(X)$ (or on X) and the category of all abelian sheaves on $\mathcal{M}(X)$.

Proof: (1) First we show that for any abelian sheaf T on $\mathcal{M}(X)$ the sheaf r^*T is overconvergent. Since sheafification preserves the property of being overconvergent it suffices to check that the presheaf inverse image of T is overconvergent. This is the presheaf P on $\mathcal{P}(X)$ given by

$$P(N) = \varinjlim_{N \subseteq r^{-1}M} T(M) \quad .$$

Let $U = R(f_0, \ldots, f_n)$ be a rational subset of X. Then

$$r(\tilde{U}) = \{a \in \mathcal{M}(X) : U \in a\} = \{a \in \mathcal{M}(X) : |f_0|_a \geq |f_1|_a, \ldots, |f_n|_a\}$$

is a closed subset in $\mathcal{M}(X)$. A fundamental system of open neighbourhoods of $r(\tilde{U})$ is given by $\{M(\varepsilon) : \varepsilon > 0\}$ with

$$M(\varepsilon) := \{a \in \mathcal{M}(X) : (1+\varepsilon)|f_0|_a > |f_1|_a, \ldots, |f_n|_a\} \quad .$$

We have $P(\tilde{U}) = \varinjlim T(M(\varepsilon))$ and this implies that

$$P(\tilde{U}) = \varinjlim P(R(\rho f_0, f_1, \ldots, f_n)^{\sim})$$

where the direct limit is taken over all $\rho \in \sqrt{|k^\times|}$, $\rho > 1$. The presheaf P therefore is overconvergent.
(2) Next we show that for any overconvergent sheaf S on $\mathcal{P}(X)$ and any analytic point a of X the natural map

$$(r_*S)_a \xrightarrow{\cong} S_a$$

is bijective. By the construction of the topology S_a is the direct limit of all $S(\tilde{U})$ with $U = R(f_0, \ldots, f_n) \in a$. Since S is overconvergent S_a is also the direct limit of all $S(\tilde{U})$ where $U = R(\rho f_0, f_1, \ldots, f_n)$ with $R(f_0, \ldots, f_n) \in a$ and $\rho \in \sqrt{|k^\times|}$, $\rho > 1$. Our claim follows since the open subset

$$M := \{b \in \mathcal{M}(X) : \rho|f_0|_b > |f_1|_b, \ldots, |f_n|_b\}$$

in $\mathcal{M}(X)$ satisfies $r^{-1}M \subseteq R(\rho f_0, f_1, \ldots, f_n)^{\sim}$ by the formula $(*)$ in the proof of Theorem 3.5.
(3) A homomorphism of overconvergent sheaves $S \to S'$ is an isomorphism if for any analytic point $a \in \mathcal{P}(X)$ the homomorphism $S_a \to S'_a$ is bijective. Indeed, let p be any prime filter contained in the maximal filter a. Then the natural map $S_p \to S_a$ is an isomorphism for any overconvergent sheaf S.
(4) For any abelian sheaf T on $\mathcal{M}(X)$ we have $(r_*r^*T)_a = (r^*T)_a$ by (1) and (2) and $(r^*T)_a = T_a$ since $r(a) = a$. Hence $r_*r^*T = T$.
For any overconvergent sheaf S on $\mathcal{P}(X)$ and any analytic point $a \in \mathcal{P}(X)$ we have $(r^*r_*S)_a = (r_*S)_a = S_a$ according to (2). Because of (1) and (3) this proves that $r^*r_*S = S$.

Corollary 3:

Let S be a sheaf on $\mathcal{P}(X)$ and a an analytic point of X; we have

$$(r^* r_* S)_a = (r_* S)_a = S_{\tilde{a}} := \varinjlim_{U \in \tilde{a}} S(\tilde{U}) \quad .$$

Proof: ($\tilde{a}$ was defined in Lemma 2.7.) Consider the two families

$$\{\tilde{U} : U \in \tilde{a}\}$$

and

$$\{r^{-1}M(f_0, \ldots, f_n) : f_0, \ldots, f_n \in \mathcal{O}(X) \text{ generating the unit ideal} \\ \text{such that } a \in M(f_0, \ldots, f_n)\}$$

of open subsets in $\mathcal{P}(X)$. It is an immediate consequence of the formula $(*)$ in the proof of Theorem 3.5 that every member of the second family contains a member of the first family. The same formula $(*)$ also shows, as noted already in the previous proof, the following: If $R(f_0, \ldots, f_n) \in a$ so that $R(\rho f_0, f_1, \ldots, f_n) \in \tilde{a}$ for any $\rho \in \sqrt{|k^\times|}$, $\rho > 1$, then

$$r^{-1}\{b \in \mathcal{M}(X) : \rho |f_0|_b > |f_i|_b \text{ for all } 1 \leq i \leq n\} \subseteq R(\rho f_0, f_1, \ldots, f_n)^\sim \quad .$$

Hence any set in the first family contains a set in the second family.

5. General rigid spaces

In this section X is an arbitrary rigid space over k. The definition of the sets $\mathcal{M}(X) \subseteq \mathcal{P}(X)$ and $\mathrm{Val}(X)$ together with the natural bijection between $\mathrm{Val}(X)$ and $\mathcal{P}(X)$ generalizes in a straightforward way. Of course filters on X now have to be formed among all admissible open subsets of X and prime filters have to be defined by the condition (p4)'. For any affinoid open subset $U \subseteq X$ there are obvious bijections

$$\mathcal{P}(U) \xrightarrow{\sim} \tilde{U} := \{p \in \mathcal{P}(X) : U \in p\}$$

and

$$\mathcal{M}(U) \xrightarrow{\sim} \underline{U} := \{a \in \mathcal{M}(X) : U \in a\} \quad .$$

The retraction map

$$\begin{array}{rcl} r_X : \mathcal{P}(X) & \longrightarrow & \mathcal{M}(X) \\ p & \longmapsto & \text{unique maximal filter containing } p \end{array}$$

still is defined. We equip $\mathcal{P}(X)$ with the topology for which the subsets $\tilde{U}$ for $U \subseteq X$ affinoid open form a base. Then $\mathcal{P}(U) \xrightarrow{\sim} \tilde{U}$ is a homeomorphism. We have

$$r_X^{-1}(\{a\}) = \text{closure of } \{a\} \text{ in } \mathcal{P}(X) \qquad \text{for any } a \in \mathcal{M}(X) \quad .$$

It remains true in general that the categories of abelian sheaves on $\mathcal{P}(X)$ and on X are naturally equivalent. We always give $\mathcal{M}(X)$ the quotient topology with respect to the map r_X.

Lemma 1:

Assume X to be affinoid; then the subsets $\mathcal{M}(X)\backslash\underline{U}$ with U running through the affinoid subdomains of X are open and generate the topology of $\mathcal{M}(X)$.

Proof: Clearly the natural map $\mathcal{M}(U) \to \mathcal{M}(X)$ is continuous. Since both sides are compact its image $\underline{U}$ is closed. Using the Berkovich topology we know that the sets $\{a \in \mathcal{M}(X) : |f|_a < \rho\}$ and $\{a \in \mathcal{M}(X) : |f|_a > \rho\}$ with $f \in \mathcal{O}(X)$ and $\rho \in \sqrt{|k^\times|}$ generate the topology of $\mathcal{M}(X)$. Those sets obviously are of the form $\mathcal{M}(X)\backslash\underline{U}$.

Lemma 2:

Assume X to be quasi-separated and let $U \subseteq X$ be affinoid open; then the natural map $\mathcal{M}(U) \to \mathcal{M}(X)$ is a homeomorphism onto its image $\underline{U}$ which is closed in $\mathcal{M}(X)$.

Proof: Let $X = \bigcup\limits_{i\in I} U_i$ be an admissible affinoid open covering. Making obvious identifications we have

$$\begin{aligned}\mathcal{P}(X)\backslash r_X^{-1}\underline{U} &= \bigcup_{i\in I} \mathcal{P}(U_i)\backslash r_X^{-1}\underline{U} = \bigcup_{i\in I} \mathcal{P}(U_i)\backslash r_{U_i}^{-1}(\underline{U} \cap \underline{U_i}) \\ &= \bigcup_{i\in I} r_{U_i}^{-1}(\mathcal{M}(U_i)\backslash(\underline{U} \cap \underline{U_i})) \quad .\end{aligned}$$

Since X is quasi-separated it follows from Lemma 1 that $\mathcal{M}(U_i)\backslash(\underline{U} \cap \underline{U_i})$ is open in $\mathcal{M}(U_i)$. This implies that $\underline{U}$ is closed in $\mathcal{M}(X)$. In this way we see that the natural continuous map $\mathcal{M}(U) \to \mathcal{M}(X)$ has the property that the image $\underline{V}$ of $\mathcal{M}(V)$ for any affinoid subdomain $V \subseteq U$ is closed in $\mathcal{M}(X)$. Again by Lemma 1 this map therefore is closed.

Lemma 3:

Let $X = \bigcup\limits_{i\in I} U_i$ be an admissible affinoid open covering; we then have: A subset M of $\mathcal{M}(X)$ is open (closed) if and only if the preimage of M in $\mathcal{M}(U_i)$ is open (closed) for any $i \in I$.

Proof: The direct implication is trivial. Also the assertion about closedness follows from the one about openness. Therefore assume that the preimage of M in $\mathcal{M}(U_i)$ is open for any $i \in I$. We have to show that $r_X^{-1}M$ is open. But (again with the obvious identifications)

$$r_X^{-1}M = \bigcup_{i\in I}(r_X^{-1}M \cap \tilde{U}_i) = \bigcup_{i\in I} r_{U_i}^{-1}(M \cap \mathcal{M}(U_i)) \quad .$$

The next results should be compared with [Be] §1.6. They say that as far as quasi-separated spaces X with the subsequent condition $(*)$ are concerned the theory of Berkovich is the theory of the space $\mathcal{M}(X)$. In the following we extend the notations $\tilde{U}$ and $\underline{U}$ in the obvious way to arbitrary admissible open subsets $U \subseteq X$.

Proposition 4:

Suppose that X is quasi-separated and that it has an admissible affinoid open covering $X = \bigcup_{i\in I} U_i$ such that

$(*)$ *for any $i \in I$ there are only finitely many $j \in I$ with $U_i \cap U_j \neq \emptyset$;*

then $\mathcal{M}(X)$ is a locally compact and paracompact Hausdorff space. Moreover any point in $\mathcal{M}(X)$ has a fundamental system of compact neighbourhoods of the form $\underline{V}$ for some quasi-compact admissible open subset $V \subseteq X$.

Proof: Let a, b denote distinct points in $\mathcal{M}(X)$. Since the $\mathcal{M}(U_i)$ are Hausdorff it follows from Lemma 2 that we can choose, for any $i \in I$, closed subsets $M_i, N_i \subseteq \underline{U_i}$ with union $\underline{U_i}$ and such that $a \notin M_i$, $b \notin N_i$. Define $M := \bigcup_{i\in I} M_i$ and $N := \bigcup_{i\in I} N_i$. The condition $(*)$ implies that the intersections of M and N with $\underline{U_i}$ are closed. Therefore M and N are closed in $\mathcal{M}(X)$ by Lemma 3; their union is $\mathcal{M}(X)$ and $a \notin M$, $b \notin N$. This shows that $\mathcal{M}(X)$ is Hausdorff. By a similar reasoning the union of $\underline{U_i}$ with i running through any subset of I is closed in $\mathcal{M}(X)$. Applying this to those $\underline{U_i}$ which do not contain a given point $a \in \mathcal{M}(X)$ we see that the union of the finitely many other $\underline{U_i}$ which do contain a is a compact neighbourhood of a. Hence $\mathcal{M}(X)$ is locally compact. It is also clear now that the $\underline{U_i}$ for $i \in I$ form a locally finite covering of $\mathcal{M}(X)$ by compact subsets. This implies that $\mathcal{M}(X)$ is paracompact ([E] 5.1.34). The above argument showed that a point $a \in \mathcal{M}(X)$ has at least one neighbourhood of the form $\underline{V}$ for some quasi-compact admissible open subset $V \subseteq X$. Using a finite affinoid open covering of V a simple topological argument therefore implies that our second assertion only has to be checked in the case of an affinoid space X. This is done in [B] 2.2.3 (iii).

Of course, in the situation of Proposition 4 the space $\mathcal{M}(U)$, for any admissible open subset $U \subseteq X$, is Hausdorff.
An abelian sheaf F on X (or $\mathcal{P}(X)$) is called *overconvergent* if its restriction to any affinoid open subset $U \subseteq X$ (or $\mathcal{P}(U)$) is overconvergent. It is shown in [S] §2 that it suffices to test this condition for an admissible affinoid open covering of X. It is immediate from the affinoid case that the functor r_X^* maps any abelian sheaf on $\mathcal{M}(X)$ to an overconvergent sheaf on $\mathcal{P}(X)$ (or X).

Lemma 5:

With the same assumptions as in Proposition 4 let $U \subseteq X$ be a quasi-compact admissible open subset and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{P}(U) & \stackrel{\varphi}{\longrightarrow} & \mathcal{P}(X) \\ {\scriptstyle r_U}\downarrow & & \downarrow{\scriptstyle r_X} \\ \mathcal{M}(U) & \stackrel{\psi}{\longrightarrow} & \mathcal{M}(X) \end{array}$$

where φ and ψ are the natural maps; for any overconvergent sheaf S on $\mathcal{P}(X)$ the base change map

$$\psi^* r_{X*} S \xrightarrow{\cong} r_{U*}\varphi^* S$$

is an isomorphism.

Proof: (The above diagram in general is not cartesian. The map φ is an open immersion whereas ψ, by Lemma 2, is a closed immersion.) It is convenient to introduce the following notation. For any two admissible open subsets $V \subseteq W \subseteq X$ let $S(V,W)$ denote the sheaf on $\tilde{W}$ which is the direct image of $S|\tilde{V}$. The proof proceeds in several steps by imposing additional assumptions which gradually will be weakened.
Step 1: U and X both are affinoid. Then the assertion is an immediate consequence of Theorem 4.2.
Step 2: X is separated, U is affinoid, and the sheaf is of the form $S(V,X)$ for some affinoid open subset $V \subseteq X$. Since by [S] 2.4 with S also $S(V,X)$ is overconvergent we may apply Step 1 to $V \cap U \subseteq V$ and obtain

$$\begin{aligned} r_{U*}\varphi^* S(V,X) &= r_{U*} S(V \cap U, U) \\ &= \text{direct image on } \underline{U} \text{ of } r_{V\cap U*}(S|\tilde{V} \cap \tilde{U}) \\ &= \text{direct image on } \underline{U} \text{ of } (r_{V*}(S|\tilde{V}))|\underline{V} \cap \underline{U} \\ &= \psi^*(\text{direct image on } \mathcal{M}(X) \text{ of } r_{V*}(S|\tilde{V})) \\ &= \psi^* r_{X*} S(V,X) \quad . \end{aligned}$$

Step 3: X is separated and quasi-compact and U is affinoid. Then we have the exact sequence of sheaves

$$0 \longrightarrow S \longrightarrow \underset{i\in I}{\oplus} S(U_i, X) \longrightarrow \underset{i,j\in I}{\oplus} S(U_i \cap U_j, X)$$

where I can be taken to be finite. We apply now Step 2 to the middle and the right hand terms and we use that the functors involved in the base change map are left exact and commute with finite direct sums.
Step 4: X is separated and quasi-compact. Let $U = V_1 \cup \ldots \cup V_r$ be a covering by affinoid open subsets $V_j \subseteq U$ and let $\varphi_j : \mathcal{P}(V_j) \to \mathcal{P}(U)$ and $\psi_j : \mathcal{M}(V_j) \to \mathcal{M}(U)$ be the natural maps. The assertion may be checked after restricting to $\mathcal{M}(V_j)$ for all $1 \leq j \leq r$. But then using Step 3 twice for $V_j \subseteq X$ and $V_j \subseteq U$ we obtain

$$\psi_j^* \psi^* r_{X*} S = (\psi\psi_j)^* r_{X*} S = r_{V_j *}(\varphi\varphi_j)^* S = r_{V_j *}\varphi_j^* \varphi^* S = \psi_j^* r_{U*} \varphi^* S \quad .$$

Step 5: The sheaf is of the form $\mathcal{S}(V, X)$ for some separated and quasi-compact admissible open subset $V \subseteq X$. Redo Step 2 but now using Step 4 instead of Step 1.
Step 6: In the general situation we consider the exact sequence of sheaves

$$0 \longrightarrow S \longrightarrow \prod_{i\in I} S(U_i, X) \longrightarrow \prod_{i,j\in I} S(U_i \cap U_j, X) \quad .$$

The condition $(*)$ implies that the infinite products appearing in this sequence coincide with the corresponding direct sums. Therefore we can redo Step 3 now based on Step 5.

Theorem 6:

With the same assumptions as in Proposition 4 we have that r_{X*} and r_X^* are quasi-inverse equivalences between the category of overconvergent sheaves on $\mathcal{P}(X)$ (or X) and the category of all abelian sheaves on $\mathcal{M}(X)$.

Proof: We have to show that the two adjunction maps are isomorphisms. This can be checked after restriction to $\mathcal{P}(U_i)$ and $\mathcal{M}(U_i)$, respectively. But applying Lemma 5 and Theorem 4.2 we obtain

$$(r_X^* r_{X*} S)|\mathcal{P}(U_i) = r_{U_i}^*((r_{X*}S)|\mathcal{M}(U_i)) = r_{U_i}^*(r_{U_i *}(S|\mathcal{P}(U_i)) = S|\mathcal{P}(U_i)$$

for any overconvergent sheaf S on $\mathcal{P}(X)$ and similarly

$$(r_{X*} r_X^* T)|\mathcal{M}(U_i) = r_{U_i *}((r_X^* T)|\mathcal{P}(U_i)) = r_{U_i *} r_{U_i}^*(T|\mathcal{M}(U_i)) = T|\mathcal{M}(U_i)$$

for any sheaf T on $\mathcal{M}(X)$.

The assumptions of Proposition 4 are satisfied by any reasonable rigid space. For example, the generic fiber of any formal scheme of finite type and flat over k^0 is quasi-separated and quasi-compact. To discuss the property $(*)$ a little further we first note that obviously any morphism $\beta : X \to Y$ of rigid spaces over k induces a continuous map

$$\begin{aligned} \mathcal{P}(\beta) : \mathcal{P}(X) &\longrightarrow \mathcal{P}(Y) \\ p &\longmapsto \{V \subseteq Y \text{ admissible open } : \beta^{-1}V \in p\} \quad . \end{aligned}$$

This map respects maximal filters: To see this we may assume X and Y to be affinoid; then it is a consequence of Lemma 2.6. Hence β also induces a continuous map

$$\mathcal{M}(\beta) : \mathcal{M}(X) \longrightarrow \mathcal{M}(Y) \quad .$$

If $\iota : U \xrightarrow{\subseteq} X$ is the inclusion of an admissible open subset then $\mathcal{M}(\iota)$ is injective but in general not open. We therefore introduce the following notion.

Definition:

An open immersion $\iota : U \to X$ is called wide open if $\mathcal{M}(\iota)$ is an open immersion, too.

In case ι is a wide open inclusion map we call U simply a wide open subset of X. By Lemma 3 this notion is local in X.

Lemma 7:

For any admissible open subset $U \subseteq X$ the following conditions are equivalent:

i. U is wide open in X;

ii. $r_X^{-1}(\underline{U}) = \tilde{U}$;

iii. $\underline{U}$ is open in $\mathcal{M}(X)$.

Proof: It is trivial that i. implies iii. Assume now iii. to hold. It is clear that $\tilde{U} \subseteq r_X^{-1}(\underline{U})$. Let $p \in r_X^{-1}(\underline{U})$ be any prime filter. Since $r_X^{-1}(\underline{U})$ is open in $\mathcal{P}(X)$ we find an affinoid open subset $V \subseteq X$ such that $V \in p$ and $\underline{\tilde{V}} \subseteq \underline{U}$. The latter implies that $V \subseteq U$ and hence $U \in p$ which means that $p \in \tilde{U}$.
The assertion ii. can be expressed by saying that the obvious diagram of topological spaces

$$\begin{array}{ccc} \mathcal{P}(U) & \xrightarrow{\mathcal{P}(\iota)} & \mathcal{P}(X) \\ r_U \downarrow & & \downarrow r_X \\ \mathcal{M}(U) & \xrightarrow{\mathcal{M}(\iota)} & \mathcal{M}(X) \end{array}$$

is cartesian. Moreover $\mathcal{P}(\iota)$ is an open immersion and r_U and r_X are quotient maps. It is straightforward that then $\mathcal{M}(\iota)$ has to be an open immersion, too.

Proposition 8:

With the same assumptions as in Proposition 4 let $U \subseteq X$ be a wide open subset which possesses a countable admissible affinoid open covering; then U is quasi-separated and satisfies the condition $(*)$ in Proposition 4.

Proof: It follows from Proposition 4 that $\mathcal{M}(X)$ is locally compact. As an open subset $\mathcal{M}(U)$ is locally compact, too. Of course U is quasi-separated. The assumption about the countable covering then implies by Lemma 2 that $\mathcal{M}(U)$ is a countable union of compact subsets or in other words is countable at infinity and in particular is paracompact.
According to [Bou] I.9.10 Cor. there is a locally finite open covering

$$\mathcal{M}(U) = \bigcup_{i \in J} M_i$$

such that all the $\overline{M_i}$ are compact. Moreover, by [Bou] IX.4.3 Thm. 3 there are open coverings

$$\mathcal{M}(U) = \bigcup_{i \in J} N_i = \bigcup_{i \in J} L_i$$

such that

$$\overline{L_i} \subseteq N_i \subseteq \overline{N_i} \subseteq M_i \quad \text{for any } i \in J \quad .$$

The $\overline{N_i}$ and $\overline{L_i}$ are compact as well. We claim that for any $i \in J$ there are only finitely many $j \in J$ with $\overline{N_i} \cap \overline{N_j} \neq \emptyset$. Fix an $i \in J$. The family $\{\overline{N_j} : j \in J\}$ being locally finite we find, for any $a \in \overline{N_i}$, an open neighbourhood M_a of a in M_i which intersects only finitely many sets $\overline{N_j}$. Since $\overline{N_i}$ is compact we have

$$\overline{N_i} \subseteq M_{a_1} \cup \ldots \cup M_{a_r} \quad \text{for some points } a_1, \ldots, a_r \quad .$$

If now $\overline{N_i} \cap \overline{N_j} \neq \emptyset$ then also $M_{a_\rho} \cap \overline{N_j} \neq \emptyset$ for some $1 \leq \rho \leq r$. Hence this can happen only for finitely many $j \in J$.
According to Proposition 4 any point in some $\overline{L_i}$ has a neighbourhood in N_i of the form $\underline{V}$ for some quasi-compact admissible open subset $V \subseteq U$. Since the $\overline{L_i}$ are compact and cover $\mathcal{M}(U)$ it follows that there exists a family $\{V_i\}_{i \in I}$ of affinoid open subsets in U such that

— $U = \bigcup_{i \in I} V_i$,

— for any $i \in I$ there are only finitely many $j \in I$ with $V_i \cap V_j \neq \emptyset$, and

— for any point $a \in \mathcal{M}(U)$ there is an $i \in I$ such that $\underline{V_i}$ is a neighbourhood of a in $\mathcal{M}(U)$.

The latter property implies that the V_i form an admissible covering of U: Let $\beta : Y \to U$ be any morphism from an affinoid space Y into U. Using the compactness of $\mathcal{M}(Y)$ it easily follows that the covering $\{\beta^{-1}V_i\}_{i \in I}$ of Y has a finite subcovering and hence is admissible.

Now let $X = Z^{an}$ be the rigid space associated to a separated scheme Z of finite type over k ([BGR] 9.3.6). Of course X is separated.

Lemma 9:

If $Z_0 \subseteq Z$ is a Zariski open subscheme then $U := Z_0^{an}$ is wide open in $X = Z^{an}$.

Proof: Using Lemma 3 and an admissible affinoid open covering of Z^{an} this follows from the fact that any Zariski open subset in an affinoid space is wide open ([S] §3 Prop. 3 (iii)).

Proposition 10:

$X = Z^{an}$ satisfies the condition $(*)$ in Proposition 4.

Proof: We want to apply Proposition 8. By Nagata Z is Zariski open in a proper scheme $\overline{Z}$ over k. Since $\overline{Z}^{an}$ is proper and hence quasi-compact it satisfies the assumptions of Proposition 4. According to the previous Lemma Z^{an} is wide open in $\overline{Z}^{an}$. By Proposition 8 it remains to check that Z^{an} has a countable admissible affinoid open covering. Writing Z as a finite union of affine open subschemes we are reduced to consider the case where Z is affine. But for the affine space and then also for any Zariski closed subscheme in the affine space our claim is obvious.

Using the properties of the Raynaud functor ([Meh] or [BL] 4.1) one can easily establish a version of Theorem 3.4 for any quasi-separated and quasi-compact space X.

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