K-types for the tempered components of a p-adic general linear group

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with an Appendix: **The definition of the tempered category**
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Wir widmen diese Arbeit dem Andenken an Jürgen Neukirch. Der erste Autor gibt dadurch seiner besonderen Dankbarkeit Ausdruck.

Let G be the group of rational points of a connected reductive group over the nonarchimedean field F and let $\mathcal{M}(G)$ denote the category of smooth G-representations. By computing explicitly the centre of $\mathcal{M}(G)$ Bernstein ([Ber], [BeR]) has in particular determined the central idempotents and has obtained a corresponding decomposition of $\mathcal{M}(G)$ into the direct product of its "connected components" $\mathcal{M}(\Omega)$. If G is a general linear group then a further important step was made by Bushnell and Kutzko ([BK1-3]) who have constructed a theory of K-types for the category $\mathcal{M}(G)$ in this case. For each component $\mathcal{M}(\Omega)$ they give a pair (J, λ), its K-type, consisting of a certain compact open subgroup $J \subseteq G$ and a certain irreducible smooth representation λ of J which characterizes $\mathcal{M}(\Omega)$ in the following sense: A representation V in $\mathcal{M}(G)$ lies in $\mathcal{M}(\Omega)$ if and only if V as a G-representation is generated by its λ-isotypic part.

A very important subclass among all smooth admissible G-representations, singled out by Harish-Chandra, is formed by the tempered admissible G-representations which are defined in terms of growth conditions on their matrix coefficients. As a consequence they carry a natural action of the Schwartz algebra $\mathcal{S}(G)$ of all uniformly locally constant and rapidly decreasing functions on G. It therefore seems natural to define the category $\mathcal{M}^t(G)$ of tempered G-representations as the category of all nondegenerate (left) $\mathcal{S}(G)$-modules (having in mind that $\mathcal{M}(G)$ coincides with the category of all nondegenerate $\mathcal{H}(G)$-modules where $\mathcal{H}(G)$ is the Hecke algebra of all locally constant functions with compact support on G). Although he did not express it this way the central idempotents of $\mathcal{M}^t(G)$ are known from Harish-Chandra's Plancherel formula. As before this gives rise to a decomposition of $\mathcal{M}^t(G)$ into "connected components" $\mathcal{M}^t(\Theta)$. The natural question which arises here is whether each $\mathcal{M}^t(\Theta)$ can be characterized by a K-type in a similar way as for the $\mathcal{M}(\Omega)$. This is the problem which we address and solve, for G a general linear group, in this paper.

The forgetful functor from $\mathcal{M}^t(G)$ to $\mathcal{M}(G)$ respects connected components in the sense that each $\mathcal{M}^t(\Theta)$ is mapped entirely into a specific $\mathcal{M}(\Omega)$. In addition the $\mathcal{M}^t(\Theta)$ which are mapped into a given $\mathcal{M}(\Omega)$ are finite in number. Let us assume that (J, λ) is a K-type for $\mathcal{M}(\Omega)$ as constructed by Bushnell and Kutzko. Our main result is that the finitely many $\mathcal{M}^t(\Theta)$ mapping to $\mathcal{M}(\Omega)$ correspond bijectively to the irreducible smooth representations σ of a certain fixed maximal compact subgroup $K \subseteq G$ which contains J and such that $\sigma|J$ contains λ. Moreover and most importantly, a tempered G-representation V mapping to $\mathcal{M}(\Omega)$ is generated as an $\mathcal{S}(G)$-module by its σ-isotypic part if and only if it lies in $\mathcal{M}^t(\Theta^{\leq})$. Here $\mathcal{M}^t(\Theta^{\leq})$ is the finite direct product of all component categories $\mathcal{M}^t(\Theta')$ with $\Theta \leq \Theta'$ where $\leq$ is a certain natural partial order on the set of all tempered components. Although technically this partial order will be derived from the dominance partial order on partitions via the Bernstein-Zelevinsky classification which allows to parametrize the set of all tempered components by certain partition valued functions the philosophical reason why it has to come into the picture is the following. The smooth dual $\tilde{G}$ of G, i.e., the set of all isomorphism classes of irreducible smooth G-representations comes equipped with the Jacobson topology. In this topology the subset of classes of irreducible tempered G-representations is dense. Hence the Jacobson closure in $\tilde{G}$ of a fixed tempered component Θ must meet other tempered components Θ' which in our case turn out to be exactly those with $\Theta' \leq \Theta$.

At this point it should come no longer as a surprise that our tempered K-types (K, σ) in fact are good for describing the strata of a natural stratification of the full category $\mathcal{M}(G)$ of all smooth G-representations (always for $G = GL_n(F)$). Since it is not really available in the literature we make some effort to develop the concept of a stratification of a module category. The technical problem one encounters is to make the strata categories well defined, i.e., independent of a possible refinement of the partial order. We circumvent this problem here by making use of the notion of a reduced subcategory due to A. Rosenberg; this is completely analogous to the notion of a reduced subvariety in an algebraic variety.

In each component category $\mathcal{M}(\Omega)$ of $\mathcal{M}(G)$ we will construct a natural stratification by finitely many subcategories parametrized by the tempered components Θ mapping into Ω such that the partial order $\leq$ corresponds to the inclusion relation between the subcategories. We will show that the structure of the reduced strata categories of this stratification is completely determined by our tempered K-types (K, σ). In fact, each of these reduced strata categories is naturally equivalent to the category of unital modules over its centre. We will embed these centers into the rings of regular algebraic functions on certain explicit quotients of complex algebraic tori.

The remarkable picture which emerges here is that Bernstein's decomposition of the category $\mathcal{M}(G)$ into its connected components refines into a stratification of $\mathcal{M}(G)$ where the strata, at least up to nilpotent elements, are module categories over commutative rings. We strongly believe that such a picture holds true for any group G. But besides the deep theories of Bernstein-Zelevinsky and of Bushnell-Kutzko on which we completely rely, the particular feature of the group $G = GL_n(F)$ in the background which makes our theory possible is the fact that the tempered dual of $GL_n(F)$ already is commutative. There is a close relation between our stratification and Lusztig's notion of cells in affine Weyl groups (compare [Rog]); we hope to return to this in future work.

Since the Schwartz algebra $\mathcal{S}(G)$ naturally comes as a topological algebra it is not entirely clear how to define a good category of tempered G-representations. Harish-Chandra only considered tempered admissible representations. The appendix written by the first author and U. Stuhler serves the purpose to justify the somewhat surprising proposal to consider the purely algebraically defined abelian category of all nondegenerate $\mathcal{S}(G)$-modules.

We want to express our sincere thanks to C. Bushnell for explaining to us certain fine points of his theory with Kutzko and to V. Nistor for numerous discussions about stratifications of module categories.

Basic notations: Throughout the paper F is a nonarchimedean locally compact field; $|\ \ |$ denotes its normalized absolute value. We let G be the group of F-rational points of a connected reductive group over F (later $G = GL_n(F)$ for some $n \in \mathbb{N}$).
For any finite field extension E/F let o_E, resp. $\mathbb{F}_E$, denote the ring of integers in E, resp. the residue class field of o_E. If $\mathfrak{A}$ is an o_E-order in an algebra over E we let $U^1(\mathfrak{A})$ denote the subgroup in $\mathfrak{A}^\times$ of those units which are congruent to 1 modulo the Jacobson radical of $\mathfrak{A}$.

1. Bernstein and Harish-Chandra decomposition

Let $\mathcal{M}(G)$ denote the category of smooth G-representations and let $\mathrm{Irr}(G)$ be the set of isomorphism classes of irreducible smooth G-representations. An important tool for understanding the category $\mathcal{M}(G)$ is the Bernstein spectrum $\Omega(G)$ of infinitesimal characters of G ([Ber]). For us the following description of $\Omega(G)$ is the most convenient one. A cuspidal pair (M, σ) (for G) consists of a Levi subgroup M of G and an irreducible supercuspidal representation σ of M. The group G acts by conjugation on the set of cuspidal pairs and $\Omega(G)$ is the set of G-orbits of this action. For any irreducible smooth G-representation V there is up to conjugation a unique cuspidal pair (M, σ), called the cuspidal support of V, such that V is a subquotient of a G-representation which is parabolically induced from (M, σ) (in this paper induction always means normalized induction). This sets up a natural map

$$\nu \ : \ \mathrm{Irr}(G) \longrightarrow \Omega(G)$$

sending V to its cuspidal support. The Bernstein spectrum $\Omega(G)$ in a natural way is a complex locally algebraic variety. Its connected components are given as follows. Fix a cuspidal pair (M, σ) and let $X_{nr}(M)$ denote the complex algebraic torus of unramified characters of M. The connected component of the G-orbit of (M, σ) is the image of the map

$$\begin{array}{rcl} X_{nr}(M) & \longrightarrow & \Omega(G) \\ \alpha & \longmapsto & G\text{-orbit of } (M, \alpha\sigma) \ . \end{array}$$

For any connected component $\Omega \subseteq \Omega(G)$ we define

$$\mathcal{M}(\Omega) := \begin{array}{l} \text{full subcategory in } \mathcal{M}(G) \text{ of all} \\ G\text{-representations whose irreducible} \\ \text{subquotients all have cuspidal} \\ \text{support in } \Omega \ . \end{array}$$

One of Bernstein's results then says that the category

$$\mathcal{M}(G) \ = \ \prod_{\Omega} \mathcal{M}(\Omega)$$

decomposes into the direct product of the subcategories $\mathcal{M}(\Omega)$ where Ω runs through the connected components of $\Omega(G)$, i.e., every smooth G-representation decomposes naturally into a direct sum

$$V = \bigoplus_{\Omega} V(\Omega) \text{ with } V(\Omega) \in \mathcal{M}(\Omega) .$$

The subcategories $\mathcal{M}(\Omega)$ are called the Bernstein components of $\mathcal{M}(G)$.

A K-type for a Bernstein component $\mathcal{M}(\Omega)$ is a pair (I, ρ) consisting of a compact open subgroup $I \subseteq G$ and an irreducible smooth representation ρ of I such that the following two equivalent conditions are satisfied:
(i) An irreducible smooth G-representation V has cuspidal support in Ω if and only if $V^\rho \neq 0$.
(ii) A smooth G-representation V lies in $\mathcal{M}(\Omega)$ if and only if as a G-representation V is generated by its subspace V^ρ.
(Here V^ρ denotes the ρ-isotypical component of V.) If such a K-type exists then the functor $V \mapsto V^\rho$ is an equivalence of categories between $\mathcal{M}(\Omega)$ and the category of modules for the scalar Hecke algebra of the K-type ([BK3] §4). The content of the Bushnell-Kutzko theory ([BK1], [BK2]) is that they construct, for $G = GL_n(F)$, such a K-type for each Bernstein component and compute the associated Hecke algebras explicitly.

In this paper we want to develop an analogous theory for the Harish-Chandra components of the category of tempered G-representations. Let $\mathcal{H} = \mathcal{H}(G)$ denote the Hecke algebra of complex valued locally constant functions with compact support on G. Recall that $\mathcal{M}(G)$ is naturally equivalent to the category of nondegenerate left $\mathcal{H}$-modules. The Hecke algebra $\mathcal{H}$ is contained in the larger Schwartz algebra $\mathcal{S}$ of uniformly locally constant and rapidly decreasing functions on G. The space $\mathcal{S}$ carries a natural locally convex (in fact ind-Fréchet) topology such that the multiplication is separately continuous and such that $\mathcal{H}$ is dense in $\mathcal{S}$. (Compare [Sil] §4.4 where $\mathcal{S}$ is denoted by $\mathcal{C}(G)$.) But in this paper we will treat $\mathcal{S}$ as an abstract algebra. We define the category of tempered G-representations to be

$$\mathcal{M}^t(G) := \text{ category of nondegenerate left } \mathcal{S}\text{-modules} .$$

Since $\mathcal{S}$ itself is a smooth G-representation via the left translation action the forgetful functor

$$\mathcal{M}^t(G) \longrightarrow \mathcal{M}(G)$$

is well defined. Any admissible G-representation in $\mathcal{M}(G)$ which is "tempered" in the traditional sense ([Sil] §4.5) that its matrix coefficients are tempered functions on G carries a unique $\mathcal{S}$-module structure which extends the given $\mathcal{H}$-module structure. Later on we will need:
– For any simple nondegenerate $\mathcal{S}$-module the underlying smooth G-representation is irreducible and tempered in the traditional sense.
– Any two simple nondegenerate $\mathcal{S}$-modules which are isomorphic as smooth

G-representations are already isomorphic as $\mathcal{S}$-modules.
In particular the set $\mathrm{Irr}^t(G)$ of isomorphism classes of tempered irreducible G-representations is defined unambiguously and is a subset of $\mathrm{Irr}(G)$. Proofs of these facts are given in the appendix.
The counterpart for the category $\mathcal{M}^t(G)$ of the Bernstein spectrum is the Harish-Chandra (or tempered) spectrum $\Omega^t(G)$ which is constructed as follows. A discrete pair (N, τ) consists of a Levi subgroup N of G and an irreducible pre-unitary smooth representation τ of N whose matrix coefficients are square-integrable modulo centre (a discrete series representation τ of N for short).
The group G acts by conjugation on the set of discrete pairs and $\Omega^t(G)$ is defined to be the set of G-orbits of this action. For any tempered irreducible G-representation V there is up to conjugation a unique discrete pair (N, τ), called the discrete support of V, such that V is a direct summand of a G-representation parabolically induced from (N, τ). This gives a map

$$\nu^t \ : \ \mathrm{Irr}^t(G) \longrightarrow \Omega^t(G)$$

which sends V to its discrete support. Similarly we have the map

$$\begin{array}{cccc} \mathfrak{z}: & \Omega^t(G) & \longrightarrow & \Omega(G) \\ & \begin{array}{c} G\text{-orbit} \\ \text{of } (N,\tau) \end{array} & \longmapsto & \begin{array}{l} \text{cuspidal} \\ \text{support of } \tau \ . \end{array} \end{array}$$

These various maps fit together into the commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}^t(G) & \xrightarrow{\subseteq} & \mathrm{Irr}(G) \\ \nu^t \Big\downarrow & & \Big\downarrow \nu \\ \Omega^t(G) & \xrightarrow{\mathfrak{z}} & \Omega(G) \ . \end{array}$$

For the group $G = GL_n(F)$ the map $\mathfrak{z}$ is injective (compare [Ze1] 7.1 and [Rod] Prop. 11) and the map ν^t is bijective (since unitary induction is irreducible for this group); in this case we therefore will sometimes drop the symbols $\mathfrak{z}$ and ν^t from the notation. The tempered spectrum $\Omega^t(G)$ is a disjoint union of orbifolds which arise as follows. Fix a discrete pair (N, τ) and let $X^1_{nr}(N)$ denote the compact torus of unitary unramified characters of N. The connected component in $\Omega^t(G)$ of the G-orbit of (N, τ) is the image of the map

$$\begin{array}{rcl} X^1_{nr}(N) & \longrightarrow & \Omega^t(G) \\ \alpha & \longmapsto & G\text{-orbit of } (N, \alpha\tau) \ . \end{array}$$

The description of the connected components in $\Omega^t(G)$ and $\Omega(G)$ implies that the image under the map $\mathfrak{z}$ of a connected component of $\Omega^t(G)$ is entirely contained

in a connected component of $\Omega(G)$. One actually has the stronger fact that, for any connected component $\Omega \subseteq \Omega(G)$, its preimage $\mathfrak{z}^{-1}(\Omega)$ is a finite (disjoint) union of connected components of $\Omega^t(G)$. We do not give a general proof here since for the group $G = GL_n(F)$ this fact is an immediate consequence of the Bernstein-Zelevinsky classification which we will review in the next section.
For any connected component $\Theta \subseteq \Omega^t(G)$ we define the Harish-Chandra (or tempered) component $\mathcal{M}^t(\Theta)$ of $\mathcal{M}^t(G)$ by

$$\mathcal{M}^t(\Theta) := \begin{array}{l}\text{full subcategory in } \mathcal{M}^t(G) \text{ of all} \\ \text{tempered representations whose} \\ \text{simple } \mathcal{S}\text{-module subquotients all} \\ \text{have discrete support in } \Theta\,.\end{array}$$

As a consequence of Harish-Chandra's Plancherel formula (compare [Mis] or [Wal]) one has the decomposition

$$\mathcal{M}^t(G) \;=\; \prod_{\Theta} \mathcal{M}^t(\Theta)$$

where Θ runs through the connected components of $\Omega^t(G)$ meaning that any tempered G-representation V decomposes naturally into

$$V \;=\; \bigoplus_{\Theta} V(\Theta) \;\text{ with }\; V(\Theta) \in \mathcal{M}^t(\Theta)\,.$$

The density of $\mathcal{H}$ in $\mathcal{S}$ implies that the centre of the category $\mathcal{M}(G)$ is naturally embedded into the centre of the category $\mathcal{M}^t(G)$. Since the Bernstein decomposition is given by central idempotents it follows that, for any tempered G-representation V, the Bernstein decomposition $V = \bigoplus_{\Omega} V(\Omega)$ actually is a decomposition of $\mathcal{S}$-modules and that

$$V(\Omega) \;=\; \bigoplus_{\mathfrak{z}(\Theta)\subseteq\Omega} V(\Theta)\,.$$

In particular, we see that, whenever $\mathfrak{z}(\Theta) \subseteq \Omega$, the forgetful functor maps $\mathcal{M}^t(\Theta)$ into $\mathcal{M}(\Omega)$.

A tempered K-type for a Harish-Chandra component $\mathcal{M}^t(\Theta)$ would be a pair (I, ρ) consisting of a compact open subgroup $I \subseteq G$ and an irreducible smooth representation ρ of I such that the following two equivalent conditions are satisfied:
(i) A tempered irreducible G-representation V has discrete support in Θ if and only if $V^\rho \neq 0$.
(ii) A tempered G-representation V lies in $\mathcal{M}^t(\Theta)$ if and only if as an $\mathcal{S}$-module V is generated by its subspace V^ρ.
It will turn out however that this naive concept has to be modified. We will construct (in case $G = GL_n(F)$) certain finite products $\mathcal{M}^t(\Theta^{\leq})$ of Harish-Chandra

components along with tempered K-types in the above sense for them.

2. The Bernstein-Zelevinsky classification

From now on for the rest of the paper our group G is assumed to be the group $GL_n(F)$ for some $n \in \mathbb{N}$. The Bernstein-Zelevinsky classification provides a description of the set $\mathrm{Irr}(G)$ in terms of the set of connected components of $\Omega^t(G)$. This description becomes most transparent if one works with all the $G_n := GL_n(F)$ simultaneously.
For convenience we fix once and for all a system of representatives $\mathcal{C}$ for the irreducible preunitary supercuspidal representations of any G_n up to unramified twist. If $\sigma \in \mathcal{C}$ is a representation of G_n we call $d(\sigma) := n$ its degree. Let $\mathrm{Div}^+(\mathcal{C})$ denote the set of effective divisors over the set $\mathcal{C}$. For any divisor $D = \sum m_\sigma \sigma$ in $\mathrm{Div}^+(\mathcal{C})$ we have

– its degree $d(D) := \sum m_\sigma \cdot d(\sigma)$,

– the Levi subgroup

$$M_D \;:=\; \prod_\sigma (G_{d(\sigma)})^{\times m_\sigma} \quad \text{of } G_{d(D)}$$

(we fix some ordering of the factors), and

– the supercuspidal M_D-representation

$$\sigma_D \;:=\; \bigotimes_\sigma \left(\sigma^{\otimes m_\sigma}\right) .$$

Fact 1: *The map*

$$\begin{array}{rcl} \mathrm{Div}^+(\mathcal{C}) & \xrightarrow{\sim} & \begin{array}{l}\text{set of all connected}\\ \text{components of any } \Omega(G_n)\end{array} \\ D & \longmapsto & \Omega_D := \begin{array}{l}\text{component of the}\\ G_{d(D)}\text{-orbit of } (M_D, \sigma_D)\end{array} \end{array}$$

is a bijection.

In a first step Bernstein-Zelevinsky parameterize the discrete series representations of G_n in the following way. The (centered) segment $\Delta(\sigma, s)$ of length $s \in \mathbb{N}$ and with centre $\sigma \in \mathcal{C}$ is the set

$$\Delta(\sigma, s) \;:=\; \left\{ |\ |^{\frac{1-s}{2}+i} \otimes \sigma \;:\; 0 \leq i \leq s-1 \right\}$$

of representations of $G_{d(\sigma)}$. We set $d(\Delta(\sigma,s)) := sd(\sigma)$. To any such segment Δ corresponds a discrete series representation $L(\Delta)$ of $G_{d(\Delta)}$ ([Rod] Prop. 9(ii) and 4.1); the cuspidal support of $L(\Delta(\sigma,s))$ lies in $\Omega_{s\sigma}$. The mapping $\Delta \longmapsto L(\Delta)$ establishes a bijection between the set $\mathcal{C} \times \mathbb{N}$ of all those segments and a set of representatives for the discrete series representations of any G_n up to unramified twist ([Ze1] 9.3 or [Rod] Prop. 11).
We always will identify the set $\mathrm{Div}^+(\mathcal{C} \times \mathbb{N})$ of effective divisors over $\mathcal{C} \times \mathbb{N}$ with the set of all partition valued functions on $\mathcal{C}$ with finite support as follows. For us a partition P is an effective divisor over $\mathbb{N}$ which we think of as being a function with finite support $P : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$; P is a partition of the number $|P| := \sum_{n\in\mathbb{N}} P(n)\cdot n$. For any divisor $\sum_{(\sigma,s)\in\mathcal{C}\times\mathbb{N}} m_{(\sigma,s)}\Delta(\sigma,s)$ in $\mathrm{Div}^+(\mathcal{C} \times \mathbb{N})$ we have the function

$$\begin{array}{rcl} \mathcal{P} : \mathcal{C} & \longrightarrow & \mathrm{Div}^+(\mathbb{N}) \\ \sigma & \longmapsto & \mathcal{P}(\sigma)(s) := m_{(\sigma,s)} \ . \end{array}$$

Vice versa if $\mathcal{P} : \mathcal{C} \longrightarrow \mathrm{Div}^+(\mathbb{N})$ is a function with finite support then

$$\sum_{(\sigma,s)\in\mathcal{C}\times\mathbb{N}} \mathcal{P}(\sigma)(s)\cdot\Delta(\sigma,s) \in \mathrm{Div}^+(\mathcal{C} \times \mathbb{N}) \ .$$

For any $\mathcal{P} \in \mathrm{Div}^+(\mathcal{C} \times \mathbb{N})$ we have:

- its degree $d(\mathcal{P}) := \sum_{\sigma\in\mathcal{C}} |\mathcal{P}(\sigma)|\cdot d(\sigma) = \sum_{(\sigma,s)\in\mathcal{C}\times\mathbb{N}} \mathcal{P}(\sigma)(s)\cdot s\cdot d(\sigma)$,
- the Levi subgroup

  $$N_{\mathcal{P}} := \prod_{(\sigma,s)} (G_{sd(\sigma)})^{\times\mathcal{P}(\sigma)(s)} \quad \text{of } G_{d(\mathcal{P})}$$

  (we fix some ordering of the factors), and
- the discrete series representation

  $$\tau_{\mathcal{P}} := \underset{(\sigma,s)}{\otimes} (L(\Delta(\sigma,s))^{\otimes\mathcal{P}(\sigma)(s)}) \quad \text{of } N_{\mathcal{P}} \ .$$

Fact 2: *The map*

$$\begin{array}{rcl} \mathrm{Div}^+(\mathcal{C} \times \mathbb{N}) & \xrightarrow{\sim} & \text{set of all connected} \\ & & \text{components of any } \Omega^t(G_n) \\[1em] \mathcal{P} & \longmapsto & \Theta_{\mathcal{P}} := \begin{array}{c}\text{component of the} \\ G_{d(\mathcal{P})}\text{-orbit of } (N_{\mathcal{P}},\tau_{\mathcal{P}})\end{array} \end{array}$$

is a bijection.

There is the obvious map

$$\begin{aligned} D : \mathrm{Div}^+(\mathcal{C} \times \mathbb{N}) &\longrightarrow \mathrm{Div}^+(\mathcal{C}) \\ \mathcal{P} &\longmapsto \sum_\sigma |\mathcal{P}(\sigma)| \cdot \sigma \ ; \end{aligned}$$

it satisfies $d(D(\mathcal{P})) = d(\mathcal{P})$.

Fact 3: *For any $D \in \mathrm{Div}^+(\mathcal{C})$ we have*

$$\Omega_D \cap \Omega^t(G) \ = \ \bigcup_{D(\mathcal{P})=D} \Theta_{\mathcal{P}} \ .$$

The reason for this is that the $N_{\mathcal{P}}$-representation $\tau_{\mathcal{P}}$ has cuspidal support in $\Omega_{D(\mathcal{P})}$. In a second step Bernstein-Zelevinsky construct maps

$$\begin{aligned} Q_{\mathcal{P}} : X_{nr}(N_{\mathcal{P}}) &\longrightarrow \mathrm{Irr}(\Omega_{D(\mathcal{P})}) \\ \alpha &\longmapsto L(\alpha\tau_{\mathcal{P}}) \ . \end{aligned}$$

Here we let $\mathrm{Irr}(\Omega)$, for any connected component $\Omega \subseteq \Omega(G_n)$, denote the subset in $\mathrm{Irr}(G_n)$ of all those irreducible G_n-representations with cuspidal support in Ω. In the notation of [Rod] our $\tau_{\mathcal{P}}$ corresponds to a multiset of segments $\{\Delta_1, \ldots, \Delta_r\}$ with r the number of blocks of the Levi subgroup $N_{\mathcal{P}}$; similarly α can be viewed as a set of unramified characters $\{\alpha_1, \ldots, \alpha_r\}$ of the blocks. Then $L(\alpha\tau_{\mathcal{P}})$ is the representation which in [Rod] is denoted by $L(\alpha_1\Delta_1, \ldots, \alpha_r\Delta_r)$ and which corresponds to the multiset of segments $\{\alpha_1\Delta_1, \ldots, \alpha_r\Delta_r\}$. The Bernstein-Zelevinsky classification says ([Rod] Thm. 3) that, for each $D \in \mathrm{Div}^+(\mathcal{C})$,

$$\mathrm{Irr}(\Omega_D) \ = \ \dot{\bigcup_{D(\mathcal{P})=D}} \ \mathrm{im}(Q_{\mathcal{P}})$$

is the disjoint union of the images of these maps $Q_{\mathcal{P}}$. Moreover the subset $Q_{\mathcal{P}}(X^1_{nr}(N_{\mathcal{P}}))$ of $\mathrm{Irr}(\Omega_{D(\mathcal{P})}) \cap \mathrm{Irr}^t(G)$ corresponds under the bijection ν^t to the connected component $\Theta_{\mathcal{P}}$.

A for our purposes very important additional feature is the following partial order on the set $\mathrm{Div}^+(\mathcal{C} \times \mathbb{N})$. First let P and P' be two partitions. We say that P arises from P' by an elementary operation if there are numbers $0 \leq s_0 < s_1, s_2$ such that

$$P \ = \ P' - P_{s_1} - P_{s_2} + P_{s_1+s_2-s_0} + P_{s_0}$$

where P_s, for $s \in \mathbb{N}$, denotes the characteristic function of the subset $\{s\} \subseteq \mathbb{N}$ and P_0 is the zero function. We write $P \leq P'$ if P is obtained from P' by finitely many successive elementary operations. It is an elementary exercise to check that this defines a partial order on the set of all partitions. For two functions $\mathcal{P}$ and $\mathcal{P}'$ in $\mathrm{Div}^+(\mathcal{C} \times \mathbb{N})$ we now define

$$\mathcal{P} \leq \mathcal{P}' \text{ if and only if } \mathcal{P}(\sigma) \leq \mathcal{P}'(\sigma) \text{ for any } \sigma \in \mathcal{C} .$$

The meaning of this partial order in terms of the Bernstein-Zelevinsky classification is the following ([Rod] Thm. 5 and Remark on p. 215).

Lemma:

$L(\alpha\tau_{\mathcal{P}})$, for any $\alpha \in X_{nr}(N_{\mathcal{P}})$, is the only irreducible constituent of the $G_{d(\mathcal{P})}$-representation parabolically induced from $\alpha\tau_{\mathcal{P}}$ which lies in $\mathrm{im}(Q_{\mathcal{P}})$; all the other irreducible constituents lie in $\bigcup\limits_{\mathcal{P}'<\mathcal{P}} \mathrm{im}(Q_{\mathcal{P}'})$.

3. The natural partial order on partitions

The partial order $\leq$ on partitions which we have introduced in the last section is in fact the reverse of the socalled natural or dominance partial order ([Knu] p. 187-189). Let us denote the latter temporarily by $\underset{n}{\leq}$. In order to recall its definition it is convenient to view two partitions P and P' of the number $k := |P| = |P'|$ as sequences $P = (l_1, \ldots, l_k)$ and $P' = (m_1, \ldots, m_k)$ such that $k = l_1 + \ldots + l_k = m_1 + \ldots m_k$ with $l_1 \geq \ldots \geq l_k \geq 0$ and $m_1 \geq \ldots \geq m_k \geq 0$. Then

$$P \underset{n}{\leq} P' \text{ if and only if } l_1 \leq m_1, l_1+l_2 \leq m_1+m_2, \ldots, l_1+\ldots+l_k \leq m_1+\ldots+m_k .$$

One checks by explicit computation that any Young raising operation is the inverse of a particular elementary operation and that the inverse of an arbitrary elementary operation can be obtained as a sequence of Young raising operations. It follows that

$$P' \leq P \text{ if and only if } P \underset{n}{\leq} P' .$$

As an example we display the natural partial order on the partitions of $k = 6$:

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & 6 \arrow[d, no head] & \\
 & 51 \arrow[d, no head] & \\
 & 42 \arrow[dl, no head] \arrow[dr, no head] & \\
411 \arrow[dr, no head] & & 33 \arrow[dl, no head] \\
 & 321 \arrow[dl, no head] \arrow[dr, no head] & \\
3111 \arrow[dr, no head] & & 222 \arrow[dl, no head] \\
 & 2211 \arrow[d, no head] & \\
 & 21111 \arrow[d, no head] & \\
 & 111111 &
\end{tikzcd}
$$

The partial order $\leq$ appears in the complex representation theory of the symmetric group in the following way. Let S_k denote the symmetric group on k letters and let $\varepsilon = \varepsilon_k$ denote the sign character on S_k. If P is a partition we have the subgroup

$$S_P \ := \ \prod_{n \in \mathbb{N}} (S_n)^{\times P(n)} \ \text{ of } S_{|P|}$$

(fixing some ordering of the factors). We form the induced representation

$$\pi_P^\circ \ := \ \mathrm{Ind}_{S_P}^{S_{|P|}}(\varepsilon) \ = \ \mathrm{Ind}_{S_P}^{S_{|P|}}(1) \ \otimes \ \varepsilon \ .$$

Up to isomorphism it does not depend on the chosen ordering of the factors of S_P. Quite generally, for a representation π of a finite group H, let $I(\pi)$ denote the set of isomorphism classes of irreducible constituents of π; if $\pi = \mathbb{C}[H]$ is the regular representation then we write $\mathrm{Irr}(H) := I(\pi)$.

Proposition:

i. *π_P° has a unique irreducible constituent σ_P° which does not occur in any $\pi_{P'}^\circ$ for $P' < P$; this σ_P° occurs with multiplicity 1 in π_P°;*

ii. *$\sigma_{P'}^\circ$ occurs in π_P° if and only if $P' \leq P$;*

iii. *the map*

$$\begin{array}{rcl} \text{partitions of } k & \longrightarrow & \mathrm{Irr}(S_k) \\ P & \longmapsto & \sigma_P^\circ \end{array}$$

is a bijection.

Proof: This is well known and can be extracted, e.g., from [Ze2] Prop. (a) on p. 45, Prop. on p. 49, Cor. on p. 50, pp. 70 and 89; in the notation there π_P°, resp. σ_P°, correspond to y_λ, resp. $\{\lambda^t\}$.

Note that $\sigma^\circ_{(1,\ldots,1)} = 1$ and $\sigma^\circ_{(k)} = \varepsilon_k$.

4. The finite field case

Next we need to review the complex representation theory of the general linear groups $\overline{G}_n := GL_n(\mathbb{F})$ over a finite field $\mathbb{F}$. This, of course, is very well known material. Our point is to set it up in a way which is as analogous to the local field case as possible. This will help later on to make our formalism of tempered K-types more transparent. On the level of notation we will make the analogy apparent by largely using the same symbols but overlined for corresponding notions.

Let $\mathcal{M}(\overline{G}_n)$ be the category of $\overline{G}_n$-representations. As in section 1 we define the spectrum $\Omega(\overline{G}_n)$ to be the set of $\overline{G}_n$-conjugacy classes of cuspidal pairs for $\overline{G}_n$. And as there one has the cuspidal support map $\overline{\nu} : \mathrm{Irr}(\overline{G}_n) \longrightarrow \Omega(\overline{G}_n)$. Of course, $\Omega(\overline{G}_n)$ now is a discrete set. The analog of the Bernstein decomposition is the obvious decomposition

$$\mathcal{M}(\overline{G}_n) = \prod_{\omega \in \Omega(\overline{G}_n)} \mathcal{M}(\omega)$$

into the categories

$$\mathcal{M}(\omega) := \begin{array}{l} \text{full subcategory in } \mathcal{M}(\overline{G}_n) \text{ of all} \\ \overline{G}_n\text{-representations whose irreducible} \\ \text{constituents all have cuspidal support } \omega \ . \end{array}$$

The group $\overline{G}_n$ being finite we have the finer decomposition of the category $\mathcal{M}(\overline{G}_n)$ into its primary components. This will turn out to be the analog of the Harish Chandra decomposition in the local field case. Let $\overline{\mathcal{C}}$ be a fixed set of representatives for the isomorphism classes of irreducible cuspidal representations of any $\overline{G}_n$ (there is no unramified twist now). In the same way as in section 2 we may define, for any divisor $\overline{D} \in \mathrm{Div}^+(\overline{\mathcal{C}})$, its degree $d(\overline{D})$ as well as the cuspidal pair $(M_{\overline{D}}, \sigma_{\overline{D}})$ for $\overline{G}_{d(\overline{D})}$.

Fact $\overline{1}$: *The map*

$$\begin{array}{rcl} \mathrm{Div}^+(\overline{\mathcal{C}}) & \xrightarrow{\sim} & \text{disjoint union of all } \Omega(\overline{G}_n) \\ \overline{D} & \longmapsto & \omega_{\overline{D}} := \overline{G}_{d(\overline{D})}\text{-orbit of } (M_{\overline{D}}, \sigma_{\overline{D}}) \end{array}$$

is a bijection.

The role of the discrete series representations will be played by the nondegenerate irreducible representations, i.e., those irreducible representations which contain a nondegenerate character of a maximal unipotent subgroup. (The characters of these nondegenerate representations are the socalled regular characters – compare [Car] p. 281.) Any cuspidal representation is nondegenerate. We formally introduce the nondegenerate spectrum $\Omega^{nd}(\overline{G}_n)$ as the set of all $\overline{G}_n$-conjugacy classes of nondegenerate pairs, i.e., of pairs consisting of a Levi subgroup in $\overline{G}_n$ and a nondegenerate irreducible representation of that Levi subgroup. There is the obvious cuspidal support map $\bar{\mathfrak{z}} : \Omega^{nd}(\overline{G}_n) \longrightarrow \Omega(\overline{G}_n)$. For any $(\overline{\sigma}, s) \in \overline{\mathcal{C}} \times \mathbb{N}$ the generalized Steinberg representation $\mathrm{st}(\overline{\sigma}, s)$ of $\overline{G}_{sd(\overline{\sigma})}$ is defined to be the unique nondegenerate irreducible representation with cuspidal support $s\overline{\sigma} \in \mathrm{Div}^+(\overline{\mathcal{C}})$. It is contained with multiplicity 1 in the parabolic induction of the exterior tensor product $\overline{\sigma}^{\otimes s}$. The mapping $(\overline{\sigma}, s) \mapsto \mathrm{st}(\overline{\sigma}, s)$ is a bijection between $\overline{\mathcal{C}} \times \mathbb{N}$ and a set of representatives for the nondegenerate irreducible representations of any $\overline{G}_n$ ([Ze2] 9.5). This $\mathrm{st}(\overline{\sigma}, s)$ is the analog of $L(\Delta(\sigma, s))$ in section 2. In the same way as in section 2 we may now define, for any partition valued function with finite support $\wp \in \mathrm{Div}^+(\overline{\mathcal{C}} \times \mathbb{N})$, the degree $d(\wp)$, the Levi subgroup $N_\wp$ of $\overline{G}_{d(\wp)}$, and the nondegenerate representation

$$\mathrm{st}(\wp) := \bigotimes_{(\overline{\sigma}, s)} \mathrm{st}(\overline{\sigma}, s)^{\otimes \wp(\overline{\sigma})(s)} \quad \text{of } N_\wp .$$

Fact $\overline{2}$: *The map*

$$\begin{aligned} \mathrm{Div}^+(\overline{\mathcal{C}} \times \mathbb{N}) \quad &\xrightarrow{\sim} \quad \text{disjoint union of all } \Omega^{nd}(\overline{G}_n) \\ \wp \quad &\longmapsto \quad \vartheta_\wp := \overline{G}_{d(\wp)}\text{-orbit of } (N_\wp, \mathrm{st}(\wp)) \end{aligned}$$

is a bijection.

As in section 2 we have the obvious map

$$\begin{aligned} \overline{D} : \mathrm{Div}^+(\overline{\mathcal{C}} \times \mathbb{N}) \quad &\longrightarrow \quad \mathrm{Div}^+(\overline{\mathcal{C}}) \\ \wp \quad &\longmapsto \quad \sum_{\overline{\sigma}} |\wp(\overline{\sigma})| \cdot \overline{\sigma} \end{aligned}$$

satisfying $d(\overline{D}(\wp)) = d(\wp)$.

Fact $\overline{3}$: *For any $\overline{D} \in \mathrm{Div}^+(\overline{\mathcal{C}})$ we have $\bar{\mathfrak{z}}^{-1}(\omega_{\overline{D}}) = \{\vartheta_\wp : \overline{D}(\wp) = \overline{D}\}$.*

For $\wp \in \mathrm{Div}^+(\overline{\mathcal{C}} \times \mathbb{N})$ let $\pi_\wp$ denote the $\overline{G}_{d(\wp)}$-representation obtained by parabolic induction from $\mathrm{st}(\wp)$. Up to (a noncanonical) isomorphism it only depends on $\vartheta_\wp$. All the irreducible constituents of $\pi_\wp$ have the cuspidal support

$\omega_{\overline{D}(\wp)}$, i.e., $\pi_\wp$ lies in the category $\mathcal{M}(\omega_{\overline{D}(\wp)})$. We introduce a partial order $\leq$ on $\mathrm{Div}^+(\overline{\mathcal{C}} \times \mathbb{N})$ by defining

$$\wp \leq \wp' \text{ if and only if } \wp(\overline{\sigma}) \leq \wp'(\overline{\sigma}) \text{ for any } \overline{\sigma} \in \overline{\mathcal{C}} \ .$$

Proposition:

Let τ be any irreducible representation of $\overline{G}_n$. Then there is precisely one $\wp = \wp(\tau) \in \mathrm{Div}^+(\overline{\mathcal{C}} \times \mathbb{N})$ such that τ is contained in $\pi_\wp$ but not in $\pi_{\wp'}$ for any $\wp' < \wp$; moreover τ occurs in $\pi_\wp$ with multiplicity 1, and τ occurs in $\pi_{\wp'}$ if and only if $\wp \leq \wp'$. Finally the map

$$\begin{array}{ccc} \dot{\bigcup\limits_n} \mathrm{Irr}(\overline{G}_n) & \xrightarrow{\sim} & \mathrm{Div}^+(\overline{\mathcal{C}} \times \mathbb{N}) \\ \tau & \longmapsto & \wp(\tau) \end{array}$$

is a bijection.

Proof: There are basically two different techniques to reduce the assertion to the corresponding assertion for symmetric groups in the previous section. One is to use Hecke algebra isomorphisms as in [HM] 1.5.3 and A.2.1. The other one, more in the spirit of our present point of view, is due to Zelevinsky. We sketch the idea very briefly. Let R_n be the Grothendieck group of $\mathcal{M}(\overline{G}_n)$ for $n \geq 1$ and $R_0 := \mathbb{Z}$. Using parabolic induction and restriction the direct sum

$$R := \bigoplus_{n \geq 0} R_n$$

possesses a natural structure as a Hopf algebra which in a certain sense is positive and self-adjoint (called a PSH-algebra in [Ze2]). For a given $\overline{\sigma} \in \overline{\mathcal{C}}$ let $R(\overline{\sigma})$ denote the sub-PSH-algebra generated by the irreducible representations with cuspidal support in $\{\omega_{s\overline{\sigma}} : s \geq 0\}$. According to [Ze2] §2 we have the tensor product decomposition

$$R \ = \bigotimes_{\overline{\sigma} \in \overline{\mathcal{C}}} R(\overline{\sigma}) \ .$$

On the other hand we put

$$R(S) \ := \bigoplus_{n \geq 0} R(S_n)$$

where $R(S_n)$ denotes the Grothendieck group of the symmetric group S_n for $n \geq 1$ and $R(S_0) := \mathbb{Z}$. This also has a natural PSH-algebra structure. The main point now is that a PSH-algebra is so rigid a structure that one can deduce the existence of exactly two isomorphisms of PSH-algebras between any $R(\overline{\sigma})$ and $R(S)$ ([Ze2] 4.19 and 6.3). We take the isomorphism which sends the sign character ε_s, for any $s \geq 1$, to the generalized Steinberg representation $\mathrm{st}(\overline{\sigma}, s)$.

Under this isomorphism the Proposition in the previous section translates into the present assertion.

In particular, calling $\vartheta_{\wp(\tau)} \in \Omega^{nd}(\overline{G}_n)$ the nondegenerate support of τ we obtain the nondegenerate support map

$$\overline{\nu}^{\mathrm{nd}} : \mathrm{Irr}(\overline{G}_n) \xrightarrow{\sim} \Omega^{nd}(\overline{G}_n)$$

which is a bijection and makes the diagram

$$\begin{tikzcd} & \mathrm{Irr}(\overline{G}_n) \arrow[dl, "\overline{\nu}^{\mathrm{nd}}"'] \arrow[dr, "\overline{\nu}"] & \\ \Omega^{nd}(\overline{G}_n) \arrow[rr, "\mathfrak{z}"'] & & \Omega(\overline{G}_n) \end{tikzcd}$$

commutative. The inverse of the bijection in the above Proposition will be denoted by $\wp \longmapsto \sigma_\wp$; the representation $\sigma_\wp$ is the analog of the representation $L(\tau_{\mathcal{P}}) = Q_{\mathcal{P}}(1)$ in section 2.
To mention an example we fix a divisor of the form $s\overline{\sigma} \in \mathrm{Div}^+(\overline{\mathcal{C}})$ of degree n. The subset $\{\wp : \overline{D}(\wp) = s\overline{\sigma}\} \subseteq \mathrm{Div}^+(\overline{\mathcal{C}} \times \mathbb{N})$ contains a unique maximal element $\wp_{\max}$ as well a unique minimal element $\wp_{\min}$ with respect to the partial order $\leq$; they are the functions supported on $\overline{\sigma}$ with value sP_1 and P_s, respectively. The map

$$\begin{array}{rcl} I(\pi_{\wp_{\max}}) & \xrightarrow{\sim} & \{\wp : \overline{D}(\wp) = s\overline{\sigma}\} \\ \tau & \longmapsto & \wp(\tau) \end{array}$$

is a bijection. The representation $\sigma_{\wp_{\max}}$ is a socalled generalized trivial representation; it is not contained in any other $\pi_{\wp'}$ for $\wp' \neq \wp_{\max}$. On the other hand $\sigma_{\wp_{\min}} = \pi_{\wp_{\min}} = \mathrm{st}(\overline{\sigma}, s)$, and the nondegenerate support of $\mathrm{st}(\overline{\sigma}, s)$ is $(\overline{G}_n, \mathrm{st}(\overline{\sigma}, s))$. If $\overline{\sigma} = 1$ is the trivial character of $\overline{G}_1$ then $\sigma_{\wp_{\max}}$, resp. $\sigma_{\wp_{\min}}$, is the trivial, resp. the Steinberg representation, of $\overline{G}_n$.
For any $\omega \in \Omega(\overline{G}_n)$ the primary decomposition of the category $\mathcal{M}(\omega)$ is given by

$$\mathcal{M}(\omega) = \prod_{\mathfrak{z}(\vartheta)=\omega} \mathcal{M}(\vartheta)$$

with

$$\mathcal{M}(\vartheta) := \begin{array}{l} \text{full subcategory in } \mathcal{M}(\overline{G}_n) \text{ of all} \\ \overline{G}_n\text{-representations whose irreducible} \\ \text{constituents all have nondegenerate} \\ \text{support } \vartheta\ . \end{array}$$

More precisely, if $\vartheta = \vartheta_\wp$ then $\mathcal{M}(\vartheta)$ is $\sigma_\wp$-primary.

5. The functor "$\kappa_{\max}$"

Bushnell and Kutzko in [BK1] and [BK2] construct K-types for the Bernstein components of $\mathcal{M}(G)$. Since our work heavily relies on their results we have to review the basic features of their construction. In this section we deal with the simple types. The starting point is the algebra $A := M_n(F)$ whose group of units is G. One considers elements $\beta \in A$ such that

- $E := F[\beta]$ is a subfield of A,
- $\beta \notin o_E$, and
- $k_{\mathrm{o}}(\beta, \mathfrak{A}(E)) < 0$ (see [BK1] (1.4.5) and (1.4.13)(ii)).

Let B denote the centralizer of β in A and fix a pair of hereditary o_E-orders $\mathfrak{B}_{\min} \subseteq \mathfrak{B}_{\max}$ in B such that $\mathfrak{B}_{\min}$ is minimal and $\mathfrak{B}_{\max}$ is maximal. For any hereditary o_E-order $\mathfrak{B}$ in B there is a unique hereditary o_F-order $\mathfrak{A} = \mathfrak{A}(\mathfrak{B})$ in A such that $E^\times$ normalizes $\mathfrak{A}$ and $\mathfrak{A} \cap B = \mathfrak{B}$. The map $\mathfrak{B} \longmapsto \mathfrak{A}(\mathfrak{B})$ is inclusion preserving. Bushnell/Kutzko associate with β and $\mathfrak{B}$ as above ([BK1] (3.1.14)) a compact open subgroup

$$J := J(\mathfrak{B}) := J(\beta, \mathfrak{A}(\mathfrak{B}))$$

of $\mathfrak{A}(\mathfrak{B})^\times$ such that

$$J = J^1 \cdot \mathfrak{B}^\times \quad \text{with} \quad J^1 := J^1(\mathfrak{B}) := J \cap U^1(\mathfrak{A}(\mathfrak{B})) .$$

The latter identity implies

$$\mathfrak{B}^\times / U^1(\mathfrak{B}) \ = \ J/J^1 .$$

We abbreviate $J_{\max} := J(\mathfrak{B}_{\max})$ and $J^1_{\max} := J^1(\mathfrak{B}_{\max})$. For the quotient of those two groups we have

$$J_{\max}/J^1_{\max} \ = \ \mathfrak{B}^\times_{\max}/U^1(\mathfrak{B}_{\max}) \ \cong \ GL_{n/[E:F]}(\mathbb{F}_E) .$$

Furthermore, $\mathfrak{B}^\times_{\min} J^1_{\max} \subseteq J_{\max}/J^1_{\max}$ is a Borel subgroup. The parabolic subgroups containing that Borel subgroup are the groups $\mathfrak{B}^\times J^1_{\max}/J^1_{\max}$ where $\mathfrak{B}$ is any hereditary o_E-order between $\mathfrak{B}_{\min}$ and $\mathfrak{B}_{\max}$. The unipotent radical, resp. the Levi quotient, of that parabolic subgroup is $U^1(\mathfrak{B})J^1_{\max}/J^1_{\max}$, resp. $\mathfrak{B}^\times/U^1(\mathfrak{B}) = J/J^1$, i.e.,

$$\begin{array}{ccccc} U^1(\mathfrak{B})J^1_{\max}/J^1_{\max} & \xrightarrow{\subseteq} & \mathfrak{B}^\times J^1_{\max}/J^1_{\max} & \twoheadrightarrow & J/J^1 . \\ & & \big\downarrow \subseteq & & \\ & & J_{\max}/J^1_{\max} & & \end{array}$$

In a next step Bushnell/Kutzko construct, depending on some additional data, families of representations $\{\kappa(\mathfrak{B})\}_{\mathfrak{B}}$ with $\mathfrak{B}_{\min} \subseteq \mathfrak{B} \subseteq \mathfrak{B}_{\max}$ where $\kappa(\mathfrak{B})$ is an irreducible smooth representation of the group $J(\mathfrak{B})$ ([BK1] (5.2)). These families satisfy a number of special properties of which we only recall the following one.

Lemma 1:

Put $\kappa := \kappa(\mathfrak{B})$, $\kappa_{\max} := \kappa(\mathfrak{B}_{\max})$, and $\mathfrak{A} := \mathfrak{A}(\mathfrak{B})$; the induced representations $\mathrm{Ind}_J^{\mathfrak{B}^\times U^1(\mathfrak{A})}(\kappa)$ and $\mathrm{Ind}_{\mathfrak{B}^\times J^1_{\max}}^{\mathfrak{B}^\times U^1(\mathfrak{A})}(\kappa_{\max})$ are irreducible and isomorphic.

Proof: [BK1] (5.2.5)(iii).

We include into the above considerations the "degenerate" case $\beta \in o_F$ by setting $B := A$, $J(\mathfrak{B}) := \mathfrak{B}^\times$, and $\kappa(\mathfrak{B}) := 1$ for such a β ([BK1] p. 184).

Definition:

A BK-type is a triple $(J, \kappa \otimes \sigma)$ where $J = J(\mathfrak{B}_{\mathrm{o}})$ and $\kappa = \kappa(\mathfrak{B}_{\mathrm{o}})$ for some choice of β, $\mathfrak{B}_{\min} \subseteq \mathfrak{B}_{\mathrm{o}} \subseteq \mathfrak{B}_{\max}$ and $\{\kappa(\mathfrak{B})\}_{\mathfrak{B}}$ and where σ is an irreducible cuspidal representation of J/J^1.

The BK-types fall into two distinct classes, the simple types ([BK1] (5.5.10)) and the split types of level $(x, 0)$ ([BK1] (8.1.2) and (8.1.4)).
We now fix a BK-type $(J, \kappa \otimes \sigma)$ where $J = J(\mathfrak{B}_{\mathrm{o}})$ and $\kappa = \kappa(\mathfrak{B}_{\mathrm{o}})$; we put $\overline{G} := J_{\max}/J^1_{\max}$ and $\kappa_{\max} := \kappa(\mathfrak{B}_{\max})$. In the following we will investigate the exact functor

$$\begin{array}{ccc} \mathcal{M}(G) & \longrightarrow & \mathcal{M}(\overline{G}) \\ V & \longmapsto & V(\kappa_{\max}) := \mathrm{Hom}_{J^1_{\max}}(\kappa_{\max}, V) \, . \end{array}$$

If V is irreducible then $V(\kappa_{\max})$ is finite dimensional. The $\overline{G}$-representation $V(\kappa_{\max})$ can be understood by studying its Jacquet modules. This means computing the coinvariants with respect to the unipotent radicals of the parabolic subgroups as representations of the Levi factors. Since we are dealing with finite group representations we may form the invariants instead since they are naturally isomorphic to the coinvariants. Let $\mathfrak{B}$ be any hereditary o_E-order between $\mathfrak{B}_{\min}$ and $\mathfrak{B}_{\max}$. Using Lemma 1 and Frobenius reciprocity and setting $\mathfrak{A} := \mathfrak{A}(\mathfrak{B})$ we compute the Jacquet module for the parabolic subgroup

$\mathfrak{B}^\times J^1_{\max}/J^1_{\max}$ as follows:

$$\begin{aligned}
V(\kappa_{\max})^{U^1(\mathfrak{B})J^1_{\max}/J^1_{\max}} &= \mathrm{Hom}_{U^1(\mathfrak{B})J^1_{\max}}(\kappa_{\max}, V) \\
&= \mathrm{Hom}_{U^1(\mathfrak{A})}(\mathrm{Ind}^{U^1(\mathfrak{A})}_{U^1(\mathfrak{B})J^1_{\max}}(\kappa_{\max}), V) \\
&= \mathrm{Hom}_{U^1(\mathfrak{A})}(\mathrm{Ind}^{\mathfrak{B}^\times U^1(\mathfrak{A})}_{\mathfrak{B}^\times J^1_{\max}}(\kappa_{\max}), V) \\
(*) \qquad &\cong \mathrm{Hom}_{U^1(\mathfrak{A})}(\mathrm{Ind}^{\mathfrak{B}^\times U^1(\mathfrak{A})}_{J(\mathfrak{B})}(\kappa(\mathfrak{B})), V) \\
&= \mathrm{Hom}_{U^1(\mathfrak{A})}(\mathrm{Ind}^{J(\mathfrak{B})U^1(\mathfrak{A})}_{J(\mathfrak{B})}(\kappa(\mathfrak{B})), V) \\
&= \mathrm{Hom}_{U^1(\mathfrak{A})}(\mathrm{Ind}^{U^1(\mathfrak{A})}_{J^1(\mathfrak{B})}(\kappa(\mathfrak{B})), V) \\
&= \mathrm{Hom}_{J^1(\mathfrak{B})}(\kappa(\mathfrak{B}), V) \ .
\end{aligned}$$

Because

$$\mathrm{Hom}_{J(\mathfrak{B})}(\kappa(\mathfrak{B}) \otimes \tau, V) \ = \ \mathrm{Hom}_{J(\mathfrak{B})/J^1(\mathfrak{B})}(\tau, \mathrm{Hom}_{J^1(\mathfrak{B})}(\kappa(\mathfrak{B}), V))$$

for any representation τ of $J(\mathfrak{B})/J^1(\mathfrak{B})$ the formula $(*)$ implies the following.

Lemma 2:

The $\overline{G}$-representation $V(\kappa_{\max})$ has an irreducible constituent with cuspidal support on the Levi factor $J(\mathfrak{B})/J^1(\mathfrak{B})$ if and only if there is an irreducible cuspidal representation τ of $J(\mathfrak{B})/J^1(\mathfrak{B})$ such that V contains the BK-type $(J(\mathfrak{B}), \kappa(\mathfrak{B}) \otimes \tau)$.

Proposition 3:

If the BK-type $(J, \kappa\otimes\sigma)$ is contained in each irreducible subquotient of the smooth G-representation V then all irreducible constituents of the $\overline{G}$-representation $V(\kappa_{\max})$ have cuspidal support σ on the Levi factor J/J^1.

Proof: Since the formation of $V(\kappa_{\max})$ is exact we may assume that V is irreducible. We then observe the following. If V contains the BK-type $(J, \kappa \otimes \sigma)$ then, according to [BK1] (7.2.17) and (8.3.3), the cuspidal support (M, π) of V can be determined as follows. Write

$$J/J^1 \cong GL_{m_1}(\mathbb{F}_E) \times \ldots \times GL_{m_r}(\mathbb{F}_E) \ \text{ and } \ \sigma \cong \sigma_1 \otimes \ldots \otimes \sigma_r \ .$$

Then up to conjugation one has $M = GL_{m_1[E:F]}(F) \times \ldots \times GL_{m_r[E:F]}(F)$ and $\pi = \pi_1 \otimes \ldots \otimes \pi_r$ where π_i is an irreducible supercuspidal representation of $GL_{m_i[E:F]}(F)$ which contains a maximal simple type of the form $(J_i, \kappa_i \otimes \sigma_i)$;

here (J_i, κ_i) only depends on (J, κ) and M and not on σ.
If V contains two BK-types $(J, \kappa \otimes \sigma)$ and $(J, \kappa \otimes \tau)$ then it follows that π_i contains the two simple types $(J_i, \kappa_i \otimes \sigma_i)$ and $(J_i, \kappa_i \otimes \tau_{w(i)})$ where $\tau \cong \tau_1 \otimes \ldots \otimes \tau_r$ is the corresponding decomposition and where w is an appropriate permutation of the factors of M. In this situation we must have $\sigma_i \cong \tau_{w(i)}$ by [BK1] (5.7.1) (more precisely, its proof) and hence $\sigma \cong w\tau$.
Consider now some irreducible constituent $\overline{\pi}$ of the $J_{\max}/J^1_{\max}$-representation $V(\kappa_{\max})$ and assume that $\overline{\pi}$ has cuspidal support τ on the Levi factor $J(\mathfrak{B})/J^1(\mathfrak{B})$ of $J_{\max}/J^1_{\max}$. According to the previous lemma V then contains the BK-type $(J(\mathfrak{B}), \kappa(\mathfrak{B}) \otimes \tau)$. As recalled above we may compute the Levi subgroup M from J as well as from $J(\mathfrak{B})$. This shows that the two Levi factors J/J^1 and $J(\mathfrak{B})/J^1(\mathfrak{B})$ are associate in $J_{\max}/J^1_{\max}$. Hence we may assume that $J = J(\mathfrak{B})$. We then are in the situation which we have discussed first and it follows that $\sigma \cong w\tau$.

From now on we assume $(J, \kappa \otimes \sigma)$ to be a simple type. Then there is a unique connected component $\Omega \subseteq \Omega(G)$ such that $(J, \kappa \otimes \sigma)$ is a K-type for the Bernstein component $\mathcal{M}(\Omega)$ ([BK1] (8.4)). In this case Prop. 3 says that we have the exact functor

$$\begin{array}{rcl} \text{“}\kappa_{\max}\text{”} : \mathcal{M}(\Omega) & \longrightarrow & \mathcal{M}(\omega) \\ V & \longmapsto & V(\kappa_{\max}) \end{array}$$

where $\omega \in \Omega(\overline{G})$ denotes the $\overline{G}$-orbit of the cuspidal pair $(J/J^1, \sigma)$.
Later on it will be important to have a description of this latter functor in terms of Hecke algebra modules. The category $\mathcal{M}(\Omega)$ is equivalent to the category $\mathrm{Mod}(\mathfrak{H}^{\mathrm{op}})$ of (left) unital $\mathfrak{H}^{\mathrm{op}}$-modules for the Hecke algebra $\mathfrak{H} := \mathcal{H}(G, J; \kappa \otimes \sigma)$ of the simple type $(J, \kappa \otimes \sigma)$. Recall that $\mathfrak{H}$ is the convolution algebra of all compactly supported functions $f : G \longrightarrow \mathrm{End}_{\mathbb{C}}(\kappa \otimes \sigma)$ such that

$$f(b_1 g b_2) = b_1 \circ f(g) \circ b_2 \quad \text{for all} \quad g \in G \text{ and } b_1, b_2 \in J \ .$$

The equivalence of categories is given by the functor

$$\begin{array}{rcl} \mathcal{M}(\Omega) & \xrightarrow{\sim} & \mathrm{Mod}(\mathfrak{H}^{\mathrm{op}}) \\ V & \longmapsto & \mathrm{Hom}_J(\kappa \otimes \sigma, V) \end{array}$$

([BK3] Thm. (4.3)(ii); note that our conventions about Hecke algebras differ slightly from those in [BK3] which is made good by working with the opposite algebras). Similarly we have the equivalence of categories

$$\begin{array}{rcl} \mathcal{M}(\omega) & \xrightarrow{\sim} & \mathrm{Mod}(\overline{\mathfrak{H}}^{\mathrm{op}}) \\ \overline{V} & \longmapsto & \mathrm{Hom}_{\overline{P}}(\sigma, \overline{V}) \end{array}$$

where $\overline{\mathfrak{H}} := \mathcal{H}(\overline{G}, \overline{P}; \sigma)$ is the Hecke algebra of σ viewed as a representation of the parabolic subgroup $\overline{P} := \mathfrak{B}^\times_{\mathrm{o}} J^1_{\max}/J^1_{\max}$. A quasi-inverse functor is given by

$\mathfrak{M} \longmapsto \mathrm{Ind}_{\overline{P}}^{\overline{G}}(\sigma) \underset{\overline{\mathfrak{H}}^{\mathrm{op}}}{\otimes} \mathfrak{M}$. The Hecke algebras $\mathfrak{H}$ and $\overline{\mathfrak{H}}$ are related in the following way. According to [BK1] (5.5.13), resp. (5.6.2) and (5.6.3), we have the natural and support preserving identifications

$$\mathcal{H}(G, J; \kappa \otimes \sigma) \;=\; \mathcal{H}(G, \mathfrak{B}_{\mathrm{o}}^{\times} J^1_{\max}; \kappa_{\max} \otimes \sigma)$$

and

$$\mathcal{H}(\overline{G}, \overline{P}; \sigma) \;=\; \mathcal{H}(J_{\max}, \mathfrak{B}_{\mathrm{o}}^{\times} J^1_{\max}; \kappa_{\max} \otimes \sigma)\ .$$

The lower right hand side is a subalgebra of the upper right hand side. Via these identifications we always can and will view $\overline{\mathfrak{H}}$ as a subalgebra of $\mathfrak{H}$. In particular we have the corresponding restriction functor $\mathrm{Mod}(\mathfrak{H}^{\mathrm{op}}) \xrightarrow{\mathrm{res}} \mathrm{Mod}(\overline{\mathfrak{H}}^{\mathrm{op}})$.

Lemma 4:

The diagram of functors

$$\begin{array}{ccc} \mathcal{M}(\Omega) & \xrightarrow{\sim} & \mathrm{Mod}(\mathfrak{H}^{\mathrm{op}}) \\ {\scriptstyle ``\kappa_{\max}"}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}} \\ \mathcal{M}(\omega) & \xrightarrow{\sim} & \mathrm{Mod}(\overline{\mathfrak{H}}^{\mathrm{op}}) \end{array}$$

is commutative (up to natural isomorphism).

Proof: As a consequence of the above identity $(*)$ we have

$$\begin{aligned} \mathrm{Hom}_{\overline{P}}(\sigma, V(\kappa_{\max})) \quad &= \mathrm{Hom}_J(\sigma, V(\kappa_{\max})^{U^1(\mathfrak{B}_{\mathrm{o}})J^1_{\max}}) \\ &= \mathrm{Hom}_J(\sigma, \mathrm{Hom}_{J^1}(\kappa, V)) \;=\; \mathrm{Hom}_J(\kappa \otimes \sigma, V)\ . \end{aligned}$$

Essentially as a consequence of Frobenius reciprocity this is compatible with the respective Hecke algebra module structures (use the interpretation of Hecke algebras as endomorphism rings as in [BK3] (2.6)).

In the case where V is a discrete series representation in $\mathcal{M}(\Omega)$ we will be able to explicitly compute $V(\kappa_{\max})$. Then V is irreducible and hence $\mathrm{Hom}_J(\kappa \otimes \sigma, V)$ is an irreducible module for the Hecke algebra $\mathfrak{H}^{\mathrm{op}}$. But the Hecke algebra of a simple type is isomorphic to a Iwahori-Hecke algebra in a discrete series preserving way ([BK1] (7.7.1) and (7.7.2)). The only discrete series representations of a general linear group with an Iwahori fixed vector are the unitary unramified twists of the Steinberg representation; this is a consequence of the Bernstein-Zelevinsky classification (compare the section 2). Their Iwahori-Hecke modules are 1-dimensional. It follows that the simple type $(J, \kappa \otimes \sigma)$ occurs in V with multiplicity 1. In other words

$$\mathrm{Hom}_J(\kappa \otimes \sigma, V) \;=\; \mathrm{Hom}_J(\sigma, \mathrm{Hom}_{J^1}(\kappa, V))$$

is 1-dimensional which means that σ occurs in $\mathrm{Hom}_{J^1}(\kappa, V)$ with multiplicity 1. On the other hand, according to Prop. 3, all the irreducible constituents of $\mathrm{Hom}_{J^1_{\max}}(\kappa_{\max}, V)$ have cuspidal support σ on J/J^1. This means that the irreducible constituents of the Jacquet module $\mathrm{Hom}_{J^1}(\kappa, V)$ all are of the form $w\sigma$ where conjugation by $w \in J_{\max}/J^1_{\max}$ induces an automorphism of the Levi factor J/J^1. Up to inner automorphisms this must be a permutation of the blocks of J/J^1. Since we are dealing with a simple type those blocks have the same size and σ is the outer tensor product of one and the same representation on each block. It follows that $w\sigma \cong \sigma$, i.e., that $\mathrm{Hom}_{J^1}(\kappa, V)$ is σ-isotypical. This shows the following.

Lemma 5:

If V is a discrete series representation of G containing the simple type $(J, \kappa \otimes \sigma)$ then $\mathrm{Hom}_{J^1}(\kappa, V)$ is isomorphic to σ as a J/J^1-representation.

At this point we want to start to use the facts which we have recalled in section 4. But to do so we have to be a little bit careful about the identification of the group $\overline{G} = J_{\max}/J^1_{\max}$ with $GL_{n/[E:F]}(\mathbb{F}_E)$. The hereditary o_E-order $\mathfrak{B}_{\min}$ defines an o_E-lattice chain

$$\mathfrak{L}_{\min} := \text{set of all } \mathfrak{B}_{\min}\text{−lattices in } F^n.$$

Corresponding to $\mathfrak{B}_{\max}$ we have in $\mathfrak{L}_{\min}$ the o_E-lattice subchain

$$\mathfrak{L}_{\max} := \text{set of all } \mathfrak{B}_{\max}\text{−lattices in } F^n.$$

There is an E-basis $\{v_1, \ldots, v_m\}$ with $m := n/[E:F]$ of F^n such that

$$\mathfrak{L}_{\min} \;=\; \{\mathfrak{p}_E^i v_1 \,+\, \ldots + \,\mathfrak{p}_E^i v_l \,+\, \mathfrak{p}_E^{i+1} v_{l+1} \,+ \ldots +\, \mathfrak{p}_E^{i+1} v_m \;:\; i \in \mathbb{Z}\;,\; 1 \le l \le m\}$$

and

$$\mathfrak{L}_{\max} = \{\mathfrak{p}_E^i v_1 + \ldots + \mathfrak{p}_E^i v_m : i \in \mathbb{Z}\}$$

where $\mathfrak{p}_E$ denotes the maximal ideal in o_E (compare [BK1] (1.1)). Clearly this basis induces an isomorphism

$$J_{\max}/J^1_{\max} \cong GL_m(\mathbb{F}_E)$$

under which $\mathfrak{B}_{\min}^{\times} J_{\max}^1 / J_{\max}^1$, resp. $\mathfrak{B}^{\times} J_{\max}^1 / J_{\max}^1$, corresponds to the upper triangular Borel subgroup, resp. to a standard parabolic subgroup. We always will use an identification arising in this way. If we choose another E-basis of F^n also describing $\mathfrak{L}_{\max} \subseteq \mathfrak{L}_{\min}$ as above then the two resulting identifications differ by conjugation with an element in the upper triangular Borel subgroup of $GL_m(\mathbb{F}_E)$. We have

$$J/J^1 \cong GL_{m/r}(\mathbb{F}_E) \times \ldots \times GL_{m/r}(\mathbb{F}_E)$$

with r factors, and σ is the outer tensor product of one and the same cuspidal representation $\overline{\sigma}_{\mathrm{o}}$ on each block $GL_{m/r}(\mathbb{F}_E)$. Hence $\omega = \omega_{r\overline{\sigma}_{\mathrm{o}}}$ in the notation of section 4, Fact $\overline{1}$.

The following more or less straightforward consequence of the theory of Bushnell-Kutzko is the basis of our later results. It was already noticed in [Vi1] III.5.13.

Proposition 6:

If V is an unramified twist of a discrete series representation of G containing the simple type $(J, \kappa \otimes \sigma)$ then $V(\kappa_{\max})$ is isomorphic to $\mathrm{st}(\overline{\sigma}_{\mathrm{o}}, \mathrm{r})$ as a $\overline{G}$-representation.

Proof: Since the assertion is concerned with the action of a compact open subgroup on which any unramified character is trivial we may assume that V is a discrete series representation. Proposition 3 and Lemma 5 together tell us that $V(\kappa_{\max})$ is an irreducible representation of $GL_m(\mathbb{F}_E)$. We have noted already that under the Hecke algebra isomorphism of Bushnell-Kutzko ([BK1] (5.6.6)) V corresponds to an unramified twist of the Steinberg representation. On the other hand it is shown in [SZ] 4.2 and 5.7 that the restriction of this isomorphism to $\overline{\mathfrak{H}}$ respects generalized Steinberg representations. We therefore are reduced to proving our assertion in the case where V is equal to the Steinberg representation. Since the Iwahori-Hecke module of the Steinberg representation is given by the sign character, the irreducible representation $V(1)$ of $GL_n(\mathbb{F}_F)$ in this case has to be $\mathrm{st}(1, \mathrm{n})$ (a direct proof of this latter fact which even works for integral coefficients is given in [SS] p. 120).

We point out that according to [BK1] (8.5.11) any discrete series representation of G contains a simple type.

As a final technical tool we need to know how the formation of $V(\kappa_{\max})$ behaves with respect to parabolic induction. A corresponding statement is contained in [Vi2] IV.5.4. But our approach will be different. Our connected component Ω contains the G-orbit of a cuspidal pair (M, π) such that M arises from a finest E-decomposition of F^n subordinate to the o_E-order $\mathfrak{B}_{\mathrm{o}}$ as described in [BK1] (7.1.11) and (7.1.13). Then

$$\overline{M} := (M \cap \mathfrak{B}_{\max}^{\times}) J_{\max}^1 / J_{\max}^1$$

is a Levi subgroup of the parabolic subgroup $\overline{P}$ in $\overline{G}$. Let now P denote the parabolic subgroup in G with Levi subgroup M and such that

$$\overline{P} = (P \cap \mathfrak{B}_{\max}^{\times})J_{\max}^{1}/J_{\max}^{1}$$

(this P is subordinate to $(J, \kappa\otimes\sigma)$ in the sense of [BK1] (7.2.18)). We then have the bijections

$$\begin{array}{ccc} \text{Levi subgroups of } G & \xrightarrow{\sim} & \text{Levi subgroups of } \overline{G} \\ \text{containing } M & & \text{containing } \overline{M} \\ N & \longmapsto & \overline{N} := (N \cap \mathfrak{B}_{\max}^{\times})J_{\max}^{1}/J_{\max}^{1} \end{array}$$

and

$$\begin{array}{ccc} \text{parabolic subgroups of } G & \longrightarrow & \text{parabolic subgroups of } \overline{G} \\ \text{containing } P & & \text{containing } \overline{P} \\ Q & \longmapsto & \overline{Q} := (Q \cap \mathfrak{B}_{\max}^{\times})J_{\max}^{1}/J_{\max}^{1}\ ; \end{array}$$

clearly, if N is a Levi subgroup of Q then $\overline{N}$ is a Levi subgroup of $\overline{Q}$. In the following we fix a Levi subgroup $M \subseteq N \subseteq G$ and let $P \subseteq Q \subseteq G$ be the parabolic subgroup with Levi factor N. The N-orbit of the cuspidal pair (M, π) lies in the unique connected component Ω_{o} of $\Omega(N)$ which under the canonical map $\Omega(N) \to \Omega(G)$ is mapped into Ω. According to [BK1] (7.2) (in particular (7.2.13)) a "simple type" for Ω_{o} is given by $(J \cap N, \kappa_U \otimes \sigma)$ where κ_U denotes the representation of $J \cap N$ which κ induces on its $(J \cap U)$-fixed vectors. Here U, resp. U^-, denotes the unipotent radical of Q, resp. of the opposite parabolic subgroup. Note that by [BK1] (7.1.15) we have $J \cap U = J^1 \cap U$ and similarly for U^- so that σ can be viewed as a cuspidal representation of $(J \cap N)/(J^1 \cap N)$. This pair $(J \cap N, \kappa_U \otimes \sigma)$ is a product of simple types on the individual blocks of the group N and therefore is a type for the category $\mathcal{M}(\Omega_{\mathrm{o}})$ in the sense of [BK3] (4.1). We want to see that the (normalized) parabolic induction functor

$$\mathrm{Ind}_Q^G(.) : \mathcal{M}(\Omega_{\mathrm{o}}) \longrightarrow \mathcal{M}(\Omega)$$

can be described through a homomorphism between the Hecke algebras of the two types involved. An appropriate axiomatic context for doing this was introduced in [BK3] (8.1) (and the comments at the end of § 8) under the notion of a G-cover. There is the small technical problem that $(J, \kappa \otimes \sigma)$ is not a G-cover of $(J \cap N, \kappa_U \otimes \sigma)$. We have to make use of the representation κ_Q of $J_Q := (J \cap Q)H^1$ which κ induces on its $(J \cap U)$-fixed vectors (here H^1 is a certain subgroup of J^1 normal in J whose precise definition is of no importance for us - compare [BK1] (3.1.15)). Clearly $\kappa_Q | J \cap N = \kappa_U$. But according to [BK1] (7.2.15) we have

$$\mathrm{Ind}_{J_Q}^{J}(\kappa_Q) = \kappa.$$

And the argument in [BK2], proof of Prop. 1.4, shows that $(J_Q, \kappa_Q \otimes \sigma)$ is a G-cover of $(J \cap N, \kappa_U \otimes \sigma)$. In this situation [BK3] (8.4) (and [BK1] (7.6.5) for the passage to normalized induction) and [BK1] (4.1.3) provide us with a canonical injective algebra homomorphism

$$\mathfrak{H}_{\mathrm{o}} := \mathcal{H}(N, J \cap N; \kappa_U \otimes \sigma) \hookrightarrow \mathcal{H}(G, J_Q; \kappa_Q \otimes \sigma) \xrightarrow{\cong} \mathcal{H}(G, J; \kappa \otimes \sigma) = \mathfrak{H}$$

such that the diagram of functors

$$\begin{array}{ccc} \mathcal{M}(\Omega_{\mathrm{o}}) & \xrightarrow{\sim} & \mathrm{Mod}(\mathfrak{H}_{\mathrm{o}}^{\mathrm{op}}) \\ {\scriptstyle \mathrm{Ind}_Q^G(.)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathfrak{H}_0^{\mathrm{op}}}(\mathfrak{H}^{\mathrm{op}},.)} \\ \mathcal{M}(\Omega) & \xrightarrow{\sim} & \mathrm{Mod}(\mathfrak{H}^{\mathrm{op}}) \end{array}$$

is commutative where the horizontal arrows are the equivalences of categories from [BK1] (8.4). Combining this with Lemma 4 we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{M}(\Omega_{\mathrm{o}}) & \xrightarrow{\sim} & \mathrm{Mod}(\mathfrak{H}_{\mathrm{o}}^{\mathrm{op}}) \\ {\scriptstyle \mathrm{Ind}_Q^G(.)(\kappa_{\max})}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}(\mathrm{Hom}_{\mathfrak{H}_0^{\mathrm{op}}}(\mathfrak{H}^{\mathrm{op}},.))} \\ \mathcal{M}(\omega) & \xrightarrow{\sim} & \mathrm{Mod}(\overline{\mathfrak{H}}^{\mathrm{op}}) \ . \end{array}$$

Next we consider the right perpendicular arrow. Let

$$F^n = \mathcal{V}^{(1)} \oplus \ldots \oplus \mathcal{V}^{(a)}$$

be the E-decomposition subordinate to the o_E-order $\mathfrak{B}_{\mathrm{o}}$ which gives rise to the Levi subgroup $N \subseteq G$. We put $A^{(i)} := \mathrm{End}_F(\mathcal{V}^{(i)})$ and $B^{(i)} := B \cap A^{(i)} = \mathrm{End}_E(\mathcal{V}^{(i)})$. Then $\mathfrak{B}^{(i)} := \mathfrak{B}_{\mathrm{o}} \cap A^{(i)}$ is a hereditary (in fact principal) o_E-order in $B^{(i)}$. We have

$$N = A^{(1)\times} \times \ldots \times A^{(a)\times} \ \text{ and } J \cap N = J(\mathfrak{B}^{(1)}) \times \ldots \times J(\mathfrak{B}^{(a)}).$$

The representations κ_U and $\sigma | J \cap N$ decompose accordingly into

$$\kappa_U = \kappa^{(1)} \otimes \ldots \otimes \kappa^{(a)} \ \text{ and } \sigma | J \cap N = \sigma^{(1)} \otimes \ldots \otimes \sigma^{(a)}$$

where $\kappa^{(i)}$ and $\sigma^{(i)}$ are representations of $J^{(i)} := J(\mathfrak{B}^{(i)})$. Moreover $\mathfrak{B}^{(i)}_{\max} := \mathfrak{B}_{\max} \cap A^{(i)}$ is a maximal hereditary o_E-order in $B^{(i)}$. Analogously as for $\mathfrak{B}_{\max}$

we have, for each $1 \le i \le a$, the group $J^{(i)}_{\max} := J(\mathfrak{B}^{(i)}_{\max})$ as well as the representation $\kappa^{(i)}_{\max}$ of $J^{(i)}_{\max}$ ([BK1] (5.2.14)). We put

$$\begin{aligned}
\mathfrak{B}(N) \; &:= \; \mathfrak{B}^{(1)} \; \times \; \ldots \; \times \; \mathfrak{B}^{(a)}, \\
J(N)_{\max} \; &:= \; J^{(1)}_{\max} \; \times \; \ldots \; \times \; J^{(a)}_{\max}, \\
J(N)^1_{\max} \; &:= \; (J^{(1)}_{\max})^1 \; \times \; \ldots \; \times \; (J^{(a)}_{\max})^1, \text{ and} \\
\kappa_{U,\max} \; &:= \; \kappa^{(1)}_{\max} \; \otimes \; \ldots \; \otimes \; \kappa^{(a)}_{\max}.
\end{aligned}$$

The image of the natural inclusion

$$\begin{array}{ccc}
J(N)_{\max}/J(N)^1_{\max} & & J_{\max}/J^1_{\max} \\
\Big\uparrow = & & \Big\uparrow = \\
\mathfrak{B}^{(1)\times}_{\max}/U^1(\mathfrak{B}^{(1)}_{\max}) \quad \times \ldots \times \quad \mathfrak{B}^{(a)\times}_{\max}/U^1(\mathfrak{B}^{(a)}_{\max}) & \hookrightarrow & \mathfrak{B}^{\times}_{\max}/U^1(\mathfrak{B}_{\max})
\end{array}$$

is the Levi subgroup $\overline{N}$ of $\overline{G}$. The preimage in the left hand side of the parabolic subgroup $\overline{P} = \mathfrak{B}^{\times}_{\mathrm{o}} J^1_{\max}/J^1_{\max}$ in the right hand side is the parabolic subgroup $\overline{N} \cap \overline{P} = \mathfrak{B}(N)^{\times} J(N)^1_{\max}/J(N)^1_{\max}$. The respective Levi factors are $J \cap N/(J \cap N)^1 \xrightarrow{\cong} J/J^1$. In the same way as the Hecke algebra $\overline{\mathfrak{H}}$ is contained in $\mathfrak{H}$ the Hecke algebra

$$\overline{\mathfrak{H}}_{\mathrm{o}} := \mathcal{H}(\overline{N}, \overline{N} \cap \overline{P}; \sigma)$$

is contained in $\mathfrak{H}_{\mathrm{o}}$. The injective algebra homomorphism $\mathfrak{H}_{\mathrm{o}} \hookrightarrow \mathfrak{H}$ restricts, since it is support preserving ([BK3] (7.2)), to a homomorphism $\overline{\mathfrak{H}}_{\mathrm{o}} \hookrightarrow \overline{\mathfrak{H}}$. The latter, of course, is nothing else than the Hecke algebra homomorphism coming from the above inclusion of a Levi subgroup. In [BK3] (11.4) it is shown that the multiplication map induces linear isomorphisms

$$\overline{\mathfrak{H}} \underset{\mathbb{C}}{\otimes} \mathfrak{H}_M \xrightarrow{\cong} \mathfrak{H} \;\text{ and }\; \overline{\mathfrak{H}}_{\mathrm{o}} \underset{\mathbb{C}}{\otimes} \mathfrak{H}_M \xrightarrow{\cong} \mathfrak{H}_{\mathrm{o}}$$

where $\mathfrak{H}_M$ is a certain Hecke algebra coming from the common supercuspidal support of Ω and Ω_{o} (the assumptions (11.2)(iii) and (11.1) are satisfied by [BK1] (5.6.6) and [BK1] (6.2.1), (6.2.2), (8.4.1), respectively). In order to see that $\overline{\mathfrak{H}}$ is indeed the algebra which in loc. cit. is denoted by $\mathcal{K}$ we only have to observe that

$$\begin{aligned}
\overline{\mathfrak{H}} \; &= \; \mathcal{H}(J_{\max}, \mathfrak{B}^{\times}_{\mathrm{o}} J^1_{\max}; \kappa_{\max} \otimes \sigma) \\
&= \; \mathcal{H}(\mathfrak{B}^{\times}_{\max} J^1_{\max}, \mathfrak{B}^{\times}_{\mathrm{o}} J^1_{\max}; \kappa_{\max} \otimes \sigma) \\
&= \; \mathcal{H}(\mathfrak{A}(\mathfrak{B}_{\max})^{\times}, J; \kappa \otimes \sigma) \\
&= \; \mathcal{K}
\end{aligned}$$

by [BK1] (5.5.11) and (5.5.13) (and similarly for $\overline{\mathfrak{H}}_{\mathrm{o}}$). This implies that a basis of $\overline{\mathfrak{H}}$ as a right $\overline{\mathfrak{H}}_{\mathrm{o}}$-module at the same time is a basis of $\mathfrak{H}$ as a right $\mathfrak{H}_{\mathrm{o}}$-module. It follows that

$$\mathrm{Hom}_{\mathfrak{H}_{\mathrm{o}}^{\mathrm{op}}}(\mathfrak{H}^{\mathrm{op}}, \mathrm{Hom}_{J\cap N}(\kappa_U \otimes \sigma, .)) \ = \ \mathrm{Hom}_{\overline{\mathfrak{H}}_{\mathrm{o}}^{\mathrm{op}}}(\overline{\mathfrak{H}}^{\mathrm{op}}, \mathrm{Hom}_{J\cap N}(\kappa_U \otimes \sigma, .))$$

as left $\overline{\mathfrak{H}}^{\mathrm{op}}$-modules. Hence we may rewrite our last commutative diagram as

$$\begin{array}{ccc}
\mathcal{M}(\Omega_{\mathrm{o}}) & \xrightarrow{\sim} & \mathrm{Mod}(\mathfrak{H}_{\mathrm{o}}^{\mathrm{op}}) \\
\mathrm{Ind}_Q^G(.)(\kappa_{\max})\Big\downarrow & & \Big\downarrow \mathrm{Hom}_{\overline{\mathfrak{H}}_0^{\mathrm{op}}}(\overline{\mathfrak{H}}^{\mathrm{op}}, \mathrm{res}(.)) \\
\mathcal{M}(\omega) & \xrightarrow{\sim} & \mathrm{Mod}(\overline{\mathfrak{H}}^{\mathrm{op}}) \ .
\end{array}$$

Denoting by $\omega_{\mathrm{o}} \in \Omega(\overline{N})$ the $\overline{N}$-orbit of the cuspidal pair $(J \cap N/(J \cap N)^1, \sigma) \cong (J/J^1, \sigma)$ the analog of Lemma 4 gives the commutative diagram

$$\begin{array}{ccc}
\mathcal{M}(\Omega_{\mathrm{o}}) & \xrightarrow{\sim} & \mathrm{Mod}(\mathfrak{H}_{\mathrm{o}}^{\mathrm{op}}) \\
“\kappa_{U,\max}“\Big\downarrow & & \Big\downarrow \mathrm{res} \\
\mathcal{M}(\omega_{\mathrm{o}}) & \xrightarrow{\sim} & \mathrm{Mod}(\overline{\mathfrak{H}}_{\mathrm{o}}^{\mathrm{op}}) \ .
\end{array}$$

Finally, on the level of finite groups, we have the commutative diagram

$$\begin{array}{ccc}
\mathcal{M}(\omega_{\mathrm{o}}) & \xrightarrow{\sim} & \mathrm{Mod}(\overline{\mathfrak{H}}_{\mathrm{o}}^{\mathrm{op}}) \\
\mathrm{Ind}_{\overline{Q}}^{\overline{G}}(.)\Big\downarrow & & \Big\downarrow \mathrm{Hom}_{\overline{\mathfrak{H}}_{\mathrm{o}}^{\mathrm{op}}}(\overline{\mathfrak{H}}^{\mathrm{op}}, .) \\
\mathcal{M}(\omega) & \xrightarrow{\sim} & \mathrm{Mod}(\overline{\mathfrak{H}}^{\mathrm{op}}) \ .
\end{array}$$

The combination of the last three diagrams amounts to the following result.

Proposition 7:

The diagram of functors

$$\begin{array}{ccc}
\mathcal{M}(\Omega_{\mathrm{o}}) & \xrightarrow{\mathrm{Ind}_Q^G(.)} & \mathcal{M}(\Omega) \\
“\kappa_{U,\max}“\Big\downarrow & & \Big\downarrow “\kappa_{\max}“ \\
\mathcal{M}(\omega_{\mathrm{o}}) & \xrightarrow[\mathrm{Ind}_{\overline{Q}}^{\overline{G}}(.)]{} & \mathcal{M}(\omega)
\end{array}$$

is commutative (up to natural isomorphism).

Corollary 8:

Let $\wp \in \mathrm{Div}^+(\overline{\mathcal{C}} \times \mathbb{N})$ be the partition valued function supported on $\overline{\sigma}_{\mathrm{o}}$ such that $N_\wp$ is conjugate to $\overline{N}$ in $\overline{G}$; if V_{o} is an unramified twist of a discrete series representation in $\mathcal{M}(\Omega_{\mathrm{o}})$ then we have

$$\mathrm{Ind}_Q^G(V_{\mathrm{o}})(\kappa_{\max}) \;\cong\; \pi_\wp \ .$$

Proof: Combine the Prop. 6 and 7.

Recall that $\omega = \omega_{\overline{D}}$ with $\overline{D} = r\overline{\sigma}_{\mathrm{o}}$. Correspondingly the component Ω is of the form (since it corresponds to a simple type) $\Omega = \Omega_D$ with $D = r\sigma_{\mathrm{o}}$ where $\sigma_{\mathrm{o}} \in \mathcal{C}$ is a supercuspidal representation of $GL_{n/r}(F)$. The irreducible representations in the category $\mathcal{M}(\omega)$ are, up to isomorphism, the $\sigma_\wp$ (as constructed in section 4) for $\wp \in \mathrm{Div}^+(\overline{\mathcal{C}} \times \mathbb{N})$ such that $\overline{D}(\wp) = r\overline{\sigma}_{\mathrm{o}}$. We use the bijection

$$\begin{aligned} \{\mathcal{P} \in \mathrm{Div}^+(\mathcal{C} \times \mathbb{N}) : D(\mathcal{P}) = r\sigma_{\mathrm{o}}\} \quad &\xrightarrow{\sim} \quad \{\wp \in \mathrm{Div}^+(\overline{\mathcal{C}} \times \mathbb{N}) : \overline{D}(\wp) = r\overline{\sigma}_{\mathrm{o}}\} \\ \mathcal{P} \quad &\longmapsto \quad \overline{\mathcal{P}} \ \text{ with } \ \overline{\mathcal{P}}(\overline{\sigma}_{\mathrm{o}}) = \mathcal{P}(\sigma_{\mathrm{o}}) \end{aligned}$$

in order to re-index the $\sigma_\wp$ and $\pi_\wp$ in $\mathcal{M}(\omega)$ by setting

$$\sigma_{\mathcal{P}} := \sigma_{\overline{\mathcal{P}}} \ \text{ and } \ \pi_{\mathcal{P}} := \pi_{\overline{\mathcal{P}}} \ .$$

Proposition 9:

Let $\mathcal{P} \in \mathrm{Div}^+(\mathcal{C} \times \mathbb{N})$ with $D(\mathcal{P}) = D$; if V in $\mathcal{M}(\Omega)$ is parabolically induced from an unramified twist of a discrete series representation on $N_{\mathcal{P}}$ then we have $V(\kappa_{\max}) \cong \pi_{\mathcal{P}}$.

Proof: This is a reformulation of Corollary 8 once one makes the observation that, if N is conjugate to $N_{\mathcal{P}}$ in G, then $\overline{N}$ is conjugate to $N_{\overline{\mathcal{P}}}$ in $\overline{G}$.

Proposition 10:

Let $\mathcal{P} \in \mathrm{Div}^+(\mathcal{C} \times \mathbb{N})$ with $D(\mathcal{P}) = D$ and let V be an irreducible representation in $\mathcal{M}(\Omega)$ whose isomorphism class lies in $\mathrm{im}(Q_{\mathcal{P}})$; we then have:

i. *$V(\kappa_{\max})$ contains $\sigma_{\mathcal{P}}$ with multiplicity one;*

ii. *if $\sigma_{\mathcal{P}'}$ is contained in $V(\kappa_{\max})$ then $\mathcal{P}' \leq \mathcal{P}$;*

iii. *if V is tempered and $\mathcal{P}' \leq \mathcal{P}$ then $\sigma_{\mathcal{P}'}$ is contained in $V(\kappa_{\max})$.*

Proof: The representation V being of the form $V = L(\alpha\tau_{\mathcal{P}})$ for some $\alpha \in X_{nr}(N_{\mathcal{P}})$ is a specific constituent of the G-representation $\mathrm{Ind}(\alpha\tau_{\mathcal{P}})$ parabolically induced from $\alpha\tau_{\mathcal{P}}$. The previous Prop. 9 implies that $V(\kappa_{\max})$ is contained in $\mathrm{Ind}(\alpha\tau_{\mathcal{P}})(\kappa_{\max}) \cong \pi_{\mathcal{P}}$. By the Prop. in section 4 the $\sigma_{\mathcal{P}'}$ contained in $\pi_{\mathcal{P}}$ are precisely those with $\mathcal{P}' \leq \mathcal{P}$. This shows the second assertion as well as the third one since, for a tempered V, the character α must be unitary and $V = L(\alpha\tau_{\mathcal{P}}) = \mathrm{Ind}(\alpha\tau_{\mathcal{P}})$. For a general α the Lemma in section 2 says that any irreducible constituent of $\mathrm{Ind}(\alpha\tau_{\mathcal{P}})$ different from $L(\alpha\tau_{\mathcal{P}})$ also is a constituent of $\mathrm{Ind}(\alpha'\tau_{\mathcal{P}'})$ for some $\mathcal{P}' < \mathcal{P}$. The $\sigma_{\mathcal{P}}$ therefore must be contained in $L(\alpha\tau_{\mathcal{P}})(\kappa_{\max})$; the multiplicity is one again by the Prop. in section 4.

In order to extend the formalism of the functor "$\kappa_{\max}$" to the case of a Bernstein component whose K-type is semisimple (which we will do in the next section) we have to modify our point of view in the following way. First let us take the opportunity to introduce some further notation. Whenever H_0 is a compact open subgroup in a locally compact and totally disconnected group H and λ is an irreducible smooth representation of H_0 we put

$\mathcal{M}_\lambda(H) :=$ full subcategory of all smooth H-representations which as an H-representation are generated by their λ-isotypic component.

For any smooth H-representation V we let V^λ denote its λ-isotypical component and, for any compact open subgroup $H_0 \subseteq H_1 \subseteq H$, we define

$H_1 \cdot V^\lambda :=$ the H_1-subrepresentation of V generated by V^λ

as well as the functor

$$\begin{aligned} T_{H_1,\lambda} : \mathcal{M}(H) &\longrightarrow \mathcal{M}_\lambda(H_1) \\ V &\longmapsto H_1 \cdot V^\lambda . \end{aligned}$$

Fact: *If H is compact then the functors $V \longmapsto \mathrm{Hom}_{H_0}(\lambda, V)$ and $\mathfrak{M} \longmapsto \mathrm{Ind}_{H_0}^{H}(\lambda) \underset{\mathcal{H}(H,H_0;\lambda)^{\mathrm{op}}}{\otimes} \mathfrak{M}$ are quasi-inverse equivalences of categories between $\mathcal{M}_\lambda(H)$ and $\mathrm{Mod}(\mathcal{H}(H, H_0; \lambda)^{\mathrm{op}})$.*

Example: *If (J, λ) is a K-type for a Bernstein component $\mathcal{M}(\Omega)$ then we have $\mathcal{M}_\lambda(G) = \mathcal{M}(\Omega)$.*

Going back to our Bernstein component $\mathcal{M}(\Omega)$ with a simple K-type $(J, \kappa \otimes \sigma)$ we put $\lambda := \kappa \otimes \sigma$ and we fix a maximal compact subgroup $K \subseteq G$ such that $\mathfrak{A}(\mathfrak{B}_{\max})^\times \subseteq K$. Then the Hecke algebra isomorphism $\overline{\mathfrak{H}} \xrightarrow{\cong} \mathcal{H}(K, J; \lambda) = \mathcal{K}$

which we discussed in the proof of Prop. 7 (recall that $JB^\times J \cap K = J\mathfrak{B}^\times_{\max} J$) shows that the induction functor

$$\mathrm{Ind}^K_{J_{\max}}(\kappa_{\max} \otimes\ .\ ) : \mathcal{M}(\omega) \ = \ \mathcal{M}_\sigma(J_{\max}/J^1_{\max}) \ \xrightarrow{\sim} \ \mathcal{M}_\lambda(K)$$

is an equivalence of categories (here σ has to be viewed as a representation of the parabolic subgroup $\mathfrak{B}^\times_{\mathrm{o}} J^1_{\max}/J^1_{\max}$). For any V in $\mathcal{M}(\Omega)$ we compute

$$\begin{aligned}
\mathrm{Ind}^K_{J_{\max}}(\kappa_{\max} \otimes V(\kappa_{\max})) &= \mathrm{Ind}^K_{J_{\max}}(\kappa_{\max} \underset{\mathbb{C}}{\otimes} \mathrm{Ind}^{J_{\max}}_{\mathfrak{B}^\times_{\mathrm{o}} J_{\max}}(\sigma) \underset{\overline{\mathfrak{H}}^{\mathrm{op}}}{\otimes} \mathrm{Hom}_J(\lambda, V)) \\
&= \mathrm{Ind}^K_{\mathfrak{B}^\times_{\mathrm{o}} J^1_{\max}}(\kappa_{\max} \otimes \sigma) \underset{\overline{\mathfrak{H}}^{\mathrm{op}}}{\otimes} \mathrm{Hom}_J(\lambda, V) \\
&= \mathrm{Ind}^K_J(\lambda) \underset{\mathcal{H}(K,J;\lambda)^{\mathrm{op}}}{\otimes} \mathrm{Hom}_J(\lambda, V) \\
&= K \cdot V^\lambda
\end{aligned}$$

where the first, resp. third, identity uses Lemma 4, resp. Lemma 1. For the purposes of the next section it will be necessary to work not with the functor "$\kappa_{\max}$" but instead with the "equivalent" functor

$$\begin{aligned}
T_{K,\lambda} : \mathcal{M}(\Omega) &\longrightarrow \mathcal{M}_\lambda(K) \\
V &\longmapsto K \cdot V^\lambda .
\end{aligned}$$

In order to reformulate Prop. 10 in terms of this new functor we define

$$\sigma_{\mathcal{P}}(\lambda) := \mathrm{Ind}^K_{J_{\max}}(\kappa_{\max} \otimes \sigma_{\mathcal{P}})$$

for any $\mathcal{P}$ with $D(\mathcal{P}) = D$. These $\sigma_{\mathcal{P}}(\lambda)$ are, up to isomorphism, the irreducible representations in the category $\mathcal{M}_\lambda(K)$.

Proposition 11:

Let $\mathcal{P} \in \mathrm{Div}^+(\mathcal{C} \times \mathbb{N})$ with $D(\mathcal{P}) = D$ and let V be an irreducible representation in $\mathcal{M}(\Omega)$ whose isomorphism class lies in $\mathrm{im}(Q_{\mathcal{P}})$; we then have:

i. V contains $\sigma_{\mathcal{P}}(\lambda)$ with multiplicity one;

ii. if $\sigma_{\mathcal{P}'}(\lambda)$ is contained in V then $\mathcal{P}' \le \mathcal{P}$;

iii. if V is tempered and $\mathcal{P}' \le \mathcal{P}$ then $\sigma_{\mathcal{P}'}(\lambda)$ is contained in V.

6. The functor $T_{K,\lambda}$

In this section $\mathcal{M}(\Omega)$ is an arbitrary but fixed Bernstein component of $\mathcal{M}(G)$. In [BK2] a K-type for $\mathcal{M}(\Omega)$ is constructed which is called semisimple. Again we have to start by briefly reviewing this construction.

As we explained in section 2 the connected component Ω is of the form $\Omega = \Omega_D$ for a unique divisor $D = \sum m_\tau \tau$ in $\mathrm{Div}^+(\mathcal{C})$ of degree n. The representations in $\mathcal{M}(\Omega)$ have cuspidal support on the Levi subgroup $M_D = \prod_\tau (G_{d(\tau)})^{\times m_\tau}$ which is contained in the larger Levi subgroup

$$\tilde{M}_D := \prod_\tau G_{m_\tau d(\tau)}$$

(this is the group denoted by M in [BK2] 1.3). Viewing $D_\tau := m_\tau \tau$ as a divisor for $G_{m_\tau d(\tau)}$ we obtain the connected component $\Omega_{D_\tau} \subseteq \Omega(G_{m_\tau d(\tau)})$. The corresponding Bernstein component $\mathcal{M}(\Omega_{D_\tau})$ of $\mathcal{M}(G_{m_\tau d(\tau)})$ has a simple type $(J^{(\tau)}, \lambda^{(\tau)})$. In order to be consistent with the formalism of G-covers one has to replace $(J^{(\tau)}, \lambda^{(\tau)})$ by a certain pair $(J_{\mathrm{o}}^{(\tau)}, \lambda_{\mathrm{o}}^{(\tau)})$ which in particular satisfies $J_{\mathrm{o}}^{(\tau)} \subseteq J^{(\tau)}$ and $\mathrm{Ind}_{J_{\mathrm{o}}^{(\tau)}}^{J^{(\tau)}}(\lambda_{\mathrm{o}}^{(\tau)}) = \lambda^{(\tau)}$ and consequently

$$\mathcal{H}(G_{m_\tau d(\tau)}, J_{\mathrm{o}}^{(\tau)}; \lambda_{\mathrm{o}}^{(\tau)}) \xrightarrow{\cong} \mathcal{H}(G_{m_\tau d(\tau)}, J^{(\tau)}; \lambda^{(\tau)})$$

([BK2] 1.4; we already dealt with such a modification in the proof of Prop. 5.7). Set

$$\tilde{J} := \prod_\tau J_{\mathrm{o}}^{(\tau)} \quad \text{and} \quad \tilde{\lambda} := \bigotimes_\tau \lambda_{\mathrm{o}}^{(\tau)} .$$

Then $(\tilde{J}, \tilde{\lambda})$ is a K-type for the Bernstein component $\mathcal{M}(\tilde{\Omega})$ of $\mathcal{M}(\tilde{M}_D)$ where $\tilde{\Omega} := \prod_\tau \Omega_{D_\tau} \subseteq \Omega(\tilde{M}_D)$ ([BK2] 1.5). The main theorem in [BK2] says that there exists a G-cover (J, λ) of $(\tilde{J}, \tilde{\lambda})$. This means in particular that:

- $\mathcal{M}(\Omega) = \mathcal{M}_\lambda(G)$;
- for any parabolic subgroup $\tilde{Q} \subseteq G$ with Levi factor $\tilde{M}_D$ there is a support preserving Hecke algebra isomorphism

  $$j_{\tilde{Q}} : \mathcal{H}(\tilde{M}_D, \tilde{J}; \tilde{\lambda}) \xrightarrow{\cong} \mathcal{H}(G, J; \lambda)$$

  which describes the parabolic induction functor

  $$\mathrm{Ind}_{\tilde{Q}}^G(.) \ : \ \mathcal{M}(\tilde{\Omega}) \xrightarrow{\sim} \mathcal{M}(\Omega) \ ;$$

  in particular this functor is an equivalence of categories.

The actual G-cover (J, λ) as constructed in [BK2] p. 46 (to which we refer as a semisimple K-type for $\mathcal{M}(\Omega)$) has the technical property that $J \subseteq \mathfrak{A}^\times$ where $\mathfrak{A}$ is a hereditary o_F-order in A coming from a certain lattice chain in F^n. In the following the maximal compact subgroup $K \subseteq G$ is always understood to be the stabilizer of one fixed lattice in this chain. Since $j_{\tilde{Q}}$ is support preserving it restricts to an isomorphism

$$\mathcal{H}(K \cap \tilde{M}_D, \tilde{J}; \tilde{\lambda}) \xrightarrow{\cong} \mathcal{H}(K, J; \lambda)$$

which in turn induces (by the Fact at the end of the last section) an equivalence of categories

$$\mathcal{J}_{\tilde{Q}} : \mathcal{M}_{\tilde{\lambda}}(K \cap \tilde{M}_D) \xrightarrow{\sim} \mathcal{M}_\lambda(K) .$$

Proposition 1:

The diagram of functors

$$\begin{array}{ccc}
\mathcal{M}(\tilde{\Omega}) & \xrightarrow{\mathrm{Ind}_{\tilde{Q}}^{G}(.)} & \mathcal{M}(\Omega_D) \\
{\scriptstyle T_{K\cap \tilde{M}_D,\tilde{\lambda}}}\big\downarrow & & \big\downarrow{\scriptstyle T_{K,\lambda}} \\
\mathcal{M}_{\tilde{\lambda}}(K \cap \tilde{M}_D) & \xrightarrow[\mathcal{J}_{\tilde{Q}}]{} & \mathcal{M}_\lambda(K)
\end{array}$$

is commutative (up to natural isomorphism).

Proof: Let V_{o} be a representation in $\mathcal{M}(\tilde{\Omega})$. By the fundamental property of the Hecke algebra isomorphism $j_{\tilde{Q}}$ recalled above we have

$$\begin{aligned}
\text{(1)} \qquad \mathrm{Hom}_J(\lambda, \mathrm{Ind}_{\tilde{Q}}^{G}(V_{\mathrm{o}})) &= \mathrm{Hom}_{\mathcal{H}(\tilde{M}_D,\tilde{J};\tilde{\lambda})^{\mathrm{op}}}(\mathcal{H}(G,J;\lambda)^{\mathrm{op}}, \mathrm{Hom}_{\tilde{J}}(\tilde{\lambda}, V_{\mathrm{o}})) \\
&= \mathrm{Hom}_{\tilde{J}}(\tilde{\lambda}, V_{\mathrm{o}}) .
\end{aligned}$$

Secondly, using the abbreviation $A := \mathcal{H}(K \cap \tilde{M}_D, \tilde{J}; \tilde{\lambda})$, the functor $\mathcal{J}_{\tilde{Q}}$ is constructed as

$$\text{(2)} \qquad \mathcal{J}_{\tilde{Q}}(.) \;=\; \mathrm{Ind}_J^K(\lambda) \underset{A^{\mathrm{op}}}{\otimes} \mathrm{Hom}_{\tilde{J}}(\tilde{\lambda}, .) .$$

Finally the Fact at the end of the last section implies that

$$\text{(3)} \qquad (K \cap \tilde{M}_D) \cdot (V_{\mathrm{o}})^{\tilde{\lambda}} \;=\; \mathrm{Ind}_{\tilde{J}}^{K\cap \tilde{M}_D}(\tilde{\lambda}) \underset{A^{\mathrm{op}}}{\otimes} \mathrm{Hom}_{\tilde{J}}(\tilde{\lambda}, V_{\mathrm{o}})$$

and that

$$\text{(4)} \qquad K \cdot V^\lambda \;=\; \mathrm{Ind}_J^K(\lambda) \underset{\mathcal{H}(K,J;\lambda)^{\mathrm{op}}}{\otimes} \mathrm{Hom}_J(\lambda, V) \;\; \text{for any } \; V \text{ in } \mathcal{M}(\Omega) .$$

Based on these four facts we now compute

$$\begin{aligned}
\mathcal{J}_{\tilde{Q}}((K\cap \tilde{M}_D)\cdot (V_{\mathrm{o}})^{\tilde{\lambda}}) &\overset{(3)}{=} \mathcal{J}_{\tilde{Q}}(\mathrm{Ind}_{\tilde{J}}^{K\cap\tilde{M}_D}(\tilde{\lambda}) \underset{A^{\mathrm{op}}}{\otimes} \mathrm{Hom}_{\tilde{J}}(\tilde{\lambda}, V_{\mathrm{o}}))\\
&\overset{(2)}{=} \mathrm{Ind}_J^K(\lambda) \underset{A^{\mathrm{op}}}{\otimes} \mathrm{Hom}_{\tilde{J}}(\tilde{\lambda}, \mathrm{Ind}_{\tilde{J}}^{K\cap\tilde{M}_D}(\tilde{\lambda}) \underset{A^{\mathrm{op}}}{\otimes} \mathrm{Hom}_{\tilde{J}}(\tilde{\lambda}, V_{\mathrm{o}}))\\
&= \mathrm{Ind}_J^K(\lambda) \underset{A^{\mathrm{op}}}{\otimes} \mathrm{Hom}_{\tilde{J}}(\tilde{\lambda}, \mathrm{Ind}_{\tilde{J}}^{K\cap\tilde{M}_D}(\tilde{\lambda})) \underset{A^{\mathrm{op}}}{\otimes} \mathrm{Hom}_{\tilde{J}}(\tilde{\lambda}, V_{\mathrm{o}})\\
&= \mathrm{Ind}_J^K(\lambda) \underset{A^{\mathrm{op}}}{\otimes} \mathrm{Hom}_{\tilde{J}}(\tilde{\lambda}, V_{\mathrm{o}})\\
&\overset{(1)}{=} \mathrm{Ind}_J^K(\lambda) \underset{A^{\mathrm{op}}}{\otimes} \mathrm{Hom}_{J}(\lambda, \mathrm{Ind}_{\tilde{Q}}^G(V_{\mathrm{o}}))\\
&\overset{(4)}{=} K\cdot \mathrm{Ind}_{\tilde{Q}}^G(V_{\mathrm{o}})^{\lambda}
\end{aligned}$$

where the fourth identity comes from Frobenius reciprocity which implies

$$\mathrm{Hom}_{\tilde{J}}(\tilde{\lambda}, \mathrm{Ind}_{\tilde{J}}^{K\cap\tilde{M}_D}(\tilde{\lambda})) \;=\; \mathrm{End}_{K\cap\tilde{M}_D}(\mathrm{Ind}_{\tilde{J}}^{K\cap\tilde{M}_D}(\tilde{\lambda})) \;=\; A\,. \qquad \square$$

By construction we have

$$K\cap\tilde{M}_D = \prod_{m_\tau\neq 0} K^{(\tau)}$$

with $K^{(\tau)}$ a maximal compact subgroup in $G_{m_\tau d(\tau)}$ of the kind we considered in section 5. There is the obvious tensor product functor

$$\prod_{m_\tau\neq 0} \mathcal{M}_{\lambda^{(\tau)}}(K^{(\tau)}) \;=\; \prod_{m_\tau\neq 0} \mathcal{M}_{\lambda_{\mathrm{o}}^{(\tau)}}(K^{(\tau)}) \;\longrightarrow\; \mathcal{M}_{\tilde{\lambda}}(K\cap\tilde{M}_D)\,.$$

On the other hand any $\mathcal{P}\in \mathrm{Div}^+(\mathcal{C}\times \mathbb{N})$ with $D(\mathcal{P}) = D$ can be decomposed into

$$\mathcal{P} = \sum_{m_\tau\neq 0} \mathcal{P}_\tau \quad \text{where } \mathcal{P}_\tau \text{ is supported on } \tau \text{ with } \mathcal{P}_\tau(\tau) = \mathcal{P}(\tau)\,.$$

We now define, for every such $\mathcal{P}$, the representation

$$\sigma_{\mathcal{P}}(\lambda) := \mathcal{J}_{\tilde{Q}}(\underset{m_\tau\neq 0}{\otimes} \sigma_{\mathcal{P}_\tau}(\lambda^{(\tau)}))$$

in $\mathcal{M}_\lambda(K)$. These $\sigma_{\mathcal{P}}(\lambda)$ constitute, up to isomorphism, all the irreducible representations in $\mathcal{M}_\lambda(K)$.

Proposition 2:

Let $\mathcal{P} \in \mathrm{Div}^+(\mathcal{C} \times \mathbb{N})$ with $D(\mathcal{P}) = D$ and let V be an irreducible representation in $\mathcal{M}(\Omega)$ whose isomorphism class lies in $\mathrm{im}(Q_{\mathcal{P}})$*; we then have:*

i. V contains $\sigma_{\mathcal{P}}(\lambda)$ with multiplicity one;

ii. if $\sigma_{\mathcal{P}'}(\lambda)$ is contained in V then $\mathcal{P}' \le \mathcal{P}$;

iii. if V is tempered and $\mathcal{P}' \le \mathcal{P}$ then $\sigma_{\mathcal{P}'}(\lambda)$ is contained in V.

Proof: This is an immediate consequence of Prop. 1 and Prop. 5.11.

7. Tempered K-types

In this section we fix a connected component $\Theta \subseteq \Omega^t(G)$ and we let $\Omega \subseteq \Omega(G)$ denote the unique connected component such that $\mathfrak{z}(\Theta) \subseteq \Omega$. In section 2 we saw that the connected components of $\Omega^t(G)$ correspond bijectively to the partition valued functions in $\mathrm{Div}^+(\mathcal{C} \times \mathbb{N})$ of degree n. In particular we have $\Theta = \Theta_{\mathcal{P}}$ for a unique $\mathcal{P} \in \mathrm{Div}^+(\mathcal{C} \times \mathbb{N})$. We also introduced a partial order $\le$ on $\mathrm{Div}^+(\mathcal{C} \times \mathbb{N})$ which, by construction, has the property that comparable elements $\mathcal{P}' \le \mathcal{P}''$ satisfy $D(\mathcal{P}') = D(\mathcal{P}'')$. For simplicity we sometimes will write

$$\Theta' \le \Theta'' \ \text{ if } \Theta' = \Theta_{\mathcal{P}'}, \Theta'' = \Theta_{\mathcal{P}''}, \ \text{ and } \mathcal{P}' \le \mathcal{P}'' \ .$$

The Θ' such that $\Theta' \ge \Theta$ are finite in number and satisfy $\mathfrak{z}(\Theta') \subseteq \Omega$. We define

$$\Theta^{\le} := \bigcup_{\Theta \le \Theta'} \Theta' \ \text{ and } \ \mathcal{M}^t(\Theta^{\le}) := \prod_{\Theta \le \Theta'} \mathcal{M}^t(\Theta') \ .$$

The forgetful functor maps the category $\mathcal{M}^t(\Theta^{\le})$ into the Bernstein component $\mathcal{M}(\Omega)$. Our aim is to construct a “tempered K-type” for the category $\mathcal{M}^t(\Theta^{\le})$ (not $\mathcal{M}^t(\Theta)$!).

Let (J, λ) be a semisimple K-type for $\mathcal{M}(\Omega)$ and fix a maximal compact subgroup $K \subseteq G$ as in section 6. Set

$$\mathfrak{P} := \{\mathcal{P}' : D(\mathcal{P}') = D(\mathcal{P})\} \ = \ \{\mathcal{P}' : \mathfrak{z}(\Theta_{\mathcal{P}'}) \subseteq \Omega\} \ .$$

The $\sigma_{\mathcal{P}'}(\lambda)$ for $\mathcal{P}' \in \mathfrak{P}$ are the irreducible objects in the category $\mathcal{M}_\lambda(K)$. By Prop. 6.2 we have:

Let V be an irreducible representation in $\mathcal{M}^t(\Theta)$ and let $\mathcal{P}' \in \mathfrak{P}$; then $\sigma_{\mathcal{P}'}(\lambda)$ is contained in V if and only if $\mathcal{P}' \le \mathcal{P}$.

Proposition 1:

For any nonzero V in $\mathcal{M}^t(G)$ which as a G-representation lies in $\mathcal{M}(\Omega)$ the following assertions are equivalent:

i. V lies in $\mathcal{M}^t(\Theta)$;

ii. V satisfies the subsequent two conditions:

(a) for any $\mathcal{P}' \in \mathfrak{P}$, $\sigma_{\mathcal{P}'}(\lambda)$ is contained in V if and only if $\mathcal{P}' \leq \mathcal{P}$;

(b) as an $\mathcal{S}$-module V is generated by its $\sigma_{\mathcal{P}}(\lambda)$-isotypical component.

Proof: We first suppose that V lies in $\mathcal{M}^t(\Theta)$. Then all simple $\mathcal{S}$-module subquotients of V lie in $\mathcal{M}^t(\Theta)$ as well. Each such subquotient is an irreducible G-representation and satisfies (a) and (b) as a consequence of Prop. 6.2. A standard argument then shows that V satisfies (a) and (b), too. Conversely, we assume now that (a) and (b) hold true for V. Consider the Harish-Chandra decomposition

$$V = \bigoplus_{\mathcal{P}'} V(\Theta_{\mathcal{P}'}) .$$

Since V, as a G-representation, lies in $\mathcal{M}(\Omega)$ we have $V(\Theta_{\mathcal{P}'}) = 0$ unless $\mathcal{P}' \in \mathfrak{P}$. Moreover, by (b), each $V(\Theta_{\mathcal{P}'})$ is generated, as an $\mathcal{S}$-module, by its $\sigma_{\mathcal{P}}(\lambda)$-isotypical component and hence is either zero or contains $\sigma_{\mathcal{P}}(\lambda)$. Since $V(\Theta_{\mathcal{P}'})$ lies in $\mathcal{M}^t(\Theta_{\mathcal{P}'})$ the implication which we have proved already shows that $V(\Theta_{\mathcal{P}'})$ must be zero unless $\mathcal{P} \leq \mathcal{P}'$. The same implication also shows that if $V(\Theta_{\mathcal{P}'})$ is nonzero for some $\mathcal{P} < \mathcal{P}'$ then $V(\Theta_{\mathcal{P}'})$ and hence V contains $\sigma_{\mathcal{P}'}(\lambda)$. But this would lead to a contradiction to (b). Hence $V = V(\Theta)$ lies in $\mathcal{M}^t(\Theta)$.

Proposition 2:

For any tempered G-representation V the following assertions are equivalent:

i. V lies in $\mathcal{M}^t(\Theta^{\leq})$;

ii. V is generated, as an $\mathcal{S}$-module, by its $\sigma_{\mathcal{P}}(\lambda)$-isotypical component.

Proof: Let us suppose that ii. holds true. Then V is also generated, as an $\mathcal{S}$-module, by its λ-isotypical component. Since the Bernstein decomposition $V = \bigoplus_{\Omega'} V(\Omega')$ is a decomposition of $\mathcal{S}$-modules we obtain that each nonzero summand contains λ. But λ is a K-type for $\mathcal{M}(\Omega)$. This shows that V, as a G-representation, lies in $\mathcal{M}(\Omega)$. The rest of the proof is completely analogous to the proof of Prop. 1.

This last result justifies calling the pair $(K, \sigma_{\mathcal{P}}(\lambda))$ a tempered K-type for $\mathcal{M}^t(\Theta^{\leq})$. Let $e_{\mathcal{P}} \in \mathcal{H}$ denote the (central) idempotent of the irreducible representation $\sigma_{\mathcal{P}}(\lambda)$ of K extended by zero to a function on G. The $\sigma_{\mathcal{P}}(\lambda)$-isotypical component of any V in $\mathcal{M}^t(G)$ is equal to $e_{\mathcal{P}}V$. Prop. 2 therefore says that V lies in $\mathcal{M}^t(\Theta^{\leq})$ if and only if $\mathcal{S}e_{\mathcal{P}}V = V$.
Since the Harish-Chandra decomposition

$$\mathcal{S} = \bigoplus_{\Theta'} \mathcal{S}(\Theta')$$

is given by idempotents in the centre of the category $\mathcal{M}^t(G)$, it is a decomposition into 2-sided ideals of $\mathcal{S}$. We define

$$\mathcal{S}(\Theta^{\leq}) := \bigoplus_{\Theta \leq \Theta'} \mathcal{S}(\Theta') .$$

Proposition 3:

$\mathcal{S}(\Theta^{\leq}) = \mathcal{S}e_{\mathcal{P}}\mathcal{S}$.

Proof: Since $\mathcal{S}(\Theta^{\leq})$ lies in $\mathcal{M}^t(\Theta^{\leq})$ we obtain from Prop. 2 that $\mathcal{S}e_{\mathcal{P}}\mathcal{S}(\Theta^{\leq}) = \mathcal{S}(\Theta^{\leq})$ and a fortiori that $\mathcal{S}(\Theta^{\leq}) \subseteq \mathcal{S}e_{\mathcal{P}}\mathcal{S}$. For the reverse inclusion we have to show that $e_{\mathcal{P}} \in \mathcal{S}(\Theta^{\leq})$. Consider first the decomposition

$$e_{\mathcal{P}} = \sum_{\Omega'} e_{\Omega'} \text{ with } e_{\Omega'} \in \mathcal{H}(\Omega') .$$

If $e_{\mathcal{P}}\mathcal{H}(\Omega') = e_{\Omega'}\mathcal{H}(\Omega') \neq 0$ then $\mathcal{H}(\Omega')$ contains λ. But λ is a K-type for $\mathcal{M}(\Omega)$. This shows that $e_{\Omega'} = 0$ for $\Omega' \neq \Omega$ and hence that $e_{\mathcal{P}} \in \mathcal{H}(\Omega)$. Because $\mathcal{H}(\Omega) \subseteq \mathcal{S}(\Omega) = \bigoplus_{\mathcal{P}' \in \mathfrak{P}} \mathcal{S}(\Theta_{\mathcal{P}'})$ we may write

$$e_{\mathcal{P}} = \sum_{\mathcal{P}' \in \mathfrak{P}} e_{\mathcal{P},\mathcal{P}'} \text{ with } e_{\mathcal{P},\mathcal{P}'} \in \mathcal{S}(\Theta_{\mathcal{P}'}) .$$

Consider now a $\mathcal{P}' \in \mathfrak{P} \setminus \{\mathcal{P}'' \geq \mathcal{P}\}$. By Prop. 6.2 we have $0 = e_{\mathcal{P}}V = e_{\mathcal{P},\mathcal{P}'}V$ for any irreducible representation V in $\mathcal{M}^t(\Theta_{\mathcal{P}'})$. It follows that $e_{\mathcal{P},\mathcal{P}'}$ being an idempotent annihilates every object in $\mathcal{M}^t(\Theta_{\mathcal{P}'})$ and in particular $\mathcal{S}(\Theta_{\mathcal{P}'})$. This only is possible if $e_{\mathcal{P},\mathcal{P}'}$ vanishes. We see that $e_{\mathcal{P}} = \sum_{\mathcal{P} \leq \mathcal{P}'} e_{\mathcal{P},\mathcal{P}'} \in \mathcal{S}(\Theta^{\leq})$.

Corollary 4:

$\mathcal{S}(\Theta) \cong \mathcal{S}e_{\mathcal{P}}\mathcal{S} / \left(\sum_{\mathcal{P} < \mathcal{P}'} \mathcal{S}e_{\mathcal{P}'}\mathcal{S} \right)$.

It appears to be an interesting problem to find, for any pair $\mathcal{P} < \mathcal{P}'$, explicit functions $\varphi, \psi \in \mathcal{S}$ such that $e_{\mathcal{P}'} = \varphi e_{\mathcal{P}} \psi$.
We now consider the functors

$$\begin{array}{rcl} \mathcal{T} : \mathcal{M}^t(\Theta^{\leq}) & \longrightarrow & \mathrm{Mod}(e_{\mathcal{P}}\mathcal{S}e_{\mathcal{P}}) \\ V & \longmapsto & e_{\mathcal{P}}V \end{array}$$

and

$$\begin{array}{rcl} \mathcal{J} : \mathrm{Mod}(e_{\mathcal{P}}\mathcal{S}e_{\mathcal{P}}) & \longrightarrow & \mathcal{M}^t(\Theta^{\leq}) \\ \mathfrak{M} & \longmapsto & \mathcal{S}e_{\mathcal{P}} \underset{e_{\mathcal{P}}\mathcal{S}e_{\mathcal{P}}}{\otimes} \mathfrak{M} . \end{array}$$

For trivial reasons we have $\mathcal{T} \circ \mathcal{J} = id$. In order to see that the other composite is naturally isomorphic to the identity functor as well we look at the natural map

$$\begin{array}{rcl} \mathcal{S}e_{\mathcal{P}} \underset{e_{\mathcal{P}}\mathcal{S}e_{\mathcal{P}}}{\otimes} e_{\mathcal{P}}V & \xrightarrow{\text{ad}} & V \\ \varphi \otimes v & \longmapsto & \varphi v \ . \end{array}$$

It is surjective because $\mathcal{S}e_{\mathcal{P}}V = V$. Moreover, since $\mathcal{T}(\text{ad})$ is the identity map and since the functor $\mathcal{T}$ is exact we have $e_{\mathcal{P}} \ker(\text{ad}) = 0$. But the subcategory $\mathcal{M}^t(\Theta^{\leq})$ of $\mathcal{M}^t(G)$ is closed with respect to the passage to $\mathcal{S}$-submodules. Hence $\ker(\text{ad})$ lies in $\mathcal{M}^t(\Theta^{\leq})$, too. Both facts together imply, by Prop. 2, that $\ker(\text{ad}) = 0$. It follows that this map "ad" is a natural isomorphism.

Corollary 5:

The functor

$$\begin{array}{rcl} \mathcal{M}^t(\Theta^{\leq}) & \xrightarrow{\sim} & \text{Mod}(e_{\mathcal{P}}\mathcal{S}e_{\mathcal{P}}) \\ V & \longmapsto & e_{\mathcal{P}}V \end{array}$$

is an equivalence of categories and induces by restriction an equivalence of categories

$$\mathcal{M}^t(\Theta) \xrightarrow{\sim} \text{Mod}(e_{\mathcal{P}}\mathcal{S}e_{\mathcal{P}}/I_{\mathcal{P}})$$

where the 2*-sided ideal $I_{\mathcal{P}}$ in $e_{\mathcal{P}}\mathcal{S}e_{\mathcal{P}}$ is defined by*

$$I_{\mathcal{P}} := \sum_{\mathcal{P} < \mathcal{P}'} e_{\mathcal{P}}\mathcal{S}e_{\mathcal{P}'}\mathcal{S}e_{\mathcal{P}} \ .$$

Proof: Above we have already discussed the first part of the assertion. Let V be a representation in $\mathcal{M}^t(\Theta^{\leq})$. We still have to show that V lies in $\mathcal{M}^t(\Theta)$ if and only if $I_{\mathcal{P}}e_{\mathcal{P}}V = 0$. Note that

$$I_{\mathcal{P}}e_{\mathcal{P}}V \ = \ \sum_{\mathcal{P} < \mathcal{P}'} e_{\mathcal{P}}\mathcal{S}e_{\mathcal{P}'}V \ .$$

If V lies in $\mathcal{M}^t(\Theta)$ then, by Prop. 1, no $\sigma_{\mathcal{P}'}(\lambda)$ for $\mathcal{P}' > \mathcal{P}$ is contained in V. Hence $I_{\mathcal{P}}e_{\mathcal{P}}V = 0$ in this case. Let us assume vice versa that $I_{\mathcal{P}}e_{\mathcal{P}}V = 0$. Then $e_{\mathcal{P}}\mathcal{S}e_{\mathcal{P}'}V = 0$ for any $\mathcal{P}' > \mathcal{P}$. But each $\mathcal{S}e_{\mathcal{P}'}V$ is an $\mathcal{S}$-submodule of V and hence belongs to $\mathcal{M}^t(\Theta^{\leq})$. It follows that $e_{\mathcal{P}'}V = 0$ for any $\mathcal{P}' > \mathcal{P}$. In the decomposition $V = \underset{\mathcal{P}' \geq \mathcal{P}}{\oplus} V(\Theta_{\mathcal{P}'})$ we therefore must have, again by Prop. 1, that $V(\Theta_{\mathcal{P}'}) = 0$ for any $\mathcal{P}' > \mathcal{P}$. This shows that V lies in $\mathcal{M}^t(\Theta)$.

8. Stratification of module categories

In the subsequent last section we will show that our tempered K-types actually are K-types for certain subquotients of the category $\mathcal{M}(G)$. More precisely we will see that the Bernstein decomposition of $\mathcal{M}(G)$ can in a natural way be refined into a stratification of $\mathcal{M}(G)$ and that the associated strata categories (in an appropriate sense) are described by the tempered K-types.
The general concept of a stratification of an abelian category is not well developed in the literature. There is some work of Cline, Parshall, Scott (compare [CPS]) but their setting is too narrow for our needs. The present section serves the purpose to introduce and develop this concept up to the point which is needed later on.
At first let $\mathcal{A}$ be any Grothendieck category and let

$$\mathcal{A}^\wedge := \text{ set of isomorphism classes of simple objects in } \mathcal{A}\ .$$

For any subset $Z \subseteq \mathcal{A}^\wedge$ we define

$$\mathcal{A}_Z := \begin{array}{l}\text{full subcategory in } \mathcal{A} \text{ of all objects all}\\ \text{of which simple subquotients lie in } Z\ .\end{array}$$

For technical reasons we also need, for any injective object E in $\mathcal{A}$, the subcategory

$$\mathcal{A}(E) := \begin{array}{l}\text{full subcategory in } \mathcal{A} \text{ of}\\ \text{all objects } V \text{ such that}\\ \mathrm{Hom}_{\mathcal{A}}(V,E) = 0\,.\end{array}$$

This is a localizing subcategory. Recall that a thick subcategory $\mathcal{L}$ in $\mathcal{A}$ is called localizing if the quotient functor $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{L}$ (or equivalently, by [Pop] 4.5.2, the inclusion functor $\mathcal{L} \hookrightarrow \mathcal{A}$) has a right adjoint functor. We actually have ([Pop] 4.6.4) that a full subcategory $\mathcal{L}$ in $\mathcal{A}$ is localizing if and only if there is a family $\{E_\nu\}_{\nu\in N}$ of injective objects E_ν in $\mathcal{A}$ such that $\mathcal{L} = \bigcap\limits_{\nu\in N} \mathcal{A}(E_\nu)$.

Lemma 1:

Let $z = [S] \in \mathcal{A}^\wedge$, let $E(S)$ be an injective envelope of the simple object S in $\mathcal{A}$, and put $Z := \mathcal{A}^\wedge \setminus \{z\}$; we then have:

$$\mathcal{A}_Z \ = \ \mathcal{A}(E(S))\ .$$

Proof: Let us first assume that the object V in $\mathcal{A}$ has no subquotients isomorphic to S. Consider any morphism $V \longrightarrow E(S)$. If it were nonzero then as $E(S)$ is an essential extension of S its image would have to contain S thereby exhibiting S as a subquotient of V. On the other hand let now $V_\circ \subseteq V$ be a subobject for which we have an epimorphism $V_\circ \twoheadrightarrow S$. By the injectivity of $E(S)$ the composite morphism $V_\circ \twoheadrightarrow S \hookrightarrow E(S)$ then extends to a nonzero morphism $V \longrightarrow E(S)$.

As a consequence we obtain, for an arbitrary subset $Z \subseteq \mathcal{A}^\wedge$, that

$$(1) \qquad \mathcal{A}_Z = \bigcap_{[S]\notin Z} \mathcal{A}(E(S)) .$$

In particular $\mathcal{A}_Z$ is a localizing subcategory. What can we say about the simple objects in the quotient category $\mathcal{A}/\mathcal{A}_Z$? Let $Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{A}_Z$ denote the quotient functor and $Q^\wedge$ its right adjoint. Note that $Q^\wedge$ is fully faithful ([Pop] p. 177) and having an exact left adjoint preserves injective objects.

Lemma 2:

For two simple objects S and S' in $\mathcal{A}$ with classes not in Z we have:

i. $Q(S)$ is a simple object in $\mathcal{A}/\mathcal{A}_Z$;

ii. $Q(E(S))$ is an injective envelope of $Q(S)$;

iii. the adjunction map $E(S) \xrightarrow{\cong} Q^\wedge \circ Q(E(S))$ is an isomorphism;

iv. if $S \ncong S'$ then $Q(S) \ncong Q(S')$.

Proof: i. This is straightforward by using [Pop] 4.3.5. ii. [Pop] 4.5.1(2). iii. [Pop] 4.4.4 and 4.5.1(1). iv. If $Q(S) \cong Q(S')$ then, by ii. and iii., $E(S) \cong E(S')$. Since $E(S)$, resp. $E(S')$, is an essential extension of S, resp. S', it follows that $S \cong S'$.

We see that the functor Q induces an injective map

$$\mathcal{A}^\wedge \setminus (\mathcal{A}_Z)^\wedge = \mathcal{A}^\wedge \setminus Z \hookrightarrow (\mathcal{A}/\mathcal{A}_Z)^\wedge .$$

To be able to say more we assume from now on that the category $\mathcal{A}$ is locally noetherian. Let $\{T_\mu\}_{\mu\in M}$ be a set of representatives for the isomorphism classes of simple objects in $\mathcal{A}/\mathcal{A}_Z$ and let $E(T_\mu)$ be an injective envelope of T_μ in $\mathcal{A}/\mathcal{A}_Z$. Then [Pop] 5.5.9 and 5.8.5 say that

$$(2) \qquad \mathcal{A}_Z = \bigcap_{\mu\in M} \mathcal{A}(Q^\wedge(E(T_\mu))) ;$$

in addition this intersection is reduced in the sense that all the intersections $\bigcap\limits_{\mu\neq\mu_\circ} \mathcal{A}(Q^\wedge(E(T_\mu)))$ are strictly bigger than $\mathcal{A}_Z$. A comparison of the two intersections (1) and (2) now shows, by using Lemma 2, that the $E(T_\mu)$ up to isomorphism coincide with the $Q(E(S)) \cong E(Q(S))$ for $[S] \notin Z$. In other words the set $\{T_\mu\}$ can be taken to be $\{Q(S) : [S] \in \mathcal{A}^\wedge \setminus Z\}$ which means that the map

$$\mathcal{A}^\wedge \setminus Z \xrightarrow{\sim} (\mathcal{A}/\mathcal{A}_Z)^\wedge$$

is a bijection.

Proposition 3:

If $\mathcal{A}$ is locally noetherian then, for any subset $Z \subseteq \mathcal{A}^\wedge$, the simple objects in $\mathcal{A}/\mathcal{A}_Z$ are, up to isomorphism, the images of the simple objects in $\mathcal{A}$ not contained in $\mathcal{A}_Z$.

We consider now two subsets $Z, Z' \subseteq \mathcal{A}^\wedge$. If $Z \subseteq Z'$ then $\mathcal{A}_Z \subseteq \mathcal{A}_{Z'}$ and the natural functor

$$(\mathcal{A}/\mathcal{A}_Z)/(\mathcal{A}_{Z'}/\mathcal{A}_Z) \xrightarrow{\sim} \mathcal{A}/\mathcal{A}_{Z'}$$

is an equivalence of categories ([Pop] 4.3 Ex. 6). In general we have a natural functor

$$\imath : \mathcal{A}_Z/\mathcal{A}_{Z\cap Z'} \longrightarrow \mathcal{A}_{Z\cup Z'}/\mathcal{A}_{Z'}$$

but which need not to be an equivalence of categories. Both sides are locally noetherian Grothendieck categories ([Pop] 5.8.4). And the functor $\imath$ is a full and faithful embedding whose essential image is closed under the formation of subquotients and arbitrary direct sums ([Ro2] VI.1.4.1; note that on both sides direct sums can be computed in $\mathcal{A}$). Moreover, by Proposition 3, both sides have up to isomorphism the same simple objects namely the respective images of the simple objects S in $\mathcal{A}$ with $[S] \in Z \setminus Z'$. In order to exploit this fact we need to introduce a few more concepts.
A full subcategory $\mathcal{B}$ in $\mathcal{A}$ is called **Jacobson closed** if:
1) The inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ has a left adjoint functor;
2) $\mathcal{B}$ is closed under the formation of subquotients and arbitrary direct sums in $\mathcal{A}$; in particular, the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ also has a right adjoint functor.
Any intersection of Jacobson closed subcategories again is Jacobson closed ([Ro1] 6.2.2). This makes it possible to define $\mathcal{A}_{\mathrm{red}}$ to be the smallest Jacobson closed subcategory which contains all the simple objects of $\mathcal{A}$.
We call a pair (Z, Z') of subsets of $\mathcal{A}^\wedge$ **excisive** if the above functor $\imath$ has a left adjoint functor or equivalently if the essential image of $\imath$ is a Jacobson closed subcategory in the right hand side.

Lemma 4:

If $\mathcal{A}$ is locally noetherian and if (Z, Z') is an excisive pair of subsets in $\mathcal{A}^\wedge$ then we have a natural equivalence of categories

$$\imath_{\mathrm{red}} : (\mathcal{A}_Z/\mathcal{A}_{Z\cap Z'})_{\mathrm{red}} \xrightarrow{\sim} (\mathcal{A}_{Z\cup Z'}/\mathcal{A}_{Z'})_{\mathrm{red}} .$$

Proof: Put $\mathcal{C} := \mathcal{A}_{Z\cup Z'}/\mathcal{A}_{Z'}$ and let $\mathcal{C}_{\mathrm{o}} \subseteq \mathcal{C}$ be the essential image of $\imath$. Then $\mathcal{C}_{\mathrm{o}}$ and $\mathcal{C}$ are Grothendieck categories with exactly the same class of simple objects. By assumption $\mathcal{C}_{\mathrm{o}}$ is Jacobson closed in $\mathcal{C}$. Since the property of being Jacobson closed evidently is transitive we must have $(\mathcal{C}_{\mathrm{o}})_{\mathrm{red}} = \mathcal{C}_{\mathrm{red}}$.

The excisive pairs of interest to us in this paper occur as follows. Let A be a possibly nonunital ring. We assume instead that A is idempotented, i.e., that for any finite subset $\{a_1, \ldots, a_m\} \subseteq A$ there is an idempotent $e \in A$ such that $ea_i = a_i e = a_i$ for any $1 \leq i \leq m$. A left A-module V is called nondegenerate if for any $v \in V$ there is an idempotent $e \in A$ such that $ev = v$. We let $\mathcal{M}(A)$ denote the category of all nondegenerate (left) A-modules. Observe that any submodule or quotient module of a nondegenerate A-module is nondegenerate and that any direct sum of nondegenerate A-modules is nondegenerate. It easily follows that $\mathcal{M}(A)$ is an abelian category with arbitrary direct sums and exact direct limits. Because A is a generator for $\mathcal{M}(A)$ we see that $\mathcal{M}(A)$ is a Grothendieck category. Since after this preliminary remarks all occuring A-modules will be nondegenerate we from now on simply say "A-module" for "nondegenerate left A-module".
By [Ro1] 6.4.1 the Jacobson closed subcategories in $\mathcal{M}(A)$ are precisely the subcategories of the form $\mathcal{M}(A/I)$ for some 2-sided ideal $I \subseteq A$. In addition one has

$$\mathcal{M}(A/I)_{\mathrm{red}} \;=\; \mathcal{M}(A/R(I))$$

where $R(I)$ is the Jacobson radical of the 2-sided ideal I, i.e., the intersection of all modular maximal left ideals of A which contain I. We write $\hat{A}$ for the set of all isomorphism classes of simple A-modules, i.e., $\hat{A} = \mathcal{M}(A)^\wedge$, and $\mathcal{M}_Z(A)$ instead of $\mathcal{M}(A)_Z$ for any subset $Z \subseteq \hat{A}$. For any 2-sided ideal $I \subseteq A$ we have the subset

$$V(I) := \{[V] \in \hat{A} : IV = 0\} \ .$$

It is easy to see that these subsets $V(I)$ form the family of closed subsets of a topology on $\hat{A}$ which is called the Jacobson topology. If the 2-sided ideal $J \subseteq A$ is idempotent (i.e., $J^2 = J$) then the subcategory $\mathcal{M}(A/J)$ is thick and bilocalizing; the latter means that the quotient functor $\mathcal{M}(A) \longrightarrow \mathcal{M}(A)/\mathcal{M}(A/J)$ has a left as well as a right adjoint functor ([Pop] 4.5.2 and 4.21.1). Closed subsets $Z \subseteq \hat{A}$ of the form $Z = V(J)$ with J idempotent will be called special. With Z and Z' also $Z \cap Z'$ is special.

Lemma 5:

Assume $\mathcal{M}(A)$ to be locally noetherian. Let $J \subseteq A$ be an idempotent 2-sided ideal and set $Z := V(J)$; we then have

$$\mathcal{M}_Z(A) \;=\; \mathcal{M}(A/J) \ .$$

In particular, the idempotent ideal J is uniquely determined by the closed subset Z.

Proof: Let V be a module in $\mathcal{M}_Z(A)$. We have to show that $JV = 0$. Otherwise we find a nonzero $v \in V$ such that $Jv \neq 0$. But Jv being a submodule of Av is finitely generated and consequently possesses a simple A-module quotient. By our assumption on V this simple quotient is annihilated by J. This leads to the contradiction that $Jv = J(Jv) \underset{\neq}{\subset} Jv$.

Proposition 6:

If $\mathcal{M}(A)$ is locally noetherian then any pair (Z, Z') of special subsets in $\hat{A}$ is excisive; in particular we have the natural equivalence of categories

$$[\mathcal{M}_Z(A)/\mathcal{M}_{Z\cap Z'}(A)]_{\mathrm{red}} \xrightarrow{\sim} [\mathcal{M}_{Z\cup Z'}(A)/\mathcal{M}_{Z'}(A)]_{\mathrm{red}} .$$

Proof: It suffices to show that for Z and Z' special the natural functor $\imath$: $\mathcal{M}_Z(A)/\mathcal{M}_{Z\cap Z'}(A) \longrightarrow \mathcal{M}(A)/\mathcal{M}_{Z'}(A)$ has a left adjoint functor. Assume that $Z = V(J)$ and $Z' = V(J')$ with idempotent 2-sided ideals $J, J' \subseteq A$. Consider now the functor

$$\begin{array}{rcl} \mathcal{M}(A) & \longrightarrow & \mathcal{M}(A/J) \\ V & \longmapsto & J'V/JJ'V \ . \end{array}$$

Let $\alpha : V_0 \longrightarrow V_1$ be any A-module homomorphism which induces an isomorphism in the quotient category $\mathcal{M}(A)/\mathcal{M}(A/J')$ and consider the induced homomorphism

$$\tilde{\alpha} \ : \ J'V_0/JJ'V_0 \ \longrightarrow \ J'V_1/JJ'V_1 \ \text{ in } \mathcal{M}(A/J) \ .$$

By [Pop] 4.3.7 the assumption on α means that $J' \cdot \ker(\alpha) = J' \cdot \mathrm{coker}(\alpha) = 0$. It follows that $J'V_1 = J'(J'V_1) = J'\alpha(V_0)$ and hence that $\mathrm{coker}(\tilde{\alpha}) = J'V_1/(JJ'V_1 + J'\alpha(V_0)) = 0$. It also follows that $JJ'V_1 = JJ'(J'V_1) = JJ'\alpha(V_0)$ $= \alpha(JJ'V_0)$ which implies that

$$\ker(\tilde{\alpha}) \ = \ [\ker(\alpha|J'V_0) + JJ'V_0]/JJ'V_0 \subseteq [\ker(\alpha) + JJ'V_0]/JJ'V_0$$

is annihilated by J'. This shows that $\tilde{\alpha}$ induces an isomorphism in the quotient category $\mathcal{M}(A/J)/\mathcal{M}(A/J + J')$. The above functor therefore induces a well defined functor

$${}^{\mathrm{ad}}\imath : \mathcal{M}(A)/\mathcal{M}(A/J') \ \longrightarrow \ \mathcal{M}(A/J)/\mathcal{M}(A/J + J') \ .$$

By a direct calculation one checks that ${}^{\mathrm{ad}}\imath$ is left adjoint to $\imath$.

We point out that this result is applicable to the category $\mathcal{M}(G) = \mathcal{M}(\mathcal{H})$ since $\mathcal{M}(\mathcal{H})$, by [BeR] III.2.2 Prop. 32 and III.4.1 Thm. 23, is locally noetherian. Assuming that $\mathcal{M}(A)$ is locally noetherian consider now a decomposition

$$\hat{A} \ = \ \dot{\bigcup_{\mu \in M}} \ Y_\mu$$

of $\hat{A}$ into finitely many disjoint subsets Y_μ where the index set M is equipped with a partial order $\leq$. A subset $N \subseteq M$ is saturated if with any $\nu \in N$ all the $\mu \in M$ with $\mu \leq \nu$ belong to N. We call the family $\{Y_\mu\}_{\mu \in M}$ a **special stratification** of $\hat{A}$ if any union $\bigcup_{\mu \in N} Y_\mu$, for N a saturated subset of M, is a special closed subset of $\hat{A}$. The "strata" Y_μ then are locally closed in $\hat{A}$.
Let us assume that $\{Y_\mu\}_{\mu \in M}$ is a special stratification of $\hat{A}$. We then introduce the special closed subsets

$$Z_\mu := \bigcup_{\mu' \leq \mu} Y_{\mu'}$$

of $\hat{A}$ for any $\mu \in M$. We think of the corresponding subcategories $\mathcal{M}_{Z_\mu}(A)$ as forming a stratification of the category $\mathcal{M}(A)$. The quotient categories $\mathcal{M}_{Z_\mu}(A)/\mathcal{M}_{Z_\mu \setminus Y_\mu}(A)$ are considered as the strata categories since their simple objects are precisely the images of the simple A-modules S with $[S] \in Y_\mu$. But in a way to be explained below only the reduced strata categories

$$[\mathcal{M}_{Z_\mu}(A)/\mathcal{M}_{Z_\mu \setminus Y_\mu}(A)]_{\mathrm{red}}$$

are really canonical. Let $\underset{\sim}{<}$ be a second partial order on M which refines $\leq$. Since any $\underset{\sim}{<}$-saturated subset of M is $\leq$-saturated the family $\{Y_\mu\}_{\mu \in M}$ also is a special stratification with respect to $\underset{\sim}{<}$. But the new strata categories $\mathcal{M}_{\tilde{Z}_\mu}(A)/\mathcal{M}_{\tilde{Z}_\mu \setminus Y_\mu}(A)$ with $\tilde{Z}_\mu := \bigcup_{\mu' \underset{\sim}{<} \mu} Y_{\mu'}$ in general will be larger than the old ones. Observing that $\tilde{Z}_\mu = Z_\mu \cup (\tilde{Z}_\mu \setminus Y_\mu)$ and $Z_\mu \setminus Y_\mu = Z_\mu \cap (\tilde{Z}_\mu \setminus Y_\mu)$ we may apply Proposition 6 though and obtain that the reduced strata categories in the old and in the new sense are naturally equivalent.

9. K-types for a stratification of $\mathcal{M}(G)$

We fix again a connected component $\Omega \subseteq \Omega(G)$ and a semisimple K-type (J, λ) for $\mathcal{M}(\Omega)$. In this section we will use our earlier results to construct a specific special stratification of the locally noetherian category $\mathcal{M}(\Omega) = \mathcal{M}(\mathcal{H}(\Omega))$ and to describe the strata categories in terms of the tempered K-types. Note that, according to our various notations, $\mathcal{M}(\Omega)^\wedge = \mathcal{H}(\Omega)^\wedge = \mathrm{Irr}(\Omega)$. In section 2 we reviewed the following facts:
1. There is a unique divisor $D \in \mathrm{Div}^+(\mathcal{C})$ such that $\Omega = \Omega_D$.
2. The finite set of partition valued functions $\mathfrak{P} := \mathfrak{P}_D := \{\mathcal{P} \in \mathrm{Div}^+(\mathcal{C} \times \mathbb{N}) : D(\mathcal{P}) = D\}$ carries a partial order $\leq$ (induced by the opposite of the dominance partial order on partitions).
3. For each $\mathcal{P} \in \mathfrak{P}$ there is a natural map

$$Q_{\mathcal{P}} \ : \ X_{nr}(N_{\mathcal{P}}) \longrightarrow \mathrm{Irr}(\Omega)$$

such that

$$(*) \qquad \mathrm{Irr}(\Omega) \;=\; \dot{\bigcup_{\mathcal{P}\in\mathfrak{P}}} \mathrm{im}(Q_{\mathcal{P}}) \;.$$

Our first objective in this section is to see that $(*)$ actually is a special stratification of $\mathrm{Irr}(\Omega)$.
Let $\mathcal{P}, \mathcal{P}' \in \mathfrak{P}$ and let V in $\mathcal{M}(\Omega)$ be an irreducible representation whose isomorphism class lies in $\mathrm{im}(Q_{\mathcal{P}})$. By Prop. 6.2 we have:
4. $\sigma_{\mathcal{P}}(\lambda)$ is contained in V.
5. If $\sigma_{\mathcal{P}'}(\lambda)$ is contained in V then $\mathcal{P}' \leq \mathcal{P}$.
We fix a maximal compact subgroup $K \subseteq G$ as in section 6 and let $e_{\mathcal{P}} \in \mathcal{H}(\Omega)$ denote the (central) idempotent of the irreducible representation $\sigma_{\mathcal{P}}(\lambda)$ of K (extended by zero to a function on G). We then have $V^{\sigma_{\mathcal{P}}(\lambda)} = e_{\mathcal{P}}V$ for any V in $\mathcal{M}(G)$. The last two facts therefore can be reformulated in the following way:
4'. $\mathcal{H}e_{\mathcal{P}}V = V$.
5'. $e_{\mathcal{P}'}V = 0$ unless $\mathcal{P}' \leq \mathcal{P}$.

Proposition 1:

$(*)$ is a special stratification; more precisely, for any saturated subset $\mathfrak{P}_{\mathrm{o}} \subseteq \mathfrak{P}$ let $Z := \bigcup_{\mathcal{P}\in\mathfrak{P}_{\mathrm{o}}} \mathrm{im}(Q_{\mathcal{P}})$ and let $J \subseteq \mathcal{H}(\Omega)$ be the 2-sided ideal generated by the $e_{\mathcal{P}'}$ for $\mathcal{P}' \in \mathfrak{P} \setminus \mathfrak{P}_{\mathrm{o}}$; we then have $Z = V(J)$.

Proof: First of all notice that the ideal J is idempotent. Let V be an irreducible representation in $\mathcal{M}(\Omega)$. We assume first that the class of V lies in Z and hence in $\mathrm{im}(Q_{\mathcal{P}})$ for some $\mathcal{P} \in \mathfrak{P}_{\mathrm{o}}$. Let $e_{\mathcal{P}'}$ be one of the generators of J. Since $\mathfrak{P}_{\mathrm{o}}$ is saturated we cannot have $\mathcal{P}' \leq \mathcal{P}$. The fact 5'. therefore says that $e_{\mathcal{P}'}V = 0$. It follows that $JV = 0$. Now let us suppose that the class of V does not lie in Z. It then must lie in $\mathrm{im}(Q_{\mathcal{P}'})$ for some $\mathcal{P}' \notin \mathfrak{P}_{\mathrm{o}}$. The fact 4'. says that $e_{\mathcal{P}'}V \neq 0$ and a fortiori that $JV \neq 0$.

Applying this result, for any $\mathcal{P} \in \mathfrak{P}$, to the Jacobson closed subset

$$Z_{\mathcal{P}} \;:=\; \bigcup_{\mathcal{P}'\leq\mathcal{P}} \mathrm{im}(Q_{\mathcal{P}'})$$

we obtain

$$Z_{\mathcal{P}} \;=\; V(J_{\mathcal{P}})$$

where

$$J_{\mathcal{P}} := \begin{array}{l} \text{the 2-sided ideal generated by} \\ \text{all } e_{\mathcal{P}'} \text{ for } \mathcal{P}' \in \mathfrak{P} \setminus \{\mathcal{P}'' : \mathcal{P}'' \leq \mathcal{P}\} \;. \end{array}$$

Similarly we have

$$Z_{\mathcal{P}} \setminus \mathrm{im}(Q_{\mathcal{P}}) \;=\; V(J_{\mathcal{P}} + \mathcal{H}e_{\mathcal{P}}\mathcal{H}) \;.$$

For the strata categories

$$\mathcal{M}_{\mathcal{P}}(\Omega) \ := \ \mathcal{M}_{Z_{\mathcal{P}}}(\mathcal{H}(\Omega))/\mathcal{M}_{Z_{\mathcal{P}}\setminus \mathrm{im}(Q_{\mathcal{P}})}(\mathcal{H}(\Omega))$$

of our stratification (∗) we therefore obtain the formula

$$\mathcal{M}_{\mathcal{P}}(\Omega) \ = \ \mathcal{M}(\mathcal{H}(\Omega)/J_{\mathcal{P}})/\mathcal{M}(\mathcal{H}(\Omega)/(J_{\mathcal{P}}+\mathcal{H}e_{\mathcal{P}}\mathcal{H})) \ .$$

Remark: *Let A be an idempotented ring and $e \in A$ be an idempotent; then the functor*

$$\begin{array}{rcl} \mathcal{M}(A)/\mathcal{M}(A/AeA) & \xrightarrow{\sim} & \mathrm{Mod}(eAe) \\ V & \longmapsto & eV \end{array}$$

is an equivalence of categories.

Proof. Let $\mathcal{I}$ denote the functor in question and let $\mathcal{J}$ be the functor in the reverse direction given by $\mathcal{J}(\mathfrak{M}) := Ae \underset{eAe}{\otimes} \mathfrak{M}$. Clearly $\mathcal{I} \circ \mathcal{J} = id$. One checks that kernel and cokernel of the natural map

$$\begin{array}{rcl} Ae \underset{eAe}{\otimes} eV & \xrightarrow{\mathrm{ad}} & V \\ a \otimes v & \longmapsto & av \end{array}$$

are annihilated by the idempotent e. In the quotient category ″ad″ therefore is a natural isomorphism between $\mathcal{J} \circ \mathcal{I}$ and the identity functor.

We apply this Remark to the ring $A := \mathcal{H}(\Omega)/J_{\mathcal{P}}$ and the idempotent $\bar{e}_{\mathcal{P}} := e_{\mathcal{P}} + J_{\mathcal{P}}$ and obtain the equivalence of categories

$$\begin{array}{rcl} \mathcal{M}_{\mathcal{P}}(\Omega) & \xrightarrow{\sim} & \mathrm{Mod}(\bar{e}_{\mathcal{P}}(\mathcal{H}(\Omega)/J_{\mathcal{P}})\bar{e}_{\mathcal{P}}) \\ V & \longmapsto & e_{\mathcal{P}}V \ . \end{array}$$

For the reduced strata category this restricts to an equivalence of categories

$$\mathcal{M}_{\mathcal{P}}(\Omega)_{\mathrm{red}} \ \xrightarrow{\sim} \ \mathrm{Mod}(\mathcal{H}_{\mathcal{P}}(\lambda))$$

where

$$\mathcal{H}_{\mathcal{P}}(\lambda) := [\bar{e}_{\mathcal{P}}(\mathcal{H}(\Omega)/J_{\mathcal{P}})\bar{e}_{\mathcal{P}}]/\text{Jacobson radical} \ .$$

We remark that the Jacobson radical of $\bar{e}_{\mathcal{P}}(\mathcal{H}(\Omega)/J_{\mathcal{P}})\bar{e}_{\mathcal{P}}$ is nilpotent (compare [KNS] Lemma 1 (iii)). Let

$$\mathcal{Z}_{\mathcal{P}}(\lambda) := \ \text{centre of } \mathcal{H}_{\mathcal{P}}(\lambda) \ .$$

Proposition 2:

$\mathcal{H}_{\mathcal{P}}(\lambda)$ is a matrix algebra over $\mathcal{Z}_{\mathcal{P}}(\lambda)$ of degree equal to $\dim_{\mathbb{C}}(\sigma_{\mathcal{P}}(\lambda))$.

Proof: The algebra $\mathcal{H}_{\mathcal{P}}(\lambda)$ is the quotient of the algebra $e_{\mathcal{P}}\mathcal{H}(\Omega)e_{\mathcal{P}}$ by the radical ideal $\mathrm{Rad}(e_{\mathcal{P}}J_{\mathcal{P}}e_{\mathcal{P}})$ of the 2-sided ideal $e_{\mathcal{P}}J_{\mathcal{P}}e_{\mathcal{P}}$. Consider now the natural isomorphism (compare [BK1] (4.2.4))

$$e_{\mathcal{P}}\mathcal{H}(\Omega)e_{\mathcal{P}} = e_{\mathcal{P}}\mathcal{H}e_{\mathcal{P}} \cong \mathcal{H}(G,K;\sigma_{\mathcal{P}}(\lambda))^{\mathrm{op}} \underset{\mathbb{C}}{\otimes} \mathrm{End}_{\mathbb{C}}(\sigma_{\mathcal{P}}(\lambda)) \ .$$

Obviously all 2-sided ideals of the right hand side have the form

$$I = I_0 \otimes \mathrm{End}(\sigma_{\mathcal{P}}(\lambda))$$

where I_0 is a 2-sided ideal of $\mathcal{H}(G,K;\sigma_{\mathcal{P}}(\lambda))^{\mathrm{op}}$. Correspondingly we must have

$$e_{\mathcal{P}}J_{\mathcal{P}}e_{\mathcal{P}} \cong J^0_{\mathcal{P}} \otimes \mathrm{End}(\sigma_{\mathcal{P}}(\lambda))$$

and

$$\mathcal{H}_{\mathcal{P}}(\lambda) \cong \mathcal{Z}^{\mathrm{op}} \otimes \mathrm{End}(\sigma_{\mathcal{P}}(\lambda))$$

where $\mathcal{Z} := \mathcal{H}(G,K;\sigma_{\mathcal{P}}(\lambda))/\mathrm{Rad}(J^0_{\mathcal{P}})$. In particular, $\mathcal{H}_{\mathcal{P}}(\lambda)$ is a matrix algebra of degree $d := \dim_{\mathbb{C}}(\sigma_{\mathcal{P}}(\lambda))$ over $\mathcal{Z}^{\mathrm{op}}$. The multiplicity one part of Prop. 6.2 says that all simple $\mathcal{H}_{\mathcal{P}}(\lambda)$-modules have dimension d over $\mathbb{C}$. It follows that all simple $\mathcal{Z}$-modules are one-dimensional. On the other hand $e_{\mathcal{P}}\mathcal{H}e_{\mathcal{P}}$ and hence $\mathcal{H}_{\mathcal{P}}(\lambda)$ and $\mathcal{Z}$ are finite type algebras (see [KNS] § 1 for this notion) by [Ber] Cor. 3.4. In this situation the Artin-Procesi theorem (compare [KNS] Lemma 2 and Thm. 2) implies that $\mathcal{Z}$ actually is commutative. We then necessarily have $\mathcal{Z} = \mathcal{Z}_{\mathcal{P}}(\lambda)$.

The above proof actually shows that $\mathcal{Z}_{\mathcal{P}}(\lambda)$ is a commutative quotient of the operator valued Hecke algebra $\mathcal{H}(G,K;\sigma_{\mathcal{P}}(\lambda))$ and that we have a natural isomorphism

$$\mathcal{H}_{\mathcal{P}}(\lambda) \cong \mathcal{Z}_{\mathcal{P}}(\lambda) \underset{\mathbb{C}}{\otimes} \mathrm{End}_{\mathbb{C}}(\sigma_{\mathcal{P}}(\lambda)) \ .$$

Corollary 3:

The reduced strata category $\mathcal{M}_{\mathcal{P}}(\Omega)_{\mathrm{red}}$ is equivalent to the category $\mathrm{Mod}(\mathcal{Z}_{\mathcal{P}}(\lambda))$ *of unital modules for the commutative ring $\mathcal{Z}_{\mathcal{P}}(\lambda)$; more precisely, the functor*

$$\begin{array}{rcl} \mathcal{M}_{\mathcal{P}}(\Omega)_{\mathrm{red}} & \xrightarrow{\sim} & \mathrm{Mod}(\mathcal{Z}_{\mathcal{P}}(\lambda)) \\ V & \longmapsto & \mathrm{Hom}_K(\sigma_{\mathcal{P}}(\lambda), V) \end{array}$$

is an equivalence of categories.

Let $N_{\mathcal{P}}^{\circ} \subseteq N_{\mathcal{P}}$ denote the simultaneous kernel of all unramified characters of $N_{\mathcal{P}}$. The evaluation map identifies $N_{\mathcal{P}}/N_{\mathcal{P}}^{\circ}$ with the group of rational characters of the complex torus $X_{nr}(N_{\mathcal{P}})$. Hence there is the obvious "universal" unramified character

$$\alpha_{\mathcal{P}} : N_{\mathcal{P}} \longrightarrow \mathbb{C}[N_{\mathcal{P}}/N_{\mathcal{P}}^{\circ}]^{\times} = \mathcal{O}(X_{nr}(N_{\mathcal{P}}))^{\times}$$

into the ring of regular functions on the torus $X_{nr}(N_{\mathcal{P}})$. The smooth G-representation $\mathcal{L}_{\mathcal{P}}$ parabolically induced from $\alpha_{\mathcal{P}} \otimes \tau_{\mathcal{P}}$ is an inductive limit of algebraic vector bundles on $X_{nr}(N_{\mathcal{P}})$ on which G acts $\mathcal{O}(X_{nr}(N_{\mathcal{P}}))$-linearly. The fibers of $\mathcal{L}_{\mathcal{P}}$ are the representations parabolically induced from $\alpha\tau_{\mathcal{P}}$ for $\alpha \in X_{nr}(N_{\mathcal{P}})$. We deduce from the lemma in section 2 that the idempotent ideal $J_{\mathcal{P}}$ annihilates all fibers of $\mathcal{L}_{\mathcal{P}}$ which means that $\mathcal{L}_{\mathcal{P}}$ lies in the category $\mathcal{M}(\mathcal{H}(\Omega)/J_{\mathcal{P}})$. It follows that $e_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}$ is a projective $\mathcal{O}(X_{nr}(N_{\mathcal{P}}))$-module of finite rank and that the action of $\overline{e}_{\mathcal{P}}(\mathcal{H}(\Omega)/J_{\mathcal{P}})\overline{e}_{\mathcal{P}}$ on $e_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}$ is $\mathcal{O}(X_{nr}(N_{\mathcal{P}}))$-linear. By the lemma in section 2 and Prop. 6.2, all fibers of $e_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}$ are simple modules for $\overline{e}_{\mathcal{P}}(\mathcal{H}(\Omega)/J_{\mathcal{P}})\overline{e}_{\mathcal{P}}$ and therefore are annihilated by the Jacobson radical of this latter ring. We see that $e_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}$ actually is an $\mathcal{H}_{\mathcal{P}}(\lambda)$-module. The argument in [Ber] Lemme 1.17 then shows that the centre $\mathcal{Z}_{\mathcal{P}}(\lambda)$ of $\mathcal{H}_{\mathcal{P}}(\lambda)$ acts on $e_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}$ through a homomorphism of algebras

$$\zeta_{\mathcal{P}} : \mathcal{Z}_{\mathcal{P}}(\lambda) \longrightarrow \mathcal{O}(X_{nr}(N_{\mathcal{P}})) \ .$$

For each $z \in \mathcal{Z}_{\mathcal{P}}(\lambda)$ and $\alpha \in X_{nr}(N_{\mathcal{P}})$ the value $\zeta_{\mathcal{P}}(z)(\alpha)$ is characterized as being the scalar by which z acts on the fibre of $e_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}$ in α. Since these fibers contain all simple $\mathcal{H}_{\mathcal{P}}(\lambda)$-modules this makes clear that $\zeta_{\mathcal{P}}$ is injective. Also, if α and α' in $X_{nr}(N_{\mathcal{P}})$ are such that $Q_{\mathcal{P}}(\alpha) = Q_{\mathcal{P}}(\alpha')$ then the two fibers $e_{\mathcal{P}}L(\alpha\tau_{\mathcal{P}}) \cong e_{\mathcal{P}}L(\alpha'\tau_{\mathcal{P}})$ of $e_{\mathcal{P}}\mathcal{L}_{\mathcal{P}}$ in α and α', respectively, are isomorphic; we therefore must have $\zeta_{\mathcal{P}}(z)(\alpha) = \zeta_{\mathcal{P}}(z)(\alpha')$ for any $z \in \mathcal{Z}_{\mathcal{P}}(\lambda)$. The Bernstein-Zelevinsky classification shows that the fibers of the map $Q_{\mathcal{P}} : X_{nr}(N_{\mathcal{P}}) \longrightarrow \mathrm{Irr}(\Omega)$ are the orbits of a certain finite group acting algebraically on $X_{nr}(N_{\mathcal{P}})$. We therefore may equip $\mathrm{im}(Q_{\mathcal{P}})$ in a natural way with the structure of an affine algebraic variety and may view the above map as an embedding

$$\zeta_{\mathcal{P}} : \mathcal{Z}_{\mathcal{P}}(\lambda) \hookrightarrow \mathcal{O}(\mathrm{im}(Q_{\mathcal{P}}))$$

of the centre $\mathcal{Z}_{\mathcal{P}}(\lambda)$ into the algebra of regular functions on $\mathrm{im}(Q_{\mathcal{P}})$. At present it remains an important open problem to determine the image of this homomorphism. If $\mathcal{P}$ is either the minimal or the maximal element in $\mathfrak{P}$ (corresponding to the closed and the open stratum, respectively) then this last map $\zeta_{\mathcal{P}}$ is an isomorphism. This is a consequence of the fact that in both cases the representation $\mathcal{L}_{\mathcal{P}}$ turns out to be a projective generator of the category $\mathcal{M}(\mathcal{H}(\Omega)/\mathrm{Rad}(J_{\mathcal{P}}))$ and to have $\mathcal{O}(\mathrm{im}(Q_{\mathcal{P}}))$ for the centre of its ring of endomorphisms. For the maximal $\mathcal{P}$ the map $\zeta_{\mathcal{P}}$ is nothing else than the Satake isomorphism (compare [Dat] Thm. 4.1).

Because of the equivalences of categories

$$\mathcal{M}(\Omega) = \mathcal{M}_\lambda(G) \xrightarrow{\sim} \mathrm{Mod}(\mathcal{H}(G, J; \lambda)^{\mathrm{op}})$$

and

$$\mathcal{M}_\lambda(K) \xrightarrow{\sim} \mathrm{Mod}(\mathcal{H}(K, J; \lambda)^{\mathrm{op}})$$

given by $V \longmapsto \mathrm{Hom}_J(\lambda, V)$ it must be possible to reformulate our above results purely in terms of the Hecke algebra $\mathcal{H}(G, J; \lambda)$ of the semisimple type (J, λ). We finish by explaining how to do this. The idempotents $e_\mathcal{P}$, for $\mathcal{P} \in \mathfrak{P}$, can be viewed as projectors in $\mathrm{End}_K(\mathrm{Ind}_J^K(\lambda))$. On the other hand the convolution

$$(f * \varphi)(h) := \sum_{g \in K/J} f(h^{-1}g)(\varphi(g)) \quad \text{for } f \in \mathcal{H}(K, J; \lambda) \text{ and } \varphi \in \mathrm{Ind}_J^K(\lambda)$$

induces a natural isomorphism of algebras

$$\mathcal{H}(K, J; \lambda) \cong \mathrm{End}_K(\mathrm{Ind}_J^K(\lambda)) \ .$$

Let $e_\mathcal{P}(\lambda) \in \mathcal{H}(K, J; \lambda)$ be the idempotent which, under this isomorphism, corresponds to the projector $e_\mathcal{P} \in \mathrm{End}_K(\mathrm{Ind}_J^K(\lambda))$. More explicitly $e_\mathcal{P}(\lambda)$ is given as

$$e_\mathcal{P}(\lambda)(g) = \pi e_\mathcal{P} g \iota \in \mathrm{End}_\mathbb{C}(\lambda) \quad \text{for} \quad g \in K$$

where $\iota : \lambda \to \mathrm{Ind}_J^K(\lambda)|J$ and $\pi : \mathrm{Ind}_J^K(\lambda)|J \to \lambda$ are the usual adjunction maps and where g and $e_\mathcal{P}$ are to be considered as endomorphisms of $\mathrm{Ind}_J^K(\lambda)$. Since the extension by zero embeds $\mathcal{H}(K, J; \lambda)$ as a subalgebra into $\mathcal{H}(G, J; \lambda)$ we may view each $e_\mathcal{P}(\lambda)$ as an idempotent in the Hecke algebra $\mathcal{H}(G, J; \lambda)$. Letting $m_\mathcal{P}$ denote the multiplicity of $\sigma_\mathcal{P}(\lambda)$ in $\mathrm{Ind}_J^K(\lambda)$ we then obtain

$$\begin{aligned} e_\mathcal{P}(\lambda)\mathcal{H}(G, J; \lambda)e_\mathcal{P}(\lambda) &\cong e_\mathcal{P}\mathcal{H}(G, K; \mathrm{Ind}_J^K(\lambda))e_\mathcal{P} \\ &\cong \mathcal{H}(G, K; e_\mathcal{P}\mathrm{Ind}_J^K(\lambda)) \\ &\cong \mathcal{H}(G, K; \sigma_\mathcal{P}(\lambda)) \otimes M_{m_\mathcal{P} \times m_\mathcal{P}}(\mathbb{C}) \ . \end{aligned}$$

Repeating our above discussion for $\mathcal{H}(G, J; \lambda)$, $e_\mathcal{P}(\lambda)$, and the 2-sided ideal $J_\mathcal{P}(\lambda) :=< e_{\mathcal{P}'}(\lambda);\ \mathcal{P}' \not\leq \mathcal{P} >$ instead of $\mathcal{H}(\Omega)$, $e_\mathcal{P}$, and $J_\mathcal{P}$ we recover $\mathcal{Z}_\mathcal{P}(\lambda)$ as the centre of the quotient algebra

$$\mathcal{H}_\mathcal{P}(G, J; \lambda) := e_\mathcal{P}(\lambda)\mathcal{H}(G, J; \lambda)e_\mathcal{P}(\lambda)/\mathrm{Rad}(e_\mathcal{P}(\lambda)J_\mathcal{P}(\lambda)e_\mathcal{P}(\lambda)) \ .$$

Moreover, $\mathcal{H}_\mathcal{P}(G, J; \lambda)$ is a matrix algebra over $\mathcal{Z}_\mathcal{P}(\lambda)$ of degree equal to $m_\mathcal{P}$ and

$$\begin{array}{ccc} \mathcal{M}_\mathcal{P}(\Omega)_{\mathrm{red}} & \xrightarrow{\sim} & \mathrm{Mod}(\mathcal{H}_\mathcal{P}(G, J; \lambda)^{\mathrm{op}}) \\ V & \longmapsto & e_\mathcal{P}(\lambda)\mathrm{Hom}_J(\lambda, V) \end{array}$$

is an equivalence of categories.

Proposition 4:

With the notations of section 6 we have an isomorphism of algebras

$$\mathcal{Z}_{\mathcal{P}}(\lambda) \cong \bigotimes_{\tau} \mathcal{Z}_{\mathcal{P}_\tau}(\lambda^{(\tau)}) \ .$$

Proof: Because $j_{\tilde{Q}}$ combined with the relation between $\lambda_o^{(\tau)}$ and $\lambda^{(\tau)}$ induces an isomorphism

$$\mathcal{H}(G, J; \lambda) \cong \bigotimes_{\tau} \mathcal{H}(G_{m_\tau d(\tau)}, J^{(\tau)}; \lambda^{(\tau)})$$

under which the central idempotents $e_{\mathcal{P}}(\lambda)$ and $\otimes_\tau e_{\mathcal{P}_\tau}(\lambda^{(\tau)})$ in $\mathcal{H}(K, J; \lambda)$ and $\otimes_\tau \mathcal{H}(K^{(\tau)}, J^{(\tau)}; \lambda^{(\tau)})$, respectively, correspond to each other, we obtain an induced isomorphism

$$\mathcal{H}(G, K; e_{\mathcal{P}}\mathrm{Ind}_J^K(\lambda)) \cong \bigotimes_{\tau} \mathcal{H}(G_{m_\tau d(\tau)}, K^{(\tau)}; e_{\mathcal{P}_\tau}\mathrm{Ind}_{J^{(\tau)}}^{K^{(\tau)}}(\lambda^{(\tau)})) \ .$$

Using the identity (2) in section 6 and again the relation between $\lambda_o^{(\tau)}$ and $\lambda^{(\tau)}$ it is easy to see that the multiplicities of $\sigma_{\mathcal{P}}(\lambda)$ in $\mathrm{Ind}_J^K(\lambda)$ and of $\sigma_{\mathcal{P}_\tau}(\lambda^{(\tau)})$ in $\mathrm{Ind}_{J^{(\tau)}}^{K^{(\tau)}}(\lambda^{(\tau)})$ satisfy

$$m_{\mathcal{P}} = \prod_{\tau} m_{\mathcal{P}_\tau} \ .$$

Hence we obtain an isomorphism

$$\mathcal{H}(G, K; \sigma_{\mathcal{P}}(\lambda)) \cong \bigotimes_{\tau} \mathcal{H}(G_{m_\tau d(\tau)}, K^{(\tau)}; \sigma_{\mathcal{P}_\tau}(\lambda^{(\tau)})) \ .$$

Our assertion follows from this by dividing out the appropriate ideals on both sides.

According to [BK 1](5.6.6) the algebras $\mathcal{H}(G_{m_\tau d(\tau)}, J^{(\tau)}; \lambda^{(\tau)})$ are Iwahori-Hecke algebras. Hence the last Proposition reduces the problem of determining the image of the homomorphism $\zeta_{\mathcal{P}}$ to the case where the Bernstein component $\mathcal{M}(\Omega)$ is the Iwahori component of those representations which are generated by their fixed vectors under the Iwahori subgroup.

Appendix: The definition of the tempered category
(by P. Schneider, U. Stuhler)

We will use the same notations as in section 1 of the main text. In particular, G is the group of F-rational points of an arbitrary connected reductive group over the nonarchimedean locally compact field F, $\mathcal{H}$ is the Hecke algebra of $\mathbb{C}$-valued locally constant functions with compact support on G, and $\mathcal{M}(G) = \mathcal{M}(\mathcal{H})$ is the category of smooth G-representations (or equivalently the category of nondegenerate left $\mathcal{H}$-modules). We also recall that the Schwartz algebra $\mathcal{S}$ of uniformly locally constant and rapidly decreasing functions on G carries a natural locally convex ind-Fréchet topology such that the multiplication is separately continuous and such that $\mathcal{H}$ is dense in $\mathcal{S}$. For any compact open subgroup $U \subseteq G$ let $e_U \in \mathcal{H}$ denote the idempotent which is the constant function with value $\mathrm{vol}(U)^{-1}$ on U and which vanishes outside of U. The subalgebra $e_U \mathcal{S} e_U$ of all U-bi-invariant rapidly decreasing functions on G is a unital Fréchet algebra. The topology on $\mathcal{S}$ is the locally convex inductive limit topology, for varying U, of these Fréchet topologies. Since the family of these U has a countable cofinal subfamily we see that $\mathcal{S}$ actually is the strict inductive limit of a sequence of unital Fréchet algebras.

Although this topology on $\mathcal{S}$ is technically important it is the point of this appendix to show that representation theoretically it means essentially no loss of information if one treats $\mathcal{S}$ as an abstract algebra. So we define the category of tempered G-representations to be

$$\mathcal{M}^t(G) := \mathcal{M}(\mathcal{S}) := \text{category of nondegenerate left } \mathcal{S}\text{-modules.}$$

Note that $\mathcal{S}$ by construction is an idempotented algebra. Since $\mathcal{S}$ itself is a smooth G-representation via the left translation action the forgetful functor

$$\mathcal{M}^t(G) \longrightarrow \mathcal{M}(G)$$

is well defined. Our first observation is that any admissible G-representation in $\mathcal{M}(G)$ which is "tempered" in the traditional sense ([Sil] §4.5) that its matrix coefficients are tempered functions on G carries a unique $\mathcal{S}$-module structure which extends the given $\mathcal{H}$-module structure. We recall that a smooth G-representation V is called admissible if, for any compact open subgroup $U \subseteq G$, the subspace $V^U = e_U V$ of U-invariant vectors is finite dimensional. For any smooth G-representation V let $\tilde{V}$ denote the smooth dual of V. One has the identity

$$\tilde{v}(\psi * v) = \int_G \psi(g)\tilde{v}(gv)dg \quad \text{for } (v, \tilde{v}) \in V \times \tilde{V} \text{ and } \psi \in \mathcal{H}.$$

Proposition 1:

If V is an admissible representation in $\mathcal{M}(G)$ then the following assertions are equivalent:

i. the matrix coefficients $g \longmapsto \tilde{v}(gv)$ for $(v, \tilde{v}) \in V \times \tilde{V}$ are tempered functions on G;

ii. for any pair $(v, \tilde{v}) \in V \times \tilde{V}$, the linear form

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathbb{C} \\ \psi & \longmapsto & \tilde{v}(\psi * v) \end{array}$$

extends continuously to $\mathcal{S}$;

iii. V carries an $\mathcal{S}$-module structure which extends the given $\mathcal{H}$-module structure and for which the structure map $\mathcal{S} \times V \longrightarrow V$ is separately continuous with respect to the finest locally convex topology on V;

iv. V carries an $\mathcal{S}$-module structure which extends the given $\mathcal{H}$-module structure.
In addition, the $\mathcal{S}$-module structures in iii. and iv. are unique (if they exist).

Proof: The equivalence of i. and ii. is a consequence of the above identity. That iii. implies ii. is trivial. For the other direction let us use, for the moment being, $\tilde{v}(\psi * v)$ as a formal notation for the value of the continuous extension of the linear form $\tilde{v}(\,.\, * v)$ in any function $\psi \in \mathcal{S}$. We then have the bilinear map

$$\begin{array}{ccc} \mathcal{S} \times V & \longrightarrow & \tilde{\tilde{V}} \\ (\psi, v) & \longmapsto & [\tilde{v} \mapsto \tilde{v}(\psi * v)] \ . \end{array}$$

Since $\tilde{\tilde{V}} = V$ there actually has to be an element $\psi * v \in V$, for any $\psi \in \mathcal{S}$ and $v \in V$, such that $\tilde{v}(\psi * v)$ is the value of the linear form $\tilde{v} \in \tilde{V}$ in $\psi * v$. This defines a bilinear map $\mathcal{S} \times V \longrightarrow V$ which by construction extends the convolution product. It is continuous in the second component since the topology on V is the finest locally convex one. For the continuity in the first component it suffices to show for any $v \in V$ and any compact open subgroup $U \subseteq G$ that the map

$$\begin{array}{ccc} \mathcal{S}^U & \longrightarrow & (\tilde{\tilde{V}})^U \\ \psi & \longmapsto & [\tilde{v} \mapsto \tilde{v}(\psi * v)] \end{array}$$

is continuous. The right hand side is the linear dual of the finite dimensional vector space $(\tilde{V})^U$. Choose a basis $\tilde{v}_1, \ldots, \tilde{v}_m$ of $(\tilde{V})^U$ and let $\tilde{\tilde{v}}_1, \ldots, \tilde{\tilde{v}}_m$ be the dual basis of $(\tilde{\tilde{V}})^U$. Then the map in question is given by

$$\psi \longmapsto \sum_{i=1}^{m} \tilde{v}_i(\psi * v) \cdot \tilde{\tilde{v}}_i$$

which visibly is continuous. The unicity of this $\mathcal{S}$-module structure is an immediate consequence of its continuity property and the density of $\mathcal{H}$ in $\mathcal{S}$.
For the equivalence of iii. and iv. and the unicity statement in iv. we have to show that any $\mathcal{S}$-module structure on V which extends the given $\mathcal{H}$-module structure automatically has the continuity property asserted in iii. It certainly suffices to show that, for each compact open subgroup $U \subseteq G$, the restricted structure map $e_U \mathcal{S} e_U \times V^U \longrightarrow V^U$ is continuous. Since V is finite dimensional, say of dimension d, this amounts to the continuity of the unital algebra homomorphism

$$\begin{array}{ccc} e_U \mathcal{S} e_U & \longrightarrow & \mathrm{End}_{\mathbb{C}}(V^U) \cong M_{d\times d}(\mathbb{C}) \\ \psi & \longmapsto & [v \mapsto \psi * v] \,. \end{array}$$

Since by [Vi0] Thm. 29.3 the set $(e_U \mathcal{S} e_U)^\times$ of invertible elements is open in the Fréchet algebra $e_U \mathcal{S} e_U$ it follows from [AN] Thm. 7 that any such homomorphism indeed is continuous.

Corollary 2:

Let $\mathcal{M}^t_{adm}(G)$ denote the full subcategory in $\mathcal{M}^t(G)$ of all those $\mathcal{S}$-modules which as a G-representation are admissible; the forgetful functor induces a fully faithful embedding $\mathcal{M}^t_{adm}(G) \hookrightarrow \mathcal{M}(G)$ whose image is closed with respect to the formation of subquotients.

Proof: As a consequence of Prop. 1 the $\mathcal{S}$-module structure map $\mathcal{S} \times V \longrightarrow V$, for any V in $\mathcal{M}^t_{adm}(G)$, is separately continuous for the finest locally convex topology on V. This together with the density of $\mathcal{H}$ in $\mathcal{S}$ and the observation that in the finest locally convex topology every subspace of V is closed implies the assertion.

Our second observation concerns the relation between the simple $\mathcal{H}$-modules and the simple $\mathcal{S}$-modules by which we mean, of course, the simple objects in the categories $\mathcal{M}(\mathcal{H})$ and $\mathcal{M}(\mathcal{S})$, respectively. A simple $\mathcal{H}$-module is the same as an irreducible smooth G-representation.

Proposition 3:

Let V be a simple $\mathcal{S}$-module; we then have:

i. V is simple as an $\mathcal{H}$-module;

ii. any simple $\mathcal{S}$-module which is isomorphic to V as an $\mathcal{H}$-module is already isomorphic to V as an $\mathcal{S}$-module.

Proof: i. We will use in a crucial way the reduced C^*-algebra $C^*_r(G)$ of the group G which is defined as the completion of $\mathcal{H}$ in the operator norm on $L^2(G)$. According to [Vi0] Prop.28 the Schwartz algebra $\mathcal{S}$ is contained in $C^*_r(G)$. With

$\mathcal{H}$ then also $\mathcal{S}$ is dense in $C_r^*(G)$. Let $U \subseteq G$ be any compact open subgroup. We need the following facts about the unital C^*-algebra $e_U C_r^*(G) e_U$. Quite generally (in fact,for any Banach algebra) one has:
(1) The group of units $(e_U C_r^*(G) e_U)^\times$ is open in $e_U C_r^*(G) e_U$.
Now let temporarily $C_r^*(G,U)$ denote the completion of $e_U \mathcal{H} e_U$ in the operator norm on $e_U L_2(G) e_U$. In [Vi0] Prop.13 and 18 it is shown that

$$\mathcal{S} \cap C_r^*(G,U)^\times \;=\; (e_U \mathcal{S} e_U)^\times \;.$$

On the other hand one easily checks that $e_U L^2(G) e_U$ is faithful as a left $e_U C_r^*(G) e_U$-module. It follows that the natural map

$$e_U C_r^*(G) e_U \xrightarrow{\cong} C_r^*(G,U)$$

is an isomorphism (this fact was pointed out to us by P.Kutzko). In this way we obtain

$$(2) \qquad \mathcal{S} \cap (e_U C_r^*(G) e_U)^\times \;=\; (e_U \mathcal{S} e_U)^\times \;.$$

After this preliminary discussion we first compare simple $\mathcal{S}$-modules with simple $C_r^*(G)$-modules. The following argument is modelled after [Sch] Thm. 1.4. Any simple $\mathcal{S}$-module up to isomorphism is of the form $\mathcal{S}/\mathfrak{m}$ for some modular maximal left ideal $\mathfrak{m} \subseteq \mathcal{S}$. The modularity amounts to the existence of an element $e \in \mathcal{S}$ such that

$$a - ae \in \mathfrak{m} \quad \text{for any } a \in \mathcal{S} \;.$$

For any small enough compact open subgroup $U \subseteq G$ we have $e_U e = e = e e_U$ and hence $e_U - e \in \mathfrak{m}$, $e_U \notin \mathfrak{m}$, and $\mathfrak{m} e_U \subseteq \mathfrak{m}$. Let $\overline{\mathfrak{m}}$ denote the closure of $\mathfrak{m}$ in $C_r^*(G)$. By the density of $\mathcal{S}$ in $C_r^*(G)$ this is a left ideal in $C_r^*(G)$ satisfying

$$a - ae \in \overline{\mathfrak{m}} \quad \text{for any } a \in C_r^*(G) \;.$$

Consider a small enough U so that in particular $e_U \notin \mathfrak{m}$. Then (2) implies

$$(\mathfrak{m} \cap e_U \mathcal{S} e_U) \cap (e_U C_r^*(G) e_U)^\times = \emptyset$$

and, by (1), even

$$\overline{(\mathfrak{m} \cap e_U \mathcal{S} e_U)} \cap (e_U C_r^*(G) e_U)^\times = \emptyset$$

holds true. This shows that $e_U \notin \overline{(\mathfrak{m} \cap e_U \mathcal{S} e_U)}$. Because of

$$e_U \overline{\mathfrak{m}} e_U = \overline{e_U \mathfrak{m} e_U} \subseteq \overline{\mathfrak{m} \cap e_U \mathcal{S} e_U}$$

it follows that $e_U \notin \overline{\mathfrak{m}}$ and hence that $e \notin \overline{\mathfrak{m}}$. We see that $\overline{\mathfrak{m}}$ is a proper modular left ideal in $C_r^*(G)$. We choose a (modular) maximal left ideal $\mathfrak{n} \subseteq C_r^*(G)$ such that $\overline{\mathfrak{m}} \subseteq \mathfrak{n}$. Then $e \notin \mathfrak{n}$ so that $\mathfrak{m} \subseteq \mathcal{S} \cap \mathfrak{n} \underset{\neq}{\subset} \mathcal{S}$. From the maximality of $\mathfrak{m}$ we conclude that $\mathfrak{m} = \mathcal{S} \cap \mathfrak{n}$ which means that the natural map

$$\mathcal{S}/\mathfrak{m} \hookrightarrow C_r^*(G)/\mathfrak{n}$$

is an injective (and dense) embedding of the simple $\mathcal{S}$-module $\mathcal{S}/\mathfrak{m}$ into the simple $C_r^*(G)$-module $C_r^*(G)/\mathfrak{n}$. The right hand side is an irreducible unitary representation of the group G. It is known (compare [Crt] Cor.2.3) that the subspace of smooth vectors in an irreducible unitary G-representation is an irreducible smooth G-representation. In our situation it follows that the above map identifies $\mathcal{S}/\mathfrak{m}$ with the subspace of smooth vectors in $C_r^*(G)/\mathfrak{n}$ and that $\mathcal{S}/\mathfrak{m}$ is a simple $\mathcal{H}$-module.
ii. Any simple $\mathcal{S}$-module being, by i., an irreducible smooth G-representation lies in $\mathcal{M}^t_{adm}(G)$. The assertion therefore is a consequence of Cor. 2 .

We see that simple $\mathcal{S}$-modules are the same as irreducible tempered G-representations.

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