Representation theory and sheaves on the Bruhat-Tits building

P.Schneider, U.Stuhler

The Bruhat-Tits building X of a connected reductive group G over a nonarchimedean local field K is a rather intriguing G-space. It displays in a geometric way the inner structure of the locally compact group G like the classification of maximal compact subgroups or the theory of the Iwahori subgroup. One might consider X not quite as a full analogue of a real symmetric space but as a kind of skeleton of such an analogue. As such it immediately turned out to be an important technical device in the smooth representation theory of the group G. As a reminder let us mention that the irreducible smooth representations of G lie at the core of the local Langlands program which aims at understanding the absolute Galois group of the local field K.

In this paper we develop a systematic and conceptional theory which allows to pass in a functorial way from smooth representations of G to equivariant objects on X. There actually will be two such constructions - a homological and a cohomological one. Since the building carries a natural CW-structure the notion of a coefficient system (or cosheaf) on X makes sense. In the homological theory we will construct functors from smooth representations to G-equivariant coefficient systems on X. It should be stressed that the definition of the coefficient system only involves the original G-representation as far as the action of certain compact open subgroups of G is concerned. One therefore might consider the whole construction as a kind of localization process. Our main result will be that the cellular chain complex naturally associated with a coefficient system provides (under mild assumptions) a functorial projective resolution of the G-representation we started with.

In the cohomological theory we will associate, again functorially, G-equivariant sheaves on X with smooth G-representations. The main task which we will achieve then is the computation of the cohomology with compact support of the sheaves coming from an irreducible smooth G-representation. The result can best be formulated in terms of a certain duality functor on the category of finite length smooth G-representations. As a major application we will prove Zelevinsky's conjecture in [Zel] that his duality map on the level of Grothendieck groups preserves irreducibility. For carrying out our computation we have to extend the sheaves under consideration in such a way to the Borel-Serre compactification $\overline{X}$ of X that the cohomology at the boundary becomes computable. Since the stabilizers of boundary points are parabolic subgroups it might not surprise that this can be achieved by using the Jacquet modules of the representation as the stalks at the boundary points. The cohomology at the boundary then is computed by adapting a strategy of Deligne and Lusztig ([DL]) for reductive groups over finite fields to our purposes.

Apart from the theory of buildings we will very much rely on the beautiful results of Bernstein on the category of smooth representations in [Ber]. All the known homological finiteness properties of this category follow already from his work. The point of our paper is rather that we construct nice projective resolutions in that category in a functorial as well as explicit manner. Exactly this explicitness enables us to apply our theory to the harmonic analysis of the group G. It turns out that Kottwitz' Euler-Poincaré function in [Kot] is a special case of a general theory of Euler-Poincaré functions for finite length smooth G-representations. Besides being pseudo-coefficients their main property is that their elliptic orbital integrals coincide with the Harish-Chandra character of the given representation. This leads to a Hopf-Lefschetz type trace formula for the Harish-Chandra character at an elliptic element. Combined with powerful results of Kazhdan in [Ka1] it also leads to a proof of the general orthogonality formula for Harish-Chandra characters as conjectured by Kazhdan.

Let us now describe the contents of the paper in some more detail. The first chapter contains most of the input which we need from the theory of buildings. The cells of the natural CW-structure of X actually are polysimplices and are called facets. For any such facet F of X let P_F denote its pointwise stabilizer in G. The technical heart of our theory is the construction of certain decreasing filtrations $P_F \supseteq U_F^{(0)} \supseteq \ldots \supseteq U_F^{(e)} \supseteq \ldots$ of P_F by compact open subgroups $U_F^{(e)}$. This is done in I.2 where also the more basic properties of these filtrations are established. Since we work with an arbitrary connected reductive group G that construction involves more or less all of the finer aspects of the theory developed in the volumes [BT]. This unfortunately makes numerous references to [BT] unavoidable so that any reader without an expert knowledge of the work [BT] might find this section hard to read. We apologize for that. In order to make it a little easier we give in I.1 a brief overview over the theory of buildings for reductive groups. In the section I.3 we give those properties of the groups $U_F^{(e)}$ which later on are needed for the computation of the (co)homology. Notably we study how the groups $U_F^{(e)}$ behave if the facet F is moved along a geodesic in the building X. In case G is absolutely quasi-simple and simply connected similar filtrations appear in [PR] and [MP].

The second chapter contains the homological theory. In II.1 we briefly recall the formalism of cellular chains. The section II.2 contains the definition of the functor γ_e from smooth G-representations to equivariant coefficient systems on X. Here $e \geq 0$ is a fixed "level". The coefficient system $\gamma_e(V)$ corresponding to a representation V is formed by associating with a facet F the subspace of $U_F^{(e)}$-invariant vectors in V. In addition properties of finite generation and projectivity of the chain complex of $\gamma_e(V)$ are discussed. The main result is shown in II.3. It says that at least for any finitely generated smooth G-representation V we can choose the level e large enough so that the chain complex of $\gamma_e(V)$ is an exact resolution of V in the category $\mathrm{Alg}(G)$ of all smooth G-representations. In the case that G is the general linear group we proved this already in [SS]. The strategy of the proof for arbitrary G is the same once the necessary properties of the groups $U_F^{(e)}$ are known.

In the third chapter we develop that part of the duality theory which uses the chain complex of $\gamma_e(V)$. Since the polysimplicial structure of the building

X is locally finite we actually can associate with the coefficient system $\gamma_e(V)$ also a complex of cochains with finite support. Let us fix a character χ of the connected center of G, let $\mathrm{Alg}_\chi(G)$ denote the category of all those smooth G-representations on which the connected center acts through χ, and let $\mathcal{H}_\chi$ denote the χ-Hecke algebra of G. If V is an admissible representation in $\mathrm{Alg}_\chi(G)$ then the functor $\mathrm{Hom}_G(.\,,\mathcal{H}_\chi)$ transforms the chain complex of $\gamma_e(V)$ into the cochain complex of $\gamma_e(\tilde{V})$ where $\tilde{V}$ is the smooth dual of V. Assume now that V even is of finite length and choose e large enough. Then we know that the chain complex of $\gamma_e(V)$ is a projective resolution of V in $\mathrm{Alg}_\chi(G)$. It follows that the cochain complex of $\gamma_e(\tilde{V})$ computes the Ext-groups $\mathcal{E}^*(V) := \mathrm{Ext}^*_{\mathrm{Alg}_\chi(G)}(V, \mathcal{H}_\chi)$. All this is shown in III.1. Later on in IV.1 we will see that the same cochain complex computes the cohomology with compact support of a certain sheaf on X associated with the representation $\tilde{V}$. This fact will enable us in chapter IV to compute the groups $\mathcal{E}^*(V)$ in the case that V is a representation which is parabolically induced from an irreducible supercuspidal representation of a Levi subgroup. In III.2 we briefly recall the theory of parabolic induction following [Cas]. Then in III.3 taking the computation of $\mathcal{E}^*(V)$ for induced V for granted we deduce the following result for an arbitrary irreducible smooth representation V: The groups $\mathcal{E}^*(V)$ vanish except in a single degree $d(V), \mathcal{E}(V) := \mathcal{E}^{d(V)}(V)$ again is an irreducible smooth representation, and moreover $\mathcal{E}(\mathcal{E}(V)) = V$. Standard techniques of homological algebra now allow to establish a general duality formalism which relates the Ext- and Tor-functors on the category $\mathrm{Alg}_\chi(G)$. Loosely speaking one might say that $\mathcal{H}_\chi$ is a "Gorenstein ring".

Since our applications to harmonic analysis all come from the exactness of the chain complex of $\gamma_e(V)$ we include them here as the section III.4 before we turn to the sheaf theory on X. For reasons of convenience we assume that the center of G is compact. Since in the paper [Ka1] the field K is assumed to be of characteristic 0 we have to make the same assumption in most of our results of this section. We obtain: A general notion of Euler-Poincaré functions, a formula for the formal degree, the existence of explicit pseudo-coefficients, the Harish-Chandra character on the elliptic set as an explicit orbital integral, the general orthogonality relation for Harish-Chandra characters, and the 0-th Chern character on the Grothendieck group of finite length representations.

In the fourth chapter we present the sheaf theory on X. In IV.1 we functorially associate a sheaf $\underset{\approx}{V}$ on X with any representation V in $\mathrm{Alg}(G)$. Of course this construction again depends on the choice of a level $e \geq 0$ which is fixed once and for all and which, for simplicity, is dropped from the notation. The sheaf $\underset{\approx}{V}$ is constant on each facet F having the $U_F^{(e)}$-coinvariants of V as stalks. As promised earlier we show that the cohomology with compact support of $\underset{\approx}{V}$ is computable from the complex of cochains with finite support of the coefficient system $\gamma_e(V)$. We also rewrite our earlier formula for the Harish-Chandra character of a finite length representation V at an elliptic element $h \in G$ as a Hopf-Lefschetz trace formula: The character value at h is equal to the trace (in the sense of linear algebra) of h on the cohomology of the sheaf $\underset{\approx}{V}$ restricted to the fixed point set X^h. In IV.2 we construct a "smooth" extension of $\underset{\approx}{V}$ to a sheaf $j_{*,\infty}\underset{\approx}{V}$ on the Borel-Serre compactification $\overline{X}$ of X. This extension is in some sense intermediate between the extension by zero $j_!\underset{\approx}{V}$ and the full direct image $j_*\underset{\approx}{V}$. It requires a rather detailed and technical investigation of the geometry of $\overline{X}$. Let $X_\infty := \overline{X} \setminus X$ be the boundary. In IV.3 we compute the cohomology of $j_{*,\infty}\underset{\approx}{V}$ restricted to X_∞ in the case where the representation V is parabolically induced from a supercuspidal representation. Then in IV.4 we show that, for any finitely generated V and any e large enough, the sheaf $j_{*,\infty}\underset{\approx}{V}$ in fact has no higher cohomology. The combination of these two results immediately leads to the computation of the cohomology with compact support of the original sheaf $\underset{\approx}{V}$ provided V is parabolically induced as above. This is the fact which we had taken for granted in chapter III. So the duality theory, i.e., the investigation of the Ext-groups $\mathcal{E}^*(V)$, now is complete. As an application we prove in IV.5 Zelevinsky's conjecture. At this point we want to mention that Bernstein has a completely different proof (unpublished) of this conjecture along with the fact that the Zelevinsky involution comes from the functor $\mathcal{E}^*(.)$.

The last chapter complements the discussion of coefficient systems. We show that a rather big subcategory of $\mathrm{Alg}(G)$ is a localization of the category of equivariant coefficient systems on X. As will be explained in a forthcoming paper of the first author the latter objects constitute something which one might call perverse sheaves on the building X. From this point of view our constructions bear a certain resemblance to the Beilinson-Bernstein localization theory from Lie algebra representations to perverse sheaves on the flag manifold.

During this work we have profitted from conversations with M.Harris, G.Henniart, M.Rapoport, M.Tadic, J.Tits, and M.-F.Vigneras for which we are grateful. We especially want to thank E.Landvogt whose expert knowledge of the building has helped us a lot. The support which we have received at various stages from the MSRI at Berkeley, the Newton Institute, the Tata Institute, and the Université Paris 7 is gratefully acknowledged.

Added in proof: As the referee has pointed out, special cases of the Zelevinsky conjecture are treated in the papers [Kat] and [Pro]. Their methods are completely different from ours. Also in the meantime Aubert has given in [Au2] a proof of the Zelevinsky conjecture in the general case by studying on the Grothendieck group a certain involution which is defined in terms of parabolic induction (compare our IV.5.2).

Contents

Notations:

K a nonarchimedean locally compact field,

o the ring of integers in K,

π a fixed prime element in o,

$\omega : K^\times \to \mathbb{Z}$ the discrete valuation normalized by $\omega(\pi) = 1$,

$\overline{K} := o/\pi o$ the residue class field of o,

for any object X over o for which it makes sense to speak about its base change to $\overline{K}$ we denote this base change by $\overline{X}$,

$\mathbf{G}$ a connected reductive group over K,

$G := \mathbf{G}(K)$

I. The groups $U_F^{(e)}$

I.1. Review of the Bruhat-Tits building

The (semisimple) Bruhat-Tits building X of the group G is the central object in this paper. Since the monumental treatise [BT] is not so easily accessible for the nonexpert we believe it to be necessary to briefly review the construction and basic properties of X. Most of the notations to be introduced for this purpose will be needed later on anyway.

We fix a maximal K-split torus S in G. (Strictly speaking S is the group of K-rational points of that torus. This kind of abuse of language will usually be made.) Let $X^*(S)$, resp. $X_*(S)$, denote the group of algebraic characters, resp. cocharacters, of S. Similarly let $X_*(C)$ denote the group of K-algebraic cocharacters of the connected center C of G. The real vector space

$$A := (X_*(S)/X_*(C)) \otimes \mathbb{R}$$

is called the basic apartment. Let Z, resp. N, be the centralizer, resp. normalizer, of S in G. The Weyl group $W := N/Z$ acts by conjugation on S; this induces a faithful linear action of W on A. On the other hand let

$$\langle\ ,\rangle : X_*(S) \times X^*(S) \longrightarrow \mathbb{Z}$$

be the obvious pairing; its $\mathbb{R}$-linear extension also is denoted by $\langle\ ,\rangle$. There is a unique homomorphism

$$\nu : Z \longrightarrow X_*(S) \otimes \mathbb{R}$$

such that

$$\langle \nu(g), \chi | S\rangle = -\omega(\chi(g))$$

for any $g \in Z$ and any K-algebraic character χ of Z. We let $g \in Z$ act on A by the translation

$$gx := x + \text{image of } \nu(g) \text{ in } A \qquad \text{for } x \in A \quad .$$

The first important observation in this theory is that this translation action of Z on A can be extended to an action of N on A by affine automorphisms ([Tit] 1.2). We fix one such extension and simply denote it by $x \mapsto nx$ for $n \in N$ and $x \in A$. One has:

— All possible other such extensions are given by $x \mapsto n(x + x_0) - x_0$ where $x_0 \in A$ is a fixed but arbitrary point.

— If $w \in W$ is the image of $n \in N$ then

$$(x \mapsto wx) = \text{ linear part of } (x \mapsto nx) \quad .$$

In order to equip A with an additional structure we need the set $\Phi \subseteq X^*(S)$ of roots of G with respect to S. Any root α obviously induces a linear form $\alpha : A \to \mathbb{R}$. Also corresponding to any $\alpha \in \Phi$ we have the coroot $\check{\alpha} \in A$ and the involution $s_\alpha \in W$ whose action on A is given by

$$s_\alpha x = x - \alpha(x) \cdot \check{\alpha} \quad \text{for } x \in A \quad .$$

For us the most important object associated to an $\alpha \in \Phi$ is its root subgroup $U_\alpha \subseteq G$ ([Bor] 21.9 where the notation $U_{(\alpha)}$ is used); in particular it is a unipotent subgroup normalized by Z. Let $\Phi^{red} := \{\alpha \in \Phi : \alpha/2 \notin \Phi\}$ be the subset of reduced roots. Crucial is the following fact ([BoT] §5): For $\alpha \in \Phi^{red}$ and each $u \in U_\alpha \backslash \{1\}$ the intersection

$$U_{-\alpha} u U_{-\alpha} \cap N = \{m(u)\}$$

consists of a single element called $m(u)$; moreover the image of $m(u)$ in W is s_α. A central assertion in the Bruhat-Tits theory now is the fact that the translation part of the affine automorphism of A corresponding to $m(u)$ is given by $-\ell(u)\cdot\check{\alpha}$ for some real number $\ell(u)$, i.e., we have

$$m(u)x = s_\alpha x - \ell(u) \cdot \check{\alpha} = x - (\alpha(x) + \ell(u)) \cdot \check{\alpha} \quad \text{for any } x \in A \quad .$$

We may view $m(u)$ as the "reflection" at the affine hyperplane $\{x \in A : \alpha(x) = -\ell(u)\}$ ([Tit] 1.4). Put $\Gamma_\alpha := \{\ell(u) : u \in U_\alpha \backslash \{1\}\} \subseteq \mathbb{R}$; this is a in both directions unbounded discrete subset in $\mathbb{R}$ and $-\Gamma_\alpha = \Gamma_{-\alpha}$ ([BT] I.6.2.16). The affine functions $\alpha(.)+\ell$ on A for $\alpha \in \Phi^{red}$ and $\ell \in \Gamma_\alpha$ are called affine roots. Two points x and y in A are called equivalent if each affine root is either positive or zero or negative at both points; the corresponding equivalence classes are called facets. This imposes an additional geometric structure on the apartment A which is respected by the action of N.

Parallel to this structure the root subgroup U_α for $\alpha \in \Phi^{red}$ possesses the filtration

$$U_{\alpha,r} := \{u \in U_\alpha \backslash \{1\} : \ell(u) \geq r\} \cup \{1\} \quad \text{for } r \in \mathbb{R} \quad .$$

This is an exhaustive and separated discrete filtration of U_α by subgroups ([BT] I.6.2.12b)); put $U_{\alpha,\infty} := \{1\}$. For any nonempty subset $\Omega \subseteq A$ we define

$$\begin{aligned} f_\Omega : \Phi &\longrightarrow \mathbb{R} \cup \{\infty\} \\ \alpha &\longmapsto -\inf_{x \in \Omega} \alpha(x) \end{aligned}$$

and

$$\begin{aligned} U_\Omega := &\text{ subgroup of } G \text{ generated by} \\ &\text{all } U_{\alpha, f_\Omega(\alpha)} \text{ for } \alpha \in \Phi^{red} \quad . \end{aligned}$$

This group has various important properties ([BT] I.6.2.10, 6.4.9, and 7.1.3):
1. $nU_\Omega n^{-1} = U_{n\Omega}$ for any $n \in N$; in particular $N_\Omega := \{n \in N : nx = x$ for any $x \in \Omega\}$ normalizes U_Ω.
2. $U_\Omega \cap N \subseteq N_\Omega$.
3. $U_\Omega \cap U_\alpha = U_{\alpha, f_\Omega(\alpha)}$ for any $\alpha \in \Phi^{red}$.
4. Let $\Phi = \Phi^+ \cup \Phi^-$ be any decomposition into positive and negative roots and put

$$U^\pm := \text{subgroup of } G \text{ generated by all } U_\alpha \text{ for } \alpha \in \Phi^\pm \cap \Phi^{red} \quad ;$$

then

$$U_\Omega = (U_\Omega \cap U^-)(U_\Omega \cap U^+)(U_\Omega \cap N) \quad ;$$

moreover the product map induces bijections

$$\prod_{\alpha \in \Phi^\pm \cap \Phi^{red}} U_{\alpha, f_\Omega(\alpha)} \xrightarrow{\sim} U_\Omega \cap U^\pm$$

whatever ordering of the factors on the left hand side we choose. Define

$$P_\Omega := N_\Omega \cdot U_\Omega$$

which contains U_Ω as a normal subgroup by 1. By 2. we have $P_\Omega \cap N = N_\Omega$. (Warning: In [BT] our groups N_Ω and P_Ω are denoted by $\hat{N}_\Omega$ and $\hat{P}_\Omega$ and our symbols have a different meaning.) In case $\Omega = \{x\}$ we write f_x, U_x, N_x, and P_x instead of $f_{\{x\}}, \ldots$

We now are ready to define the Bruhat-Tits building X. Consider the relation $\sim$ on $G \times A$ defined by

$$(g, x) \sim (h, y) \text{ if there is a } n \in N \text{ such that } nx = y \text{ and } g^{-1}hn \in U_x \quad ;$$

it is easily checked that this is an equivalence relation. We put

$$X := G \times A / \sim \quad .$$

It is straightforward to see that G acts on X via

$$g \cdot \text{ class of } (h, y) := \text{ class of } (gh, y) \text{ for } g \in G \text{ and } (h, y) \in G \times A$$

and that the map

$$\begin{aligned} A &\longrightarrow X \\ x &\longmapsto \text{ class of } (1, x) \end{aligned}$$

is injective and N-equivariant. Viewing the latter map as an inclusion we can write gx for the class of (g, x). A first basic fact ([BT] I.7.4.4) is that, for $\Omega \subseteq A$ nonempty,

$$P_\Omega = \{g \in G : gx = x \text{ for any } x \in \Omega\}$$

holds true. The relation between the facet structure of A and the subgroup filtration in U_α, for $\alpha \in \Phi^{red}$, is given by the fact that for $u \in U_\alpha \setminus \{1\}$ we have

$$\{x \in A : ux = x\} = \{x \in A : \alpha(x) + \ell(u) \geq 0\}$$

([BT] I.7.4.5). The subsets of X of the form gA with $g \in G$ are called apartments. A very important technical property of the G-action on X is the following:
5. For any $g \in G$ there exists a $n \in N$ such that $gx = nx$ for any $x \in A \cap g^{-1}A$ ([BT] I.7.4.8).
For example it implies that the partition into facets can be extended from A to all of X in the following way: A subset $F' \subseteq X$ is called a facet if it is of the form $F' = gF$ for some $g \in G$ and some facet $F \subseteq A$. It also implies:
6. For any nonempty $\Omega \subseteq A$ the group U_Ω acts transitively on the set of all apartments which contain Ω.
From the Bruhat decomposition

$$G = U_x N U_y \quad \text{for } x, y \in A$$

one concludes:
7. Any two points and even any two facets in X are contained in a common apartment ([BT] I.7.4.18).
For any nonempty subset $\Omega \subseteq X$ we define

$$P_\Omega := \{g \in G : gz = z \quad \text{for any } z \in \Omega\}$$

and

$$P_\Omega^\dagger := \{g \in G : g\Omega = \Omega\}$$

and we abbreviate $P_z := P_{\{z\}} = P_{\{z\}}^\dagger$ for any $z \in X$.
Finally we fix once and for all a W-invariant euclidean metric d on A. The action of N on A then automatically is isometric. As a simple consequence of 5.-7. this metric extends in a unique G-invariant way to a metric d on all of X.

The metric space (X, d) together with its isometric G-action and its partition into facets is called the Bruhat-Tits building of G. Further properties of this very rich structure will be recalled when they are needed.

I.2. Definition of the groups $U_F^{(e)}$

For any facet F in A let $\mathbf{G}_F^0$ be the smooth affine o-group scheme with general fiber $\mathbf{G}$ constructed in [BT] II.5.1.30. By [BT] II.5.2.4 the group $\mathbf{G}_F^0(o)$ is the subgroup of G generated by U_F and $\mathscr{Z}^0(o)$ where $\mathscr{Z}^0$ is the connected component of the "canonical" extension $\mathscr{Z}$ of Z to a smooth affine o-group scheme ([BT] II.5.2.1). Put

$$H := \{n \in N : nx = x \ \text{ for all } x \in A\}$$

and

$$H^1 := \{n \in H : \omega(\chi(n)) = 0 \text{ for any } K\text{-algebraic character } \chi \text{ of } G\} \quad .$$

According to [BT] II.5.2.1 we have

$$\mathscr{Z}(o) = H^1 \quad .$$

Therefore $\mathscr{Z}^0(o)$ is of finite index in H^1. Since $H \subseteq N_F$ we see that

$$U_F \subseteq \mathbf{G}_F^0(o) = U_F \cdot \mathscr{Z}^0(o) \subseteq P_F \quad .$$

It follows from [BT] II.4.6.17 that any interior automorphism $g \mapsto ngn^{-1}$ of $\mathbf{G}$ with $n \in N$ extends to an isomorphism of o-group schemes

$$\mathbf{G}_F^0 \xrightarrow{\cong} \mathbf{G}_{nF}^0 \quad .$$

The closed fiber $\overline{\mathbf{G}}_F^0$ is a connected smooth algebraic group over $\overline{K}$; let $\overline{\mathbf{R}}_F$ denote its unipotent radical. Put

$$R_F := \{g \in \mathbf{G}_F^0(o) : (g \bmod \pi) \in \overline{\mathbf{R}}_F(\overline{K})\} \quad ;$$

this is a compact open subgroup of G. Because of $nR_Fn^{-1} = R_{nF}$ for $n \in N$ it is normalized by

$$N_F^\dagger := \{n \in N : nF = F\} \quad .$$

The property 1.5 implies that

$$P_F^\dagger = N_F^\dagger P_F = N_F^\dagger U_F \quad .$$

Hence R_F is a normal subgroup of $P_F^\dagger$. In the following we will construct a specific decreasing filtration

$$\mathbf{G}_F^0(o) \supseteq R_F =: U_F^{(0)} \supseteq U_F^{(1)} \supseteq \dots$$

by subgroups $U_F^{(e)}$ which are normal in $P_F^\dagger$ and compact open in G. For doing this we need the concept of a concave function in [BT] I.6.4.1-5.

First we have to introduce the totally ordered commutative monoid

$$\tilde{\mathbb{R}} := \mathbb{R} \cup \{r+ : r \in \mathbb{R}\} \cup \{\infty\} \quad .$$

Its total order is given by the usual total order on $\mathbb{R}$ and by

$$r \leq r+ \leq s \leq \infty \quad \text{if } r < s \quad ;$$

its monoid structure extends the addition on $\mathbb{R}$ and is given by

$$\begin{aligned} &r + (s+) = (r+) + (s+) = (r+s)+ \quad \text{and} \\ &r + \infty = (r+) + \infty = \infty + \infty = \infty \quad . \end{aligned}$$

We put $\frac{1}{2} \cdot (r+) := (\frac{1}{2}r)+$ and $\frac{1}{2} \cdot \infty := \infty$. A function $f : \Phi \to \tilde{\mathbb{R}}$ is called concave if

$$\begin{array}{ll} f(\alpha) + f(\beta) \geq f(\alpha+\beta) & \text{for any } \alpha, \beta, \alpha+\beta \in \Phi \text{ , and} \\ f(\alpha) + f(-\alpha) \geq 0 & \text{for any } \alpha \in \Phi \end{array}$$

hold. For $\alpha \in \Phi^{red}$ and $r \in \mathbb{R}$ we define

$$U_{\alpha,r+} := \bigcup_{s \in \mathbb{R}, s > r} U_{\alpha,s} \quad .$$

Then, for any concave function f, the group

$$\begin{aligned} U_f := \ &\text{subgroup of } G \text{ generated by} \\ &\text{all } U_{\alpha,f(\alpha)} \text{ for } \alpha \in \Phi^{red} \text{ and} \\ &\text{all } U_{2\alpha} \cap U_{\alpha,\frac{1}{2}f(2\alpha)} \text{ for } \alpha, 2\alpha \in \Phi \end{aligned}$$

has properties completely analogous to 1.1-1.4 ([BT] I.6.4.9). Observe that $U_\Omega = U_{f_\Omega}$.

Starting from the concave function f_F, for a facet F in A, we define a new function $f_F^* : \Phi \to \tilde{\mathbb{R}}$ by

$$f_F^*(\alpha) := \begin{cases} f_F(\alpha)+ & \text{if } \alpha|F \text{ is constant} \quad , \\ f_F(\alpha) & \text{otherwise} \quad ; \end{cases}$$

it is concave, too, by [BT] I.6.4.23. In case $\alpha, 2\alpha \in \Phi$ we have $f_F^*(2\alpha) = 2f_F^*(\alpha)$ so that $U_{f_F^*}$ is the subgroup generated by all $U_{\alpha,f_F^*(\alpha)}$ for $\alpha \in \Phi^{red}$.

Lemma I.2.1:

$R_F \cap U_{\alpha,f_F(\alpha)} = U_{\alpha,f_F^*(\alpha)}$ *for any* $\alpha \in \Phi^{red}$.

Proof: We have to introduce further notations. In case $2\alpha \in \Phi$ put

$$\Gamma'_{2\alpha} := \{2\ell(u) : u \in U_{2\alpha}\backslash\{1\}\}$$

(recall that $U_{2\alpha} \subseteq U_\alpha$) and

$$\Gamma'_\alpha := \{\ell(u) \in \Gamma_\alpha : u \in U_\alpha \backslash U_{2\alpha} \text{ and } \ell(u) = \sup \ell(uU_{2\alpha})\} \quad ;$$

one has $\Gamma_\alpha = \Gamma'_\alpha \cup \frac{1}{2}\Gamma'_{2\alpha}$ ([BT] I.6.2.2) and $\Gamma'_\alpha \neq \emptyset$ ([BT] II.4.2.21). In case $2\alpha \notin \Phi$ put $\Gamma'_\alpha := \Gamma_\alpha$. The "optimization" $g_F : \Phi \to \mathbb{R}$ of the function f_F is defined by

$$g_F(\beta) := \inf\{\ell \in \Gamma'_\beta : \ell \geq f_F(\beta)\} \quad .$$

Moreover we put

$$g_F^*(\beta) := \begin{cases} g_F(\beta)+ & \text{if } g_F(\beta) + g_F(-\beta) = 0 \quad , \\ g_F(\beta) & \text{otherwise} \quad . \end{cases}$$

(The functions g_F and g_F^* in general are no longer concave but only quasi-concave in the sense of [BT].) In [BT] II.4.6.10 (compare in particular the third paragraph on p. 321) and 5.1.31 it is proved that

$$R_F \cap U_{\alpha,f_F(\alpha)} = \begin{cases} U_{\alpha,g_F^*(\alpha)} & \text{if } 2\alpha \notin \Phi \quad , \\ U_{\alpha,g_F^*(\alpha)} \cdot (U_{2\alpha} \cap U_{\alpha,\frac{1}{2}g_F^*(2\alpha)}) & \text{if } 2\alpha \in \Phi \end{cases}$$

holds true. Let us first consider the case $2\alpha \notin \Phi$. If $g_F(\alpha) + g_F(-\alpha) = 0$ then clearly also $f_F(\alpha) + f_F(-\alpha) = 0$, i.e., $\alpha|F$ is constant and $g_F(\alpha) = f_F(\alpha)$; we obtain

$$U_{\alpha,g_F^*(\alpha)} = U_{\alpha,g_F(\alpha)+} = U_{\alpha,f_F^*(\alpha)} \quad .$$

Assume now that $g_F(\alpha) + g_F(-\alpha) \neq 0$. If $\alpha|F$ is not constant then by the definitions we have

$$U_{\alpha,g_F^*(\alpha)} = U_{\alpha,g_F(\alpha)} = U_{\alpha,f_F(\alpha)} = U_{\alpha,f_F^*(\alpha)} \quad ;$$

the same holds if $\alpha|F$ is constant since then $f_F(\alpha) \notin \Gamma_\alpha$ which implies the last identity.
We turn to the case $2\alpha \in \Phi$. There are the following four possibilities:

1)	$g_F^*(\alpha) = g_F(\alpha)+$	and	$g_F^*(2\alpha) = g_F(2\alpha)+$	,
2)	$g_F^*(\alpha) = g_F(\alpha)+$	and	$g_F^*(2\alpha) = g_F(2\alpha)$	,
3)	$g_F^*(\alpha) = g_F(\alpha)$	and	$g_F^*(2\alpha) = g_F(2\alpha)$	,
4)	$g_F^*(\alpha) = g_F(\alpha)$	and	$g_F^*(2\alpha) = g_F(2\alpha)+$	.

In case 1) we have

$$\alpha|F \text{ is constant }, \ f_F(\alpha) = g_F(\alpha) \in \Gamma'_\alpha \cap \frac{1}{2}\Gamma'_{2\alpha'} \text{ and } g_F(2\alpha) = 2f_F(\alpha)$$

and hence

$$U_{\alpha,g^*_F(\alpha)} \cdot (U_{2\alpha} \cap U_{\alpha,\frac{1}{2}g^*_F(2\alpha)}) = U_{\alpha,f^*_F(\alpha)} \cdot (U_{2\alpha} \cap U_{\alpha,f^*_F(\alpha)}) = U_{\alpha,f^*_F(\alpha)} \quad .$$

In case 2) we have

$$\alpha|F \text{ is constant }, \ f_F(\alpha) = g_F(\alpha) \in \Gamma'_\alpha \ , \text{ and } \frac{1}{2}g_F(2\alpha) > f_F(\alpha)$$

and hence

$$U_{\alpha,g^*_F(\alpha)} \cdot (U_{2\alpha} \cap U_{\alpha,\frac{1}{2}g^*_F(2\alpha)}) = U_{\alpha,f^*_F(\alpha)} \cdot (U_{2\alpha} \cap U_{\alpha,\frac{1}{2}g_F(2\alpha)}) = U_{\alpha,f^*_F(\alpha)} \quad .$$

In case 3) we have

$$g^*_F(\alpha) = \inf\{\ell \in \Gamma'_\alpha : \ell \geq f_F(\alpha)\} \text{ and}$$
$$\frac{1}{2}g^*_F(2\alpha) = \inf\{\ell \in \frac{1}{2}\Gamma'_{2\alpha} : \ell \geq f_F(\alpha)\} \quad .$$

Let us first assume that $\frac{1}{2}g^*_F(2\alpha) \geq g^*_F(\alpha)$; then

$$g^*_F(\alpha) = \inf\{\ell \in \Gamma_\alpha : \ell \geq f_F(\alpha)\} \quad .$$

This implies

$$U_{\alpha,g^*_F(\alpha)} \cdot (U_{2\alpha} \cap U_{\alpha,\frac{1}{2}g^*_F(2\alpha)}) = U_{\alpha,g^*_F(\alpha)} = U_{\alpha,f_F(\alpha)} \quad .$$

Now assume that $\frac{1}{2}g^*_F(2\alpha) < g^*_F(\alpha)$; then

$$\frac{1}{2}g^*_F(2\alpha) = \inf\{\ell \in \Gamma_\alpha : \ell \geq f_F(\alpha)\} \quad .$$

We are going to use the following general fact which is a straightforward consequence of the definition of the set Γ'_α: If $r < s$ are values in Γ_α such that

$$r \notin \Gamma'_\alpha \text{ and } s = \inf\{\ell \in \Gamma'_\alpha : \ell \geq r\}$$

then

$$(*) \qquad U_{\alpha,r} \subseteq U_{\alpha,s} \cdot (U_{2\alpha} \cap U_{\alpha,r}) \quad .$$

Applied to our situation this leads to

$$U_{\alpha,g_F^*(\alpha)} \cdot (U_{2\alpha} \cap U_{\alpha,\frac{1}{2}g_F^*(2\alpha)}) = U_{\alpha,\frac{1}{2}g_F^*(2\alpha)} = U_{\alpha,f_F(\alpha)} \quad .$$

Moreover in case 3) we always have $U_{\alpha,f_F(\alpha)} = U_{\alpha,f_F^*(\alpha)}$ since if $\alpha|F$ is constant then both $g_F^*(\alpha)$ and $\frac{1}{2}g_F^*(2\alpha)$ are strictly bigger than $f_F(\alpha)$.
Finally in case 4) we have

$$\alpha|F \text{ is constant }, \ f_F(\alpha) = \frac{1}{2}g_F(2\alpha) \in \frac{1}{2}\Gamma'_{2\alpha} \ , \text{ and } g_F(\alpha) > f_F(\alpha)$$

and hence

$$U_{\alpha,g_F^*(\alpha)} \cdot (U_{2\alpha} \cap U_{\alpha,\frac{1}{2}g_F^*(2\alpha)}) = U_{\alpha,g_F(\alpha)} \cdot (U_{2\alpha} \cap U_{\alpha,f_F(\alpha)+}) = U_{\alpha,f_F^*(\alpha)}$$

where the second identity again is a consequence of $(*)$ since $f_F(\alpha) \notin \Gamma'_\alpha$. □

There is a scheme theoretic version of 1.4 ([BT] II.5.2.2-4): $\mathbf{G}_F^0$ possesses smooth closed o-subgroup schemes $\mathcal{U}_{\alpha,F}$ for $\alpha \in \Phi^{red}$ and $\mathcal{U}_F^\pm$ for any fixed decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots such that

$$\mathcal{U}_{\alpha,F}(o) = U_{\alpha,f_F(\alpha)} \text{ and } \mathcal{U}_F^\pm(o) = U_F \cap U^\pm \quad .$$

(We have simplified the notation a little by writing $\mathcal{U}_{\alpha,F}$ for $\mathcal{U}_{\alpha,(f_F(\alpha),f_F(2\alpha))}$ in loc. cit.) Moreover the product map induces an isomorphism of o-schemes

$$\prod_{\alpha\in\Phi^\pm\cap\Phi^{red}} \mathcal{U}_{\alpha,F} \xrightarrow{\cong} \mathcal{U}_F^\pm$$

(whatever ordering of the factors on the left hand side we choose) as well as an open immersion of o-group schemes

$$\mathcal{U}_F^- \times \mathcal{Z}^0 \times \mathcal{U}_F^+ \hookrightarrow \mathbf{G}_F^0 \quad .$$

We put

$$Z^{(0)} := \{g \in \mathcal{Z}^0(o) : (g \bmod \pi) \in R_u(\overline{\mathcal{Z}}^0)(\overline{K})\}$$

where $R_u(\overline{\mathcal{Z}}^0)$ denotes the unipotent radical of $\overline{\mathcal{Z}}^0$.

Proposition I.2.2:

The product map induces a bijection

$$\left(\prod_{\alpha\in\Phi^-\cap\Phi^{red}} U_{\alpha,f_F^*(\alpha)}\right)\times Z^{(0)}\times\left(\prod_{\alpha\in\Phi^+\cap\Phi^{red}} U_{\alpha,f_F^*(\alpha)}\right)\xrightarrow{\simeq} R_F \quad .$$

Proof: We recall from [BT] II.4.6.4 and 5.1.31: If $\mathcal{S}^0$ denotes the connected component of the Néron model over o of S then $\overline{\mathcal{S}}^0$ is a maximal $\overline{K}$-split torus in $\overline{\mathbf{G}}_F^0$ and $\overline{\mathcal{Z}}^0$ is its centralizer. Also the $\overline{\mathcal{U}}_{\alpha,F}$ are the root subgroups in $\overline{\mathbf{G}}_F^0$. By [BT] II.1.1.11 the above open immersion therefore induces an isomorphism

$$\left(\prod_{\alpha\in\Phi^-\cap\Phi^{red}} \overline{\mathcal{U}}_{\alpha,F}\cap\overline{\mathbf{R}}_F\right)\times\overline{R}_u(\overline{\mathcal{Z}}^0)\times\left(\prod_{\alpha\in\Phi^+\cap\Phi^{red}} \overline{\mathcal{U}}_{\alpha,F}\cap\overline{\mathbf{R}}_F\right)\xrightarrow{\simeq}\overline{\mathbf{R}}_F$$

and hence also an isomorphism between formal completions

$$(\prod\mathcal{U}_{\alpha,F}^R)\times{\mathcal{Z}^0}^R\times(\prod\mathcal{U}_{\alpha,F}^R)\xrightarrow{\simeq}{\mathbf{G}_F^0}^R$$

where $?^R$ denotes the formal completion of ? along $\overline{?}\cap\overline{\mathbf{R}}_F$. It remains to observe that $\mathcal{U}_{\alpha,F}^R(o)=R_F\cap U_{\alpha,f_F(\alpha)}=U_{\alpha,f_F^*(\alpha)}$, ${\mathcal{Z}^0}^R(o)=Z^{(0)}$, and ${\mathbf{G}_F^0}^R(o)=R_F$. $\square$

Corollary I.2.3:

$R_F=U_{f_F^*}\cdot Z^{(0)}$.

With f_F^* also the functions f_F^*+e, for any integer $e\geq 0$, are concave. Hence we have the descending sequence of subgroups

$$U_{f_F^*}\supseteq U_{f_F^*+1}\supseteq U_{f_F^*+2}\supseteq\ldots$$

We also need a corresponding filtration

$$Z^{(0)}\supseteq Z^{(1)}\supseteq\ldots\supseteq Z^{(e)}\supseteq\ldots$$

The subgroups we are looking for then will be defined to be

$$U_F^{(e)}:=U_{f_F^*+e}\cdot Z^{(e)}\quad ;$$

note that H normalizes U_f for any concave function f ([BT] I.6.2.10(iii)). The properties which we want the subgroups $U_F^{(e)}$ to have impose certain conditions on the possible shape of the filtration $Z^{(\cdot)}$. These conditions are axiomatized in [BT] I.6.4 in the following way.

First of all it is notationally convenient to define

$$U_{2\alpha} := \{1\} \text{ in case } \alpha \in \Phi \text{ but } 2\alpha \notin \Phi$$

and

$$U_{2\alpha,k} := U_{2\alpha} \cap U_{\alpha,\frac{1}{2}k} \text{ for any } \alpha \in \Phi \text{ and any } k \in \tilde{\mathbb{R}} \quad .$$

For any $k \in \tilde{\mathbb{R}}$ put

$$\begin{aligned} H_{(k)} := {} & \text{set of all } h \in H \text{ such that} \\ & (h, U_{\alpha,r}) := \{(h,u) : u \in U_{\alpha,r}\} \subseteq U_{\alpha,r+k} \cdot U_{2\alpha,2r+k} \\ & \text{for any } \alpha \in \Phi \text{ and any } r \in \mathbb{R} \quad . \end{aligned}$$

The $H_{(k)}$ form a decreasing family of subgroups in H which are normal in N; obviously $H_{(k)} = H$ for $k \leq 0$. Another such family denoted by $H_{[k]}$ is given as follows: For $k \leq 0$ put

$$\begin{aligned} H_{[k]} := {} & \text{subgroup generated by all} \\ & H \cap \langle U_{\alpha,r} \cup U_{-\alpha,-r} \rangle \text{ for } \alpha \in \Phi \text{ and } r \in \mathbb{R} \quad . \end{aligned}$$

In case $0 < k < \infty$ the commutator (u, u') for $u \in U_{\alpha,r}, u' \in U_{-\alpha,s}$ with $r+s = k$ or $u' \in U_{-2\alpha,s}$ with $2r + s = k, r, s \in \tilde{\mathbb{R}}$, and any $\alpha \in \Phi$ lies in a double coset $U_\alpha h_{u,u'} U_{-\alpha}$ with a uniquely determined element $h_{u,u'} \in H$ ([BT] I.6.3.9); we put

$$H_{[k]} := \text{ subgroup generated by all those } h_{u,u'} \quad .$$

Finally we set

$$H_{[\infty]} := \bigcap_{k<\infty} H_{[k]} \quad .$$

The $H_{[k]}$ again form a decreasing family of subgroups of H which are normal in N. One has $H_{[r+]} = \bigcup_{s>r} H_{[s]}$ for any $r \in \mathbb{R}$. The key property of this latter family is the following. Let us call a function $f : \Phi \cup \{0\} \to \tilde{\mathbb{R}}$ concave if $f|\Phi$ is concave and if

$$f(\alpha) + f(-\alpha) \geq f(0) \geq 0 \text{ for any } \alpha \in \Phi$$

holds. In this situation we have

$$H \cap U_{f|\Phi} \subseteq H_{[f(0)]}$$

([BT] I.6.4.17).

A *good filtration* of H now by definition is a family of subgroups $H_r \subseteq H$ for $r \in \mathbb{R}$ such that

— $H_r = H$ for $r \leq 0$,

— $H_r \subseteq H_s$ if $r \geq s$,

— $H_{[r]} \subseteq H_r \subseteq H_{(r)}$ for any $r \in \mathbb{R}$, and

— $(H_r, H_s) \subseteq H_{r+s}$ for any $r, s \in \mathbb{R}$.

These properties together with [BT] I.6.4.33 imply that the H_r are normal in N. A necessary and sufficient condition for the existence of a good filtration is, according to [BT] I.6.4.39, that

$$H_{[k]} \subseteq H_{(k)} \quad \text{for any } k \in \tilde{\mathbb{R}}$$

holds true. In [BT] I.6.4.15 it is stated that this condition actually is fulfilled — see Proposition 6 below. For the moment we assume that a good filtration is given. We pose

$$H_{r+} := \bigcup_{s>r} H_s \quad \text{for } r \in \mathbb{R} \text{ and } H_\infty := \bigcap_{r \in \mathbb{R}} H_r \quad .$$

Also for any concave function $f : \Phi \cup \{0\} \to \tilde{\mathbb{R}}$ we define the subgroup

$$U_f := U_{f|\Phi} \cdot H_{f(0)} \quad .$$

Lemma I.2.4:

Let $f, g : \Phi \cup \{0\} \to \tilde{\mathbb{R}}$ be two concave functions such that $g(p\alpha + q\beta) \leq pg(\alpha) + qf(\beta)$ for any $\alpha, \beta \in \Phi \cup \{0\}$ and $p, q \in \mathbb{N}$ such that $p\alpha + q\beta \in \Phi \cup \{0\}$; then U_f normalizes U_g.

Proof: [BT] I.6.4.43. □

Lemma I.2.5:

i. $H_{[0+]} \subseteq Z^{(0)} \subseteq H_{(0+)}$;

ii. $Z^{(0)}$ is normal in N.

Proof: i. $H_{[0+]}$ is the subgroup generated by the $h_{u,u'}$ for $u \in U_{\alpha,r+}$ and $u' \in U_{-\alpha,-r}$ with $\alpha \in \Phi^{red}$ and $r \in \mathbb{R}$. Fix such u and u' and choose a vertex x in A (i.e., $\{x\}$ is a facet in A) such that $U_{\alpha,r} = U_{\alpha,-\alpha(x)}$. Then $u \in R_{\{x\}}$ by Lemma 1 and $u' \in U_x \subseteq P_x$. Since $R_{\{x\}}$ is normal in P_x the commutator (u, u') lies in $R_{\{x\}}$. It follows now from Proposition 2 that $h_{u,u'} \in Z^{(0)}$.
On the other hand we have to show that $(Z^{(0)}, U_{\alpha,r}) \subseteq U_{\alpha,r+}$ for any $\alpha \in \Phi^{red}$ and $r \in \mathbb{R}$. But choosing the vertex x as before we have

$$(Z^{(0)}, U_{\alpha,r}) = (Z^{(0)}, U_{\alpha,f_x(\alpha)}) \subseteq R_F \cap U_{\alpha,f_x(\alpha)} = U_{\alpha,f_x(\alpha)+} \subseteq U_{\alpha,r+} \quad .$$

ii. By the very construction of the o-group scheme $\mathcal{Z}^0$ any automorphism $g \to ngn^{-1}$ of Z with $n \in N$ extends to an automorphism of the o-group scheme $\mathcal{Z}^0$. □

Proposition I.2.6:

There exists a good filtration $(H_r)_{r\in\mathbb{R}}$ *of* H *such that:*

i. $H_{0+} = Z^{(0)}$,

ii. $H_\infty = \{1\}$, *and*

iii. H_{r+} *is open in* H *for any* $r \in \mathbb{R}$.

Proof: As mentioned already this is a slightly sharpened version of [BT] I.6.4.15. We are indebted to J. Tits for explaining to us the proof which is missing in [BT] and which we briefly sketch in the following.
First of all we note that it suffices to find a good filtration H'_r which fulfills ii., iii., and the weaker condition

i'. $\quad Z^{(0)} \subseteq H'_{0+}$.

Because then

$$H_r := \begin{cases} H & \text{for } r \leq 0 \\ Z^{(0)} \cap H'_r & \text{for } r > 0 \end{cases} ,$$

is a good filtration satisfying i. — iii. This follows from Lemma 5.i and the fact that $Z^{(0)}$ is open in Z.

Step 1: The split case. If G is split then we have $S = Z \cong (K^\times)^n$ and $\mathcal{Z}^0 \cong \mathbb{G}^n_{m/o}$. We define

$$H_r := ker(\mathcal{Z}^0(o) \longrightarrow \mathcal{Z}^0(o/\pi^{m+1}o))$$

if $m < r \leq m+1$ with $m \in \mathbb{N} \cup \{0\}$. The only thing which has to be checked is

$$H_{[r]} \subseteq H_r \subseteq H_{(r)} \quad \text{for } r > 0 \quad .$$

The left inclusion, by [BT] II.3.2.1, can be checked in $SL_2(K)$ where it is straightforward. For the right inclusion we use the following two identities. Let $\alpha \in \Phi$ be a root.

— ([BT] II.3.2.1)
$(h, u) = (\alpha(h) - 1)u$ for $h \in Z$ and $u \in U_\alpha$.

— ([BT] I.6.1.3 b) and 6.2.3 b))
$\ell(au) = \omega(a) + \ell(u)$ for $a \in K$ and $u \in U_\alpha$ (here $\ell(1) := \infty$).

By definition any $h \in H_r$ satisfies $\omega(\alpha(h)-1) \geq m+1$. For $u \in U_{\alpha,s}$ we therefore obtain

$$\ell((h,u)) = \omega(\alpha(h) - 1) + \ell(u) \geq m + 1 + s \geq r + s \text{ , i.e., } (h,u) \in U_{\alpha,r+s} \quad .$$

This shows that $h \in H_{(r)}$.

To deduce the general case we observe that Bruhat and Tits proceed by applying the descent theory in [BT] I.9 in two steps: First from the split to the quasi-split case ([BT] II.4.2.3) and then from the latter to the general case ([BT] II.5.1.20). Hence we may use [BT] I.9.1.15 in order to see that our assertion descends as well. In each of the two steps we have to check that the assumption (DP) in loc.cit. is fulfilled and that the descent preserves the properties i'., ii., and iii.
Step 2: From the split to the quasi-split case. If G is quasi-split then Z is a maximal torus in G and $\mathcal{Z}$ is that part of the Néron model of Z over o which in the closed fibre consists of the connected components of finite order. The condition (DP) follows from the explicit computations in [BT] II.4.3.5. The filtration of H by construction is the intersection of H with a corresponding filtration over a splitting field of Z. From this it is obvious that the properties i'., ii., and iii. are preserved.
Step 3: From the quasi-split to the general case. Note that the descent is along an unramified extension L/K. The condition (DP) (with $t = 0$) holds by [BT] II.5.2.2. The descent of the properties i.-iii. is deduced from the following fact: Applying the argument in the proof of Proposition 2 to the group scheme $\mathcal{Z}^0$ (compare [BT] II.5.2.1) we obtain the decomposition

$$Z^{(0)} = Z \cap \left(\prod_{\tilde{\alpha} \in \tilde{\Phi}_0^-} U_{\tilde{\alpha},0+} \times Z_L^{(0)} \times \prod_{\tilde{\alpha} \in \tilde{\Phi}_0^+} U_{\tilde{\alpha},0+} \right) \quad ;$$

here $Z_L^{(0)}$ denotes the analog of $Z^{(0)}$ for $\mathbf{G}(L)$, $\tilde{\Phi}$ is the root system of $\mathbf{G}(L)$ with respect to some maximal L-split torus which contains S, and $\tilde{\Phi}_0$ is the subset of those reduced roots which restrict to 0 on S. □

We fix once and for all a good filtration of H as in Proposition 6. Define

$$Z^{(e)} := H_{e+} \quad \text{and} \quad U_F^{(e)} := U_{f_F^* + e} \cdot Z^{(e)} \quad \text{for } e \geq 0 \quad .$$

In other words we have

$$U_F^{(e)} = U_{h_F + e}$$

where the concave function $h_F : \Phi \cup \{0\} \to \tilde{\mathbb{R}}$ is defined by

$$h_F | \Phi := f_F^* \quad \text{and} \quad h_F(0) := 0+ \quad .$$

The functions h_F and f_F extended by $f_F(0) := 0$ fulfill the assumption of Lemma 4: This is straightforward if one of the $\alpha, \beta, p\alpha + q\beta$ is equal to 0; otherwise it is shown in the proof of [BT] I.6.4.23. Therefore U_F normalizes $U_F^{(e)}$ for any $e \geq 0$. Since N normalizes $Z^{(e)}$ and $N_F^\dagger$ normalizes $U_{f_F^* + e}$ ([BT] I.6.2.10 (iii)) we obtain that

$$U_F^{(e)} \quad \text{, for any } e \geq 0 \text{ , is normal in } P_F^\dagger \quad .$$

The same argument shows that

$$nU_F^{(e)}n^{-1} = U_{nF}^{(e)} \quad \text{for } n \in N \text{ and } e \geq 0 \quad .$$

Proposition I.2.7:

For any $e \geq 0$ the product map induces a bijection

$$\left(\prod_{\alpha \in \Phi^- \cap \Phi^{red}} U_{f_F^* + e} \cap U_\alpha \right) \times Z^{(e)} \times \left(\prod_{\alpha \in \Phi^+ \cap \Phi^{red}} U_{f_F^* + e} \cap U_\alpha \right) \xrightarrow{\simeq} U_F^{(e)} \quad ;$$

moreover we have

$$U_{f_F^* + e} \cap U_\alpha = U_{\alpha, f_F^*(\alpha) + e} \cdot U_{2\alpha, 2f_F^*(\alpha) + e} \quad \textit{for any } \alpha \in \Phi^{red} \quad .$$

Proof: Proposition 2 and [BT] I.6.9(i). □

Corollary I.2.8:

$U_F^{(e)} = (U_F^{(e)} \cap U^-)(U_F^{(e)} \cap Z)(U_F^{(e)} \cap U^+)$ *for any $e \geq 0$.*

Corollary I.2.9:

The $U_F^{(e)}$ for $e \geq 0$ (and F fixed) form a fundamental system of compact open neighbourhoods of 1 in G.

Proof: Since

$$\mathcal{U}_F^- \times \mathcal{Z}^0 \times \mathcal{U}_F^+ \hookrightarrow \mathbf{G}_F^0$$

is an open immersion the subset

$$\left(\prod_{\alpha \in \Phi^- \cap \Phi^{red}} U_{\alpha, f_F(\alpha)} \right) \times \mathcal{Z}^0(o) \times \left(\prod_{\alpha \in \Phi^+ \cap \Phi^{red}} U_{\alpha, f_F(\alpha)} \right)$$

is compact open in G. Clearly the $U_{f_F^* + e} \cap U_\alpha$ form a fundamental system of compact open neighbourhoods of 1 in U_α for any $\alpha \in \Phi^{red}$. Similarly Proposition 6 implies that the $Z^{(e)}$ form a fundamental system of compact open neighbourhoods of 1 in $\mathcal{Z}^0(o)$. □

Using 1.5 we may define, for any facet F' in X and any $e \geq 0$, a compact open subgroup

$$U_{F'}^{(e)} := gU_F^{(e)}g^{-1} \text{ if } F' = gF \text{ with } g \in G \text{ and } F \text{ a facet in } A$$

in G. By construction we have

$$gU_{F'}^{(e)}g^{-1} = U_{gF'}^{(e)} \text{ for any } g \in G \quad .$$

If x is a vertex of X, i.e., $\{x\}$ is a facet then we replace similarly as before $\{x\}$ by x in all our notations; e.g., we write $U_x^{(e)}$ instead of $U_{\{x\}}^{(e)}$.

Lemma I.2.10:

There is a point $y_0 \in A$ such that we have

$$\Gamma_\alpha = \alpha(y_0) + \frac{1}{n_\alpha}\mathbb{Z} \quad \textit{for any } \alpha \in \Phi^{red}$$

where $n_\alpha \in \mathbb{N}$ is a natural number which moreover is even in case $2\alpha \in \Phi$.

Proof: Step 1: According to [BT] I.6.2.23 the statement at least holds with some real number $\varepsilon_\alpha > 0$ instead of $\frac{1}{n_\alpha}$. (The point $-y_0$ has to be a special point; in loc. cit. it is assumed to be the origin and therefore does not appear. Also note that $\Phi' = \Phi$ by [BT] II.4.2.21 and 5.1.19.) Step 2: It suffices to show that Γ_α contains a subset of the form $c_\alpha + \frac{1}{n'_\alpha}\mathbb{Z}$ with some $n'_\alpha \in \mathbb{N}$ which is even in case $2\alpha \in \Phi$ and some $c_\alpha \in \mathbb{R}$. Because then there has to be a map $\nu : \mathbb{Z} \to \mathbb{Z}$ such that

$$c_\alpha + \frac{1}{n'_\alpha}m = \alpha(y_0) + \varepsilon_\alpha\nu(m) \text{ for any } m \in \mathbb{Z} \quad .$$

If $m = 0$ this means that $c_\alpha = \alpha(y_0) + \varepsilon_\alpha\nu(0)$ which inserted back implies

$$\varepsilon_\alpha\nu(0) + \frac{1}{n'_\alpha}m = \varepsilon_\alpha\nu(m) \text{ for any } m \in \mathbb{Z} \quad .$$

We obtain

$$\varepsilon_\alpha^{-1} = n'_\alpha \cdot (\nu(1) - \nu(0))$$

so that our assertion holds with $n_\alpha := n'_\alpha \cdot (\nu(1) - \nu(0))$.
Step 3: Let K^{sh} be the strict Henselization of K. The quasi-split group $\mathbf{G}_{/K^{sh}}$ possesses a maximal K^{sh}-split torus T which is defined over K and contains S ([BT] II.5.1.12); let Φ^{sh} be the set of roots of $\mathbf{G}_{/K^{sh}}$ with respect to T. Restricting characters defines a surjective map $\Phi^{sh} \cup \{0\} \to \Phi \cup \{0\}$. By [BT]

II.5.1.19 we have $\Gamma'_{\tilde{\alpha}} \subseteq \Gamma'_\alpha \subseteq \Gamma_\alpha$ whenever $\tilde{\alpha} \in \Phi^{sh}$ restricts to $\alpha \in \Phi$. The sets $\Gamma'_{\tilde{\alpha}}$ are explicitly computed in [BT] II.4.2.21 and 4.3.4: For any $\tilde{\alpha} \in \Phi^{sh}$ one has

$$\Gamma'_{\tilde{\alpha}} = c_{\tilde{\alpha}} + \frac{1}{n_{\tilde{\alpha}}}\mathbb{Z}$$

with appropriate constants $c_{\tilde{\alpha}} \in \mathbb{R}$ and $n_{\tilde{\alpha}} \in \mathbb{N}$. Let now an $\alpha \in \Phi^{red}$ be given. If $2\alpha \notin \Phi$ then we choose an $\tilde{\alpha} \in \Phi^{sh}$ restricting to α and we obtain

$$\Gamma_\alpha \supseteq \Gamma'_{\tilde{\alpha}} = c_{\tilde{\alpha}} + \frac{1}{n_{\tilde{\alpha}}}\mathbb{Z} \quad .$$

If $2\alpha \in \Phi$ then we choose a $\tilde{\beta} \in \Phi^{sh}$ restricting to 2α and we obtain

$$\Gamma_\alpha \supseteq \frac{1}{2}\Gamma'_{2\alpha} \supseteq \frac{1}{2}\Gamma'_{\tilde{\beta}} = \frac{1}{2}c_{\tilde{\beta}} + \frac{1}{2n_{\tilde{\beta}}}\mathbb{Z} \quad . \qquad \square$$

Proposition I.2.11:

i. $U^{(e)}_{F'} \subseteq U^{(e)}_F$ *for any two facets* F, F' *in* X *such that* $F' \subseteq \overline{F}$*;*

ii. $U^{(e)}_F = \prod_{\text{x vertex in}\overline{\text{F}}} U^{(e)}_x$ *for any facet* F *in* X *and any ordering of the factors on the right hand side.*

Proof: We may assume that $F \subseteq A$. First we consider the case that $F' = \{x\}$ is a vertex. Then

$$U^{(e)}_x \cap U_\alpha = U_{\alpha,(-\alpha(x)+e)+} \cdot U_{2\alpha,(-2\alpha(x)+e)+} \quad \text{for } \alpha \in \Phi^{red} \quad .$$

If $\alpha(x) = \inf\limits_{y \in F} \alpha(y)$ then clearly $(-\alpha(x))+ \geq f^*_F(\alpha)$ and hence $U^{(e)}_x \cap U_\alpha \subseteq U^{(e)}_F \cap U_\alpha$. If $\alpha(x) > \inf\limits_{y \in F} \alpha(y)$ then $\alpha|F$ is not constant so that

$$-\alpha(x) < f_F(\alpha) = f^*_F(\alpha) \quad .$$

Furthermore, by the definition of facets, we then have

$$-\alpha(y) \notin \Gamma_\alpha \quad \text{for any } y \in F \quad .$$

Hence

$$\inf\{\ell \in \Gamma_\alpha : \ell > -\alpha(x)\} \geq f_F(\alpha)$$

and because of Lemma 10 also

$$\inf\{\ell \in \Gamma_\alpha : \ell > -\alpha(x) + e\} \geq f_F(\alpha) + e$$

and

$$\inf\{\ell \in \Gamma_\alpha : \ell > -\alpha(x) + \frac{e}{2}\} \geq f_F(\alpha) + \frac{e}{2} \quad \text{in case } 2\alpha \in \Phi \quad .$$

This implies that again $U_x^{(e)} \cap U_\alpha \subseteq U_F^{(e)} \cap U_\alpha$. Using Proposition 7 we obtain

$$\begin{aligned} U_F^{(e)} \supseteq\ & \text{subgroup generated by all} \\ & U_x^{(e)} \text{ for } x \in \overline{F} \text{ a vertex} \quad . \end{aligned}$$

To get the reverse inclusion we fix an $\alpha \in \Phi^{red}$ and consider $U_F^{(e)} \cap U_\alpha$. Let $x \in \overline{F}$ be a vertex such that $\alpha(x) = \sup_{y \in F} \alpha(y)$. Then

$$(-\alpha(x)) + \left\{ \begin{array}{ll} \leq & -\inf\limits_{y\in F} \alpha(y) \\ = & (-\inf\limits_{y \in F} \alpha(y)) + \end{array} \right\} = f_F^*(\alpha) \quad \begin{array}{l} \text{if} \quad \alpha|F \text{ is not constant,} \\ \text{if} \quad \alpha|F \text{ is constant;} \end{array}$$

hence $U_F^{(e)} \cap U_\alpha \subseteq U_x^{(e)} \cap U_\alpha$. Again using Proposition 7 we see that

$$\begin{aligned} U_F^{(e)} =\ & \text{subgroup generated by all} \\ & U_x^{(e)} \text{ for } x \in \overline{F} \text{ a vertex} \quad . \end{aligned}$$

This implies in particular the assertion i. for an arbitrary facet $F' \subseteq \overline{F}$. For ii. it remains to show that for any two vertices $x, y \in \overline{F}$ the subgroup $U_x^{(e)}$ normalizes the subgroup $U_y^{(e)}$. But by i. we have

$$U_x^{(e)} \subseteq U_F^{(e)} \subseteq P_F \subseteq P_y$$

and $U_y^{(e)}$ is normal in P_y. □

Finally we define, for any $z \in X$,

$$U_z^{(e)} := U_F^{(e)} \quad \text{if } z \text{ lies in the facet } F \text{ of } X \quad .$$

Note that $U_z = U_F$ in this situation if $F \subseteq A$ ([BT] I.7.1.2).

I.3. Properties of the groups $U_F^{(e)}$

Here we will establish those properties of the groups $U_F^{(e)}$ which are responsible for our later results about the cohomology of the Bruhat-Tits building. We recommend the reader to skip this section at first reading and only come back to it when the results are needed. Fix an $e \geq 0$.

A first technical clue is the observation that the following representation theoretic fact is at our disposal. The notion of a smooth G-representation will be recalled in II.2. A vertex x in A is called special if $\alpha(x) \in -\Gamma_\alpha$ for any $\alpha \in \Phi^{red}$. There always exists a special vertex ([BT] I.6.2.15).

Theorem: (Bernstein)

Let x be a special vertex in A. The category of smooth G-representations V which are generated (as a G-representation) by their $U_x^{(e)}$-fixed vectors $V^{U_x^{(e)}}$ is stable with respect to the formation of G-equivariant subquotients.

Proof: This is [Ber] 3.9 (i). We only have to check that our group $U_x^{(e)}$ fulfills the assumptions made there. Since the vertex x is special the Iwasawa decomposition

$$G = U_x P = \mathbf{G}_x^0(o) \cdot P \quad \text{for any parabolic subgroup } P \subseteq G$$

holds true ([BT] I.7.3.2 (ii)). Moreover the decomposition property (3.5.1) in [Ber] is a consequence of 2.7. $\square$

Next we need some control over how $U_z^{(e)}$ changes if z varies along a geodesic line in X. We fix two different points x and x' in A. The geodesic $[xx']$ joining x and x' is

$$[xx'] = \{(1-r)x + rx' : 0 \le r \le 1\} \quad .$$

Proposition I.3.1:

Assume x to be a special vertex; for any point $z \in [xx']$ we have

$$U_z^{(e)} \subseteq U_x^{(e)} \cdot U_{x'}^{(e)} \quad .$$

Proof: We may assume z to be different from x and x'. Let F, resp. F', denote the facet in A which contains z, resp. x'. Define

$$\Psi := \{\alpha \in \Phi^{red} : \alpha(x) < \alpha(x')\} \quad .$$

There certainly exists a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots such that $\Psi \subseteq \Phi^+$. As a consequence of 2.7 it suffices to check that

$$U_{\alpha, f_F^*(\alpha)+e} \cdot U_{2\alpha, 2f_F^*(\alpha)+e} \subseteq \begin{cases} U_{\alpha,(-\alpha(x)+e)+} \cdot U_{2\alpha,(-2\alpha(x)+e)+} & \text{if } \alpha \in \Phi^{red} \backslash \Psi \ , \\ U_{\alpha, f_{F'}^*(\alpha)+e} \cdot U_{2\alpha, 2f_{F'}^*(\alpha)+e} & \text{if } \alpha \in \Psi \ . \end{cases}$$

Assume first that $\alpha \in \Phi^{red} \backslash \Psi$. Since x is special we have $\ell := -\alpha(x) \in \Gamma_\alpha$. If $\alpha(x) = \alpha(x')$ then $-\alpha(y) = \ell$ for any $y \in \{x\} \cup F \cup F'$ and hence $f_F^*(\alpha) = \ell+$. If $\alpha(x) > \alpha(z) > \alpha(x')$ then $-\alpha(y) > \ell$ for any $y \in F$ and hence $f_F^*(\alpha) \ge f_F(\alpha) \ge \ell+$.
Now assume that $\alpha \in \Psi$, i.e., that $\alpha(x) < \alpha(z) < \alpha(x')$. If there exists a $\ell' \in \Gamma_\alpha$ such that $-\alpha(x') < \ell' \le -\alpha(z)$ then

$$f_F^*(\alpha) \ge f_F(\alpha) \ge \ell' \ge f_{F'}^*(\alpha) \quad .$$

Otherwise there are two successive values $\ell' < \ell''$ in Γ_α such that

$$\ell' \leq -\alpha(x') < -\alpha(z) < \ell'' \quad ;$$

hence

$$\ell' < f^*_F(\alpha) \leq \ell'' \ \text{ and } \ \ell' \leq f^*_{F'}(\alpha) \leq \ell'' \quad .$$

Using 2.10 we see that in this case

$$\begin{aligned} U_{\alpha, f^*_F(\alpha)+e} \cdot U_{2\alpha, 2f^*_F(\alpha)+e} &= U_{\alpha,\ell''+e} \cdot U_{2\alpha,2\ell''+e} \\ &\subseteq U_{\alpha, f^*_{F'}(\alpha)+e} \cdot U_{2\alpha, 2f^*_{F'}(\alpha)+e} \quad . \end{aligned}$$ □

Because of 1.7 the assumption that x and x' are contained in the basic apartment A is unnecessary. Also the statement remains true even if x is not a special vertex. Since it is not needed we do not go into this. But see the proof of III.4.14.
Consider the half-line

$$\mathfrak{s} := \{(1-r)x + rx' : r \geq 0\}$$

in A and put

$$\begin{aligned} U_{\mathfrak{s}} := \ & \text{subgroup generated by all } U_\alpha \\ & \text{for } \alpha \in \Phi \text{ such that } \alpha(x') > \alpha(x) \quad . \end{aligned}$$

As will be explained in IV.2 this group is the unipotent radical of some parabolic subgroup of G.

Proposition I.3.2:

Assume x to be a special vertex; for any point $z \in \mathfrak{s}$ we have

$$U^{(e)}_z \subseteq U_{\mathfrak{s}} \cdot U^{(e)}_x \quad .$$

Proof: This is a straighforward (actually simpler) variant of the previous proof. The only additional fact to use is that the product map induces a bijection

$$\prod_{\alpha\in\Psi} U_\alpha \xrightarrow{\sim} U_{\mathfrak{s}}$$

whatever ordering of the factors on the left hand side we choose ([Bor] 21.9). □

II. The homological theory

II.1. Cellular chains

Through its partition into facets the Bruhat-Tits building X acquires the structure of a d-dimensional locally finite polysimplicial complex ([BT] I.2.1.12 and II.5.1.32) where $d := \dim A$ is the semisimple K-rank of $\mathbf{G}$. For $0 \leq q \leq d$ put

$$X_q := \text{ set of all } q\text{-dimensional facets of } X \quad .$$

In particular we may view X as a d-dimensional CW-complex the q-cells of which are the facets in X_q. The G-action on X is cellular. Let

$$X^q := \bigcup_{F \in X_q} \overline{F}$$

denote the q-skeleton of X; also put $X^{-1} := \emptyset$. With the composed maps

$$\partial_q : H_{q+1}(X^{q+1}, X^q; \mathbb{Z}) \xrightarrow{\partial} H_q(X^q, \mathbb{Z}) \longrightarrow H_q(X^q, X^{q-1}; \mathbb{Z})$$

as boundary maps the augmented complex

$$H_d(X^d, X^{d-1}; \mathbb{Z}) \xrightarrow{\partial_{d-1}} \ldots \xrightarrow{\partial_0} H_0(X^0, \mathbb{Z}) = \bigoplus_{F \in X_0} \mathbb{Z} \xrightarrow{\Sigma} \mathbb{Z}$$

computes the (singular) homology of X ([Dol] V.1.3). It is G-equivariant and it is exact since G is contractible ([BT] I.2.5.16).

In order to motivate later constructions we want to give a more combinatorial description of that complex. By [Dol] V.4.4 and V.6.2 we have the direct sum decomposition

$$H_q(X^q, X^{q-1}; \mathbb{Z}) = \bigoplus_{F \in X_q} H_q(X^q, X^q \backslash F; \mathbb{Z}) \quad .$$

Consider, for any $F \in X_{q+1}$ and $F' \in X_q$, the composed map

$$\begin{array}{rl} \partial^F_{F'} : H_{q+1}(X^{q+1}, X^{q+1}\backslash F; \mathbb{Z}) \hookrightarrow H_{q+1}(X^{q+1}, X^q; \mathbb{Z}) \xrightarrow{\partial_q} & H_q(X^q, X^{q-1}; \mathbb{Z}) \\ & \downarrow \\ & H_q(X^q, X^q \backslash F'; \mathbb{Z}) \ . \end{array}$$

One has:
— $H_q(X^q, X^q\backslash F; \mathbb{Z}) \cong \mathbb{Z}$ for $F \in X_q$ (but for $q > 0$ no canonical such isomorphism exists).
— $\partial^F_{F'}$ is an isomorphism if $F' \subseteq \overline{F}$ and is the zero map otherwise. (Using [Dol] V.6.11 this follows from the fact that in our case the characteristic map Φ_F of F can be chosen to be injective and hence to be a homeomorphism onto $\overline{F}$.)

Define now an oriented q-facet to be, in case $q > 0$, a pair (F, c) where $F \in X_q$ and c is a generator of $H_q(X^q, X^q \backslash F, \mathbb{Z})$; then $(F, -c)$ is another oriented q-facet. In case $q = 0$ an oriented 0-facet simply is a 0-facet F which we sometimes also think of as the pair $(F, 1)$ where 1 is the canonical generator of $H_0(X^0, X^0 \backslash F; \mathbb{Z}) = \mathbb{Z}$. Let $X_{(q)}$ denote the set of all oriented q-facets. Observe that for any $(F, c) \in X_{(q+1)}$ with $q \geq 1$ and any $F' \in X_q$ such that $F' \subseteq \overline{F}$ we have

$$(F', \partial^F_{F'}(c)) \in X_{(q)} \quad .$$

The group of oriented cellular q-chains of X by definition is

$$\begin{aligned} C_c^{or}(X_{(q)}, \mathbb{Z}) := & \text{ group of all maps } \omega : X_{(q)} \to \mathbb{Z} \\ & \text{ such that} \\ & \text{ — } \omega \text{ has finite support, and, if } q \geq 1, \\ & \text{ — } \omega((F, -c)) = -\omega((F, c)) \text{ for any } (F, c) \in X_{(q)} \quad . \end{aligned}$$

Clearly

$$\begin{aligned} C_c^{or}(X_{(q)}, \mathbb{Z}) & \xrightarrow{\cong} H_q(X^q, X^{q-1}; \mathbb{Z}) \\ \omega & \longmapsto 2^{\varepsilon} \cdot \sum_{(F,c) \in X_{(q)}} \omega((F, c)) \cdot c \end{aligned}$$

with $\varepsilon = -1$, resp. 0, in case $q > 0$, resp. $= 0$, is an isomorphism which is G-equivariant if G acts on the left hand side by

$$(g\omega)((F, c)) := \omega((g^{-1}F, g^{-1}c)) \quad .$$

The boundary map ∂_q becomes

$$\begin{aligned} \partial_q : C_c^{or}(X_{(q+1)}, \mathbb{Z}) & \longrightarrow C_c^{or}(X_{(q)}, \mathbb{Z}) \\ \omega & \longmapsto ((F', c') \mapsto \sum_{\substack{(F,c) \in X_{(q+1)} \\ F' \subseteq \overline{F} \\ \partial^F_{F'}(c) = c'}} \omega((F, c))) \quad . \end{aligned}$$

The augmentation map becomes

$$\begin{aligned} \varepsilon : C_c^{or}(X_{(0)}, \mathbb{Z}) & \longrightarrow \mathbb{Z} \\ \omega & \longmapsto \sum_{F \in X_{(0)}} \omega(F) \quad . \end{aligned}$$

II.2. Representations as coefficient systems

A smooth (or algebraic) representation V of G is a complex vector space V together with a linear action of G such that the stabilizer of each vector is open in G. Let $\mathrm{Alg}(G)$ denote the category of those smooth representations.

On the other hand a coefficient system (of complex vector spaces) $\underline{\underline{V}}$ on the Bruhat-Tits building X consists of

— complex vector spaces V_F for each facet $F \subseteq X$, and

— linear maps $r^F_{F'} : V_F \to V_{F'}$ for each pair of facets $F' \subseteq \overline{F}$ such that $r^F_F = \mathrm{id}$ and $r^F_{F''} = r^{F'}_{F''} \circ r^F_{F'}$ whenever $F'' \subseteq \overline{F'}$ and $F' \subseteq \overline{F}$.

In an obvious way the coefficient systems form a category which we denote by $\mathrm{Coeff}(X)$.

We fix now an integer $e \geq 0$. For any representation V in $\mathrm{Alg}(G)$ we then have the coefficient system $\underline{\underline{V}} := (V^{U_F^{(e)}})$ of subspaces of fixed vectors

$$V_F := V^{U_F^{(e)}} := \{v \in V : gv = v \text{ for all } g \in U_F^{(e)}\} \quad ;$$

because of $U_{F'}^{(e)} \subseteq U_F^{(e)}$ for $F' \subseteq \overline{F}$ the transition maps $r^F_{F'}$ are the obvious inclusions. Since the $U_F^{(e)}$ are profinite groups the functor

$$\begin{aligned} \gamma_e : \mathrm{Alg}(G) &\longrightarrow \mathrm{Coeff}(X) \\ V &\longmapsto (V^{U_F^{(e)}})_F \end{aligned}$$

is exact. For any $0 \leq q \leq d$ the space of oriented (cellular) q-chains of $\gamma_e(V)$ by definition is

$$\begin{aligned} C_c^{or}(X_{(q)}, \gamma_e(V)) := \ & \mathbb{C}\text{-vector space of all maps } \omega : X_{(q)} \to V \\ & \text{such that} \\ & \text{— } \omega \text{ has finite support,} \\ & \text{— } \omega((F,c)) \in V^{U_F^{(e)}}, \text{ and, if } q \geq 1, \\ & \text{— } \omega((F,-c)) = -\omega((F,c)) \text{ for any } (F,c) \in X_{(q)}. \end{aligned}$$

The group G acts smoothly on these spaces via

$$(g\omega)((F,c)) := g(\omega((g^{-1}F, g^{-1}c))) \quad .$$

A straightforward computation shows that the boundary map

$$\begin{aligned} \partial : C_c^{or}(X_{(q+1)}, \gamma_e(V)) &\longrightarrow C_c^{or}(X_{(q)}, \gamma_e(V)) \\ \omega &\longmapsto ((F', c') \mapsto \sum_{\substack{(F,c) \in X_{(q+1)} \\ F' \subseteq \overline{F} \\ \partial^F_{F'}(c) = c'}} \omega((F,c))) \end{aligned}$$

fulfills $\partial \circ \partial = 0$. In this way we obtain the augmented chain complex

$$C_c^{or}(X_{(d)}, \gamma_e(V)) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_c^{or}(X_{(0)}, \gamma_e(V)) \xrightarrow{\varepsilon} V$$

where the augmentation map is given by

$$\begin{aligned} \varepsilon : C_c^{or}(X_{(0)}, \gamma_e(V)) &\longrightarrow V \\ \omega &\longmapsto \sum_{F \in X_{(0)}} \omega(F) \quad . \end{aligned}$$

The homology of this chain complex could be called the (cellular) homology of the coefficient system $\gamma_e(V)$ on the space X. We will not use this terminology since in the next section it will be shown that these complexes under a rather weak assumption are exact. That assumption has to do with the surjectivity of the augmentation map. For any open subgroup $U \subseteq G$ we have the full subcategory

$$\begin{aligned} \mathrm{Alg}^U(G) := {} & \text{category of those smooth } G\text{-representations } V \\ & \text{which (as } G\text{-representation) are generated by} \\ & \text{their } U\text{-fixed vectors } V^U \end{aligned}$$

of $\mathrm{Alg}(G)$. If the representation V lies in $\mathrm{Alg}^{U_x^{(e)}}(G)$ for some vertex x in X then the augmentation map ε clearly is surjective. The subsequent two statements are immediate generalizations of the Propositions 1 and 2 in [SS] §3. Recall that a representation V in $\mathrm{Alg}(G)$ is called admissible if the subspace V^U, for any open subgroup $U \subseteq G$, is finite-dimensional.

Proposition II.2.1:

If V in $\mathrm{Alg}(G)$ is admissible then the complex $C_c^{or}(X_{(.)}, \gamma_e(V))$ consists of finitely generated G-representations.

Proof: By the admissibility assumption the subspace in $C_c^{or}(X_{(q)}, \gamma_e(V))$ of q-chains supported on $\{(F, \pm c)\}$, for a given facet $F \in X_q$, is finite dimensional. If F runs over a set of representatives for the finitely many G-orbits in X_q then the corresponding subspaces together generate $C_c^{or}(X_{(q)}, \gamma_e(V))$ as a G-representation. $\square$

For any continuous character $\chi : C \to \mathbb{C}^\times$ of the connected center C of G we define the full subcategory

$$\begin{aligned} \mathrm{Alg}_\chi(G) := {} & \text{category of those smooth } G\text{-representations } V \\ & \text{such that } gv = \chi(g) \cdot v \text{ for all } g \in C \text{ and } v \in V \end{aligned}$$

of $\mathrm{Alg}(G)$. Since C acts trivially on X the complex $C_c^{or}(X_{(.)}, \gamma_e(V))$ lies in $\mathrm{Alg}_\chi(G)$ if V does.

Proposition II.2.2:

For any representation V in $\mathrm{Alg}_\chi(G)$ the complex $C_c^{or}(X_{(.)}, \gamma_e(V))$ consists of projective objects in $\mathrm{Alg}_\chi(G)$.

Proof: This relies on the fact that the group $P_F^\dagger / C U_F^{(e)}$ is finite for any facet $F \subseteq X$. As a consequence of I.2.9 the group $\mathbf{G}_F^0(o)/U_F^{(e)}$ is finite. On the other hand $P_F^\dagger / C\mathbf{G}_F^0(o)$ is finite according to [BT] II.4.6.28. $\square$

II.3. Homological resolutions

In order to formulate the main result of this chapter let $e \geq 0$ be an integer and let x be a special vertex in A.

Theorem II.3.1:

For any representation V in $\mathrm{Alg}^{U_x^{(e)}}(G)$ the augmented complex

$$C_c^{or}(X_{(.)}, \gamma_e(V)) \to V$$

is an exact resolution of V in $\mathrm{Alg}(G)$.

Proof: In the case $G = GL_{d+1}(K)$ this result was established in [SS]. The proof in the general case in its most parts is a direct generalization of the arguments in [SS]. Insofar we will only indicate the main steps.
Step 1: Let $C_c(T)$ denote the space of complex valued functions with finite support on the G-set $T := G/U_x^{(e)}$. This is a smooth representation of G which acts by left translations. It lies in $\mathrm{Alg}^{U_x^{(e)}}(G)$ and one has the surjective G-homomorphism

$$\begin{aligned} C_c(T) \otimes V^{U_x^{(e)}} &\twoheadrightarrow V \\ \psi \otimes v &\longmapsto \sum_{g \in G/U_x^{(e)}} \psi(g) \cdot g(v) \quad . \end{aligned}$$

Bernstein's theorem (I.3) now ensures the existence of an exact resolution in $\mathrm{Alg}^{U_x^{(e)}}(G)$ of the form

$$\ldots \longrightarrow \bigoplus_{I_1} C_c(T) \longrightarrow \bigoplus_{I_0} C_c(T) \longrightarrow V \longrightarrow 0$$

with appropriate index sets $I_0, I_1, \ldots$ Since the functor γ_e is exact a standard double complex argument reduces therefore our assertion in case V to the "universal" case $C_c(T)$.

Step 2: A slightly more sophisticated double complex argument for $C_c(T)$ ([SS] §1) further reduces our assertion to a geometric property of the Bruhat-Tits building X. For any facet F we put

$$T_F := U_F^{(e)} \backslash T \quad .$$

It follows from I.2.11.ii that

$$T_F = T_{x_0} \coprod_T \ldots \coprod_T T_{x_r}$$

if $\{x_0, \ldots, x_r\}$ are the vertices in $\overline{F}$. The natural projection $T \to T_F$ is finite and induces an isomorphism

$$C_c(T_F) \xrightarrow{\cong} C_c(T)^{U_F^{(e)}}$$

which we will view as an identification. More generally we have the simplicial set

$$T.^F : \ldots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} T \underset{T_F}{\times} T \underset{T_F}{\times} T \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} T \underset{T_F}{\times} T \rightrightarrows T$$

all face maps of which are finite. There are obvious commutative diagrams

$$\begin{array}{ccccc} & & T.^F & & \\ & \nearrow & & \searrow & \\ T & & \longrightarrow & & T_F \end{array}$$

and

$$\begin{array}{ccc} T.^{F'} & \longrightarrow & T_{F'} \\ \downarrow & & \downarrow \\ T.^F & \longrightarrow & T_F \end{array} \qquad \text{for } F' \subseteq \overline{F} \quad .$$

By passing to functions we obtain the double complex

(A)

$$\begin{array}{ccccccc}
 & 0 & & & 0 & & 0 \\
 & \downarrow & & & \downarrow & & \downarrow \\
0 \to & \bigoplus\limits_{F\in X_d} C_c(T_F) & \overset{\partial}{\to} & \ldots \overset{\partial}{\to} & \bigoplus\limits_{F\in X_0} C_c(T_F) & \to & C_c(T) \to 0 \\
 & \downarrow & & & \downarrow & & \| \\
0 \to & \bigoplus\limits_{F\in X_d} C_c(T) & \to & \ldots \to & \bigoplus\limits_{F\in X_0} C_c(T) & \to & C_c(T) \to 0 \\
 & \downarrow & & & \downarrow & & \downarrow 0 \\
0 \to & \bigoplus\limits_{F\in X_d} C_c(T \underset{T_F}{\times} T) & \to & \ldots \to & \bigoplus\limits_{F\in X_0} C_c(T \underset{T_F}{\times} T) & \to & C_c(T) \to 0 \\
 & \downarrow & & & \downarrow & & \| \\
0 \to & \bigoplus\limits_{F\in X_d} C_c(T \underset{T_F}{\times} T \underset{T_F}{\times} T) & \to & \ldots \to & \bigoplus\limits_{F\in X_0} C_c(T \underset{T_F}{\times} T \underset{T_F}{\times} T) & \to & C_c(T) \to 0 \\
 & \downarrow & & & \downarrow & & \downarrow 0 \\
 & \vdots & & & \vdots & & \vdots
\end{array}$$

All its columns are exact. This follows from the fact that each $T.^F$ is the disjoint union

$$T.^F = \dot{\bigcup_{t\in T_F}} T.^{(t)}$$

of the simplicial finite sets

$$T.^{(t)} : \ldots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} T_t \times T_t \times T_t \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} T_t \times T_t \substack{\longrightarrow\\\longrightarrow} T_t$$

where T_t denotes the fiber of the map $T \to T_F$ in t. Simplicial sets of the form $T.^{(t)}$ are well-known to be contractible. The top row in (A) is the complex whose exactness we want to establish. Here we have fixed for simplicity an orientation of the building X. Next we have to study, for a fixed $m \geq 0$, the row

$$(A_m) \qquad 0 \to \bigoplus_{F\in X_d} C_c(T_m^F) \to \ldots \to \bigoplus_{F\in X_0} C_c(T_m^F) \to C_c(T) \to 0$$

from (A). If we view each T_m^F, resp. T, as a subset of $T^{m+1} := T \times \ldots \times T$ ($m+1$ factors) in the obvious way, resp. diagonally, then the differentials in the above complex are induced by the inclusions

$$T \subseteq T_m^{F'} \subseteq T_m^F \quad \text{for } F' \subseteq \overline{F} \quad .$$

In order to rewrite this row in a more suitable form we introduce certain subcomplexes of the Bruhat-Tits building. For a fixed $(t_0, \ldots, t_m) \in T^{m+1}$ we put

$$X^{(t_0,\ldots,t_m)} := \text{union of all facets } F \subseteq X \text{ such that } t_0, \ldots, t_m$$
$$\text{are not mapped to the same element in } U_F^{(e)} \backslash T \quad .$$

It is easy to see that

$$\bigcap_{F \in X_0} T_m^F = T \quad ,$$

i.e., that $X^{(t_0,\ldots,t_m)}$ is empty if and only if $t_0 = \ldots = t_m$. If $\{t_0, \ldots, t_m\}$ has cardinality at least 2 than $X^{(t_0,\ldots,t_m)}$ is a nonempty closed CW-subspace of X (I.2.11.i and [Dol] V.2.7). We now have

$$\bigoplus_{F \in X.} C_c(T_m^F) = \bigoplus_{(t_0,\ldots,t_m) \in T^{m+1}} C_c(X. \backslash X.^{(t_0,\ldots,t_m)})$$

and the decomposition on the right hand side is compatible with the differentials. As a result of this discussion we obtain that

$$(A_m) = \bigoplus_{(t_0,\ldots,t_m) \in T^{m+1}} \begin{array}{l}\text{(augmented complex of relative} \\ \text{chains of the pair } (X, X^{(t_0,\ldots,t_m)})\end{array} \quad .$$

Since X is contractible this means that we are reduced to show the contractibility of the subspaces $X^{(t_0,\ldots,t_m)}$ for any $\{t_0, \ldots, t_m\}$ of cardinality at least 2.
Step 3: The special vertex

$$x_0 := g_0 x \quad \text{where} \quad t_0 = g_0 U_x^{(e)}$$

is contained in $X^{(t_0,\ldots,t_m)}$ ([SS] §2 Remark 1). We show that with any point $y \in X^{(t_0,\ldots,t_m)}$ the whole geodesic $[x_0 y]$ lies in $X^{(t_0,\ldots,t_m)}$. This of course implies the wanted contractibility. Fix a point $z \in [x_0 y]$ and let F, resp. F', denote the facet in X which contains z, resp. y. Clearly $F' \subseteq X^{(t_0,\ldots,t_m)}$ so that there exists a $1 \leq j \leq m$ such that

$$g t_j \neq t_0 \quad \text{for all } g \in U_{F'}^{(e)} \quad .$$

If F is not contained in $X^{(t_0,\ldots,t_m)}$ then we find a $g_1 \in U_F^{(e)}$ with $g_1 t_j = t_0$. According to I.3.1 we have

$$U_F^{(e)} \subseteq U_{x_0}^{(e)} \cdot U_{F'}^{(e)}$$

so that

$$g_1 \in h U_{F'}^{(e)} \quad \text{for some } h \in U_{x_0}^{(e)} \quad .$$

Put $g := h^{-1} g_1 \in U_{F'}^{(e)}$. Then $g t_j = h^{-1} t_0 = t_0$ which is a contradiction. □

Corollary II.3.2:

For any representation V in $\mathrm{Alg}^{U_x^{(e)}}(G) \cap \mathrm{Alg}_\chi(G)$ the augmented complex

$$C_c^{or}(X_{(.)}, \gamma_e(V)) \longrightarrow V$$

is a projective exact resolution of V in $\mathrm{Alg}_\chi(G)$.

Proof: Combine Theorem 1 and 2.2. $\square$

Corollary II.3.3:

Let V, resp. V', be an admissible representation in $\mathrm{Alg}^{U_x^{(e)}}(G) \cap \mathrm{Alg}_\chi(G)$, resp. $\mathrm{Alg}_\chi(G)$; then the vector spaces $\mathrm{Ext}^*_{\mathrm{Alg}_\chi(G)}(V, V')$ are finite-dimensional and vanish for $* > d$.

Proof: Combine Corollary 2 and 2.1 (compare [SS] §3 Cor. 3). $\square$

Note that because of I.2.9 any finitely generated smooth G-representation lies in $\mathrm{Alg}^{U_x^{(e)}}(G)$ if only e is chosen large enough.

III. Duality theory

III.1. Cellular cochains

An element of the space $C_c^{or}(X_{(q)}, \gamma_e(V))$ also can be viewed as an oriented cellular cochain with finite support on X. This suggests that there is a cohomological differential, too. Indeed we have, for any pair of facets $F, F' \subseteq X$ such that $F' \subseteq \overline{F}$, the projection map

$$\begin{aligned} pr_F^{F'} : V^{U_{F'}^{(e)}} &\longrightarrow V^{U_F^{(e)}} \\ v &\longmapsto \frac{1}{[U_F^{(e)} : U_{F'}^{(e)}]} \cdot \sum_{g \in U_F^{(e)}/U_{F'}^{(e)}} gv \quad . \end{aligned}$$

Since the partition of X into facets is locally finite the coboundary map

$$\begin{aligned} d : C_c^{or}(X_{(q)}, \gamma_e(V)) &\longrightarrow C_c^{or}(X_{(q+1)}, \gamma_e(V)) \\ \omega &\longmapsto ((F,c) \mapsto \sum_{\substack{F' \in X_q \\ F' \subseteq \overline{F}}} pr_F^{F'}(\omega((F', \partial_{F'}^F(c))))) \end{aligned}$$

is well defined; in case $q = 0$ the summands on the right hand side have to be interpreted as $\partial_{F'}^F(c) \cdot \omega(F')$. A standard computation ([Dol] VI.7.11) shows that

$$C_c^{or}(X_{(0)}, \gamma_e(V)) \xrightarrow{d} \ldots \xrightarrow{d} C_c^{or}(X_{(d)}, \gamma_e(V))$$

is a complex in $\mathrm{Alg}(G)$ — the cochain complex (with finite support) of $\gamma_e(V)$. We will see in Chapter IV that this complex computes the cohomology with compact support of a certain sheaf on X. Here we are interested in the relation between the chain and the cochain complex.

Again let $\chi : C \to \mathbb{C}^\times$ be a continuous character. In $\mathrm{Alg}_\chi(G)$ there is the "universal" representation

$\mathcal{H}_\chi :=$ space of all locally constant functions
$\psi : G \to \mathbb{C}$ such that
— $\psi(g^{-1}h) = \chi(g) \cdot \psi(h)$ for all $g \in C$, $h \in G$,
— there is a compact subset $\Sigma \subseteq G$ such that
ψ vanishes outside $\Sigma \cdot C$

where G acts by left translations. This is the χ-Hecke algebra of G; its algebra structure will be recalled later on. Note that G also acts smoothly on $\mathcal{H}_\chi$ by right translations. Both actions commute with each other. In the second action the connected center C acts through the character χ^{-1}.

Fix now a representation V in $\mathrm{Alg}_\chi(G)$. Its smooth dual $\tilde{V}$ lies in $\mathrm{Alg}_{\chi^{-1}}(G)$. For any $0 \le q \le d$ we have the pairing

$$C_c^{or}(X_{(q)}, \gamma_e(\tilde{V})) \times C_c^{or}(X_{(q)}, \gamma_e(V)) \longrightarrow \mathcal{H}_\chi$$
$$(\eta, \omega) \longmapsto \Psi_{\eta,\omega}$$

defined by

$$\Psi_{\eta,\omega}(g) := \sum_{(F,c)\in X_{(q)}} \eta((F,c))[(g^{-1}\omega)((F,c))] \quad .$$

One easily verifies that

$$\Psi_{\eta,h\omega} = h\Psi_{\eta,\omega} \ \text{ and } \ \Psi_{h\eta,\omega} = \Psi_{\eta,\omega}(.h) \ \text{ for any } h \in G \quad .$$

This means that the above pairing induces a homomorphism

$$\Psi : C_c^{or}(X_{(q)}, \gamma_e(\tilde{V})) \longrightarrow \mathrm{Hom}_G(C_c^{or}(X_{(q)}, \gamma_e(V)), \mathcal{H}_\chi)$$
$$\eta \longmapsto (\omega \mapsto \Psi_{\eta,\omega})$$

which moreover is G-equivariant if the action on the right hand side is the one induced by the right translation action on $\mathcal{H}_\chi$. Next one checks that

$$\Psi_{\partial\eta,\omega} = \Psi_{\eta,d\omega} \ \text{ and } \ \Psi_{d\eta,\omega} = \Psi_{\eta,\partial\omega} \quad .$$

In other words Ψ is a homomorphism of complexes from the chain, resp. cochain, complex of $\gamma_e(\tilde{V})$ into the $\mathrm{Hom}_G(.,\mathcal{H}_\chi)$-dual of the cochain, resp. chain, complex of $\gamma_e(V)$. We claim that Ψ is injective. Define, for any $(F,c) \in X_{(q)}$ and any $v \in V$, an oriented q-chain $\omega_{(F,c),v}$ of $\gamma_e(V)$ by

$$\omega_{(F,c),v}((F',c')) := \begin{cases} pr_F(v) & \text{if } (F',c') = (F,c), \\ -pr_F(v) & \text{if } q \ge 1 \text{ and } (F',c') = (F,-c), \\ 0 & \text{otherwise;} \end{cases}$$

here pr_F denotes the projection map

$$pr_F : V \longrightarrow V^{U_F^{(e)}}$$
$$v \longmapsto \frac{1}{[U_F^{(e)} : U]} \cdot \sum_{g\in U_F^{(e)}/U} gv \qquad \text{if } v \in V^U \text{ for some open subgroup } U \subseteq U_F^{(e)}.$$

We then have

$$\Psi_{\eta,\omega_{(F,c),v}}(1) = 2^\varepsilon \cdot \eta((F,c))[pr_F(v)] = 2^\varepsilon \cdot \eta((F,c))[v]$$

with $\varepsilon = 1$, resp. 0, in case $q \ge 1$, resp. $= 0$. The second identity comes from the observation that any linear form in $\tilde{V}^{U_F^{(e)}}$ factorizes through pr_F. This clearly implies the injectivity of Ψ.

Lemma III.1.1:

Let V be an admissible representation in $\mathrm{Alg}_\chi(G)$; then the G-equivariant linear map

$$\Psi : C_c^{or}(X_{(.)}, \gamma_e(\tilde{V})) \xrightarrow{\cong} \mathrm{Hom}_G(C_c^{or}(X_{(.)}, \gamma_e(V)), \mathcal{H}_\chi)$$

is an isomorphism; under this identification we have

$$\mathrm{Hom}_G(\partial, \mathcal{H}_\chi) = d \quad \textit{and} \quad \mathrm{Hom}_G(d, \mathcal{H}_\chi) = \partial \quad .$$

Proof: Only the surjectivity of Ψ remains to be established. Let Ψ_o be an element of the right hand side. Define a map $\eta : X_{(q)} \to \tilde{V}$ by

$$\eta((F,c))[v] := \Psi_o(\omega_{(F,c),v})(1) \quad .$$

For a fixed (F,c) the function $\Psi_o(\omega_{(F,c),v})$ in $\mathcal{H}_\chi$ only depends on $pr_F(v)$. Therefore the admissibility assumption guarantees the existence of a compact subset $\Sigma \subseteq G$ such that all functions $\Psi_o(\omega_{(F,c),v})$ for $v \in V$ vanish outside $\Sigma \cdot C$. Because of

$$\eta(g^{-1}(F,c))[v] = \Psi_o(\omega_{(F,c),gv})(g)$$

it follows that

$$\eta(g^{-1}(F,c)) = 0 \quad \text{for } g^{-1} \notin \Sigma \cdot C \quad .$$

Since G has only finitely many orbits in $X_{(q)}$ we obtain that the map η has finite support. It is now straightforward to see that $\eta \in C_c^{or}(X_{(q)}, \gamma_e(\tilde{V}))$ is a q-chain such that $\Psi(\eta) = 2^\varepsilon \cdot \Psi_o$ with the same ε as above. □

This duality between chain and cochain complexes is perfectly suited to analyze the Ext-groups

$$\mathcal{E}^*(V) := \mathrm{Ext}^*_{\mathrm{Alg}_\chi}(V, \mathcal{H}_\chi)$$

in the category $\mathrm{Alg}_\chi(G)$. Through the right translation action of G upon $\mathcal{H}_\chi$ the space $\mathcal{E}^*(V)$ in a natural way is a G-representation which in general might not be smooth. As before we fix a special vertex x in A.

Lemma III.1.2:

For any representation V in $\mathrm{Alg}^{U_x^{(e)}}(G) \cap \mathrm{Alg}_\chi(G)$ we have

$$\mathcal{E}^*(V) = h^*(\mathrm{Hom}_G(C_c^{or}(X_{(.)}, \gamma_e(V)), \mathcal{H}_\chi), \mathrm{Hom}_G(\partial, \mathcal{H}_\chi)) \quad .$$

Proof: II.3.2. □

Proposition III.1.3:

For any admissible representation V in $\mathrm{Alg}^{U_x^{(e)}}(G) \cap \mathrm{Alg}_\chi(G)$ *we have*

$$\mathcal{E}^*(V) = h^*(C_c^{or}(X_{(.)}, \gamma_e(\tilde{V})), d) \quad .$$

Proof: Lemmata 1 and 2. □

Remark III.1.4:

i. The category of finitely generated smooth G-representations is stable with respect to the formation of G-equivariant subquotients;
ii. a smooth G-representation is finitely generated and admissible if and only if it is of finite length;
iii. let V be an admissible representation in $\mathrm{Alg}^{U_x^{(e)}}(G) \cap \mathrm{Alg}_\chi(G)$*; then $\tilde{V}$ is an admissible representation in* $\mathrm{Alg}^{U_x^{(e)}}(G) \cap \mathrm{Alg}_{\chi^{-1}}(G)$ *and we have $\tilde{\tilde{V}} = V$.*

Proof: i. and ii. [Ber] 3.12. iii. [Cas] 2.1.10 and 2.2.3 together with Bernstein's theorem (I.3). □

In particular it follows from these considerations together with II.2.1 that the spaces $\mathcal{E}^*(.)$ form functors

$$\mathcal{E}^* \quad : \quad \begin{array}{l}\text{finitely generated and}\\ \text{admissible representations}\\ \text{in } \mathrm{Alg}_\chi(G)\end{array} \longrightarrow \begin{array}{l}\text{finitely generated}\\ \text{representations}\\ \text{in } \mathrm{Alg}_{\chi^{-1}}(G)\end{array} \quad .$$

If $* > d$ then $\mathcal{E}^* = 0$. For later use we need the following technical consequence of the above results.

Lemma III.1.5:

Let V be a representation of finite length in $\mathrm{Alg}_\chi(G)$ *and assume that there is an integer $0 \leq d(V) \leq d$ such that $\mathcal{E}^*(V) = 0$ for $* \neq d(V)$; we then have:*
i. For e big enough the complex

$$\begin{array}{ccc} C_c^{or}(X_{(0)}, \gamma_e(\tilde{V})) \xrightarrow{d} \ldots \xrightarrow{d} C_c^{or}(X_{(d(V)-1)}, \gamma_e(\tilde{V})) \xrightarrow{d} & \ker d^{d(V)} \\ & \downarrow \\ & \mathcal{E}^{d(V)}(V) \end{array}$$

is an exact projective resolution of $\mathcal{E}^{d(V)}(V)$ in $\mathrm{Alg}_{\chi^{-1}}(G)$*;*
ii. $\mathcal{E}^*(\mathcal{E}^{d(V)}(V)) = \begin{cases} V & \text{if } * = d(V) \quad , \\ 0 & \text{otherwise} \quad . \end{cases}$

III.2. Parabolic induction

The computation of the spaces $\mathcal{E}^*(V)$ in an essential way makes use of the theory of parabolic induction. We fix a decomposition $\Phi = \Phi^+ \cup \Phi^-$ of the set of roots Φ into positive and negative roots. Let $\Delta \subseteq \Phi^+$ be the corresponding subset of simple roots. The subsets $\Theta \subseteq \Delta$ parametrize the conjugacy classes of parabolic subgroups of G in the following way. First we have the torus

$$S_\Theta := \text{ connected component of } \bigcap_{\alpha \in \Theta} \ker \alpha$$

of dimension $\dim S - \#\Theta$ and the Levi subgroup

$$M_\Theta := \text{ centralizer of } S_\Theta \text{ in } G \quad .$$

Second there is the unipotent subgroup

$$\begin{aligned} U_\Theta := &\text{ subgroup of } G \text{ generated by all} \\ &\text{ root subgroups } U_\alpha \text{ for } \alpha \in \Phi^+ \backslash \langle \Theta \rangle \end{aligned}$$

where $\langle \Theta \rangle := \{\alpha \in \Phi : \alpha \text{ is a linear combination of roots in } \Theta\}$. The product

$$P_\Theta := M_\Theta U_\Theta$$

is a parabolic subgroup of G; its unipotent radical is U_Θ.

Let

$$\begin{aligned} \delta_\Theta : P_\Theta \overset{pr}{\twoheadrightarrow} M_\Theta &\longrightarrow \mathbb{R}_+^\times \\ h &\longmapsto |\det(Ad(h); Lie\, U_\Theta)|^{-1} \end{aligned}$$

denote the modulus character of P_Θ. Because of $C \subseteq M_\Theta$ the category $\mathrm{Alg}_\chi(M_\Theta)$ of all smooth M_Θ-representations on which C acts through the character χ is defined. We have the "normalized" induction functor

$$\begin{aligned} \mathrm{Alg}_\chi(M_\Theta) &\longrightarrow \mathrm{Alg}_\chi(G) \\ E &\longmapsto Ind(E) \end{aligned}$$

where

$$\begin{aligned} Ind(E) := &\text{ space of all locally constant functions} \\ &\varphi : G \to E \text{ such that} \\ &\varphi(ghu) = \delta_\Theta^{1/2}(h) \cdot h^{-1}(\varphi(g)) \\ &\text{for all } g \in G, h \in M_\Theta, \text{ and } u \in U_\Theta \end{aligned}$$

with G acting by left translations. The reason for introducing the character δ_Θ is the formula

$$Ind(E)^\sim = Ind(\tilde{E})$$

([Cas] 3.1.2) for the smooth dual $\tilde{E}$. An irreducible representation V in $\mathrm{Alg}_\chi(G)$ is called of type Θ if there is an irreducible supercuspidal representation E in $\mathrm{Alg}_\chi(M_\Theta)$ such that V is isomorphic to a subquotient of $Ind(E)$. We put

$$\mathrm{Alg}^{fl}_{\chi,\Theta}(G) := \text{category of all smooth } G\text{-representations of finite length all of whose irreducible subquotients are of type } \Theta.$$

Any irreducible representation in $\mathrm{Alg}_\chi(G)$ has a type, i.e., lies in some category $\mathrm{Alg}^{fl}_{\chi,\Theta}(G)$ ([Cas] 5.1.2). Also $Ind(E)$ lies in $\mathrm{Alg}^{fl}_{\chi,\Theta}(G)$ if E is irreducible supercuspidal in $\mathrm{Alg}_\chi(M_\Theta)$ ([Cas] 6.3.7). Technically very important is the fact that

$$\mathrm{Alg}^{fl}_{\chi,\Theta}(G) = \mathrm{Alg}^{fl}_{\chi,\Theta'}(G) \quad \text{if } \Theta \text{ and } \Theta' \text{ are associated}$$

([Cas] 6.3.11). We recall that two subsets Θ and Θ' of Δ are called associated if S_Θ and $S_{\Theta'}$ are conjugate in G; in this case M_Θ and $M_{\Theta'}$ are conjugate in G, too, and $\#\Theta = \#\Theta'$. We actually need a more precise version of that fact. Fix a $g \in G$ such that $M_\Theta = gM_{\Theta'}g^{-1}$ and let E be an irreducible supercuspidal representation in $\mathrm{Alg}_\chi(M_\Theta)$. Via the map

$$\begin{aligned} M_{\Theta'} &\longrightarrow M_\Theta \\ h &\longmapsto ghg^{-1} \end{aligned}$$

E can be considered as an irreducible supercuspidal representation in $\mathrm{Alg}_\chi(M_{\Theta'})$ which we denote by ${}^gE$. Obviously the isomorphism class of ${}^gE$ does not depend on the choice of g.

Proposition III.2.1:

$Ind(E)$ and $Ind({}^gE)$ have (up to isomorphism) the same irreducible subquotients; moreover given an irreducible subquotient V of $Ind(E)$ there is a subset $\Theta' \subseteq \Delta$ associated to Θ such that V is a homomorphic image of $Ind({}^gE)$.

Proof: [Cas] 6.3.7 and 6.3.11. □

A key result of this paper which will be established in the Chapter IV (IV.4.18) is the following.

Theorem III.2.2:

Let E be an irreducible supercuspidal representation in $\mathrm{Alg}_\chi(M_\Theta)$; there is a subset $\Theta' \subseteq \Delta$ associated to Θ such that

$$\mathcal{E}^*(Ind(E)) \cong \begin{cases} Ind({}^g\tilde{E}) & \text{if } * = d - \#\Theta\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

III.3. The involution

Theorem III.3.1:

For any representation V in $\mathrm{Alg}^{fl}_{\chi,\Theta}(G)$ we have:
i. $\mathcal{E}^*(V) = 0$ for $* \neq d - \#\Theta$;
ii. $\mathcal{E}^{d-\#\Theta}(V)$ lies in $\mathrm{Alg}^{fl}_{\chi^{-1},\Theta}(G)$.

Proof: We prove the vanishing of $\mathcal{E}^*(V)$ for $0 \leq * < d - \#\Theta$ and all V in $\mathrm{Alg}^{fl}_{\chi,\Theta}(G)$ by induction with respect to $*$. Fix an integer q such that $0 \leq q < d - \#\Theta$ and assume that $\mathcal{E}^*(V) = 0$ for all $* < q$ and all V. We have to show that $\mathcal{E}^q(V) = 0$ for all V. By induction with respect to a Jordan-Hölder series we obviously may assume that V is irreducible. Then 2.1 says that there is an exact sequence of G-representations

$$0 \longrightarrow V' \longrightarrow Ind(E) \longrightarrow V \longrightarrow 0$$

where E is an irreducible supercuspidal representation in $\mathrm{Alg}_\chi(M_{\Theta'})$ for some subset $\Theta' \subseteq \Delta$ associated to Θ. We obtain an exact sequence

$$\mathcal{E}^{q-1}(V') \longrightarrow \mathcal{E}^q(V) \longrightarrow \mathcal{E}^q(Ind(E)) \quad .$$

Because of $\mathrm{Alg}^{fl}_{\chi,\Theta'}(G) = \mathrm{Alg}^{fl}_{\chi,\Theta}(G)$ (and $\#\Theta' = \#\Theta$) the left term vanishes by the induction hypothesis and the right term by 2.2.
For proving ii. we again may assume that V is irreducible. Similarly as above we then obtain an injection

$$\mathcal{E}^{d-\#\Theta}(V) \hookrightarrow \mathcal{E}^{d-\#\Theta}(Ind(E)) = Ind({}^g\tilde{E})$$

where the right hand equality comes from 2.2. This shows that $\mathcal{E}^{d-\#\Theta}(V)$ lies in $\mathrm{Alg}^{fl}_{\chi^{-1},\Theta'}(G) = \mathrm{Alg}^{fl}_{\chi^{-1},\Theta}(G)$.
The remaining vanishing assertion in i. also follows by induction. We already know that $\mathcal{E}^*(V) = 0$ for $* > d$. Since, quite generally, $Ind(E)^\sim = Ind(\tilde{E})$ holds 2.1 can be dualized to the statement that in case V is irreducible we find a monomorphism of G-representations $V \hookrightarrow Ind(E')$ where again E' is an irreducible supercuspidal representation in $\mathrm{Alg}_\chi(M_{\Theta'})$ for some subset $\Theta' \subseteq \Delta$ associated to Θ. Therefore an induction argument similar to the above one but downwards from $* = d+1$ to $* = d+1-\#\Theta$ is possible. $\square$

This result together with 1.5.ii implies that

$$\begin{aligned}\mathcal{E} : \mathrm{Alg}^{fl}_{\chi,\Theta}(G) &\longrightarrow \mathrm{Alg}^{fl}_{\chi^{-1},\Theta}(G)\\ V &\longmapsto \mathcal{E}^{d-\#\Theta}(V)\end{aligned}$$

is an exact (contravariant) functor such that $\mathcal{E} \circ \mathcal{E} = \mathrm{id}$.

Corollary III.3.2:

If V is an irreducible representation in $\mathrm{Alg}_\chi(G)$ *then* $\mathcal{E}(V)$ *is irreducible, too.*

Corollary III.3.3:

For any representation V in $\mathrm{Alg}^{fl}_{\chi,\Theta}(G)$ *we have:*
i. V has an exact projective resolution in $\mathrm{Alg}_\chi(G)$ *of length* $d - \#\Theta$*;*
ii. $\mathrm{Ext}^*_{\mathrm{Alg}_\chi(G)}(V, V') = 0$ *for* $* > d - \#\Theta$ *and any representation V' in* $\mathrm{Alg}_\chi(G)$*.*

Proof: Theorem 1 and 1.5.i. □

Remark III.3.4:

For any representation V in $\mathrm{Alg}^{fl}_{\chi,\Delta}(G)$ *we have* $\mathcal{E}(V) = \mathrm{Hom}_G(V, \mathcal{H}_\chi) = \tilde{V}$*.*

Proof: We may V assume to be irreducible. Then $\mathrm{Hom}_G(V, \mathcal{H}_\chi)$ is irreducible, too, by Corollary 2. On the other hand the matrix coefficients of V provide an embedding $\tilde{V} \hookrightarrow \mathrm{Hom}_G(V, \mathcal{H}_\chi)$ ([Cas] 5.2.1). □

Fix an invariant measure $d\overline{g}$ on G/C. Then $\mathcal{H}_\chi$ becomes an associative $\mathbb{C}$-algebra (without unit) via the convolution product

$$(\psi * \phi)(h) := \int_{G/C} \psi(g)\phi(g^{-1}h)d\overline{g} \quad \text{for } \psi, \phi \in \mathcal{H}_\chi \quad .$$

Also this algebra $\mathcal{H}_\chi$ acts from the left on each representation V' in $\mathrm{Alg}_\chi(G)$ through

$$\psi * v := \int_{G/C} \psi(g) \cdot gv d\overline{g} \quad \text{for } \psi \in \mathcal{H}_\chi, v \in V' \quad .$$

The antiautomorphism $g \mapsto g^{-1}$ of G induces an algebra antiisomorphism

$$\mathcal{H}_\chi \xrightarrow{\cong} \mathcal{H}_{\chi^{-1}} \quad .$$

Hence any representation in $\mathrm{Alg}_{\chi^{-1}}(G)$ can be viewed as a right $\mathcal{H}_\chi$-module. In this way the tensor product $V'' \underset{\mathcal{H}_\chi}{\otimes} V'$ and its left derived functors $\mathrm{Tor}^{\mathcal{H}_\chi}_*(V'', V')$ are defined for any $V'' \in \mathrm{Alg}_{\chi^{-1}}(G)$ and $V' \in \mathrm{Alg}_\chi(G)$.

Duality theorem:

Let V be a representation in $\mathrm{Alg}^{fl}_{\chi,\Theta}(G)$; we then have a natural isomorphism of functors

$$\mathrm{Ext}^*_{\mathrm{Alg}_\chi(G)}(V,.) = \mathrm{Tor}^{\mathcal{H}_\chi}_{d-\#\Theta-*}(\mathcal{E}(V),.)$$

on $\mathrm{Alg}_\chi(G)$.

Proof: For any V' in $\mathrm{Alg}_\chi(G)$ consider the smooth representation $\mathcal{H}_\chi \underset{\mathbb{C}}{\otimes} V'$ where G acts only on the first factor. The map

$$\begin{aligned} \mathcal{H}_\chi \underset{\mathbb{C}}{\otimes} V' &\twoheadrightarrow V' \\ \psi \otimes v &\longmapsto \psi * v \end{aligned}$$

is a G-equivariant epimorphism. According to (a slight generalization of) a result in [Bla] (compare also [Ca2] A.4) the representation $\mathcal{H}_\chi$ and hence $\mathcal{H}_\chi \underset{\mathbb{C}}{\otimes} V'$ is a projective object in $\mathrm{Alg}_\chi(G)$. We see that the objects in $\mathrm{Alg}_\chi(G)$ have a functorial projective resolution by representations of the form $\mathcal{H}_\chi \underset{\mathbb{C}}{\otimes} V'$. It follows from Theorem 1 that

$$\mathrm{Ext}^*_{\mathrm{Alg}_\chi(G)}(V, \mathcal{H}_\chi \underset{\mathbb{C}}{\otimes} V') = \mathrm{Ext}^*_{\mathrm{Alg}_\chi(G)}(V, \mathcal{H}_\chi) \underset{\mathbb{C}}{\otimes} V' = 0$$

for $* \neq d - \#\Theta$; here the first equality is an immediate consequence of the fact that V has a projective resolution by finitely generated G-representations (II.2.1 and II.3.2). These two properties imply by a standard homological algebra argument ([Har] I.7.4) that $\mathrm{Ext}^*_{\mathrm{Alg}_\chi(G)}(V,.)$ is the left derived functor of $\mathrm{Ext}^{d-\#\Theta}_{\mathrm{Alg}_\chi(G)}(V,.)$. In order to establish our assertion it therefore remains to exhibit a natural isomorphism

$$\mathrm{Ext}^{d-\#\Theta}_{\mathrm{Alg}_\chi(G)}(V,V') \cong \mathcal{E}(V) \underset{\mathcal{H}_\chi}{\otimes} V' = \mathrm{Ext}^{d-\#\Theta}_{\mathrm{Alg}_\chi(G)}(V,\mathcal{H}_\chi) \underset{\mathcal{H}_\chi}{\otimes} V' \quad .$$

Using the projective resolution in II.3.2 (for an e large enough) to compute the Ext's on both sides we see that a natural homomorphism

$$\mathrm{Ext}^*_{\mathrm{Alg}_\chi(G)}(V,\mathcal{H}_\chi) \underset{\mathcal{H}_\chi}{\otimes} V' \longrightarrow \mathrm{Ext}^*_{\mathrm{Alg}_\chi(G)}(V,V')$$

is induced by the homomorphism of complexes

$$\begin{aligned} \mathrm{Hom}_G(C^{or}_c(X_{(\cdot)},\gamma_e(V)),\mathcal{H}_\chi) \underset{\mathcal{H}_\chi}{\otimes} V' &\longrightarrow \mathrm{Hom}_G(C^{or}_c(X_{(\cdot)},\gamma_e(V)),V') \\ \Psi \otimes v' &\longmapsto (\omega \mapsto \Psi(\omega) * v') \quad . \end{aligned}$$

In order to establish that the induced homomorphism in degree $d - \#\Theta$ in fact is an isomorphism it suffices, again by applying the above constructed resolution of V', to consider the case $V' = \mathcal{H}_\chi$. But then the map in question simply is

$$\begin{aligned} \mathcal{E}(V) \underset{\mathcal{H}_\chi}{\otimes} \mathcal{H}_\chi &\longrightarrow \mathcal{E}(V) \\ w \otimes \phi &\longmapsto w * \phi \end{aligned}$$

which, although the ring $\mathcal{H}_\chi$ has no unit element, is bijective ([BW] XII.0.3(i)).$\square$

In a more elegant but less precise way these results can be formulated on the level of derived categories. Let $D^b_{fl}(\mathrm{Alg}_\chi(G))$ denote the bounded derived category of complexes in $\mathrm{Alg}_\chi(G)$ whose cohomology objects all are of finite length. Then the functor

$$\begin{aligned} \mathbb{D}_\chi : D^b_{fl}(\mathrm{Alg}_\chi(G)) &\longrightarrow D^b_{fl}(\mathrm{Alg}_{\chi^{-1}}(G)) \\ V^{\cdot} &\longmapsto R\,\mathrm{Hom}_{\mathrm{Alg}_\chi(G)}(V^{\cdot}, \mathcal{H}_\chi) \end{aligned}$$

is well-defined and is an anti-equivalence such that $\mathbb{D}_{\chi^{-1}} \circ \mathbb{D}_\chi = \mathrm{id}$. The Duality theorem becomes the statement that

$$R\,\mathrm{Hom}_{\mathrm{Alg}_\chi(G)}(V^{\cdot}, V'^{\cdot}) = \mathbb{D}_\chi(V^{\cdot}) \overset{\mathbb{L}}{\underset{\mathcal{H}_\chi}{\otimes}} V'^{\cdot} \quad \text{for } V^{\cdot}, V'^{\cdot} \text{ in } D^b_{fl}(\mathrm{Alg}_\chi(G)) \quad .$$

These facts constitute a kind of "Gorenstein property" ([Har] V.9.1) for the noncommutative ring $\mathcal{H}_\chi$.

III.4. Euler-Poincaré functions

In this section we assume that the connected center C of G is anisotropic and hence compact. Then all our previous results hold true without fixing a specific central character χ in advance. Dropping χ from a notation has the obvious meaning; e.g., $\mathcal{H}$ is the Hecke algebra of all locally constant functions with compact support on G. We fix a representation V in $\mathrm{Alg}(G)$ of finite length. By II.3.3 we have, for any other admissible representation V' in $\mathrm{Alg}(G)$, the Euler-Poincaré characteristic

$$EP(V, V') := \sum_{q \geq 0} (-1)^q \cdot \dim \mathrm{Ext}^q_{\mathrm{Alg}(G)}(V, V') \quad .$$

It will be a consequence of our theory that this Euler-Poincaré characteristic is a character value of V'. The character of V' is the linear form

$$\begin{aligned} tr_{V'} : \mathcal{H} &\longrightarrow \mathbb{C} \\ \psi &\longmapsto trace(\psi * .; V') \end{aligned}$$

which exists since the operator $\psi * .$ on V' has finite rank. In order to see this consequence we fix, for any $0 \leq q \leq d$, a set $\mathcal{F}_q$ of representatives for the G-orbits in X_q. By our assumption on G the stabilizer $P_F^\dagger$ of a facet F is a compact open subgroup in G. Let $\varepsilon_F : P_F^\dagger \to \{\pm 1\}$ be the unique character such that

$$g((F,c)) = (F, \varepsilon_F(g) \cdot c) \quad \text{for } (F,c) \in X_{(q)} \quad \text{and any } g \in P_F^\dagger \quad .$$

We also fix a special vertex x in A and an integer $e \geq 0$ such that V lies in $\mathrm{Alg}^{U_x^{(e)}}(G)$. Since $U_F^{(e)}$ is normal in $P_F^\dagger$ the finite group $P_F^\dagger / U_F^{(e)}$ acts on the finite-dimensional space $V^{U_F^{(e)}}$. The character of this latter representation will be denoted by $\tau_{F,e}^V : P_F^\dagger \to \mathbb{C}$. We extend the functions ε_F and $\tau_{F,e}^V$ by zero to functions on G. With these notations we define

$$f_{EP}^V := \sum_{q=0}^{d} \sum_{F \in \mathcal{F}_q} (-1)^q \cdot vol(P_F^\dagger)^{-1} \cdot \overline{\tau_{F,e}^V} \cdot \varepsilon_F \quad .$$

The volume $vol(P_F^\dagger)$ is formed with respect to a fixed Haar measure dg on G. In order to be consistent with our earlier conventions we always let the invariant measure $d\overline{g}$ on G/C be the quotient measure of dg by the unique Haar measure of total volume one on C. Then the operator $\psi * .$ on V', for $\psi \in \mathcal{H}$, can be written

$$\psi * v = \int_G \psi(g) \cdot gv dg \quad \text{for } v \in V' \quad .$$

The function f_{EP}^V obviously lies in $\mathcal{H}$. It depends on our choices which, for simplicity, we do not indicate in the notation. We call f_{EP}^V an Euler-Poincaré function for the representation V. If $V = \mathbb{C}$ is the trivial representation then $f_{EP}^{\mathbb{C}}$ is the Euler-Poincaré function of Kottwitz ([Kot]). Our subsequent results generalize corresponding results in [Kot] §2.

Proposition III.4.1:

For any admissible representation V' in Alg(G) *we have:*

$$tr_{V'}(f_{EP}^V) = EP(V, V') \quad .$$

Proof: According to II.3.2 we have

$$\mathrm{Ext}^*_{\mathrm{Alg}(G)}(V, V') = h^*(\mathrm{Hom}_G(C_c^{or}(X_{(.)}, \gamma_e(V)), V')) \quad .$$

But each $C_c^{or}(X_{(q)}, \gamma_e(V))$ decomposes as a G-representation into

$$C_c^{or}(X_{(q)}, \gamma_e(V)) = \bigoplus_{F \in \mathcal{F}_q} C_c^{or}(F, \gamma_e(V))$$

where

$$C_c^{or}(F,\gamma_e(V)) := \text{subspace of all those chains with support in the union of the } G\text{-orbits of the oriented facets with underlying facet } F.$$

We therefore obtain

$$EP(V,V') = \sum_{q=0}^{d} \sum_{F\in\mathcal{F}_q} (-1)^q \cdot \dim \operatorname{Hom}_G(C_c^{or}(F,\gamma_e(V)),V') \quad .$$

Consider now a single facet $F \in \mathcal{F}_q$ and fix an oriented facet (F,c). Using the oriented q-chains $\omega_{(F,c),v}$ introduced in III.1 we have the isomorphism

$$\begin{aligned} \operatorname{Hom}_G(C_c^{or}(F,\gamma_e(V)),V') &\xrightarrow{\cong} \operatorname{Hom}_{\mathbb{C}}(V^{U_F^{(e)}}, V'^{U_F^{(e)}})^{\varepsilon_F} \\ \Psi &\longmapsto (v \mapsto \Psi(\omega_{(F,c),v})) \end{aligned}$$

where the exponent ε_F on the right hand side stands for the ε_F-eigenspace of $P_F^\dagger$. It is a standard fact from the representation theory of finite groups ([CR] (32.8)) that

$$\dim \operatorname{Hom}_{\mathbb{C}}(V^{U_F^{(e)}}, V'^{U_F^{(e)}})^{\varepsilon_F} = vol(P_F^\dagger)^{-1} \cdot \int_{P_F^\dagger} \overline{\tau_{F,e}^V} \cdot \tau_{F,e}^{V'} \cdot \varepsilon_F dg \quad .$$

Therefore our assertion finally comes down to the following general observation: For any function $\psi \in \mathcal{H}$ which is supported on $P_F^\dagger$ and is constant on the cosets modulo $U_F^{(e)}$ one has

$$tr_{V'}(\psi) = \int_{P_F^\dagger} \psi \cdot \tau_{F,e}^{V'} dg$$

([Car] p. 120). $\square$

The real number

$$f_{EP}^V(1) = \sum_{q=0}^{d} \sum_{F\in\mathcal{F}_q} (-1)^q \cdot vol(P_F^\dagger)^{-1} \cdot \dim V^{U_F^{(e)}}$$

is independent of the choice of the sets $\mathcal{F}_q$. Moreover the invariant measure

$$d^V g := f_{EP}^V(1) \cdot dg$$

on G does not depend on the choice of the Haar measure dg. We call it an Euler-Poincaré measure for V. The corresponding volume function is denoted by vol_V. In case of the trivial representation $V = \mathbb{C}$ the measure $d^{\mathbb{C}} g$ is the canonical measure of G in the sense of Serre ([Ser] 3.3). We recall that $d^{\mathbb{C}} g$ is nonzero with sign $(-1)^d$ ([Ser] Prop. 28). Serre's "Euler-Poincaré property" of $d^{\mathbb{C}} g$ has a counterpart for any V.

Proposition III.4.2:

For any cocompact and torsionfree discrete subgroup Γ *in* G *we have*

$$vol_V(\Gamma\backslash G) = \sum_{q\geq 0}(-1)^q \cdot \dim H_q(\Gamma, V) = \sum_{q\geq 0}(-1)^q \cdot \dim \operatorname{Ext}^q_{\mathbb{C}[\Gamma]}(V, \mathbb{C}) \quad .$$

Proof: Since V is admissible the spaces $V^{U_F^{(e)}}$ are finite-dimensional. Moreover Γ being torsionfree and cocompact acts freely on X_q with finitely many orbits. Hence $C_c^{or}(X_{(q)}, \gamma_e(V))$ is a finitely generated free $\mathbb{C}[\Gamma]$-module. We therefore can use the resolution in II.3.2 in order to compute the homology groups $H_*(\Gamma, V)$ and we see that those groups are finite-dimensional and vanish in degrees $> d$. The second identity is also clear from that. We obtain (compare [Ser] p. 140)

$$\begin{aligned}
\sum_{q\geq 0}(-1)^q \cdot \dim H_q(\Gamma, V) &= \sum_{q\geq 0}(-1)^q \cdot \sum_{F\in\Gamma\backslash X_q} \dim V^{U_F^{(e)}} \\
&= \sum_{q\geq 0}(-1)^q \cdot \sum_{F\in\mathcal{F}_q} \dim V^{U_F^{(e)}} \cdot \#\Gamma\backslash G/P_F^\dagger \\
&= vol(\Gamma\backslash G) \cdot \sum_{q\geq 0}(-1)^q \cdot \sum_{F\in\mathcal{F}_q} vol(P_F^\dagger)^{-1} \cdot \dim V^{U_F^{(e)}} \\
&= f_{EP}^V(1) \cdot vol(\Gamma\backslash G) = vol_V(\Gamma\backslash G) \quad .
\end{aligned}$$

□

If K has characteristic 0 then a discrete subgroup Γ as in Proposition 2 always exists ([BH]Thm.A and Remark 2.3). Hence in this case the measure $d^V g$ is uniquely determined by the representation V (and does not depend on the choice of $U_x^{(e)}$).
We also introduce the rational number

$$d_{EP}(V) := \frac{f_{EP}^V(1)}{f_{EP}^{\mathbb{C}}(1)} \quad ;$$

it fulfills

$$d^V g = d_{EP}(V) \cdot d^{\mathbb{C}} g \quad .$$

The denominator of $d_{EP}(V)$ is bounded independently of V. If K has characteristic 0 then as a consequence of Proposition 2 the number $d_{EP}(V)$ only depends on V; in the rare case that V is finite-dimensional we have $d_{EP}(V) = \dim V$ ([Ser] p. 85). We therefore call $d_{EP}(V)$ the formal dimension of V. In a moment it will be seen that this is compatible with the notion of the formal dimension (or degree) of a square-integrable representation.

Remark III.4.3:

If V lies in $\mathrm{Alg}_{\Theta}^{fl}(G)$ then we have:
i. $EP(\mathcal{E}(\tilde{V}), V') = (-1)^{d-\#\Theta} \cdot EP(V, V')$ *for any admissible representation* V';
ii. $\sum_{q \geq 0} (-1)^q \cdot \dim H_q(\Gamma, \mathcal{E}(\tilde{V})) = (-1)^{d-\#\Theta} \cdot \sum_{q \geq 0} (-1)^q \cdot \dim H_q(\Gamma, V)$ *for any cocompact and torsionfree discrete subgroup* Γ *in* G;
iii. $d_{EP}(\mathcal{E}(\tilde{V})) = (-1)^{d-\#\Theta} \cdot d_{EP}(V)$ *if* K *has characteristic* 0.

Proof: In the proofs of the above two Propositions we used the chain complex $(C_c^{or}(X_{(.)}, \gamma_e(V)), \partial)$ whose only nonvanishing homology is V in degree 0. On the other hand we know from 1.5.i and 3.1 that the only nonvanishing homology of the cochain complex $(C_c^{or}(X_{(.)}, \gamma_e(V)), d)$ is $\mathcal{E}(\tilde{V})$ in degree $d - \#\Theta$. □

Associated with any irreducible square-integrable representation V is a unique Haar measure $d_V g$ on G which makes the Schur orthogonality relations hold (compare [Car] p. 122 and p. 131); $d_V g$ is called the formal degree of V.

Proposition III.4.4:

If either V is irreducible supercuspidal or V is irreducible square-integrable and K has characteristic 0 then we have $d^V g = d_V g$.

Let us draw immediately the following consequence which reproves results of Harish-Chandra, Howe, and Vigneras.

Corollary III.4.5:

If either V is supercuspidal or V is square-integrable and K has characteristic 0 then the rational number $d_{EP}(V)$ only depends on V and has sign $(-1)^d$; its denominator is bounded independently of V.

Proof: The only additional observation which we have to make is that $d_{EP}(\cdot)$ is additive in short exact sequences. □

The proof of Proposition 4 requires the abstract Plancherel formula. Let $\hat{G}$ denote the unitary dual of G, i.e. ([Car] p. 133), the set (of isomorphism classes) of preunitary irreducible representations in $\mathrm{Alg}(G)$. For any $\psi \in \mathcal{H}$ the Fourier transform $\hat{\psi}$ is the function on $\hat{G}$ defined by

$$\hat{\psi}(V') := tr_{V'}(\psi) \quad .$$

Since the group G is of type I ([Car] p. 133) the abstract Plancherel formula ([Dix] 18.8) is available; it says:

— $\psi(1) = \int_{\hat{G}} \hat{\psi}(\hat{g})d\hat{g}$ for any $\psi \in \mathcal{H}$ where $d\hat{g}$ denotes the Plancherel measure (corresponding to dg) on $\hat{G}$;
— a V' in $\hat{G}$ is square-integrable if and only if $vol_{d\hat{g}}(\{V'\}) > 0$ in which case we have $d_{V'}g = vol_{d\hat{g}}(\{V'\}) \cdot dg$.

Let us now first consider the case of an irreducible supercuspidal representation V. Since V is a projective object in Alg(G) ([Cas] 5.4.1) Proposition 1 implies that

$$(f_{EP}^V)\hat{}(V') = \begin{cases} 1 & \text{if } V' \cong V \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Inserting the function f_{EP}^V into the abstract Plancherel formula therefore gives

$$d^V g = f_{EP}^V(1) \cdot dg = vol_{d\hat{g}}(\{V\}) \cdot dg = d_V g$$

which proves Proposition 4 in the special case under consideration. The argument in the other case is the same once we use the following two additional facts. Firstly the support of the Plancherel measure is contained in the tempered irreducible representations ([Be2] Example 4.3.1). Secondly we have the following result.

Theorem III.4.6:

Assume that K has characteristic 0. If V is irreducible square-integrable and V' is irreducible tempered then

$$EP(V, V') = \begin{cases} 1 & \text{if } V' \cong V, \\ 0 & \text{otherwise} \ . \end{cases}$$

Proof: The vanishing assertion is a consequence of the subsequent Theorem 21 and [Ka1] Cor. on p. 29. In the case $V' = V$ we have to show, again by Theorem 21, that $\int_{C^{ell}} \theta_V(c^{-1}) \cdot \theta_V(c)dc = 1$; here C^{ell} denotes the set of regular elliptic conjugacy classes in G and dc the natural measure on it ([Ka1] §3 Lemma 1). But this can easily be deduced from [Ka1] Thm. F and [Ka1] §5 Prop. 3. □

There are two more consequences of this type of arguments.

Corollary III.4.7:

Assume that K has characteristic 0. If V is irreducible tempered but not square-integrable then $d_{EP}(V) = 0$.

Proof: More generally Theorem 21 and [Ka1] Cor. on p. 29 imply that, for V and V' irreducible tempered, we have

$$EP(V, V') = 0$$

unless V and V' are relatives in the sense of [Ka1] p. 10; the latter means that there is a representation which is parabolically induced from a square-integrable representation of a Levi subgroup and of which V and V' both are constituents. None of these finitely many relatives is square-integrable ([Ka1] Lemma 1.4). Using Proposition 1 we see that the Fourier transform $(f_{EP}^V)\hat{}$ has support in a set of Plancherel measure 0. Hence the abstract Plancherel formula says that $f_{EP}^V(1) = 0$.

Recall that if V is irreducible square-integrable then a function $\psi \in \mathcal{H}$ is called a pseudo-coefficient for V if

$$tr_{V'}(\psi) = \begin{cases} 0 & \text{for } V' \text{ irreducible tempered but } V' \not\cong V \quad , \\ 1 & \text{for } V' \cong V \quad . \end{cases}$$

Corollary III.4.8:

Assume that K has characteristic 0. If V is irreducible square-integrable and lies in $\mathrm{Alg}_\Theta^{fl}(G)$ then f_{EP}^V and $(-1)^{d-\#\Theta} \cdot f_{EP}^{\mathcal{E}(\tilde{V})}$ are pseudo-coefficients for V.

Proof: Proposition 1, Remark 3.i, and Theorem 6. □

It is very likely that the restriction to characteristic 0 is unnecessary in all of the above statements. Actually we strongly believe that

$$\mathrm{Ext}^*_{\mathrm{Alg}(G)}(V, V') = 0 \text{ for any } * \geq 0$$

holds true whenever V and V' are two irreducible tempered representations which are not relatives in the sense of [Ka1] p. 10. A possible strategy to prove this would be the following. Define an appropriate category $\mathrm{Temp}(G)$ of tempered representations and show that the Ext-groups in $\mathrm{Alg}(G)$ and in $\mathrm{Temp}(G)$ of any two admissible tempered representations (which naturally belong to both categories) coincide.

Next we will discuss the orbital integrals of the Euler-Poincaré functions f_{EP}^V and their relation to the character tr_V as a locally constant function on the regular elliptic set. Recall that an element of G is regular elliptic if its connected centralizer in G is a compact torus. For the sake of completeness let us include the following well-known fact.

Lemma III.4.9:

If $h \in G$ is regular elliptic then the map

$$\begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1}hg \end{aligned}$$

is proper.

Proof: It suffices to show that the preimage of a compact subset of the form Ug_0U where $U \subseteq G$ is a compact open subgroup is compact. This preimage equals

$$\bigcup_{\substack{g \in G_h \backslash G/U \\ g^{-1}hg \in Ug_0U}} G_h g U$$

where G_h denotes the centralizer of h in G. The element h being regular elliptic its centralizer is compact. Hence the double cosets $G_h g U$ are compact. On the other hand it is a particular case of Lemma 19 in [HCD] that the set over which the above union is taken is finite. □

We denote by G^{ell} the open subset of all regular elliptic elements in G. The above Lemma says that for each $h \in G^{ell}$ and each $\psi \in \mathcal{H}$ the integral

$$\overset{\vee}{\psi}(h) = \int_G \psi(g^{-1}hg)dg$$

exists. As another consequence of that Lemma the set X^h of fixed points of a given element $h \in G^{ell}$ in the Bruhat-Tits building X is compact (compare also [Rog] Lemma 1). To see this fix a facet $F \in \mathcal{F}_q$. The Lemma says that the set

$$\begin{aligned} \{g \in G : g^{-1}hg \in P_F^\dagger\}/P_F^\dagger &= \{g \in G : h \in P_{gF}^\dagger\}/P_F^\dagger \\ &= \{g \in G : X^h \cap gF \neq \emptyset\}/P_F^\dagger \end{aligned}$$

is finite. It follows that X^h is covered by finitely many facets and hence is compact. We have $hF = F$ for a facet $F \in X_q$ if and only if $F(h) := F \cap X^h \neq \emptyset$; moreover, as explained in [Kot], $F(h)$ then is a polysimplex whose dimension fulfills

$$\varepsilon_F(h) = (-1)^{q - \dim F(h)} \quad .$$

In this way X^h is a finite polysimplicial complex; we denote its set of q-dimensional facets by $(X^h)_q$.

Lemma III.4.10:

For any $h \in G^{ell}$ we have

$$(f_{EP}^V)^\vee(h) = \sum_{q=0}^{d} \sum_{F(h) \in (X^h)_q} (-1)^q \cdot \overline{\tau_{F,e}^V}(h) \quad .$$

Proof: We first quite generally consider a function $\psi \in \mathcal{H}$ whose restriction to $P_F^\dagger$, for some facet $F \in X_q$, is a class function and which is zero outside of $P_F^\dagger$.

Let $(X_q)^h$ denote the set of fixed points of h in X_q and let $(G \cdot F)^h$ be the intersection of $(X_q)^h$ with the G-orbit $G \cdot F$ of F in X_q. We compute

$$\begin{aligned}
\int_G \psi(g^{-1}hg)dg &= \sum_{\substack{g \in G_h \backslash G / P_F^\dagger \\ g^{-1}hg \in P_F^\dagger}} \psi(g^{-1}hg) \cdot \mathrm{vol}(G_h g P_F^\dagger) \\
&= \sum_{gF \in G_h \backslash (G \cdot F)^h} \psi(g^{-1}hg) \cdot \mathrm{vol}(P_F^\dagger) \cdot [G_h : G_h \cap P_{gF}^\dagger] \\
&= \mathrm{vol}(P_F^\dagger) \cdot \sum_{gF \in (G \cdot F)^h} \psi(g^{-1}hg) \quad .
\end{aligned}$$

Applying this to each summand of our function f_{EP}^V we obtain

$$\begin{aligned}
(f_{EP}^V)^\vee(h) &= \sum_{q=0}^{d} \sum_{F \in \mathcal{F}_q} (-1)^q \cdot \sum_{gF \in (G \cdot F)^h} (\overline{\tau_{F,e}^V} \cdot \varepsilon_F)(g^{-1}hg) \\
&= \sum_{q=0}^{d} \sum_{F \in \mathcal{F}_q} \sum_{gF \in (G \cdot F)^h} (-1)^q \cdot \varepsilon_{gF}(h) \cdot \overline{\tau_{gF,e}^V}(h) \\
&= \sum_{q=0}^{d} \sum_{F \in (X_q)^h} (-1)^q \cdot \varepsilon_F(h) \cdot \overline{\tau_{F,e}^V}(h) \\
&= \sum_{q=0}^{d} \sum_{F(h) \in (X^h)_q} (-1)^q \cdot \overline{\tau_{F,e}^V}(h) \quad .
\end{aligned}$$

□

An element in G is called noncompact if it is not contained in any compact subgroup.

Remark III.4.11: (The Selberg principle for Euler-Poincaré functions)

For any noncompact semisimple element $h \in G$ we have

$$\int_{G_h \backslash G} f_{EP}^V(g^{-1}hg) \frac{dg}{dg'} = 0$$

where dg' is any Haar measure on the centralizer G_h of h in G.

Proof: Note that since semisimple orbits are closed their orbital integrals exist in any characteristic. The Euler-Poincaré function f_{EP}^V vanishes on noncompact elements. □

Lemma III.4.12:
The function $(f_{EP}^{V})^{\vee}$ on G^{ell} is locally constant.

Proof: This is a consequence of Lemma 10 once we know that for any given $h \in G^{ell}$ there is an open subgroup $U \subseteq G$ such that

$$X^{h'} = X^h \quad \text{for any } h' \in hU \cap G^{ell} \quad .$$

First note that X^h never is empty ([Tit] 2.3.1). We choose a point $y \in X^h$. Since X^h is compact we find a constant $r > 0$ such that

$$X^h \subseteq \{z \in X : d(y,z) < r\} \quad .$$

Consider now the open subgroup

$$U := \{g \in G : gz = z \text{ for any } z \in X \text{ with } d(y,z) \leq r\} \quad .$$

Clearly, for any $h' \in hU$, we have

$$X^h \subseteq X^{h'}$$

and

$$X^{h'} \backslash X^h \subseteq \{z \in X : d(y,z) > r\} \quad .$$

Since with any $z \in X^{h'}$ the whole geodesic $[yz]$ is contained in $X^{h'}$ the latter inclusion forces $X^{h'} \backslash X^h$ to be empty.

We define an equivalence relation on G^{ell} by

$$h \sim h' \quad \text{if} \quad X^h = X^{h'} \quad .$$

In the preceding proof we have seen that the corresponding equivalence classes are open. Hence any function $\psi \in \mathcal{H}$ with support in G^{ell} can in a unique way be written as

$$\psi = \sum_{h \in G^{ell}/\sim} \psi_h$$

where $\psi_h \in \mathcal{H}$ has support in the equivalence class of h. Also the function

$$\varepsilon_h := \sum_{q=0}^{d} \sum_{F(h) \in (X^h)_q} (-1)^q \cdot \varepsilon_{U_F^{(e)}}$$

in $\mathcal{H}$ only depends on the equivalence class of h; here ε_U denotes, for any compact open subgroup $U \subseteq G$, the idempotent

$$\varepsilon_U(g) := \begin{cases} \mathrm{vol}(U)^{-1} & \text{if } g \in U \ , \\ 0 & \text{otherwise} \end{cases}$$

in $\mathcal{H}$. □

Lemma III.4.13:

For any $\psi \in \mathcal{H}$ with support in G^{ell} we have

$$\int_G \psi(g)(f_{EP}^V)^\vee(g^{-1})dg = tr_V(\sum_{h\in G^{ell}/\sim} \psi_h * \varepsilon_h) \quad .$$

Proof: It is clear ([Car] p. 120) that

$$\tau_{F,e}^V(h) = tr_V(h(\varepsilon_{U_F^{(e)}}))$$

holds true for any facet F in X such that $F \cap X^h \neq \emptyset$. Using this together with Lemma 10 we obtain

$$\begin{aligned}(f_{EP}^V)^\vee(h^{-1}) &= tr_V(h(\varepsilon_h)) \\ &= trace(v \longmapsto \int_G \varepsilon_h(h^{-1}g)gvdg) \\ &= trace(v \longmapsto \int_G \varepsilon_h(g)hgvdg) \\ &= trace(v \longmapsto h(\varepsilon_h * v)) \quad .\end{aligned}$$

Therefore the left hand side in our assertion becomes

$$\int_G \psi(g) \cdot trace(v \mapsto g(\varepsilon_g * v))dg = trace(v \mapsto \int_G \psi(g) \cdot g(\varepsilon_g * v)dg) \quad .$$

If ψ has support in the equivalence class of some $h \in G^{ell}$ the last expression obviously is equal to

$$trace(v \mapsto \int_G \psi(g) \cdot g(\varepsilon_h * v)dg) = trace(v \mapsto \psi * (\varepsilon_h * v)) = tr_V(\psi * \varepsilon_h) \quad . \quad \square$$

Lemma III.4.14:

For any $h \in G^{ell}$ and any $F_0(h) \in (X^h)_0$ we have

$$\varepsilon_{U_{F_0}^{(e)}} * \varepsilon_h = \varepsilon_{U_{F_0}^{(e)}} \quad .$$

Proof: Let $F_0(h) = \{y\}$ be the given vertex of X^h. We introduce a relation between facets F and F' in X as follows: We write $F \xrightarrow{y} F'$ if

— $F(h) \neq \emptyset$, $F'(h) \neq \emptyset$,

— $F' \subseteq \overline{F}$ (equivalently $F'(h) \subseteq \overline{F(h)}$), $F' \neq F$, and

— there are points $z \in F(h)$ and $z' \in F'(h)$ such that $z \in [yz']$;

moreover in this situation F, resp. F', is called y-large, resp. y-small. These notions have the following elementary geometric properties:

1. Any facet $F \neq F_0$ with $F(h) \neq \emptyset$ is either y-large or y-small.
2. A y-large facet is not y-small and vice versa.
3. For any y-small facet F' there is a unique y-large facet F such that $F \xrightarrow{y} F'$.
4. If F is y-large we have

$$(-1)^{\dim F(h)} + \sum_{F \xrightarrow{y} F'} (-1)^{\dim F'(h)} = 0 \quad .$$

In order to see these properties consider an arbitrary point $y' \in X^h$ different from y. The whole geodesic $[yy']$ then belongs to X^h. In addition one has:
5. There are only finitely many facets $F_0, \ldots, F_m$ in X such that $F_i \cap [yy'] \neq \emptyset$.

> X is locally finite.

6. Each intersection $F_i \cap [yy']$ either consists of one point or is an open convex subset of $[yy']$.

> Choose an apartment $A' \subseteq X$ which contains $[yy']$ and therefore each F_i. Let $\langle F_i \rangle$ denote the affine subspace of A' generated by F_i; note that F_i is open in $\langle F_i \rangle$. If the intersection $F_i \cap [yy']$ consists of more than one point then $[yy'] \subseteq \langle F_i \rangle$.

The enumeration $F_0, \ldots, F_m$ obviously can be made in such a way that

$$d(y, z_i) < d(y, z_{i+1}) \quad \text{whenever } z_i \in F_i \cap [yy'] \text{ and } z_{i+1} \in F_{i+1} \cap [yy'] \quad .$$

Then the intersections consisting of one point precisely are the

$$F_{2i} \cap [yy'] \quad \text{for } 0 \leq i \leq \frac{m}{2}$$

and we have

$$F_0 \subseteq \overline{F}_1 \supseteq F_2 \subseteq \overline{F}_3 \supseteq F_4 \subseteq \ldots$$

It is clear that $F_{m-1} \xrightarrow{y} F_m$ if $F_m \cap [yy'] = \{y'\}$. The converse holds in the following stronger form.
7. If $F \xrightarrow{y} F_m$ then $F = F_{m-1}$ and hence $F_m \cap [yy'] = \{y'\}$.

> Let $z \in F(h)$ and $z' \in F_m(h)$ be points such that $z \in [yz']$. Choose A' as above so that $F_m \subseteq A'$ and hence $[yz'] \cup F \subseteq A'$. Applying the preceding discussion to z' we obtain that $\langle F \rangle$ contains $[yz'] \cup F_m$ and hence $[yy']$ and F_{m-1}. We also obtain that $F_m \cap [yz'] = \{z'\}$ so that $[yy']$ cannot belong to $\langle F_m \rangle$; this means that $F_m \cap [yy'] = \{y'\}$ or in other words that $\overline{F}_{m-1} \supseteq F_m$. As a consequence $\langle F_{m-1} \rangle$ contains $[yy'] \cup F_m$ and hence

$[yz']$ and F. Therefore F and F_{m-1} must be facets of the same dimension both having F_m in their boundary. Assuming F and F_{m-1} to be different there would exist an affine root for A' which is 0 on F_m but has different signs on F and F_{m-1}. On the other hand because of $F \cap [yz'] \neq \emptyset$, resp. $F_{m-1} \cap [yy'] \neq \emptyset$, this affine root has the same sign on y and on F, resp. F_{m-1}, which is a contradiction.

This implies 3. together with the following characterization.
8. A facet $F' \neq F_0$ in X such that $F'(h) \neq \emptyset$ is y-small if and only if $F' \cap [yz'] = \{z'\}$ for some (or any) $z' \in F'(h)$.
It also implies the direct implication in the following analogous characterization of y-large facets.
9. A facet $F \neq F_0$ in X such that $F(h) \neq \emptyset$ is y-large if and only if $F \cap [yz]$ is open in $[yz]$ for some (or any) $z \in F(h)$.

For the reverse implication let $A' \subseteq X$ be an apartment which contains $[yz]$ and let $L \subseteq A'$ denote the affine line generated by $[yz]$. It follows from I.1.5 that $\overline{F} \cap L \subseteq X^h$. The intersection $(\overline{F} \backslash F) \cap L$ consists of exactly two points and those belong to X^h. Taking as y' that one of bigger distance to y we obtain, with the previous notations, that $F \xrightarrow{y} F_m$.

Clearly 6., 8., and 9. imply 1. and 2. It remains to discuss the property 4.
Fix a y-large facet F. In particular $F \neq F_0$.
10. Let F'' and $F' \neq F$ be facets such that $F'' \subseteq \overline{F'}, F' \subseteq \overline{F}$, and $F'(h) \neq \emptyset$; if $F \xrightarrow{y} F''$ then $F \xrightarrow{y} F'$.

We choose points $z \in F(h)$ and $z'' \in F''(h)$ such that $z \in [yz'']$. We also choose an apartment $A' \subseteq X$ containing y and F and therefore also F' and F''. Fix a point $\tilde{z}' \in F'(h)$ and consider the euclidean triangle in A' with vertices y, z'', and $\tilde{z}'$. It follows that $(z\tilde{z}') := [z\tilde{z}'] \backslash \{z, \tilde{z}'\}$ is nonempty and is contained in $F(h)$. Fix a point $\tilde{z} \in (z\tilde{z}')$. The two affine lines in A' through y and $\tilde{z}$ and through z'' and $\tilde{z}'$ intersect in a point $z' \in F'(h)$. Then $\tilde{z} \in [yz']$ and hence $F \xrightarrow{y} F'$.

The last statement means that

$$Y := \text{ union of all } F'(h) \text{ where } F' \subseteq \overline{F}, F' \neq F, \text{ and not } F \xrightarrow{y} F'$$

is a subcomplex of $\overline{F} \cap X^h$. Since the latter is contractible we have

$$\sum_{\substack{F' \subseteq \overline{F} \\ F'(h) \neq \emptyset}} (-1)^{\dim F'(h)} = 1 \quad .$$

The property 4. therefore is equivalent to

$$\sum_{\emptyset \neq F'(h) \subseteq Y} (-1)^{\dim F'(h)} = 1 \quad .$$

The latter certainly holds if we show Y to be contractible. We may assume that y and F are contained in the standard apartment A. Let $\langle F \rangle$ denote the affine subspace of A generated by F. Since F is y-large we necessarily have $y \in \langle F \rangle$. Let $H_1, \ldots, H_s \subseteq A$ be affine root hyperplanes such that the $F_1, \ldots, F_s$ defined by

$$\overline{F_i} = H_i \cap \overline{F}$$

precisely are the codimension 1 facets of $\overline{F}$. For each H_i fix a defining affine root $\alpha_i(.) + \ell_i$ in such a way that

$$F = \{x \in \langle F \rangle : \alpha_i(x) + \ell_i > 0 \ \text{ for any } 1 \leq i \leq s\} \quad .$$

Since $F \neq F_0$ the set

$$I := \{1 \leq i \leq s : \alpha_i(y) + \ell_i \leq 0\}$$

is nonempty. Using [Bou] V.3.9.8 (ii) one sees that the α_i for $i \in I$ restricted to the linear subspace parallel to $\langle F \rangle$ are linearly independent. Hence there is a facet $F' \subseteq \overline{F}$ such that

$$\overline{F'} = \bigcap_{i \in I} \overline{F_i} \quad .$$

Since h fixes y and F it permutes the $\overline{F_i}$ with $i \in I$ and fixes F'. This means that $F'(h) \neq \emptyset$. Once we show that

$$Y = \bigcup_{i \in I} \overline{F_i} \cap X^h$$

it is then clear that Y can be contracted to any point in $F'(h)$. Consider first a point $x \in \overline{F_i} \cap X^h$ for some $i \in I$. Then

$$\alpha_i(x) + \ell_i = 0 \ \text{ and } \ \alpha_i(y) + \ell_i \leq 0 \quad .$$

This implies $(\alpha_i(.) + \ell_i) | [yx] \leq 0$ and hence $[yx] \cap F = \emptyset$. If $\tilde{F} \subseteq \overline{F}$ is the facet containing x then it follows that not $F \xrightarrow{y} \tilde{F}$. We conclude that $x \in \tilde{F}(h) \subseteq Y$. Now let x be a point in Y. Then $[yx] \cap F = \emptyset$. Moreover we have

$$(\alpha_i(.) + \ell_i) | [yx] \backslash \{x\} > 0 \ \text{ for any } i \notin I \quad .$$

The case $x = y$ is clear since then $x \in \overline{F_i}$ for all $i \in I$. Assume therefore that $x \neq y$ and that x is not contained in the right hand side of our claimed equality. Then $\alpha_i(x) + \ell_i > 0$ for any $i \in I$. Hence we would find a $x' \in [yx]$, $x' \neq x$, such that $\alpha_i(x') + \ell_i > 0$ for any $1 \leq i \leq s$. The latter means that $x' \in F$ which is a

contradiction.
This finishes the proof of the properties 1.-4. It follows from 1.-3. that

$$\varepsilon_h = \varepsilon_{U^{(e)}_{F_0}} + \sum_{F\ y-\text{large}} ((-1)^{\dim F(h)} \varepsilon_{U^{(e)}_F} + \sum_{F \xrightarrow{y} F'} (-1)^{\dim F'(h)} \varepsilon_{U^{(e)}_{F'}}) \quad .$$

Because of 4. it is therefore sufficient to show that

$$\varepsilon_{U^{(e)}_{F_0}} * \varepsilon_{U^{(e)}_F} = \varepsilon_{U^{(e)}_{F_0}} * \varepsilon_{U^{(e)}_{F'}} \quad \text{whenever } F \xrightarrow{y} F' \quad .$$

Fixing a pair $F \xrightarrow{y} F'$ we have to check that

$$U^{(e)}_{F_0} \cdot U^{(e)}_F = U^{(e)}_{F_0} \cdot U^{(e)}_{F'}$$

holds true. The inclusion $U^{(e)}_{F'} \subseteq U^{(e)}_F$ is clear from I.2.11. Hence it remains to establish the other inclusion

$$U^{(e)}_F \subseteq U^{(e)}_{F_0} \cdot U^{(e)}_{F'} \quad .$$

This is a variant of I.3.1. We may and will assume that the facets F_0, F, and F' lie in the basic apartment A. Let $z \in F(h)$ and $z' \in F'(h)$ be points such that $z \in [yz']$. It is trivial to see that, for $\alpha \in \Phi^{red}$, we have

$$f^*_F(\alpha) \geq f^*_{F'}(\alpha) \quad \text{and hence} \quad U^{(e)}_F \cap U_\alpha \subseteq U^{(e)}_{F'}$$

except possibly in the case

$$f_F(\alpha) = f_{F'}(\alpha) \ , \ \alpha|F \text{ not constant, but } \alpha|F' \text{ constant} \quad .$$

In that case we have

$$-\alpha(y) < -\alpha(z) < -\alpha(z') = f^*_F(\alpha) \quad .$$

If $f_{F_0}(\alpha) \leq -\alpha(z')$ then $f^*_F(\alpha) \geq f^*_{F_0}(\alpha)$ and hence $U^{(e)}_F \cap U_\alpha \subseteq U^{(e)}_{F_0}$. Otherwise there are two consecutive values $\ell < \ell'$ in Γ_α such that

$$\ell < -\alpha(y) < -\alpha(z') < \ell' \quad .$$

Using I.2.10 we obtain

$$U^{(e)}_{F_0} \cap U_\alpha = U_{\alpha,\ell'} \cdot U_{2\alpha,2\ell'+e} = U^{(e)}_F \cap U_\alpha \quad . \qquad \square$$

To get further we need the subsequent result which doubtlessly holds true in general but which we can establish, at present, only under some additional assumption. Let $[\mathcal{H}, \mathcal{H}]$ be the additive subgroup of $\mathcal{H}$ generated by all commutators $\psi * \phi - \phi * \psi$ for $\phi, \psi \in \mathcal{H}$ and put $\mathcal{H}^{ab} := \mathcal{H}/[\mathcal{H}, \mathcal{H}]$.

Proposition III.4.15:

If G is split or if K has characteristic 0 then the class of f_{EP}^V in $\mathcal{H}^{ab}$ and hence the function $(f_{EP}^V)^\vee$ is uniquely determined by the representation V (and the Haar measure dg and does not depend on the choice of $U_x^{(e)}$).

Proof: Proposition 1, [Ka1] Thm. 0, and [Ka2] Thm. B. □

Theorem III.4.16:

If G is split or if K has characteristic 0 then we have

$$tr_V(\psi) = \int_G \psi(g)(f_{EP}^V)^\vee(g^{-1})dg$$

for any $\psi \in \mathcal{H}$ with support in G^{ell}.

Proof: By Proposition 15 we may choose the number e as large as we want. Let $h_1, \ldots, h_m \in G^{ell}$ be representatives of those equivalence classes which meet the support of ψ and, for each $1 \leq i \leq m$, fix a $F_i(h_i) \in (X^{h_i})_0$. We now choose e large enough so that ψ_{h_i} is $U_{F_i}^{(e)}$-right invariant for any $1 \leq i \leq m$. This means that $\psi_{h_i} * \varepsilon_{U_{F_i}^{(e)}} = \psi_{h_i}$ and hence, by Lemma 14, that $\psi_{h_i} * \varepsilon_{h_i} = \psi_{h_i}$. The statement follows then from Lemma 13. □

Of course we know from Harish-Chandra that the distribution tr_V is given by a locally constant function on the regular semisimple subset G^{reg} in G; let θ_V denote the restriction of that function to G^{ell}. The last Theorem then can be rephrased by saying that under the assumption made there we have

$$\theta_V(h) = (f_{EP}^V)^\vee(h^{-1}) \quad .$$

This might be viewed as a kind of explicit formula for the character values on the regular elliptic set; compare also the Hopf-Lefschetz type formula in IV.1.5.

Corollary III.4.17:

If G is split or K has characteristic 0 then we have:

i. $\theta_V(h^{-1}) = \overline{\theta_V(h)}$ *for $h \in G^{ell}$;*

ii. $\theta_{\mathcal{E}(\tilde{V})} = (-1)^{d-\#\Theta} \cdot \theta_V$ *if V lies in* $\mathrm{Alg}_\Theta^{fl}(G)$.

Proof: i. We obviously have $f_{EP}^V(g^{-1}) = \overline{f_{EP}^V(g)}$ for any $g \in G$. ii. It follows from Proposition 1 and Remark 3.i that

$$tr_{V'}(f_{EP}^{\mathcal{E}(\tilde{V})} - (-1)^{d-\#\Theta} \cdot f_{EP}^V) = 0 \text{ for any admissible } V' \quad .$$

Hence that function is contained in $[\mathcal{H}, \mathcal{H}]$ by [Ka1] Thm. 0 and [Ka2] Thm. B, respectively. □

Lemma III.4.18:

i. $EP(V, V') = EP(V', V)$ *for* V *and* V' *of finite length in* $\mathrm{Alg}(G)$*;*

ii. let $\Theta \underset{\neq}{\subset} \Delta$ *be a proper subset and let* E *be a representation of finite length in* $\mathrm{Alg}(M_\Theta)$*; then* $EP(V, Ind(E)) = 0$.

Proof: i. The symmetry is obvious from the expression

$$EP(V, V') = \sum_{q=0}^{d} \sum_{F \in \mathcal{F}_q} (-1)^q \cdot \mathrm{vol}(P_F^\dagger)^{-1} \cdot \int_{P_F^\dagger} \overline{\tau_{F,e}^V} \cdot \tau_{F,e}^{V'} \cdot \varepsilon_F dg$$

which was given in the proof of Proposition 1. Of course $e \geq 0$ here should be chosen in such a way that both V and V' lie in $\mathrm{Alg}^{U_x^{(e)}}(G)$.
ii. The subsequent argument is due to Kazhdan. The set of unramified characters of M_Θ is a complex algebraic torus of dimension $d - \#\Theta$ (compare [Car]3.2). The function

$$\xi \longmapsto tr_{Ind(E \otimes \xi)}(f_{EP}^V)$$

on this torus is regular according to [BDK] §1.2. Using Proposition 1 we see that the function

$$\xi \longmapsto EP(V, Ind(E \otimes \xi))$$

is regular and integral valued; it therefore is constant. On the other hand it is shown in [Ca2] A.12 that

$$\mathrm{Ext}^*_{\mathrm{Alg}(G)}(V, Ind(E \otimes \xi)) = \mathrm{Ext}^*_{\mathrm{Alg}(M_\Theta)}(V_{U_\Theta}, E \otimes \xi)$$

holds true. But for any character ξ_0 such that the central torus S_Θ in M_Θ acts on the Jacquet module V_{U_Θ} and on $E \otimes \xi_0$ by different characters we have

$$\mathrm{Ext}^*_{\mathrm{Alg}(M_\Theta)}(V_{U_\Theta}, E \otimes \xi_0) = 0$$

([BW] IX.1.9). □

Lemma III.4.19:

If K has characteristic 0 then we have

$$\int_{G_h \backslash G} f^V_{EP}(g^{-1}hg)\frac{dg}{dg'} = 0 \quad \textit{for any } h \in G^{reg} \backslash G^{ell} \quad .$$

Proof: Proposition 1, Lemma 18, and [Ka1] Thm. A. □

Following [Ka1] we put

$$A(G) := \{\psi \in \mathcal{H} : \int_{G_h \backslash G} \psi(g^{-1}hg)\frac{dg}{dg'} = 0 \ \text{ for any } h \in G^{reg} \backslash G^{ell}\}$$

and

$$\overline{A}(G) := A(G)/[\mathcal{H}, \mathcal{H}] \quad .$$

Let

$R(G) :=$ Grothendieck group of representations of finite length in Alg(G) (w.r.t. exact sequences) tensorized by $\mathbb{C}$.

The induction functor $Ind(.)$ induces a homomorphism

$$R(M_\Theta) \longrightarrow R(G)$$

for any subset $\Theta \subseteq \Delta$. We put

$$R_I(G) := \sum_{\Theta \underset{\neq}{\subset} \Delta} \text{image of } R(M_\Theta) \quad \text{and} \quad \overline{R}(G) := R(G)/R_I(G) \quad .$$

If K has characteristic 0 then it follows from Proposition 15 and Lemma 19 that

$$\begin{aligned} R(G) &\longrightarrow \overline{A}(G) \\ \text{class of } V &\longmapsto \text{class of } f^V_{EP} \end{aligned}$$

is a well-defined homomorphism; as a consequence of Proposition 1, Lemma 18, and [Ka1] Thm. 0 this map is trivial on $R_I(G)$.

Proposition III.4.20:

If K has characteristic 0 then the map

$$\overline{R}(G) \xrightarrow{\cong} \overline{A}(G)$$
$$\text{class of } V \longmapsto \text{class of } f_{EP}^{V}$$

is an isomorphism.

Proof: It follows from Theorem 16 that the map in question is up to the substitution $h \mapsto h^{-1}$ the inverse of the isomorphism in [Ka1] Thm. E. □

Our approach allows to establish a kind of orthogonality formula for characters which was conjectured by Kazhdan and which generalizes [Ka1] Cor. on p. 29. Let C^{ell} denote the set of all regular elliptic conjugacy classes in G; then $\overset{\vee}{\psi}$, for any $\psi \in \mathcal{H}$, as well as θ_V can be viewed as functions on C^{ell}. According to [Ka1] §3 Lemma 1 there is a unique measure dc on C^{ell} such that

$$\int_G \psi dg = \int_{C^{ell}} \overset{\vee}{\psi}(c)dc \quad \text{for any } \psi \in \mathcal{H} \text{ with support in } G^{ell} \quad .$$

Theorem III.4.21: (Orthogonality)

If K has characteristic 0 then, for any two representations V and V' of finite length in Alg(G)*, we have*

$$\int_{C^{ell}} \theta_V(c^{-1}) \cdot \theta_{V'}(c)dc = EP(V, V') \quad .$$

Proof: According to [Ka1] Thm. F we have the identity

$$tr_{V'}(\psi) = \int_{C^{ell}} \theta_{V'}(c) \cdot \overset{\vee}{\psi}(c)dc$$

for any function $\psi \in A(G)$. The Lemma 19 allows to apply this identity to the function $\psi = f_{EP}^{V}$. Using Theorem 16 we obtain

$$tr_{V'}(f_{EP}^{V}) = \int_{C^{ell}} \theta_V(c^{-1}) \cdot \theta_{V'}(c)dc \quad .$$

It remains to apply Proposition 1 to the left hand side. □

By Lemma 18 the Euler-Poincaré characteristic induces a symmetric bilinear form

$$EP(.,.) : \overline{R}(G) \times \overline{R}(G) \longrightarrow \mathbb{C} \quad .$$

Because of Theorem 21 this form coincides with the form $[.,.]$ considered on p. 5 in [Ka1] provided K has characteristic 0; hence it is nondegenerate in this case. As is pointed out in [Clo] §5 a better understanding of this form is tied up with the study of the L-packets.

Finally we want to relate our concepts to the notion of the rank of V introduced by Vigneras ([Vig]). She extends the formalism of the Hattori-Stallings trace to the context of smooth representations. The technical difficulty which arises is that the Hecke algebra $\mathcal{H}$ has no unit element in general. For us it is most natural to work with the subalgebras

$$\mathcal{H}(e) := \varepsilon_{U_x^{(e)}} * \mathcal{H} * \varepsilon_{U_x^{(e)}} \quad \text{for } e \geq 0$$

in which $\varepsilon_{U_x^{(e)}}$ is the unit element (recall that x is a fixed special vertex); by I.2.9 we have

$$\mathcal{H} = \bigcup_{e \geq 0} \mathcal{H}(e) \quad .$$

The point is that $U_x^{(e)}$ fulfills the assumptions of [Ber] 3.9 as we noted already in I.3 so that the functor

$$\begin{aligned} \mathrm{Alg}^{U_x^{(e)}}(G) &\xrightarrow{\sim} \begin{array}{l}\text{category of unital}\\ \text{left } \mathcal{H}(e)\text{-modules}\end{array} \\ V' &\longmapsto V'(e) := (V')^{U_x^{(e)}} \end{aligned}$$

is an equivalence of categories which in addition ([Ber] 3.3) respects the property of being finitely generated. First let V' be a finitely generated projective representation in $\mathrm{Alg}^{U_x^{(e)}}(G)$; then $V'(e)$ is a finitely generated projective $\mathcal{H}(e)$-module and we have the obvious isomorphism

$$\mathrm{Hom}_{\mathcal{H}(e)}(V'(e), \mathcal{H}(e)) \underset{\mathcal{H}(e)}{\otimes} V'(e) \xrightarrow{\cong} \mathrm{End}_{\mathcal{H}(e)}(V'(e)) \quad .$$

If $\sum_i v_i^* \otimes v_i$ is the element in the left hand side which corresponds to the identity endomorphism in the right hand side then the <u>rank</u> of V' is defined as

$$r_{V'} := \text{class of } \sum_i v_i^*(v_i) \quad \text{in } \mathcal{H}^{ab} \quad .$$

Now consider an arbitrary finitely generated representation V' in $\mathrm{Alg}(G)$. Bernstein has shown ([Vig] Prop. 37; alternatively we can use II.3.2) that V' has a resolution

$$0 \longrightarrow V'_d \longrightarrow \ldots \longrightarrow V'_0 \longrightarrow V' \longrightarrow 0$$

by finitely generated projective representations V_i' in $\mathrm{Alg}(G)$. Choosing e large enough so that the whole resolution lies in $\mathrm{Alg}^{U_x^{(e)}}(G)$ the rank of V' then is defined to be

$$r_{V'} := \sum_{i=0}^{d} (-1)^i \cdot r_{V_i'} \quad .$$

It is shown in [Vig] Prop. 39 that if G is split or K has characteristic 0 then the class $r_{V'} \in \mathcal{H}^{ab}$ only depends on V' and is characterized by the property that

$$tr_{V''}(r_{V'}) = EP(V', V'')$$

holds true for any irreducible representation V'' in $\mathrm{Alg}(G)$. Combining this with our Proposition 1 we see that the Euler-Poincaré functions f_{EP}^V of our representation of finite length V are representatives of its rank r_V.

Proposition III.4.22:

If G is split or if K has characteristic 0 then we have

$$r_V = \textit{class of } f_{EP}^V \ \textit{ in } \mathcal{H}^{ab} \quad .$$

Actually a more precise result holds true. Each individual summand of f_{EP}^V is the rank of a corresponding direct summand of our projective resolution in II.3.2: As already used in the proof of Proposition 1 there is the decomposition

$$C_c^{or}(X_{(q)}, \gamma_e(V)) = \bigoplus_{F \in \mathcal{F}_q} C_c^{or}(F, \gamma_e(V)) \quad .$$

Fix a facet $F \in \mathcal{F}_q$ and put $V' := C_c^{or}(F, \gamma_e(V))$. Since V' is projective its rank is characterized by the property that

$$tr_{V''}(r_{V'}) = \dim \mathrm{Hom}_G(V', V'')$$

for any irreducible representation V''. But in the proof of Proposition 1 it was shown that the latter dimension is equal to

$$tr_{V''}(\mathrm{vol}(P_F^\dagger)^{-1} \cdot \overline{\tau_{F,e}^V} \cdot \varepsilon_F) \quad .$$

Hence we obtain that

$$\text{rank of } C_c^{or}(F, \gamma_e(V)) = \text{class of } \mathrm{vol}(P_F^\dagger)^{-1} \cdot \overline{\tau_{F,e}^V} \cdot \varepsilon_F \ \text{ in } \mathcal{H}^{ab} \quad .$$

Propositions 2 and 22 together give a different proof, for a representation of finite length, of the dimension formula in the Main Theorem 36 in [Vig]; the positivity statement in loc. cit. is, by the above discussion, trivial for the projective representations appearing in our resolution II.3.2. More importantly we obtain a relation between the rank r_V and the trace θ_V which is entirely similar to the case of a finite group ([Hat]). Note that the function $\overset{\vee}{\psi}$ on G^{ell} only depends on the class of ψ in $\mathcal{H}^{ab}$.

Theorem III.4.23:

If G is split or if K has characteristic 0 then we have

$$\theta_V(h) = (r_V)^\vee(h^{-1}) \quad \text{for } h \in G^{ell} \quad .$$

Proof: Theorem 16 and Proposition 22. □

IV. Representations as sheaves on the Borel-Serre compactification

IV.1. Representations as sheaves on the Bruhat-Tits building

Let V be a smooth representation of G. For any open subgroup $U \subseteq G$ we have the space

$$V_U := \text{maximal quotient of } V \text{ on which the } U\text{-action is trivial}$$

of U-coinvariants of V. We write $v \bmod U$ for the image in V_U of a vector $v \in V$. Fix an integer $e \geq 0$. In order to simplify the notation we sometimes will suppress indicating the dependence on e in the notions to be introduced. Let F be a facet of X. Since $U_F^{(e)}$ is profinite the projection map $pr_F : V \twoheadrightarrow V^{U_F^{(e)}}$ in III.1 induces an isomorphism

$$V_{U_F^{(e)}} \xrightarrow{\cong} V^{U_F^{(e)}} \quad .$$

Whenever F' is another facet such that $F' \subseteq \overline{F}$ then we have the commutative square

$$\begin{array}{ccc} V^{U_{F'}^{(e)}} & \xrightarrow{pr_{F'}^F} & V^{U_F^{(e)}} \\ {\scriptstyle pr_{F'}}\uparrow\cong & & \cong\uparrow{\scriptstyle pr_F} \\ V_{U_{F'}^{(e)}} & \xrightarrow{pr} & V_{U_F^{(e)}} \end{array}$$

where $pr_{F'}^F$ is the other projection map from III.1 and pr is the obvious quotient map (coming from the fact that $U_{F'}^{(e)} \subseteq U_F^{(e)}$).

The representation V gives rise to a sheaf $\underset{\approx}{V}$ on the Bruhat-Tits building X in the following way: For any open subset $\Omega \subseteq X$ put

$$\underset{\approx}{V}(\Omega) := \mathbb{C}\text{-vector space of all maps } s : \Omega \to \dot{\bigcup_{z\in\Omega}} V_{U_z^{(e)}}$$

such that

— $s(z) \in V_{U_z^{(e)}}$ for any $z \in \Omega$,

— there is an open covering $\Omega = \bigcup_{i\in I} \Omega_i$ and vectors $v_i \in V$ with $s(z) = v_i \bmod U_z^{(e)}$ for any $z \in \Omega_i$ and $i \in I$.

The stalks of the sheaf $\underset{\approx}{V}$ are the expected ones as we will see in a moment. The star of a facet F' in X is the subset of X defined by

$$St(F') := \text{ union of all facets } F \subseteq X \text{ such that } F' \subseteq \overline{F} \quad .$$

These stars form a locally finite open covering of X.

Lemma IV.1.1:

i. $(\underset{\approx}{V})_z = V_{U_z^{(e)}}$ *for any* $z \in X$*;*
ii. the restriction of $\underset{\approx}{V}$ *to any facet* F *of* X *is the constant sheaf with value* $V_{U_F^{(e)}}$*.*

Proof: There is the obvious map

$$\begin{aligned} (\underset{\approx}{V})_z &\longrightarrow V_{U_z^{(e)}} \\ \text{germ of } s &\longmapsto s(z) \quad . \end{aligned}$$

It is an isomorphism since if z lies in the facet F then $St(F)$ is an open neighbourhood of z with the property that $U_z^{(e)} \subseteq U_{z'}^{(e)}$ for any $z' \in St(F)$ (by I.2.11.i). The same argument shows more generally that, for any nonempty subset Σ open in F, we have

$$\varinjlim_{\substack{\Omega \subseteq St(F) \\ open, \\ \Omega \cap F = \Sigma}} \underset{\approx}{V}(\Omega) = \text{ locally constant } V_{U_F^{(e)}}\text{-valued functions on } \Sigma \quad . \qquad \square$$

Lemma IV.1.2:

Let F *be any facet in* X*; then*

$$H^*(St(F), \underset{\approx}{V}|St(F)) = H^*(F, \underset{\approx}{V}|F) = \begin{cases} V_{U_F^{(e)}} & \textit{if } * = 0 \quad , \\ 0 & \textit{if } * > 0 \quad . \end{cases}$$

Proof: This is (a polysimplicial version of) [KS] 8.1.4. $\square$

It follows from Lemma 1 that the functor

$$\begin{aligned} \mathrm{Alg}(G) &\longrightarrow \text{sheaves on } X \\ V &\longmapsto \underset{\approx}{V} \end{aligned}$$

is exact. Our aim in this Chapter is to compute the cohomology with compact support $H_c^*(X, \underset{\approx}{V})$ of the sheaf $\underset{\approx}{V}$. The interest in this comes from the fact that this cohomology can be calculated from the cochain complex of $\gamma_e(V)$ considered in III.1.

Proposition IV.1.3:

$H_c^*(X, \underset{\approx}{V}) = h^*(C_c^{or}(X_{(.)}, \gamma_e(V)), d)$.

Proof: The filtration $X = \Omega^0 \supseteq \Omega^1 \supseteq \ldots \supseteq \Omega^d$ of X by the open subsets $\Omega^n := X \backslash X^{n-1}$ induces the filtration

$$\Gamma_c(X, .) \supseteq \Gamma_c(\Omega^1, .) \supseteq \ldots \supseteq \Gamma_c(\Omega^d, .) \quad .$$

Because of [God] II.4.10.1 the spectral sequence of this filtration reads

$$E_1^{n,m} := \underset{F \in X_n}{\oplus} H_c^{n+m}(F, \underset{\approx}{V}|F) \Longrightarrow H_c^{n+m}(X, \underset{\approx}{V}) \quad .$$

According to Lemma 1.ii we have

$$H_c^*(F, \underset{\approx}{V}|F) = \begin{cases} H_c^n(F, \mathbb{Z}) \otimes V_{U_F^{(e)}} & \text{if } F \in X_n \text{ and } * = n \quad , \\ 0 & \text{otherwise} \quad . \end{cases}$$

Inserting this into the spectral sequence we obtain

$$H_c^*(X, \underset{\approx}{V}) = h^* \left[\underset{F \in X_0}{\oplus} H_c^0(F, \mathbb{Z}) \otimes V_{U_F^{(e)}} \xrightarrow{d_1^{0,0}} \ldots \xrightarrow{d_1^{d-1,0}} \underset{F \in X_d}{\oplus} H_c^d(F, \mathbb{Z}) \otimes V_{U_F^{(e)}} \right] \quad .$$

The description of the cellular coboundary in [Dol] V.6 and VI.7.11 shows that the complex on the right hand side coincides with $(C_c^{or}(X_{(\cdot)}, \gamma_e(V)), d)$. □

Corollary IV.1.4:

Let V be a representation of finite length in $\mathrm{Alg}_\chi(G)$; if e is chosen big enough then we have

$$\mathcal{E}^*(V) = H_c^*(X, \underset{\approx}{\tilde{V}}) \quad .$$

Proof: Proposition 3 and III.1.3. □

The sheaf $\underset{\approx}{V}$ at hand we can reformulate III.4.16 as a trace formula.

Proposition IV.1.5: (Hopf-Lefschetz trace formula)

We assume that the connected center C of G is anisotropic and that either G is split or K has characteristic 0*. Let V be a representation of finite length in* $\mathrm{Alg}(G)$ *and choose e big enough. For any $h \in G^{ell}$ we have*

$$\theta_V(h) = \sum_{q=0}^{d} (-1)^q \cdot trace(h; H^q(X^h, \underset{\approx}{V})) \quad .$$

Proof: First of all note that since X^h is compact and V is admissible the cohomology $H^*(X^h, \underset{\approx}{V})$ has finite dimension. By III.4.16 (as explained in the paragraph after that Theorem) we have

$$\theta_V(h) = (f_{EP}^V)^\vee(h^{-1}) \quad .$$

Moreover III.4.10 says that

$$(f_{EP}^V)^\vee(h^{-1}) = \sum_{q=0}^{d} \sum_{F(h) \in (X^h)_q} (-1)^q \cdot trace(h; V^{U_F^{(e)}}) \quad .$$

Proposition 3 of course is completely formal and applies to the finite polysimplicial complex X^h as well. The right hand side in the last identity therefore is equal to

$$\sum_{q=0}^{d} (-1)^q \cdot trace(h; C^{or}((X^h)_{(q)}, \gamma_e(V))) = \sum_{q=0}^{d} (-1)^q \cdot trace(h; H^q(X^h, \underset{\approx}{V})) \quad . \square$$

IV.2. Extension to the boundary

In [BS] Borel and Serre have constructed a compactification $\overline{X}$ of the Bruhat-Tits building X with the help of which they could determine the cohomology with compact support of a constant sheaf on X. Our strategy for computing the cohomology with compact support of our sheaves $\underset{\approx}{V}$ will be similar. In this section we will define an appropriate "smooth" extension $j_{*,\infty}\underset{\approx}{V}$ of $\underset{\approx}{V}$ to a sheaf on $\overline{X}$. The boundary cohomology of that extension will be discussed in the next section. Finally in the section after the next one it will be shown, for V of finite length at least, that $j_{*,\infty}\underset{\approx}{V}$ is cohomologically trivial on $\overline{X}$. The result about the cohomology with compact support of $\underset{\approx}{V}$ then will be obtained from the long exact cohomology sequence.

We first give a description of the Borel-Serre compactification $\overline{X}$ which is adapted to our purposes. Let $\overline{A}$ denote the compactification of the basic apartment A by "the directions of half-lines" ([BS] 5.1). As an explicit model one can take

$$\overline{A} := \{x \in A : d(0,x) \leq 1\}$$

together with the embedding

$$\begin{aligned} j : A &\longrightarrow \overline{A} \\ x &\longmapsto \begin{cases} \frac{1-e^{-d(0,x)}}{d(0,x)} \cdot x & \text{if } x \neq 0 \ , \\ 0 & \text{if } x = 0 \ . \end{cases} \end{aligned}$$

A boundary point $x \in A_\infty := \overline{A} \backslash j(A)$ then corresponds to the half-line $[0x) := \{rx : r \geq 0\}$ in A. The N-action on A extends uniquely to a continuous action of N on $\overline{A}$. Note that Z acts trivially on the boundary A_∞. For any boundary point $x \in A_\infty$ we have the parabolic subgroup

$$\begin{aligned} P_x := &\text{ subgroup generated by } Z \text{ and all} \\ &U_\alpha \text{ for } \alpha \in \Phi \text{ such that } \alpha(x) \geq 0 \end{aligned}$$

in G; its unipotent radical is

$$\begin{aligned} U_x := &\text{ subgroup generated by all} \\ &U_\alpha \text{ for } \alpha \in \Phi \text{ such that } \alpha(x) > 0 \ ; \end{aligned}$$

clearly

$$nP_xn^{-1} = P_{nx} \quad \text{for } n \in N$$

holds. Moreover [BoT] 5.17 and 5.20 imply that

$$P_x \cap N = N_x := \{n \in N : nx = x\} \quad \text{for any } x \in A_\infty \quad .$$

These two properties allow to formally imitate the definition of X by setting

$$\overline{X} := G \times \overline{A} / \sim$$

with the equivalence relation $\sim$ on $G \times \overline{A}$ defined by

$$\begin{aligned} (g,x) \sim (h,y) &\text{ if there is a } n \in N \text{ such that} \\ &nx = y \text{ and } g^{-1}hn \in P_x \ . \end{aligned}$$

The group G acts on $\overline{X}$ through left multiplication on the first factor. The map

$$\begin{aligned} \overline{A} &\longrightarrow \overline{X} \\ x &\longmapsto \text{class of } (1,x) \end{aligned}$$

is injective and N-equivariant. There is an obvious G-equivariant map

$$j : X \longrightarrow \overline{X}$$

which is injective. The latter fact follows from the observation that, because of $P_x = U_x \cdot N_x$ for $x \in A$, we could have used the groups P_x instead of U_x in the definition of X. On the other hand the boundary $X_\infty := \overline{X} \backslash X$ is

$$X_\infty = G \times A_\infty / \sim \quad .$$

Hence X_∞ as a G-set coincides with the Tits building of parabolic subgroups in G (compare [CLT] 6.1). We see that at least as a G-set $\overline{X}$ is the Borel-Serre compactification of X. We equip $\overline{X}$ with the quotient topology of the product topology on $\overline{A}$ given by the natural topology on $\overline{A}$ and the π-adic topology on G.

Lemma IV.2.1:

$\overline{X}$ is the Borel-Serre compactification of X.

Proof: Without loss of generality we may assume that the origin 0 in A is a special vertex. Also fix a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots; this corresponds to fixing a fundamental Weyl chamber

$$D := \{x \in A : \alpha(x) \geq 0 \text{ for any } \alpha \in \Phi^+\}$$

in A. Let $\overline{D}$ denote the closure of D in $\overline{A}$. Then the obvious map

$$P_0 \times \overline{D} \longrightarrow \overline{X}$$

is surjective. The Borel-Serre topology on $\overline{X}$ is the quotient topology with respect to this map if the left hand term is equipped with the product topology of the π-adic topology on P_0 and the natural topology on $\overline{D}$ (see [BS] 5.4.1). Therefore the Borel-Serre topology is finer than our topology. But it is also shown in loc. cit. that the former one induces on $\overline{D}$ its natural topology and that the G-action on $\overline{X}$ is continuous in the Borel-Serre topology. This implies that the two topologies under consideration actually coincide. $\square$

We have ([BS] 5.4):

— $\overline{X}$ is compact and contractible;

— X is open in $\overline{X}$ and the topology induced by $\overline{X}$ on X is the metric topology of X;

— X_∞ with the topology induced by $\overline{X}$ is the π-adic Tits building of G ([BS] §1);

— the topology induced by $\overline{X}$ on $\overline{A}$ is the natural topology of $\overline{A}$;

— the G-action on $\overline{X}$ is continuous.

In the following we keep the assumptions and notations introduced in the proof of Lemma 1. One advantage of viewing $\overline{X}$ as the quotient of $P_0 \times \overline{D}$ is that since $\overline{D}$ is a fundamental domain for P_0 in $\overline{X}$ ([BS] 4.9 (iii)) the equivalence relation $\sim$ for (g, x) and $(h, y) \in P_0 \times \overline{D}$ simplifies to

$$(g, x) \sim (h, y) \text{ if and only if } x = y \text{ and } g^{-1}h \in P_0 \cap P_x \quad .$$

For later purposes it is necessary to explicitly construct appropriate neighbourhoods in $\overline{X}$ of any point in the boundary X_∞. Since $D_\infty := \overline{D} \backslash D$ is a fundamental domain for P_0 in X_∞ it suffices to consider a point $x \in D_\infty$ which is fixed throughout the following. The set

$$St_D(x) := \{x' \in D_\infty : P_{x'} \subseteq P_x\}$$

is an open neighbourhood of x in D_∞. Put

$$\Phi(x) := \{\alpha \in \Phi : \alpha(x) > 0\} \subseteq \Phi^+ \quad .$$

We also fix an open normal subgroup U in P_0 and a real number $r \geq 0$ such that

$$U \cap U_\alpha \supseteq U_{\alpha,r} \text{ for any } \alpha \in \Phi(x) \quad .$$

Lemma IV.2.2:

Let $\Omega \subseteq St_D(x) \cup \{y \in D : \alpha(y) > r$ for any $\alpha \in \Phi(x)\}$ be any subset; then the subset $U(P_0 \cap P_x) \times \Omega$ is $\sim$-saturated in $P_0 \times \overline{D}$.

Proof: Consider a point $y \in \Omega$ and elements $g \in P_0$ and $h \in U(P_0 \cap P_x)$ such that $g^{-1}h \in P_0 \cap P_y$. We have to prove that then necessarily $g \in U(P_0 \cap P_x)$. Since by assumption

$$g \in h(P_0 \cap P_y) \subseteq U(P_0 \cap P_x)(P_0 \cap P_y)$$

it suffices to show that

$$P_0 \cap P_y \subseteq U(P_0 \cap P_x)$$

holds true. In case $y \in St_D(x)$ we even have $P_y \subseteq P_x$. We therefore may assume that $y \in D$ with $\alpha(y) > r$ for any $\alpha \in \Phi(x)$. According to I.1.2 and I.1.4 we have

$$P_0 \cap P_y = P_{[0y]} = \prod_{\alpha \in \Phi^{red}} U_{\alpha, f_{[0y]}(\alpha)} \cdot N_{[0y]}$$

for an appropriate ordering of the factors on the right hand side. Since N acts on A by affine automorphisms $N_{[0y]}$ is contained in $N_{[0x']}$ where $x' \in D_\infty$ is such that y lies on the half-line $[0x')$. But $x' \in St_D(x)$; this follows from $\alpha(x') > 0$ for any $\alpha \in \Phi(x)$ which amounts to $U_x \subseteq U_{x'}$. We obtain

$$N_{[0y]} \subseteq N_{[0x')} \subseteq P_0 \cap P_{x'} \subseteq P_0 \cap P_x \quad .$$

For a root $\alpha \in \Phi^{red}$ we distinguish two cases: $\alpha \in \Phi \backslash (-\Phi(x))$ or $\alpha \in -\Phi(x)$. In the first case we have $\alpha(x) \geq 0$ which means $U_\alpha \subseteq P_x$ and hence

$$U_{\alpha, f_{[0y]}(\alpha)} \subseteq U_{\alpha,0} \subseteq P_0 \cap P_x \quad .$$

In the second case we have $-\alpha(y) > r$ and hence

$$U_{\alpha, f_{[0y]}(\alpha)} \subseteq U_{\alpha,-\alpha(y)} \subseteq U_{\alpha,r} \subseteq U \quad .$$ □

We now define, for any subset $\Omega_\infty \subseteq St_D(x)$, the subset

$$C_r(\Omega_\infty) := \Omega_\infty \cup \{y \in D : y \in [0x') \text{ for some } x' \in \Omega_\infty \text{ and } \alpha(y) > r \text{ for any } \alpha \in \Phi(x)\}$$

in $\overline{D}$. For any $x' \in \Omega_\infty$ there is a unique point $y' \in [0x')$ such that

$$[0x') \cap C_r(\Omega_\infty) = [0x') \backslash [0y'] \quad .$$

Lemma IV.2.3:

Let $\Omega_\infty \subseteq St_D(x)$ be an open neighbourhood of x in D_∞; then $U(P_0 \cap P_x) \cdot C_r(\Omega_\infty)$ is an open neighbourhood of x in $\overline{X}$.

Proof: It is easy to see that $C_r(\Omega_\infty)$ is open in $\overline{D}$. Hence $U(P_0 \cap P_x) \times C_r(\Omega_\infty)$ is open and $\sim$-saturated in $P_0 \times \overline{D}$. □

These neighbourhoods have the disadvantage not to reflect the cellular structure of X. We therefore define

$$C'_r(\Omega_\infty) := \Omega_\infty \cup \bigcup \{St(y) \cap D : y \in C_r(\Omega_\infty) \cap D \text{ a vertex such that } St(y) \cap D \subseteq C_r(\Omega_\infty)\} \quad .$$

Lemma IV.2.4:

Let $\Omega_\infty \subseteq St_D(x)$ be an open neighbourhood of x in D_∞ and put $\Omega := U(P_0 \cap P_x) \cdot C'_r(\Omega_\infty)$; we then have:

i. $C'_r(\Omega_\infty)$ *is an open neighbourhood of x in $\overline{D}$;*

ii. Ω *is an open neighbourhood of x in $\overline{X}$;*

iii. $\Omega \cap X = \bigcup_{\substack{y \in \Omega \cap X \\ \text{vertex}}} St(y)$.

Proof: The set $U(P_0 \cap P_x) \times C'_r(\Omega_\infty)$ is open in $P_0 \times \overline{D}$ if we assume i. and is $\sim$-saturated by Lemma 2. Hence ii. is a consequence of i. Moreover iii. follows from ii.: By construction $\Omega \cap X$ is a union of facets. But any open subset of X which is a union of facets contains with any facet F the whole star $St(F)$. This in particular shows that the right hand side in iii. lies in $\Omega \cap X$. To see the reverse inclusion first note that the right hand side is invariant under $U(P_0 \cap P_x)$. It therefore suffices to consider a point $z \in C'_r(\Omega_\infty) \cap D$. Then by definition there is a vertex $y \in C'_r(\Omega_\infty) \cap D \subseteq \Omega \cap X$ such that $z \in St(y)$.
The crucial assertion to establish is i. Since $St(y) \cap D$ for any vertex $y \in D$ is open in $\overline{D}$ it remains to ensure that any point $x' \in \Omega_\infty$ has an open neighbourhood in $\overline{D}$ which is contained in $C'_r(\Omega_\infty)$. For this it is convenient to use certain standard neighbourhoods of x' in D_∞. Thinking of A_∞ as being the unit sphere in A (as we do in our explicit model) we have, for any $0 < \varepsilon < 1$, the open neighbourhood

$$\Omega_\varepsilon := \{x'' \in D_\infty : d(x', x'') < \varepsilon\}$$

of x' in D_∞. We may choose ε small enough so that $\Omega_\varepsilon \subseteq \Omega_\infty$. Then $C_r(\Omega_\varepsilon)$ is an open neighbourhood of x' in $\overline{D}$ which lies in $C_r(\Omega_\infty)$. Let now $c > 0$ be a fixed real constant. It is an elementary computation to show that by decreasing ε and increasing r appropriately we obtain a $C_{r'}(\Omega_{\varepsilon'}) \subseteq C_r(\Omega_\varepsilon)$ with the property that

$$\{z' \in D : d(z, z') < c\} \subseteq C_r(\Omega_\varepsilon) \text{ for any } z \in C_{r'}(\Omega_{\varepsilon'}) \cap D \quad .$$

We choose the constant c in such a way that $d(z, z') < c$ whenever $z \in D$ and $z' \in \bigcup\{St(y) \cap D$: y any vertex of the facet containing $z\}$; this is possible by I.2.10. It follows easily that then $C_{r'}(\Omega_{\varepsilon'}) \subseteq C'_r(\Omega_\infty)$. □

Lemma IV.2.5:

Let $\Omega \subseteq \overline{X}$ be an open neighbourhood of x; then we can choose U and r in such a way that $U(P_0 \cap P_x) \cdot C'_r(\Omega_\infty) \subseteq \Omega$ for some open neighbourhood $\Omega_\infty \subseteq St_D(x)$ of x in D_∞.

Proof: Consider the quotient map $\mu : P_0 \times \overline{D} \to \overline{X}$. The subset $\mu^{-1}(\Omega)$ is open in $P_0 \times \overline{D}$ and contains $(P_0 \cap P_x) \times \{x\}$. We therefore find, for any $h \in P_0 \cap P_x$, an open normal subgroup $U(h) \subseteq P_0$ and an open neighbourhood $\Omega_0(h)$ of x in $\overline{D}$ such that

$$U(h)h \times \Omega_0(h) \subseteq \mu^{-1}(\Omega) \quad .$$

By the compactness of $(P_0 \cap P_x)/C$ we have

$$U(h_1)h_1C \cup \ldots \cup U(h_m)h_mC \supseteq P_0 \cap P_x$$

for finitely many appropriate elements $h_1, \ldots, h_m \in P_0 \cap P_x$. Now put

$$\Omega_0 := \Omega_0(h_1) \cap \ldots \cap \Omega_0(h_m) \ \text{ and } \ U := U(h_1) \cap \ldots \cap U(h_m) \quad .$$

We then obtain

$$U(P_0 \cap P_x) \times \Omega_0 \subseteq \mu^{-1}(\Omega) \ \text{ and hence } U(P_0 \cap P_x) \cdot \Omega_0 \subseteq \Omega \quad .$$

It remains to observe the elementary geometric fact that for any open neighbourhood Ω_0 of x in $\overline{D}$ we find an open neighbourhood $\Omega_\infty \subseteq St_D(x)$ of x in D_∞ and a $r \geq 0$ such that $C_r(\Omega_\infty) \subseteq \Omega_0$. □

Lemma IV.2.6:

Any boundary point in X_∞ has a fundamental system of open neighbourhoods Ω in $\overline{X}$ such that

$$\Omega \cap X = \bigcup_{\substack{y \in \Omega \cap X \\ \text{vertex}}} St(y) \quad .$$

Proof: Lemmata 4 and 5. □

Lemma IV.2.7:

Let $\mathfrak{c} \subseteq U_x$ be a compact subset; then there is an open neighbourhood Ω of x in $\overline{A}$ such that

$$\mathfrak{c} \subseteq U_F^{(e)} \quad \textit{for any facet } F \subseteq \Omega \cap X \quad .$$

Proof: (Recall that $e \geq 0$ is fixed throughout this Chapter.) Fixing an enumeration of the roots in $\Phi^{red} \cap \Phi(x)$ any element $g \in \mathfrak{c}$ can be written in a unique way as

$$g = \prod_{\alpha \in \Phi^{red} \cap \Phi(x)} g_\alpha \quad \text{where } g_\alpha \in U_\alpha \quad .$$

The compactness of $\mathfrak{c}$ implies that for any such root α the set

$$\mathfrak{c}_\alpha := \{\ell(g_\alpha) : g \in \mathfrak{c} \ \text{ such that } g_\alpha \neq 1\}$$

is bounded below; we put $\ell_\alpha := \min\{\ell : \ell \in \mathfrak{c}_\alpha\}$. Define now

$$\Omega_0 := \{y \in A : \alpha(y) > e + 1 - \ell_\alpha \ \text{ for any } \alpha \in \Phi^{red} \cap \Phi(x)\} \quad .$$

Clearly there is an open neighbourhood Ω of x in $\overline{A}$ such that $\Omega \cap A = \Omega_0$. It therefore suffices to show that

$$\mathfrak{c} \subseteq U_F^{(e)} \ \text{ for any facet } F \subseteq \Omega_0 \quad .$$

Fix a root $\alpha \in \Phi^{red} \cap \Phi(x)$, an element $g \in \mathfrak{c}$, and a facet $F \subseteq \Omega_0$. We actually check that

$$g_\alpha \in U_{\alpha, f_F^*(\alpha)+e} \subseteq U_F^{(e)} \cap U_\alpha$$

holds true. The case $g_\alpha = 1$ is trivial. Otherwise we have

$$f_F(\alpha) \leq -\inf_{y \in \Omega_0} \alpha(y) \leq -(e + 1 - \ell_\alpha) \leq -e - 1 + \ell(g_\alpha)$$

and hence

$$f_F^*(\alpha) + e < \ell(g_\alpha) \quad . \qquad \square$$

The sheaf $\underset{\approx}{V}$ has the two obvious extensions $j_!\underset{\approx}{V} \subseteq j_*\underset{\approx}{V}$ to sheaves on $\overline{X}$. We will work with a third "intermediate" or "smooth" extension

$$j_!\underset{\approx}{V} \xrightarrow{\subseteq} j_{*,\infty}\underset{\approx}{V} \longrightarrow j_*\underset{\approx}{V}$$

which is constructed as follows. Let $i : X_\infty \to \overline{X}$ denote the inclusion of the boundary. The stabilizer P_z of any boundary point $z \in X_\infty$ is a parabolic subgroup of G; let U_z denote the unipotent radical of P_z. For any $z \in X_\infty$ we then may form the Jacquet module V_{U_z} of U_z-coinvariants of V; similarly as before we write $v \bmod U_z$ for the image in V_{U_z} of a vector $v \in V$. Analogously to $\underset{\approx}{V}$ we can define a sheaf $\underline{\underline{V}}$ on X_∞ in the following way: For any open $\Omega \subseteq X_\infty$ put

$$\underline{\underline{V}}(\Omega) := \mathbb{C}\text{-vector space of all maps } s : \Omega \to \dot{\bigcup_{z \in \Omega}} V_{U_z}$$

such that

— $s(z) \in V_{U_z}$ for any $z \in \Omega$,

— there is an open covering $\Omega = \bigcup_{i \in I} \Omega_i$ and vectors $v_i \in V$ with

$s(z) = v_i \bmod U_z$ for any $z \in \Omega_i$ and $i \in I$.

It was mentioned already that X_∞ is a simplicial complex but equipped with a topology which is coarser than the simplicial one. For any point $z \in X_\infty$ we put

$$St(z) := \{z' \in X_\infty : P_{z'} \subseteq P_z\} \quad .$$

Remark IV.2.8:

Let $z \in X_\infty$ be a boundary point; for any open subgroup $U \subseteq G$ the subset $U \cdot St(z)$ is an open neighbourhood of z in X_∞.

Proof: Because of $St(z') \subseteq St(z)$ for any $z' \in St(z)$ it suffices to prove that there is a subset $\Omega \subseteq St(z)$ containing z such that $U \cdot \Omega$ is open in X_∞. We may assume that $z \in D_\infty$. Choose an open neighbourhood $\Omega_\infty \subseteq St_D(z) = St(z) \cap D_\infty$ of z in D_∞. According to Lemma 3 the subset $U(P_0 \cap P_z) \cdot \Omega_\infty$ is open in X_∞. Therefore $\Omega := (P_0 \cap P_z) \cdot \Omega_\infty$ meets the requirement. $\square$

Lemma IV.2.9:

$\underline{\underline{V}}_z = V_{U_z}$ *for any* $z \in X_\infty$.

Proof: There is the obvious map

$$\begin{aligned} \underline{\underline{V}}_z &\longrightarrow V_{U_z} \\ \text{germ of } s &\longmapsto s(z) \end{aligned}$$

which clearly is surjective. In order to see the injectivity let s be a section of $\underline{\underline{V}}$ in a neighbourhood $\Omega \subseteq X_\infty$ of z such that $s(z) = 0$. By shrinking the neighbourhood we may assume that s is represented by a single vector $v \in V$, i.e., $s(z') = v \bmod U_{z'}$ for any $z' \in \Omega$. For $z' \in St(z)$ we have $U_z \subseteq U_{z'}$ and hence $v \bmod U_{z'} = 0$. Let U be the stabilizer of v in G. It easily follows that actually $v \bmod U_{z'} = 0$ for any $z' \in U \cdot St(z)$. We obtain $s|U \cdot St(z) \cap \Omega = 0$. $\square$

Since the formation of Jacquet modules is exact ([Car] p. 128) Lemma 9 implies that the functor

$$\begin{aligned} \mathrm{Alg}(G) &\longrightarrow \text{sheaves on } X_\infty \\ V &\longmapsto \underline{\underline{V}} \end{aligned}$$

is exact. By construction the sheaf $\underset{\approx}{V}$, resp. $\underline{\underline{V}}$, is a quotient

$$V \twoheadrightarrow \underset{\approx}{V} \quad , \text{ resp. } \; V \twoheadrightarrow \underline{\underline{V}} \quad ,$$

of the constant sheaf with value V on X, resp. X_∞. The first arrow induces by adjunction and restriction a not necessarily surjective homomorphism

$$V \longrightarrow i^* j_* \underset{\approx}{V} \quad .$$

Lemma IV.2.10:
We have the commutative triangle

$$\begin{array}{ccc} & V & \\ \swarrow & & \searrow \\ \underline{\underline{V}} & \longrightarrow & i^* j_* \underset{\approx}{V} \end{array}$$

where the upper term is the constant sheaf with value V on X_∞ and the oblique arrows are the natural sheaf homomorphisms.

Proof: We have to show that, for any point $z \in X_\infty$, the natural map

$$V \longrightarrow (j_* \underset{\approx}{V})_z$$

contains in its kernel all vectors of the form $gv - v$ for some $g \in U_z$ and some $v \in V$. By the G-equivariance of this assertion we may assume that $z \in D_\infty$. Choose an open subgroup $U \subseteq G$ such that $v, gv \in V^U$. By Lemma 7 and the fact that $(P_0 \cap P_z)/C$ is compact we find an open neighbourhood Ω_0 of z in $\overline{A}$ such that

$$\{h^{-1}gh : h \in P_0 \cap P_z\} \subseteq U_{y_0}^{(e)} \text{ for any vertex } y_0 \in \Omega_0 \cap X \quad .$$

Consider now a vertex $y \in U(P_0 \cap P_z) \cdot \Omega_0 \cap X$, say,

$$y = uhy_0 \text{ with } u \in U \ , \ h \in P_0 \cap P_z \ , \text{ and } y_0 \in \Omega_0 \cap X \quad .$$

We then have

$$U_y^{(e)} = uhU_{y_0}^{(e)}h^{-1}u^{-1} \text{ and } g' := h^{-1}gh \in U_{y_0}^{(e)}$$

and hence

$$gv - v = u(gu^{-1}v - u^{-1}v) = uhg'h^{-1}u^{-1}v - v = 0 \operatorname{mod} U_y^{(e)} \quad .$$

It is quite clear that Ω_0 contains a subset of the form $C'_r(\Omega_\infty)$ as considered above. Using Lemma 4 we therefore see that z has an open neighbourhood Ω in $\overline{X}$ such that

$$\Omega \cap X = \bigcup_{\substack{y \in \Omega \cap X \\ \text{vertex}}} St(y) \text{ and}$$

$$gv - v = 0 \operatorname{mod} U_y^{(e)} \text{ for any vertex } y \in \Omega \cap X \quad .$$

If $y' \in \Omega \cap X$ is an arbitrary point, say, $y' \in St(y)$ for some vertex $y \in \Omega \cap X$ then $U_{y'}^{(e)} \supseteq U_y^{(e)}$ by I.2.11. We obtain that

$$gv - v = 0 \operatorname{mod} U_{y'}^{(e)} \text{ for any } y' \in \Omega \cap X \quad .$$

This means that the image of $gv - v$ in $(j_* \underset{\approx}{V})(\Omega)$ and a fortiori its image in $(j_* \underset{\approx}{V})_z$ is zero. $\square$

We now define $j_{*,\infty}\underset{\approx}{V}$ to be that sheaf on $\overline{X}$ which makes the diagram

$$\begin{array}{ccc} j_{*,\infty}\underset{\approx}{V} & \longrightarrow & j_*\underset{\approx}{V} \\ \downarrow & & \downarrow \\ i_*\underline{\underline{V}} & \longrightarrow & i_*i^*j_*\underset{\approx}{V} \end{array}$$

Cartesian; the right perpendicular arrow hereby is given by adjunction and the lower horizontal arrow is the direct image of the arrow in Lemma 10. By construction we have

$$j^*j_{*,\infty}\underset{\approx}{V} = \underset{\approx}{V} \quad \text{and} \quad i^*j_{*,\infty}\underset{\approx}{V} = \underline{\underline{V}} \quad .$$

In particular the functor

$$\begin{aligned} \mathrm{Alg}(G) &\longrightarrow \text{sheaves on } \overline{X} \\ V &\longmapsto j_{*,\infty}\underset{\approx}{V} \end{aligned}$$

is exact; of course this functor depends on the choice of the number e whereas the sheaf $\underline{\underline{V}}$ doesn't. We also obtain the short exact sequence of sheaves

$$0 \longrightarrow j_!\underset{\approx}{V} \longrightarrow j_{*,\infty}\underset{\approx}{V} \longrightarrow i_*\underline{\underline{V}} \longrightarrow 0 \quad .$$

Since $H^*(\overline{X}, j_!\underset{\approx}{V}) = H^*_c(X, \underset{\approx}{V})$ ([KS] 2.5.4 (i) and (2.6.6)) the associated long exact cohomology sequence reads

$$\begin{array}{ccccccc} \ldots & \longrightarrow & H^i(\overline{X}, j_{*,\infty}\underset{\approx}{V}) & \longrightarrow & H^i(X_\infty, \underline{\underline{V}}) & \longrightarrow & H^{i+1}_c(X, \underset{\approx}{V}) \\ & \longrightarrow & H^{i+1}(\overline{X}, j_{*,\infty}\underset{\approx}{V}) & \longrightarrow & \ldots & & . \end{array}$$

Later on it is technically important that for the representation $V := C_c(G/U_x^{(e)})$, x a special vertex in A, our "smooth" extension has a simpler description.

Proposition IV.2.11:

For the representation $V = C_c(G/U_x^{(e)})$, x a special vertex $x \in A$, we have

$$j_{*,\infty}\underset{\approx}{V} = \mathrm{image}(V \longrightarrow j_*\underset{\approx}{V}) \quad .$$

Proof: By Lemma 10 we quite generally have natural homomorphisms

$$V \twoheadrightarrow j_{*,\infty}\underset{\approx}{V} \longrightarrow j_*\underset{\approx}{V}$$

the first one of which is surjective. We therefore have to show that in case of our particular V the second one is injective, i.e., that for any boundary point $z \in X_\infty$ the natural map between stalks

$$V_{U_z} \longrightarrow (j_* \underset{\approx}{V})_z$$

is injective. Let us first make this map more explicit. Put

$$\begin{aligned}\mathfrak{U}(z) := \text{system of all open neighbourhoods } \Omega \\ \text{of } z \text{ in } \overline{X} \text{ with the property that} \\ \Omega \cap X = \bigcup_{\substack{y \in \Omega \cap X \\ \text{vertex}}} St(y) \quad .\end{aligned}$$

Because of Lemma 6 we have

$$(j_* \underset{\approx}{V})_z = \varinjlim_{\Omega \in \mathfrak{U}(z)} \underset{\approx}{V}(\Omega \cap X) \quad .$$

The sheaf axiom says that, for any $\Omega \in \mathfrak{U}(z)$, the restriction map

$$\underset{\approx}{V}(\Omega \cap X) \hookrightarrow \prod_{\substack{y \in \Omega \cap X \\ \text{vertex}}} \underset{\approx}{V}(St(y))$$

is injective. According to 1.2 the terms on the right hand side are

$$\underset{\approx}{V}(St(y)) = V_{U_y^{(e)}} \quad .$$

Putting this together we see that what we have to show is the injectivity of the natural map

$$V_{U_z} \longrightarrow \varinjlim_{\Omega \in \mathfrak{U}(z)} \prod_{\substack{y \in \Omega \cap X \\ \text{vertex}}} V_{U_y^{(e)}} \quad .$$

In other words fix a function $\psi \in V = C_c(G/U_x^{(e)})$ and a neighbourhood $\Omega \in \mathfrak{U}(z)$ such that

$$\psi = 0 \bmod U_y^{(e)} \ \text{ for any vertex } y \in \Omega \cap X \quad .$$

We have to check that then necessarily $\psi = 0 \bmod U_z$. We write ψ as a sum

$$\psi = \sum_{h \in U_z \backslash G / U_x^{(e)}} \psi_h$$

of functions $\psi_h \in C_c(G/U_x^{(e)})$ in such a way that ψ_h has support in $U_z h U_x^{(e)}/U_x^{(e)}$ and we will show that each summand fulfills $\psi_h = 0 \bmod U_z$. Consider an individual element $h \in G$. There is an apartment which contains both points x and $h^{-1}z$ in its closure ([Bro] Thm. VI.8). By I.1.6 we find an element $g \in P_x$ such that $x, gh^{-1}z \in \overline{A}$. Using the G-equivariance of our problem and replacing z and ψ by $gh^{-1}z$ and $gh^{-1}\psi$, respectively, we see that it suffices to deal with the case $z \in \overline{A}$ and $h \in P_x$. We choose elements $u_1, \ldots, u_m \in U_z$ such that

$$\mathrm{supp}(\psi_h) = \{u_1 h U_x^{(e)}, \ldots, u_m h U_x^{(e)}\} \quad .$$

Note that $u_i h U_x^{(e)} = u_i U_x^{(e)} h$ since P_x normalizes $U_x^{(e)}$. According to Lemma 7 we find an open neighbourhood Ω' of z in $\overline{A}$ such that $\Omega' \subseteq \Omega \cap \overline{A}$ and

$$u_1, \ldots, u_m \in U_y^{(e)} \quad \text{for any vertex } y \in \Omega' \cap A \quad .$$

In particular we have

$$\psi = 0 \bmod U_y^{(e)} \quad \text{for any vertex } y \in \Omega' \cap A \quad .$$

On the other hand it was shown in I.3.2 that

$$U_{x'}^{(e)} \subseteq U_z \cdot U_x^{(e)} \quad \text{for any } x' \in [xz) \quad .$$

Choosing $x' \in [xz)$ close enough to z in such a way that the closure of the facet F which contains x' still lies in Ω' and choosing a vertex $y_0 \in \overline{F}$ we obtain

1. $u_1, \ldots, u_m \in U_{y_0}^{(e)}$,
2. $\psi = 0 \bmod U_{y_0}^{(e)}$, and
3. $U_{y_0}^{(e)} \subseteq U_F^{(e)} = U_{y'}^{(e)} \subseteq U_z \cdot U_x^{(e)}$.

The properties 1. and 3. imply that

4. $U_{y_0}^{(e)} h U_x^{(e)} \cap \mathrm{supp}(\psi) = U_z h U_x^{(e)} \cap \mathrm{supp}(\psi) = (U_{y_0}^{(e)} \cap U_z) h U_x^{(e)} \cap \mathrm{supp}(\psi)$.

It follows from 2. and 4. that

$$\psi_h = 0 \bmod U_{y_0}^{(e)} \quad \text{and even} \quad \psi_h = 0 \bmod U_{y_0}^{(e)} \cap U_z \quad . \qquad \square$$

Appendix: Geodesics in $\overline{X}$

Implicitly in our thinking about the compactification $\overline{X}$ is the existence of a unique "half-line" in $\overline{X}$ between any given point in X and any given boundary point. In the section after the next one we actually will have to make explicit use of this fact. Also it is the link between [BS] and [Bro] VI.9. Since we could not find an appropriate reference this will be justified in the following.

We fix a point $x \in X$ and a boundary point $z \in X_\infty$. According to [BT] I.7.4.18 (ii) or [Bro] Thm.VI.8 there is an apartment $A' \subseteq X$ such that

$$x \in A' \quad \text{and} \quad z \in (A')_\infty \quad .$$

Let $[xz)_{A'}$ denote the half-line in A' in direction z emanating from x and put

$$[xz]_{A'} := [xz)_{A'} \cup \{z\} \quad .$$

The subsequent result allows to simply write $[xz)$ and to view the latter as the geodesic between x and z in $\overline{X}$.

Proposition:

$[xz)_{A'}$ does not depend on the choice of the apartment A'.

Proof: Let $A'' \subseteq X$ be a second apartment such that $x \in A''$ and $z \in (A'')_\infty$. We have to show that $[xz)_{A'} = [xz)_{A''}$. We first treat in several steps special cases where additional assumptions about A' and A'' are made.
Step 1: Here we assume that x is a special vertex. For notational simplicity we may assume by G-equivariance that $A'' = A$ is our standard apartment and that $z \in D_\infty$. By I.1.6 we find an element $h \in P_x$ such that $A' = hA$. Write $z = hz_0$ with $z_0 \in A_\infty$. Since x is a special vertex we find a $n \in N_x$ such that $nz_0 \in D_\infty$. We then must have $hn^{-1} \in P_z$ because D_∞ is a fundamental domain for P_x in X_∞ ([BS] 4.9 (iii)). Therefore replacing h by hn^{-1} we may assume that

$$A' = hA \quad \text{with} \quad h \in P_x \cap P_z \quad .$$

Again since x is a special vertex we obtain from [BS] 4.10 that h fixes the half-line $[xz)_A$ pointwise. We now conclude that

$$[xz)_{A'} = [hx, hz)_{hA} = h[xz)_A = [xz)_A \quad .$$

Step 2: Here we assume that the intersection $A' \cap A''$ contains a special vertex x_0. From the first step we know that $[x_0z)_{A'} = [x_0z)_{A''}$. Hence $[xz)_{A'}$ and $[xz)_{A''}$ are parallel rays in the sense of [Bro] VI.9A and therefore have to coincide by the Lemma 1 in loc.cit.

Step 3: Here we assume that the intersection $A' \cap A''$ contains a sector D' such that $z \in (D')_\infty$. Choose any $x_0 \in D'$. Then clearly $[x_0 z)_{A'} = [x_0 z)_{A''}$. Hence $[xz)_{A'}$ and $[xz)_{A''}$ again are parallel and therefore equal.
In order to establish the general case we will show that there is an apartment $\tilde{A} \subseteq X$ such that

> $x \in \tilde{A}$, $z \in \tilde{A}_\infty$, $A' \cap \tilde{A}$ contains a special vertex, and $A'' \cap \tilde{A}$ contains a sector $\tilde{D}$ with $z \in \tilde{D}_\infty$.

Using steps 2 and 3 we then obtain

$$[xz)_{A'} = [xz)_{\tilde{A}} = [xz)_{A''} \quad .$$

In order to find $\tilde{A}$ we choose

> a $h \in G$ with $A' = hA''$,
> a facet of maximal dimension $F \subseteq A''$ with $h^{-1}x \in \overline{F}$,
> a special vertex $x_1 \in \overline{F}$ ([BT] I.1.3.7), and
> a sector $D'' \subseteq A''$ such that $z \in (D'')_\infty$.

By [BT] I.7.4.18 (ii) or [Bro] Thm.VI.8 there exists an apartment $\tilde{A} \subseteq X$ which contains hF and an appropriate subsector $\tilde{D} \subseteq D''$. Then $x \in h\overline{F} \subseteq \tilde{A}$, $z \in \tilde{D}_\infty \subseteq \tilde{A}_\infty$, $\tilde{D} \subseteq A'' \cap \tilde{A}$, and $x_0 := hx_1 \in A' \cap \tilde{A}$. □

Corollary:

Any element in $P_x \cap P_z$ fixes $[xz]$ pointwise.

IV.3. Cohomology on the boundary

In this section we explicitly compute the boundary cohomology $H^*(X_\infty, \underline{\underline{V}})$ in the case of an induced representation. Throughout the notations introduced in III.2 will be in order. In particular $\Delta \subseteq \Phi$ is a fixed choice of simple roots. Corresponding to Δ we had defined in IV.2 the subset $D_\infty \subseteq X_\infty$; it is a $(d-1)$-dimensional simplex whose simplicial structure is given by the subsets

$$D_\infty^\Theta := \{x \in D_\infty : U_x = U_\Theta\} \ \text{ for any proper subset } \Theta \subset \Delta \quad .$$

The closure $\overline{D_\infty^\Theta}$ of D_∞^Θ in D_∞ (equivalently in X_∞) is a $(d-1-\#\Theta)$-dimensional simplex. Since D_∞ is a fundamental domain for the G-action on X_∞ ([BS] §1) we have an obvious projection map

$$\tau : X_\infty \twoheadrightarrow D_\infty = G\backslash X_\infty \quad ;$$

it is proper and has totally disconnected fibers. The proper base change theorem ([God] II.4.17.1) therefore implies

$$H^*(X_\infty, \underline{\underline{V}}) = H^*(D_\infty, \tau_* \underline{\underline{V}}) \quad .$$

For any subset $\Theta \subseteq \Delta$ we introduce the space

$$\begin{aligned} Ind^G_{P_\Theta}(V_{U_\Theta}) := \ & \text{space of all locally constant functions} \\ & \varphi : G \to V_{U_\Theta} \text{ such that} \\ & \qquad \varphi(ghu) = h^{-1}(\varphi(g)) \\ & \text{for all } g \in G, h \in M_\Theta, \text{ and } u \in U_\Theta \end{aligned}$$

on which G acts smoothly by left translations. (This is unnormalized induction!)

Lemma IV.3.1:

For any proper subset $\Theta \subset \Delta$ the restriction of $\tau_* \underline{\underline{V}}$ to D^Θ_∞ is the constant sheaf with value $Ind^G_{P_\Theta}(V_{U_\Theta})$.

Proof: The map

$$\begin{aligned} G/P_\Theta \times D^\Theta_\infty &\xrightarrow{\sim} \tau^{-1} D^\Theta_\infty \\ (gP_\Theta, x) &\longmapsto gx \end{aligned}$$

is a homeomorphism and τ corresponds to the second projection map on the left hand side. The inverse image of $\underline{\underline{V}}$ on the left hand space can be computed as follows. For any $x \in D^\Theta_\infty$ and any $g \in G$ we have the obvious map

$$V_{U_\Theta} = V_{U_x} \xrightarrow{g} V_{U_{gx}} \quad .$$

Also let

$$\rho := pr \times \mathrm{id} : G \times D^\Theta_\infty \longrightarrow G/P_\Theta \times D^\Theta_\infty \quad .$$

An argument as in the proof of 2.9 shows that the inverse image in question can be identified with the sheaf on $G/P_\Theta \times D^\Theta_\infty$ whose space of sections in an open subset Ω is the

$$\begin{aligned} & \mathbb{C}\text{-vector space of all locally constant} \\ & \text{maps } \varphi : \rho^{-1}\Omega \longrightarrow V_{U_\Theta} \text{ such that} \\ & \qquad \varphi(gh, x) = h^{-1}\varphi(g, x) \\ & \text{for any } (g, x) \in \rho^{-1}\Omega \text{ and } h \in P_\Theta. \end{aligned}$$

On the level of sections this identification is given by

$$s(gx) = g\varphi(g, x) \quad .$$

It is quite clear that the direct image of this latter sheaf under the projection map to D^Θ_∞ is the constant sheaf with value $Ind^G_{P_\Theta}(V_{U_\Theta})$. Our assertion follows now by an application of the proper base change theorem. □

Specializing to the case of an induced representation we once and for all fix a subset $\Theta_0 \subseteq \Delta$ and an irreducible supercuspidal representation E of M_{Θ_0} and we put $V := Ind(E)$.
For any $\Theta \subseteq \Delta$ we need the subgroup $W_\Theta := \langle s_\alpha : \alpha \in \Theta \rangle$ of W; moreover let $[W/W_{\Theta_0}]$, resp. $[W_\Theta \backslash W/W_{\Theta_0}]$, denote the subset in W of representatives of minimal length for the cosets in W/W_{Θ_0}, resp. the double cosets in $W_\Theta \backslash W/W_{\Theta_0}$. The Weyl group W acts on the set of roots Φ. According to [Cas] 1.3.4 two subsets Θ and Θ' in Δ are associated if and only if $\Theta' = w\Theta$ for some $w \in W$.

Lemma IV.3.2:

For any proper subset $\Theta \subset \Delta$ the following assertions are equivalent:

i. $V_{U_\Theta} \neq 0$;

ii. $w\Theta_0 \subseteq \Theta$ *for some* $w \in [W_\Theta \backslash W/W_{\Theta_0}]$;

iii. Θ *contains a subset which is associated to* Θ_0.

Proof: The equivalence of i. and ii. follows from [Cas] 6.3.5. The third assertion is a trivial consequence of the second one. To see the reverse implication assume that $w\Theta_0 \subseteq \Theta$ for some $w \in W$. We then have $w \in [W_{w\Theta_0} \backslash W/W_{\Theta_0}]$ by [Cas] 1.1.3 and hence $V_{U_{w\Theta_0}} \neq 0$. But $U_\Theta \subseteq U_{w\Theta_0}$ so that $V_{U_\Theta} \neq 0$, too. □

Corollary IV.3.3:

The support of the sheaf $\tau_* \underline{\underline{V}}$ is equal to the $(d-1-\#\Theta_0)$-dimensional simplicial subcomplex

$$D_\infty(\Theta_0) := \bigcup \{D_\infty^\Theta : w\Theta_0 \subseteq \Theta \subset \Delta \ \text{ for some } w \in W\}$$

of D_∞.

Let $\leq$ on W denote the Bruhat order. We now fix an enumeration

$$[W/W_{\Theta_0}] = \{1 = w_0, w_1, w_2 \ldots\}$$

in such a way that

$$m \leq n \ \text{ if } w_m \leq w_n \quad .$$

This allows us to define, for any proper subset $\Theta \subset \Delta$, a decreasing filtration

$$F_\Theta^n V := \{\varphi \in Ind(E) : \varphi | P_\Theta w_m P_{\Theta_0} = 0 \ \text{ for any } m < n\}$$

of V by P_Θ-invariant subspaces. It induces corresponding filtrations

$$(F_\Theta^n V)_{U_\Theta} \ \text{ of } V_{U_\Theta}$$

(forming the Jacquet module is exact!) and

$$F^n Ind^G_{P_\Theta}(V_{U_\Theta}) := Ind^G_{P_\Theta}((F^n_\Theta V)_{U_\Theta}) \text{ of } Ind^G_{P_\Theta}(V_{U_\Theta}) \quad .$$

The latter clearly is G-equivariant. Most importantly we obtain a G-equivariant filtration $F^n\tau_*\underline{\underline{V}}$ of the sheaf $\tau_*\underline{\underline{V}}$ defined by

$$\begin{aligned} F^n\tau_*\underline{\underline{V}} := \text{ subsheaf of all sections } s \text{ such that} \\ s(z) \in F^n Ind^G_{P_\Theta}(V_{U_\Theta}) \\ \text{for any } z \in D^\Theta_\infty \text{ and any } \Theta \subset \Delta. \end{aligned}$$

Our further computation is based on the associated G-equivariant spectral sequence

$$E_1^{n,m} := H^{m+n}(D_\infty, gr^n_F\tau_*\underline{\underline{V}}) \Longrightarrow H^{m+n}(D_\infty, \tau_*\underline{\underline{V}}) = H^{m+n}(X_\infty, \underline{\underline{V}}) \quad .$$

First of all we have

$$(gr^n_F\tau_*\underline{\underline{V}})_z = Ind^G_{P_\Theta}((gr^n_{F_\Theta}V)_{U_\Theta}) \text{ for any } z \in D^\Theta_\infty \text{ and } \Theta \subset \Delta \quad .$$

(The functor $Ind^G_{P_\Theta}(.)$ is exact by [Car] I.1.8.) By construction

$$gr^n_{F_\Theta}V = 0 \text{ if } w_n \notin [W_\Theta\backslash W/W_{\Theta_0}] \quad .$$

Let us fix, for any $w_n \in [W/W_\Theta]$, a lifting $g_n \in N$. If $w_n \in [W_\Theta\backslash W/W_{\Theta_0}]$ then the computation in [Cas] 6.3.1 and 6.3.4 shows that

$$\begin{aligned} (gr^n_{F_\Theta}V)_{U_\Theta} &= 0 \text{ if and only if } w_n\Theta_0 \not\subseteq \Theta \text{ and} \\ &= \delta_\Theta^{-1/2} \otimes \begin{bmatrix} \text{normalized induction of } {}^{g_n^{-1}}E \\ \text{from } g_nP_{\Theta_0}g_n^{-1} \cap M_\Theta \text{ to } M_\Theta \end{bmatrix} \text{ if } w_n\Theta_0 \subseteq \Theta \quad . \end{aligned}$$

Since we have, by [Cas] 1.3.3, the Levi decomposition

$$g_nP_{\Theta_0}g_n^{-1} \cap M_\Theta = M_{w_n\Theta_0} \cdot (g_nU_{\Theta_0}g_n^{-1} \cap M_\Theta)$$

the last formula simplifies to

$$(gr^n_{F_\Theta}V)_{U_\Theta} = \delta_\Theta^{-1/2} \otimes \begin{bmatrix} \text{normalized parabolic induction} \\ \text{of } {}^{g_n^{-1}}E \text{ from } M_{w_n\Theta_0} \text{ to } M_\Theta \end{bmatrix} \text{ if } w_n\Theta_0 \subseteq \Theta \quad .$$

Using the transitivity of parabolic induction ([BZ] 1.9.(c)) we therefore obtain that

$$(gr^n_F\tau_*\underline{\underline{V}})_z = \begin{cases} Ind({}^{g_n^{-1}}E) & \text{if } w_n \in [W_\Theta\backslash W/W_{\Theta_0}] \text{ and } w_n\Theta_0 \subseteq \Theta \quad , \\ 0 & \text{otherwise} \end{cases}$$

for any $z \in D_\infty^\Theta$ and $\Theta \subset \Delta$. Put

$$\begin{aligned} D_\infty(n) &:= \cup\{D_\infty^\Theta : w_n\Theta_0 \subseteq \Theta \subset \Delta \text{ and } w_n \in [W_\Theta\backslash W/W_{\Theta_0}]\} \\ &= \cup\{D_\infty^\Theta : \Theta \subset \Delta \text{ and } w_n\Theta_0 \subseteq \Theta \subseteq w_n\Phi^+\}\ ; \end{aligned}$$

the equality is a consequence of [Cas] 1.1.3. This set is empty if $w_n\Theta_0$ is not properly contained in Δ; otherwise it is an open subset in $\overline{D_\infty^{w_n\Theta_0}}$ which contains $D_\infty^{w_n\Theta_0}$. Altogether this establishes the following fact.

Lemma IV.3.4:

$gr_F^n\tau_*\underline{\underline{V}}$ is the constant sheaf with value $Ind({}^{g_n^{-1}}E)$ on $D_\infty(n)$ extended by zero to all of D_∞.

For the corresponding cohomology groups this has the consequence that

$$\begin{aligned} H^*(D_\infty, gr_F^n\tau_*\underline{\underline{V}}) &= H_c^*(D_\infty(n), \mathbb{Z}) \otimes Ind({}^{g_n^{-1}}E) \\ &= \begin{cases} H^{*-1}(\overline{D_\infty^{w_n\Theta_0}}\backslash D_\infty(n), \mathbb{Z}) \otimes Ind({}^{g_n^{-1}}E) & \text{if } * \geq 2\ , \\ \operatorname{coker}(\mathbb{Z} \longrightarrow H^0(\overline{D_\infty^{w_n\Theta_0}}\backslash D_\infty(n), \mathbb{Z})) \otimes Ind({}^{g_n^{-1}}E) & \text{if } * = 1\ , \\ \ker(\qquad - \text{''} - \qquad) \otimes Ind({}^{g_n^{-1}}E) & \text{if } * = 0 \end{cases} \end{aligned}$$

provided $D_\infty(n) \neq \emptyset$.

Lemma IV.3.5:

Assume that $w_n\Theta_0$ is a proper subset of Δ and that $n \neq 0$, $\#W/W_{\Theta_0}$; then $\overline{D_\infty^{w_n\Theta_0}}\backslash D_\infty(n)$ is contractible.

Proof: $\overline{D_\infty^{w_n\Theta_0}}$ is the geometric realization of the abstract simplex given by the poset (w.r.t. inclusion) of all nonempty subsets of $\Delta\backslash w_n\Theta_0$ (note that by assumption $\Delta\backslash w_n\Theta_0 \neq \emptyset$). $\overline{D_\infty^{w_n\Theta_0}}\backslash D_\infty(n)$ is the geometric realization of the subcomplex given by the subposet of all those subsets which do not contain $\Delta\backslash w_n\Phi^+$. Our assumption that $n \neq 0$, resp. $\neq \#W/W_{\Theta_0}$, implies that $\Delta\backslash w_n\Phi^+ \neq \emptyset$, resp. $\neq \Delta\backslash w_n\Theta_0$. The first implication is a consequence of [Bor] 21.3. In order to see the second implication assume $n \neq \#W/W_{\Theta_0}$ and put $w := w_n$. Let w_Δ, resp. w_{Θ_0}, be the unique maximal (w.r.t. the Bruhat order) element in W, resp. W_{Θ_0}; then $w \neq w_\Delta w_{\Theta_0}$ so that $ww_{\Theta_0} \neq w_\Delta$. Hence there is an $\alpha_0 \in \Delta$ such that

$$s_{\alpha_0}ww_{\Theta_0} > ww_{\Theta_0} \quad \text{and a fortiori } s_{\alpha_0}w > w$$

which means $\alpha_0 \in w\Phi^+$. On the other hand denoting by $\ell(.)$ the length function on W w.r.t. Δ we have, for any $\alpha \in \Theta_0$,

$$\begin{aligned}\ell(s_{w(\alpha)}ww_{\Theta_0}) &= \ell(ws_\alpha w_{\Theta_0}) = \ell(w) + \ell(s_\alpha w_{\Theta_0}) = \ell(w) + \ell(w_{\Theta_0}) - 1\\ &= \ell(ww_{\Theta_0}) - 1 \quad .\end{aligned}$$

It follows that $\alpha_0 \notin w\Theta_0$.
What we have to convince ourselves of therefore is the following. Let C be a nonempty finite set, let $\emptyset \subset C_0 \subset C$ be a nonempty proper subset, and denote by Π the poset (w.r.t. inclusion) of all nonempty subsets of C which do not contain C_0. Then the geometric realization $|\Pi|$ of the abstract simplicial complex given by Π is contractible. But this is clear: Fix an element $c \in C \backslash C_0$. The subset $\{c\}$ corresponds to a vertex in $|\Pi|$. Since for any $C' \in \Pi$ also $C' \cup \{c\} \in \Pi$ it follows that $|\Pi|$ can be contracted onto that vertex. □

Lemma IV.3.6:
i. $D_\infty(0) = \overline{D_\infty^{\Theta_0}}$ *if* $\Theta_0 \subset \Delta$;
ii. $D_\infty(n) = D_\infty^{w_n\Theta_0}$ *if* $\Theta_0 \subset \Delta$ *and* $n = \#W/W_{\Theta_0}$.

Proof: i. Obvious. ii. This follows from the fact that $w_n\Theta_0 = w_n\Phi^+ \cap \Delta$ if w_n is the unique maximal element in $[W/W_{\Theta_0}]$ ([Cas] 1.1.4). □

Theorem IV.3.7:
Assume that $V = Ind(E)$ *for some irreducible supercuspidal representation* E *of* M_{Θ_0}; *let* $g^{-1} \in N$ *be a lifting of the unique maximal element in* $[W/W_{\Theta_0}]$; *we then have*

$$H^*(X_\infty, \underline{\underline{V}}) \cong \begin{cases} V \oplus Ind({}^gE) & \text{if } * = 0 \ , \ \#\Theta_0 = d-1 \quad , \\ V & \text{if } * = 0 \ , \ \#\Theta_0 < d-1 \quad , \\ Ind({}^gE) & \text{if } * = d-1-\#\Theta_0 > 0 \quad , \\ 0 & \text{otherwise} \quad . \end{cases}$$

Proof: In case $\Theta_0 = \Delta$ we have $\underline{\underline{V}} = 0$. In the following we therefore assume that $\Theta_0 \subset \Delta$. According to the previous results the only nonzero E_1-terms in our spectral sequence then are

$$E_1^{n,m} \cong \begin{cases} V & \text{if } n = m = 0 \quad , \\ Ind({}^gE) & \text{if } n = \#W/W_{\Theta_0} \quad , \ n+m = d-1-\#\Theta_0 \quad . \end{cases}$$

Moreover since $\underline{\underline{V}}$ is a quotient of the constant sheaf with value V on X_∞ we have a natural augmentation map $V \to H^0(X_\infty, \underline{\underline{V}})$ which splits the edge homomorphism

$$H^0(X_\infty, \underline{\underline{V}}) = H^0(D_\infty, \tau_*\underline{\underline{V}}) \longrightarrow H^0(D_\infty, gr_F^0\tau_*\underline{\underline{V}}) = E_1^{0,0} = V \quad .$$

Hence the spectral sequence degenerates and the assertion follows. □

IV.4. Cohomology with compact support

Our main aim in this section is to establish the following result. In the proof we will follow the strategy developped for II.3.1; but the tools used there have to be analyzed in more depth.

Theorem IV.4.1:

Let x be a special vertex in A and let $e \geq 0$ be an integer; for any representation V in $\mathrm{Alg}^{U_x^{(e)}}(G)$ we have:

i. the natural map $V \xrightarrow{\cong} H^0(\overline{X}, j_{*,\infty}\underset{\approx}{V})$ is an isomorphism;

ii. $H^*(\overline{X}, j_{*,\infty}\underset{\approx}{V}) = 0$ for $* > 0$.

In the proof we twice will make use of homological resolutions. This is made possible by the following observation.

Lemma IV.4.2:

i. $cd(X) = d$;

ii. $cd(X_\infty) = d - 1$ if $d \geq 1$;

iii. $cd(\overline{X}) = d$.

Proof: (Here $cd(.)$ refers to the cohomology with compact support.) iii. is a consequence of i. and ii. by the additivity of the cohomological dimension. Concerning i. it is a standard fact that a d-dimensional locally finite polysimplicial complex has cohomological dimension $= d$. Finally ii. follows from the existence of a proper map from X_∞ onto a $(d-1)$-simplex whose fibers are compact and totally disconnected ([BS] 3.1). □

The functor $V \mapsto j_{*,\infty}\underset{\approx}{V}$ is exact and commutes with arbitrary direct sums as can be seen most easily from the description of the stalks. Moreover $\overline{X}$ being compact the cohomology functor $H^*(\overline{X}, .)$ commutes with arbitrary direct sums.
In the proof of II.3.1 we had seen that any V as in Theorem 1 has an exact homological resolution in $\mathrm{Alg}^{U_x^{(e)}}(G)$ by representations which are direct sums of the "universal" representation $C_c(T)$ with $T := G/U_x^{(e)}$. Using Lemma 2 and the facts given in the above paragraph we conclude by standard arguments of homological algebra that in order to prove Theorem 1 it suffices to treat the case $V = C_c(T)$.
For the rest of the proof V always denotes the representation $C_c(T)$. We begin

by constructing a very convenient simplicial resolution of the sheaf $\underset{\approx}{V}$ on X. Fix an integer $m \geq 0$ and consider the $(m+1)$-fold product

$$T_m := T^{m+1} := T \times \ldots \times T \quad .$$

As in the proof of II.3.1 we put, for any facet F in X,

$$T_F := U_F^{(e)} \backslash T \quad \text{and} \quad T_m^F := T \underset{T_F}{\times} \ldots \underset{T_F}{\times} T \quad (m+1 \text{ factors}).$$

The latter is a subset in T^{m+1}. For facets $F' \subseteq \overline{F}$ we have $T_m^{F'} \subseteq T_m^F$. Extending functions by zero therefore induces inclusions

$$C_c(T_m^F) \subseteq C_c(T^{m+1})$$

in such a way that

$$C_c(T_m^{F'}) \subseteq C_c(T_m^F) \quad \text{if } F' \subseteq \overline{F} \quad .$$

We recall that $C_c(.)$ stands for the space of complex valued functions with finite support. If the point $z \in X$ is contained in the facet F we write

$$T_z := T_F \quad \text{and} \quad T_m^z := T_m^F \quad .$$

A sheaf $\mathcal{T}_m$ on X can now be defined in the following way: For any open subset $\Omega \subseteq X$ put

$$\begin{aligned} \mathcal{T}_m(\Omega) := \ & \mathbb{C}\text{-vector space of all maps } s : \Omega \to \dot{\bigcup_{z \in \Omega}} C_c(T_m^z) \\ & \text{such that} \\ & \text{— } s(z) \in C_c(T_m^z) \text{ for any } z \in \Omega \quad , \\ & \text{— there is an open covering } \Omega = \bigcup_{i \in I} \Omega_i \text{ and} \\ & \quad \text{functions } \psi_i \in C_c(T^{m+1}) \text{ with} \\ & \quad\ s(z) = \psi_i \text{ for any } z \in \Omega_i \text{ and } i \in I \quad . \end{aligned}$$

Lemma IV.4.3:

i. $(\mathcal{T}_m)_z = C_c(T_m^z)$ *for any* $z \in X$;

ii. the restriction of $\mathcal{T}_m$ *to any facet* F *of* X *is the constant sheaf with value* $C_c(T_m^F)$;

iii. for any facet F *in* X *we have*

$$H^*(St(F), \mathcal{T}_m | St(F)) = H^*(F, \mathcal{T}_m | F = \begin{cases} C_c(T_m^F) & \text{if } * = 0 \quad , \\ 0 & \text{if } * > 0 \quad . \end{cases}$$

Proof: Entirely analogous to 1.1 and 1.2. □

In order to distinguish various constant sheaves in the following it is convenient to follow the convention that $M_{/Y}$, for any abelian group M and any topological space Y, denotes the constant sheaf with value M on Y. Obviously

$$T. : \ldots \overrightarrow{\overrightarrow{\overrightarrow{\rightarrow}}} T \times T \times T \overrightarrow{\overrightarrow{\rightarrow}} T \times T \rightrightarrows T$$

as well as

$$T.^F : \ldots \overrightarrow{\overrightarrow{\overrightarrow{\rightarrow}}} T \underset{T_F}{\times} T \underset{T_F}{\times} T \overrightarrow{\overrightarrow{\rightarrow}} T \underset{T_F}{\times} T \rightrightarrows T$$

for any facet F in a natural way are simplicial sets. The pushforward of functions with finite support with respect to these face maps commutes with extension by zero. In this way we obtain simplicial sheaves

$$C_c(T.)_{/X} : \ldots \overrightarrow{\overrightarrow{\overrightarrow{\rightarrow}}} C_c(T_2)_{/X} \overrightarrow{\overrightarrow{\rightarrow}} C_c(T_1)_{/X} \rightrightarrows C_c(T)_{/X}$$

and

$$\mathcal{T}. : \ldots \overrightarrow{\overrightarrow{\overrightarrow{\rightarrow}}} \mathcal{T}_2 \overrightarrow{\overrightarrow{\rightarrow}} \mathcal{T}_1 \rightrightarrows \mathcal{T}_0$$

on X together with an inclusion

$$\mathcal{T}. \subseteq C_c(T.)_{/X}$$

which in degree 0 is an equality $\mathcal{T}_0 = C_c(T)_{/X}$. The obvious surjection $C_c(T)_{/X} \twoheadrightarrow \underset{\approx}{V}$ defines an augmentation

$$\mathcal{T}. \longrightarrow \underset{\approx}{V} \quad .$$

Applying j_* we obtain the augmented simplicial sheaf

$$j_*\mathcal{T}. \longrightarrow j_*\underset{\approx}{V}$$

on $\overline{X}$.

Lemma IV.4.4:

For any abelian group M we have

$$j_*(M_{/X}) = M_{/\overline{X}} \quad .$$

Proof: We will establish a slightly stronger fact. Fix a boundary point $z \in X_\infty$. We will show that z has a fundamental system of open neighbourhoods Ω in $\overline{X}$ such that both Ω and $\Omega \cap X$ are path-connected. The tool to construct such neighbourhoods is the notion of the angle between two intersecting geodesics in

X ([Bro] VI.7 Ex. 1). We first need some notation. Let $x \in X$ be any point. For any other point $y \in \overline{X}$ different from x we put

$$[xy) := \begin{cases} [xy]\backslash\{y\} & \text{if } y \in X \quad , \\ \text{half-line emanating from } x \text{ in direction } y & \text{if } y \in X_\infty \end{cases}$$

([Bro] VI.9A). In either case $[xy] := [xy) \cup \{y\}$ then is a path from x to y in $\overline{X}$. We also put

$$(xy] := [xy]\backslash\{x\} \quad \text{and} \quad (xy) := [xy]\backslash\{x, y\} \quad .$$

There is a unique facet $F(x;y)$ in X such that

$$x \in \overline{F(x;y)} \quad \text{and} \quad (xy) \cap F(x;y) \neq \emptyset \quad ;$$

clearly the latter intersection is of the form

$$(xy) \cap F(x;y) = (xy_x) \quad \text{for some } y_x \in (xy] \quad .$$

Given now two points $y, y' \in \overline{X}$ different from x then the two geodesics $[xy_x]$ and $[xy'_x]$ lie in a common apartment (which is euclidean) so that the angle $0 \leq \gamma(x; y, y') \leq 3,14\ldots$ between them is defined (and, in fact, is independent of the chosen apartment). For any real number $0 < \varepsilon < 1$ we consider the subset

$$\Omega(x; z; \varepsilon) := \{y \in \overline{X}\backslash\{x\} : \gamma(x; y, z) < \varepsilon\}$$

of $\overline{X}$ which contains z. We will successively prove:
1. $\Omega(x; z; \varepsilon)$ and $\Omega(x; z; \varepsilon) \cap X$ are path-connected.
2. The function

$$\begin{aligned} X\backslash\{x\} &\longrightarrow \mathbb{R}_+ \\ y &\longmapsto \gamma(x; y, z) \end{aligned}$$

is continuous.
3. There is a constant $0 < \varepsilon(x; z) < 1$ such that

$$F(x;z) \subseteq \overline{F(x;y)} \quad \text{for any } y \in \Omega(x; z; \varepsilon(x;z)) \quad .$$

4. For any $z' \in \Omega(x; z; \varepsilon) \cap X_\infty$ and any $\varepsilon' < \min(\varepsilon - \gamma(x; z', z), \varepsilon(x; z'))$ we have

$$\Omega(x; z'; \varepsilon') \subseteq \Omega(x; z; \varepsilon) \quad .$$

5. $\Omega(x; z; \varepsilon)$ is open in $\overline{X}$.
6. For $x' \in [xz)$ and $0 < \varepsilon' \leq \varepsilon$ we have

$$\Omega(x'; z; \varepsilon') \subseteq \Omega(x; z; \varepsilon) \quad .$$

7. Let $\varepsilon(.) : \mathbb{R}_+ \to (0,1)$ be a decreasing function; then

$$\bigcap_{x' \in [xz)} \overline{\Omega(x'; z; \varepsilon(d(x,x')))} = \{z\} \quad .$$

8. The $\Omega(x;z;\varepsilon)$ for varying x and ε form a fundamental system of neighbourhoods of z in $\overline{X}$.

The assertions 1., 5., and 8. contain what we wanted to establish.

Ad 1: Let y and y' be points in $\Omega(x;z;\varepsilon)$. Then obviously $[y_x y] \cup [y'_x y'] \subseteq \Omega(x;z;\varepsilon)$. By looking at an apartment which contains $F(x;y)$ and $F(x;z)$ and hence the convex hull of $\{x, y_x, z_x\}$ one sees that $[y_x z_x] \subseteq \Omega(x;z;\varepsilon)$. We similarly have $[y'_x z_x] \subseteq \Omega(x;z;\varepsilon)$.

Ad 2: The sets $\overline{F}\backslash\{x\}$ with F running through all the facets of X form a locally finite closed covering of $X\backslash\{x\}$. It therefore suffices to check the continuity of the function restricted to each such set. But the latter is clear again by looking at an apartment which contains $F(x;z)$ and F.

Ad 3: This follows from [BT] I.2.5.11 and the elementary geometry of an apartment containing $F(x;z)$ and $F(x;y)$.

Ad 4: Let y be a point in $\Omega(x;z';\varepsilon')$. Looking at an apartment which contains $F(x;z)$ and $F(x;y)$ and hence, by the assumption on ε', also $F(x;z')$ we find

$$\gamma(x;y,z) \leq \varepsilon' + \gamma(x;z',z) < \varepsilon \quad .$$

Ad 5: As a consequence of 2, we know already that $\Omega(x;z;\varepsilon) \cap X$ lies in the interior of $\Omega(x;z;\varepsilon)$. It remains to consider a boundary point. Because of 4. it is actually sufficient to show that $\Omega(x;z;\varepsilon)$ is a neighbourhood of z where in addition we may assume that $z \in D_\infty$. Suppose this would not be true. Then we find a sequence $(z_n)_{n\in\mathbb{N}}$ of points in $\overline{X}\backslash(\Omega(x;z;\varepsilon) \cup \{x\})$ which converges to z. This means in particular that $\gamma(x;z_n,z) \geq \varepsilon$ for all $n \in \mathbb{N}$. On the other hand we will show below that

$$\lim_{n\to\infty} \gamma(x;z_n,z) = 0 \quad \text{whenever} \quad \lim_{n\to\infty} z_n = z \quad .$$

This gives a contradiction and proves our claim.

By the construction of $\overline{X}$ we have $z_n = h_n y_n$ with $h_n \in P_0$ and $y_n \in \overline{D}$. Since P_0/C and $\overline{D}$ are compact we may assume by passing to a subsequence that the $h_n C$ converge to hC for some $h \in P_0$ and that the y_n converge to some $y \in \overline{D}$. It follows that $hy = z$ and hence even $y = z$ so that $h \in P_0 \cap P_z$. Passing again to a subsequence we further may assume that all hh_n^{-1} fix $\overline{F(x;z)}$ pointwise. Using that the G-action on $\overline{X}$ respects angles ([BT] I.7.4.11) we conclude that

$$\gamma(x;z_n,z) = \gamma(hh_n^{-1}x; hy_n, hh_n^{-1}z) = \gamma(x;hy_n,z) \quad .$$

This argument shows that it suffices (replace D by hD) to consider a sequence z_n contained in $\overline{D}$. Now using [BT] I.7.4.18 (ii) or [Bro] VI.8 Thm. we find a subsector $D' := x' + D \subseteq D$ such that x and D' are contained in a common apartment A'. In particular $z \in \overline{D'}$ because of $(D')_\infty = D_\infty$; moreover $(x' + z_n)_{n\in\mathbb{N}}$ is a sequence in $\overline{D'}$ which converges to z (note that $x' + z_n = z_n$ if $z_n \in D_\infty$). It is clear from the definition of the topology of $\overline{A'}$ that

$$\lim_{n\to\infty} \gamma(x; x' + z_n, z) = 0 \quad .$$

From the cosine inequality in [Bro] VI.7 Ex. 2 it is also clear that

$$\lim_{n\to\infty} \gamma(x; z_n, x' + z_n) = 0 \quad .$$

We finally pass for a last time to subsequences of (z_n) and $(x' + z_n)$ in such a way that the facet $F := F(x; z_n)$, resp. $F' := F(x; x' + z_n)$, is independent of $n \in \mathbb{N}$. Then necessarily $F(x; z) \subseteq \overline{F'}$. In this situation we see by working in an apartment which contains F and F' that

$$\gamma(x; z_n, z) \leq \gamma(x; z_n, x' + z_n) + \gamma(x; x' + z_n, z) \quad .$$

Combining the last three formulas we obtain $\lim\limits_{n\to\infty} \gamma(x; z_n, z) = 0$.
Ad 6: Let y be any point in $\Omega(x'; z; \varepsilon')$. Assume that $y \notin [xz)$ because otherwise y for trivial reasons lies in $\Omega(x; z; \varepsilon)$. Then the function

$$x' \longmapsto \gamma(x'; y, z) \quad \text{on } [xz)$$

is defined. We show that it is increasing as a function of $d(x, x')$. Choose an apartment A' which contains $F(x'; z) \cup \{y\}$. Looking at the convex hull of $\{x', z_{x'}, y\}$ in A' it is clear that

$$\gamma(x''; y, z) \geq \gamma(x'; y, z) \quad \text{for any } x'' \in [x' z_{x'}] \quad .$$

Ad 7: Let y be any point in the intersection on the left hand side. Then $y \notin [xz)$ and $\gamma(x'; y, z) \leq \varepsilon(d(x, x'))$ for any $x' \in [xz)$. The left hand side is increasing with $d(x, x')$ by the previous argument whereas the right hand side is decreasing by assumption. It follows that $\gamma(x'; y, z) = 0$ for any $x' \in [xz)$. Would y and z be different then we would have

$$[xz] \cap [xy] = [xx'] \quad \text{for some } x' \in [xz) \quad .$$

On the other hand one easily deduces from $\gamma(x'; y, z) = 0$ that $(x' y_{x'}) \cap (x' z_{x'}) \neq \emptyset$. Hence we would obtain a contradiction.
Ad 8: Choose a sequence of points $x_1, x_2, \ldots$ in $[xz)$ such that $d(x, x_i)$ is increasing and tends to ∞ and choose a decreasing sequence of real numbers $0 < \varepsilon_1, \varepsilon_2, \ldots < 1$ which converges to 0. Then the subsets $\Omega(x_i; z; \varepsilon_i)$ form, by 5. and 6., a decreasing sequence of open neighbourhoods of z such that the intersection of their closures is, by 6. and 7., equal to $\{z\}$. It is a general fact about compact Hausdorff spaces that such a sequence has to be a fundamental system of neighbourhoods. $\square$

This has two consequences: First of all we obtain an inclusion

$$j_*\mathcal{T}. \subseteq C_c(T.)_{/\overline{X}}$$

of simplicial sheaves on $\overline{X}$. Secondly because of $j_*\mathcal{T}_0 = V_{/\overline{X}}$ it follows from 2.11 that $j_{*,\infty}\underset{\approx}{V}$ is the image of the augmentation map. Therefore we actually have an augmented simplicial sheaf

$$j_*\mathcal{T}. \twoheadrightarrow j_{*,\infty}\underset{\approx}{V}$$

with surjective augmentation map.

Proposition IV.4.5:

The associated complex of sheaves

$$\ldots \longrightarrow j_*\mathcal{T}_1 \longrightarrow j_*\mathcal{T}_0 \longrightarrow j_{*,\infty}\underset{\approx}{V} \longrightarrow 0$$

is exact.

Proof: This is shown stalkwise. First let z be a point in X. By 1.1.i and Lemma 3.i the sequence of stalks in z is the complex of functions with finite support associated with the augmented simplicial set $T.^z \to T_z$. It is exact since the fibers of that augmentation are contractible simplicial sets (compare [SS] p. 22). The same reasoning works for a boundary point $z \in X_\infty$ where we use the unipotent subgroup U_z in order to analogously define

$$T_z := U_z \backslash T \quad \text{and} \quad T_m^z := T \underset{T_z}{\times} \ldots \underset{T_z}{\times} T \quad (m+1 \text{ factors})$$

once we show that the augmented simplicial vector spaces

$$(j_*\mathcal{T}.)_z \longrightarrow C_c(T)_{U_z}$$

and

$$C_c(T.^z) \longrightarrow C_c(T_z)$$

coincide. Both simplicial vector spaces are contained in $C_c(T.)$ so that the comparison can be done termwise. We need the subsequent two facts.

Lemma IV.4.6:

Let $z \in X_\infty$ be a boundary point; for any open neighbourhood Ω of z in $\overline{X}$ we have

$$\bigcap_{y \in \Omega \cap X} T_m^y \subseteq T_m^z \quad .$$

Proof: Assume that $(t_0, \ldots, t_m)$ is a tuple which is contained in the left hand side but not in the right hand side. Then there is a $1 \leq j \leq m$ such that

$$gt_0 \neq t_j \text{ for any } g \in U_z \quad .$$

Let $g_0 \in G$ be a coset representative of $t_0 = g_0 U_x^{(e)}$. Choose a point $y \in \Omega \cap [g_0(x)z)$. By assumption we have

$$ht_0 = t_j \text{ for some } h \in U_y^{(e)} \quad .$$

In I.3.2 it was shown that

$$U_y^{(e)} \subseteq U_z \cdot U_{g_0 x}^{(e)}$$

or equivalently

$$U_y^{(e)} g_0 U_x^{(e)} \subseteq U_z g_0 U_x^{(e)}$$

holds true. (Observe that by [Bro] Thm. VI.8 there is an apartment which contains the special point $g_0 x$ and the boundary point z in its closure.) But this implies $U_y^{(e)} \cdot t_0 \subseteq U_z \cdot t_0$ which is a contradiction. $\square$

Lemma IV.4.7:

Let $z \in X_\infty$ be a boundary point; for any tuple $(t_0, \ldots, t_m) \in T_m^z$ there is an open neighbourhood Ω of z in $\overline{X}$ such that

$$(t_0, \ldots, t_m) \in T_m^y \text{ for any } y \in \Omega \cap X \quad .$$

Proof: By G-equivariance we may assume that $z \in D_\infty$ where D is the fundamental Weyl chamber introduced in 2.1. Let $U' \subseteq G$ be an open subgroup such that

$$gt_j = t_j \text{ for all } g \in U' \text{ and } 0 \leq j \leq m \quad .$$

By assumption there are elements $u_j \in U_z$ such that

$$t_j = u_j t_0 \text{ for } 1 \leq j \leq m \quad .$$

The subset

$$\mathfrak{c} := \bigcup_{1 \leq j \leq m} \{h u_j h^{-1} : h \in P_0 \cap P_z\}$$

of U_z is compact. Hence we find, by 2.7, an open neighbourhood Ω_0 of z in $\overline{A}$ such that

$$\mathfrak{c} \subseteq U_F^{(e)} \quad \text{for any facet } F \subseteq \Omega_0 \cap X \quad .$$

According to 2.5 there is now an open neighbourhood Ω of z in $\overline{X}$ of the form

$$\Omega = U(P_0 \cap P_z) \cdot \Omega_1$$

where

— $U \subseteq U'$ is an open subgroup,

— Ω_1 is an open neighbourhood of z in $\overline{D}$ such that $\Omega_1 \cap X$ is a union of facets, and

— $\Omega_1 \subseteq \Omega_0$.

In particular we have

$$\mathfrak{c} \subseteq U_{y'}^{(e)} \quad \text{for any } y' \in \Omega_1 \cap X \quad .$$

Consider an arbitrary point

$$y = ghy' \quad \text{with } g \in U \ , \ h \in P_0 \cap P_z \ , \text{ and } y' \in \Omega_1 \cap X$$

in $\Omega \cap X$. By construction we obtain

$$\begin{aligned} t_j = gt_j = gu_jt_0 = gu_jg^{-1}t_0 = gh(h^{-1}u_jh)h^{-1}g^{-1}t_0 \\ \in ghU_{y'}^{(e)}h^{-1}g^{-1} \cdot t_0 = U_y^{(e)} \cdot t_0 \end{aligned}$$

for any $1 \leq j \leq m$ which means that $(t_0, \ldots, t_m) \in T_m^y$. □

Returning to the proof of Proposition 5 let first ψ be a function in $(j_*\mathcal{T}_m)_z \subseteq C_c(T_m)$. This means that there is an appropriate open neighbourhood Ω of z in $\overline{X}$ such that $\psi \in C_c(T_m^y)$ for any $y \in \Omega \cap X$. In other words the support of ψ is contained in $\bigcap_{y \in \Omega \cap X} T_m^y$ and hence in T_m^z by Lemma 6.

Conversely let ψ be a function in $C_c(T_m^z)$. Then it follows from Lemma 7 that ψ can be viewed as a section in $\mathcal{T}_m(\Omega \cap X) = j_*\mathcal{T}_m(\Omega)$ for some neighbourhood Ω of z in $\overline{X}$.

Finally in order to compare the two augmentation maps we have to check that the pushforward of functions with respect to the projection map $T \to T_z$ induces an isomorphism

$$C_c(T)_{U_z} \xrightarrow{\cong} C_c(T_z) \quad .$$

The surjectivity is trivial. For the injectivity it suffices to consider a function $\psi \in C_c(T)$ whose image in $C_c(T_z)$ vanishes and which is supported on a single U_z-orbit in T. We then find a compact open subgroup U in U_z such that ψ even is supported on a single U-orbit in T. Hence the image of ψ in $C_c(U \backslash T)$ vanishes which implies that $\psi = 0 \bmod U$. This finishes the proof of Proposition 5. □

The computation of the cohomology of a sheaf $j_*\mathcal{T}_m$ will be based on the observation that this sheaf has a natural direct sum decomposition. For each tuple $(t_0, \ldots, t_m) \in T_m$ we introduce the subset

$$\overline{X}^{(t_0,\ldots,t_m)} := \{z \in \overline{X} : (t_0, \ldots, t_m) \notin T_m^z\}$$

of $\overline{X}$ and we put

$$X^{(t_0,\ldots,t_m)} := \overline{X}^{(t_0,\ldots,t_m)} \cap X \quad \text{and}$$

$$X_\infty^{(t_0,\ldots,t_m)} := \overline{X}^{(t_0,\ldots,t_m)} \cap X_\infty \quad .$$

Clearly $\overline{X}^{(t_0,\ldots,t_m)} = \emptyset$ if $t_0 = \ldots = t_m$. Assuming therefore that $\{t_0, \ldots, t_m\}$ has cardinality at least 2 let us first collect a number of properties of the subspace $\overline{X}^{(t_0,\ldots,t_m)}$ which will be of use later on.

IV.4.8. $X^{(t_0,\ldots,t_m)}$ is a nonempty closed geodesically contractible CW-subspace of X.
(This was already noted in the proof of II.3.1.)

IV.4.9. For any open subset $\Omega \subseteq \overline{X}$ such that $\Omega \cap X_\infty^{(t_0,\ldots,t_m)} \neq \emptyset$ we have $\Omega \cap X^{(t_0,\ldots,t_m)} \neq \emptyset$.
(This follows from Lemma 6.)

IV.4.10. Any boundary point $z \in X_\infty \backslash X_\infty^{(t_0,\ldots,t_m)}$ has an open neighbourhood Ω in $\overline{X}$ such that $\Omega \cap X^{(t_0,\ldots,t_m)} = \emptyset$.
(This follows from Lemma 7.)

IV.4.11. Choose $g_0 \in G$ such that $t_0 = g_0 U_x^{(e)}$; for any $z \in X_\infty^{(t_0,\ldots,t_m)}$ we have $[g_0(x)z) \subseteq \overline{X}^{(t_0,\ldots,t_m)}$.
(This is proved literally in the same way as Lemma 6.)

IV.4.12. We have

$$\overline{X}^{(t_0,\ldots,t_m)} = \overline{X^{(t_0,\ldots,t_m)}} \quad ;$$

in particular $\overline{X}^{(t_0,\ldots,t_m)}$ is closed in $\overline{X}$.
(This is a consequence of 10. and 11.)

For an arbitrary tuple we now define the sheaf $\mathbb{C}_{(t_0,\ldots,t_m)}$ on X to be the constant sheaf with value $\mathbb{C}$ on the open subset $X \backslash X^{(t_0,\ldots,t_m)}$ extended by zero to all of X.

Lemma IV.4.13:

$$\mathcal{T}_m = \bigoplus_{(t_0,\ldots,t_m)\in T_m} \mathbb{C}_{(t_0,\ldots,t_m)}.$$

Proof: Straightforward. □

Lemma IV.4.14:

$j_*\mathbb{C}_{(t_0,\dots,t_m)}$ is the constant sheaf with value $\mathbb{C}$ on the open subset $\overline{X}\backslash\overline{X}^{(t_0,\dots,t_m)}$ extended by zero to all of $\overline{X}$.

Proof: This is a consequence of 12 and Lemma 4. □

By comparing stalks we obtain

$$j_*\mathcal{T}_m = \bigoplus_{(t_0,\dots,t_m)\in T_m} j_*\mathbb{C}_{(t_0,\dots,t_m)}$$

and hence

$$\begin{aligned} H^*(\overline{X}, j_*\mathcal{T}_m) &= \bigoplus_{(t_0,\dots,t_m)\in T_m} H^*(\overline{X}, j_*\mathbb{C}_{(t_0,\dots,t_m)}) \\ &= \bigoplus_{(t_0,\dots,t_m)\in T_m} H^*(\overline{X}, \overline{X}^{(t_0,\dots,t_m)}; \mathbb{C}) \quad . \end{aligned}$$

Proposition IV.4.15:

$$H^*(\overline{X}, j_*\mathcal{T}_m) = \begin{cases} C_c(T) & \textit{if } * = 0 \quad , \\ 0 & \textit{if } * > 0 \quad . \end{cases}$$

Proof: $\overline{X}$ is contractible by [BS] 5.4.5. The same argument together with 8. and 11. shows that $\overline{X}^{(t_0,\dots,t_m)}$, for $\{t_0,\dots,t_m\}$ of cardinality at least 2, is contractible as well. □

Because of Lemma 2 the resolution in Proposition 5 gives rise to the hypercohomology spectral sequence

$$E_1^{r,s} = H^s(\overline{X}, j_*\mathcal{T}_{-r}) \Longrightarrow H^{r+s}(\overline{X}, j_{*,\infty}\underset{\approx}{V}) \quad .$$

It degenerates by Proposition 15 and exhibits $H^*(\overline{X}, j_{*,\infty}\underset{\approx}{V})$ as the homology of the complex

$$\dots \xrightarrow{0} V \xrightarrow{id} V \xrightarrow{0} V \quad .$$

This proves Theorem 1.

Corollary IV.4.16:

Let V be a representation of finite length in $\mathrm{Alg}(G)$*; if e is chosen large enough then we have*

$$H_c^*(X, \underset{\approx}{V}) = \begin{cases} \ker(V \to \underline{\underline{V}}(X_\infty)) & \text{if } * = 0 \quad , \\ \mathrm{coker}(V \to \underline{\underline{V}}(X_\infty)) & \text{if } * = 1 \quad , \\ H^{*-1}(X_\infty, \underline{\underline{V}}) & \text{if } * \geq 2 \quad . \end{cases}$$

We fix a subset $\Theta \subseteq \Delta$. As in the proof of 3.5 let w_Δ, resp. w_Θ, denote the unique maximal (w.r.t. the Bruhat order) element in W, resp. W_Θ. We also fix a lifting $g \in N$ of $w_\Theta w_\Delta$; note that $(w_\Theta w_\Delta)^{-1}$ is the unique maximal element in $[W/W_\Theta]$.

Theorem IV.4.17:

Assume that $V = Ind(E)$ for some irreducible supercuspidal representation E of M_Θ; if e is chosen large enough then we have

$$H_c^*(X, \underset{\approx}{V}) \cong \begin{cases} Ind({}^g E) & \text{if } * = d - \#\Theta \quad , \\ 0 & \text{otherwise} \quad . \end{cases}$$

Proof: Corollary 16 and 3.7. □

Corollary IV.4.18:

Let E be an irreducible supercuspidal representation in $\mathrm{Alg}_\chi(M_\Theta)$*; we then have*

$$\mathcal{E}^*(Ind(E)) \cong \begin{cases} Ind({}^g \tilde{E}) & \text{if } * = d - \#\Theta \quad , \\ 0 & \text{otherwise} \quad . \end{cases}$$

Proof: Theorem 17 and 1.4. □

Corollary IV.4.19:

Let V be a representation in $\mathrm{Alg}_{\chi,\Theta}^{fl}(G)$*; if e is chosen large enough then we have $H_c^*(X, \underset{\approx}{V}) = 0$ for $* \neq d - \#\Theta$.*

Proof: This follows from Theorem 17 by an induction argument as in the proof of III.3.1. (Because of 1.4 the present assertion and III.3.1.i actually are equivalent.) □

IV.5. The Zelevinsky involution

We fix a central character χ and let $R_{\mathbb{Z}}(G;\chi)$ be the Grothendieck group of representations of finite length in $\mathrm{Alg}_\chi(G)$ (w.r.t. exact sequences); the class in $R_{\mathbb{Z}}(G;\chi)$ of a representation V is denoted by $[V]$. It follows from III.3.1 that

$$\begin{aligned}\iota : R_{\mathbb{Z}}(G;\chi) &\longrightarrow R_{\mathbb{Z}}(G;\chi)\\ [V] &\longmapsto \sum_{i\geq 0}(-1)^i \cdot [\mathcal{E}^i(\tilde{V})]\end{aligned}$$

is a well-defined homomorphism such that

$$\iota([V]) = (-1)^{d-\#\Theta_0}\cdot[\mathcal{E}(\tilde{V})] \text{ for any } V \text{ in } \mathrm{Alg}^{fl}_{\chi,\Theta_0}(G) \quad .$$

Proposition IV.5.1:

ι respects up to sign the classes of irreducible representations.

Proof: This is III.3.2. □

Consider a representation V in $\mathrm{Alg}^{fl}_{\chi,\Theta_0}(G)$. It is a consequence of 3.1 that there is an augmented complex

$$\begin{array}{c} \bigoplus\limits_{\substack{\Theta\subseteq\Delta\\ \#\Theta=d}} Ind^G_{P_\Theta}(V_{U_\Theta}) = V \\ \downarrow \\ \bigoplus\limits_{\substack{\Theta\subseteq\Delta\\ \#\Theta=d-1}} Ind^G_{P_\Theta}(V_{U_\Theta}) \longrightarrow \ldots \longrightarrow \bigoplus\limits_{\substack{\Theta\subseteq\Delta\\ \#\Theta=0}} Ind^G_{P_\Theta}(V_{U_\Theta}) \end{array}$$

which computes the cohomology $H^*(X_\infty, \underline{\underline{V}})$. By combining 1.4 and 4.16 we see that the only nonvanishing homology group of that complex is $\mathcal{E}(\tilde{V})$; it sits in degree $d-1-\#\Theta_0$ if the complex is put in degree -1 up to $d-1$. Since the formation of Jacquet modules as well as the parabolic induction respect representations of finite length ([Ber] 3.1) each term in the above complex has a well-defined class in $R_{\mathbb{Z}}(G;\chi)$.

Proposition IV.5.2:

For any representation V of finite length in $\mathrm{Alg}^{fl}_\chi(G)$ we have

$$\iota([V]) = \sum_{\Theta\subseteq\Delta}(-1)^{d-\#\Theta}\cdot[Ind^G_{P_\Theta}(V_{U_\Theta})] \quad .$$

Proof: Obvious from the preceding discussion. □

For any $\Theta \subseteq \Delta$ let $P_{-\Theta}$ be the parabolic subgroup of G which contains M_Θ and is opposite to P_Θ; then $P_{-\Theta} \cap P_\Theta = M_\Theta$. Let $U_{-\Theta}$ denote the unipotent radical of $P_{-\Theta}$. The modulus character of $P_{-\Theta}$ is

$$P_{-\Theta} \overset{pr}{\twoheadrightarrow} M_\Theta \xrightarrow{\delta_\Theta^{-1}} \mathbb{R}_+^\times$$

([Cas] 1.6).

Lemma IV.5.3:

Let E be a representation of finite length in $\mathrm{Alg}_\chi(M_\Theta)$ for some $\Theta \subseteq \Delta$; we then have

$$[Ind_{P_{-\Theta}}^G(E)] = [Ind(\delta_\Theta^{-1/2} \otimes E)] \quad .$$

Proof: If $g \in N$ lifts w_Δ then $g^{-1}P_{-\Theta}g = P_{w_\Delta w_\Theta \Theta}$. We obtain

$$Ind_{P_{-\Theta}}^G(E) \cong Ind_{P_{w_\Delta w_\Theta \Theta}}({}^gE) = Ind(\delta_{w_\Delta w_\Theta \Theta}^{1/2} \otimes {}^gE) = Ind({}^g(\delta_\Theta^{-1/2} \otimes E)) \quad .$$

Because of $g^{-1}M_\Theta g = M_{w_\Delta w_\Theta \Theta}$ we may apply III.2.1 and we see that the latter representation has the same irreducible constituents as $Ind(\delta_\Theta^{-1/2} \otimes E)$. □

Proposition IV.5.4:

For any representation V in $\mathrm{Alg}_{\chi,\Theta_0}^{fl}(G)$ we have

$$[\mathcal{E}(\tilde{V})] = [\mathcal{E}(V)^\sim] \quad .$$

Proof: Dualizing the discussion preceding Proposition 2 we obtain

$$\begin{aligned}[\mathcal{E}(V)^\sim] &= \sum_{\Theta \subseteq \Delta} (-1)^{\#\Theta + \#\Theta_0} \cdot [Ind_{P_\Theta}^G(\tilde{V}_{U_\Theta})^\sim] \\ &= \sum_{\Theta \subseteq \Delta} (-1)^{\#\Theta + \#\Theta_0} \cdot [Ind(\delta_\Theta^{-1/2} \otimes V_{U_{-\Theta}})]\end{aligned}$$

where the second equality even holds termwise by [Cas] 4.2.5. On the other hand Proposition 2 holds true, of course, for any choice of simple roots, e.g., $-\Delta$. Hence we have

$$[\mathcal{E}(\tilde{V})] = \sum_{\Theta \subseteq \Delta} (-1)^{\#\Theta + \#\Theta_0} \cdot [Ind_{P_{-\Theta}}^G(V_{U_{-\Theta}})] \quad .$$

Apply now Lemma 3. □

Corollary IV.5.5:

ι is an involution, i.e., $\iota \circ \iota = \mathrm{id}$.

Proof: Combine Proposition 4 and III.1.5. $\square$

The Proposition 2 shows that $(-1)^d \cdot \iota$ coincides with the involution D_G studied in [Au1] 5.24. It therefore follows from [Au1] 5.36 that in case $G = GL_{d+1}(K)$ the Zelevinsky involution i considered in [Zel] 9.16 is equal to $-\iota$. Hence Proposition 1 proves the Duality Conjecture 9.17 in [Zel]. It also follows that the orthogonality property discussed in [Au1] 5.D holds true.

Assume K to have characteristic 0 and the center of G to be compact. Then the above results hold without having specified a central character. In III.4 after Thm. 21 we had seen that the Euler-Poincaré characteristic $EP(.,.)$ induces a nondegenerate symmetric bilinear form on the quotient $\overline{R}(G) = R(G)/R_I(G)$ of the Grothendieck group $R(G) = R_{\mathbb{Z}}(G) \otimes \mathbb{C}$. It follows from Proposition 2 or from III.4.3.i that $EP(\iota.,.) = EP(.,.)$. Hence the involution ι respects the subgroup $R_I(G)$ and induces the identity on the quotient $\overline{R}(G)$ or, equivalently, $(\mathrm{id} - \iota)(R(G)) \subseteq R_I(G)$.

V. The functor from equivariant coefficient systems to representations

In this final Chapter we want to examine more closely the relation between representations and coefficient systems. The category Coeff(X) of coefficient systems (of complex vector spaces) on X was introduced in II.2. We say the group G acts on the coefficient system $(V_F)_F$ if, for any $g \in G$ and any facet $F \subseteq X$, there is given a linear map

$$g_F : V_F \longrightarrow V_{gF}$$

in such a way that

— $g_{hF} \circ h_F = (gh)_F$ for any $g, h \in G$ and any F,

— $1_F = \mathrm{id}_{V_F}$ for any F, and

— the diagram

$$\begin{array}{ccc} V_F & \xrightarrow{g_F} & V_{gF} \\ {\scriptstyle r^F_{F'}}\downarrow & & \downarrow{\scriptstyle r^{gF}_{gF'}} \\ V_{F'} & \xrightarrow{g_{F'}} & V_{gF'} \end{array}$$

is commutative for any $g \in G$ and any pair of facets $F' \subseteq \overline{F}$.
In particular the stabilizer $P^\dagger_F$, for any facet F, acts linearly on V_F.

Definition:

An equivariant coefficient system on X is a coefficient system $(V_F)_F$ on X together with a G-action on it which has the property that, for any facet F, the stabilizer $P^\dagger_F$ acts on V_F through a discrete quotient.

Let Coeff$_G(X)$ denote the category of all equivariant coefficient systems on X. This is an abelian category.

Fix an object $\mathcal{V} = (V_F)_F$ in Coeff$_G(X)$. For any $0 \leq q \leq d$ the space of oriented q-chains of $\mathcal{V}$ by definition is

$$C^{or}_c(X_{(q)}, \mathcal{V}) := \mathbb{C}\text{-vector space of all maps } \omega : X_{(q)} \longrightarrow \dot{\bigcup_{F \in X_q}} V_F$$

such that

— ω has finite support,

— $\omega((F, c)) \in V_F$, and, if $q \geq 1$,

— $\omega((F, -c)) = -\omega((F, c))$ for any $(F, c) \in X_{(q)}$.

The group G acts smoothly on these spaces via

$$(g\omega)((F,c)) := g_{g^{-1}F}(\omega((g^{-1}F, g^{-1}c))) \quad .$$

The boundary map

$$\begin{aligned} \partial : C_c^{or}(X_{(q+1)}, \mathcal{V}) &\longrightarrow C_c^{or}(X_{(q)}, \mathcal{V}) \\ \omega &\longmapsto ((F',c') \mapsto \sum_{\substack{(F,c)\in X_{(q+1)} \\ F'\subseteq \overline{F} \\ \partial_{F'}^F(c)=c'}} r_{F'}^F(\omega((F,c)))) \end{aligned}$$

is G-equivariant. Hence we obtain the chain complex

$$C_c^{or}(X_{(d)}, \mathcal{V}) \xrightarrow{\partial} \ldots \xrightarrow{\partial} C_c^{or}(X_{(0)}, \mathcal{V})$$

in $\mathrm{Alg}(G)$. Its homology is denoted by $H_*(X, \mathcal{V})$. It is not difficult to see that the above complex as well as its homology actually lies in the full subcategory

$$\begin{aligned} \mathrm{Alg}^c(G) := {} & \text{category of those smooth } G\text{-representations } V \\ & \text{which are generated by } V^U \text{ for some open} \\ & \text{subgroup } U \subseteq G. \end{aligned}$$

As a consequence of Bernstein's theorem (I.3) the category $\mathrm{Alg}^c(G)$ is stable with respect to the formation of G-equivariant subquotients; moreover it is closed under extensions. In the following only the right exact functor

$$H_0(X, .) : \mathrm{Coeff}_G(X) \longrightarrow \mathrm{Alg}^c(G)$$

will be of importance for us. Let Σ be the class of morphisms s in $\mathrm{Coeff}_G(X)$ such that $H_0(X, s)$ is an isomorphism. We then have a unique commutative diagram of functors

$$\begin{array}{ccc} \mathrm{Coeff}_G(X) & \xrightarrow{H_0(X,.)} & \mathrm{Alg}^c(G) \\ {\scriptstyle Q} \searrow & & \nearrow {\scriptstyle \rho} \\ & \mathrm{Coeff}_G(X)[\Sigma^{-1}] & \end{array}$$

where Q is the canonical functor into the category of fractions with respect to Σ.

Theorem V.1:

The functor

$$\rho : \mathrm{Coeff}_G(X)[\Sigma^{-1}] \xrightarrow{\sim} \mathrm{Alg}^c(G)$$

is an equivalence of categories.

Proof: The following properties are a consequence of the right exactness of $H_0(X, .)$ ([GZ] I.3):

1. Σ admits a calculus of left fractions.
2. $\mathrm{Coeff}_G(X)[\Sigma^{-1}]$ is additive and has finite direct limits.
3. The functors Q and ρ are additive and respect finite direct limits.
4. The functor ρ detects isomorphisms.

The latter two properties imply:

5. The functor ρ is faithful.

Namely, let a and b be two morphisms in $\mathrm{Coeff}_G(X)[\Sigma^{-1}]$ such that $\rho(a) = \rho(b)$. Using 3. we have that

$$\rho(\mathrm{coker}(a - b)) = \mathrm{coker}(\rho(a - b)) = \mathrm{coker}(\rho(a) - \rho(b)) = \mathrm{coker}(0)$$

is an isomorphism. By 4. then $\mathrm{coker}(a - b)$ is an isomorphism, too; hence $a = b$. Fixing a special vertex x in A we have

$$\mathrm{Alg}^c(G) = \bigcup_{e \geq 0} \mathrm{Alg}^{U_x^{(e)}}(G) \quad .$$

In II.2 we have constructed, for any $e \geq 0$, an exact functor

$$\gamma_e : \mathrm{Alg}^{U_x^{(e)}}(G) \longrightarrow \mathrm{Coeff}_G(X) \quad ;$$

moreover there is an obvious natural transformation

$$\gamma_e \longrightarrow \gamma_{e+1} | \, \mathrm{Alg}^{U_x^{(e)}}(G) \quad .$$

It follows from II.3.1 that the latter induces a natural isomorphism in homology

$$H_*(X, \gamma_e(.)) \xrightarrow{\cong} H_*(X, \gamma_{e+1}(.)) \ \text{ on } \ \mathrm{Alg}^{U_x^{(e)}}(G) \quad .$$

After composing with the functor Q the above natural transformation therefore becomes a natural isomorphism

$$Q \circ \gamma_e \xrightarrow{\cong} Q \circ \gamma_{e+1} | \, \mathrm{Alg}^{U_x^{(e)}}(G) \quad .$$

Hence we obtain in the direct limit the functor

$$\gamma := \varinjlim_{e \geq 0} Q \circ \gamma_e : \mathrm{Alg}^c(G) \longrightarrow \mathrm{Coeff}_G(X)[\Sigma^{-1}] \quad .$$

Again by II.3.1 we have

6. $\rho \circ \gamma \cong \mathrm{id}_{\mathrm{Alg}^c(G)}$.

It is an immediate consequence of 5. and 6. that ρ and γ are quasi-inverse to each other. □

Because of their practical importance let us state separately the following facts which were established in the course of the previous proof.

Lemma V.2:

i. Σ *admits a calculus of left fractions;*

ii. the functor $\gamma : \mathrm{Alg}^c(G) \to \mathrm{Coeff}_G(X)[\Sigma^{-1}]$ *is quasi-inverse to* ρ.

Index of Notations

K	a nonarchimedean locally compact field
o	the ring of integers in K
π	a fixed prime element in o
ω	the discrete valuation of K normalized by $\omega(\pi) = 1$
$\overline{K}$	the residue class field of o
$\overline{X}$	the base change to $\overline{K}$ of some object X over o
$\mathbf{G}$	a connected reductive group over K
G	the group of K-rational points of $\mathbf{G}$
C	the center of G
S	a maximal K-split torus in G
W	the Weyl group of G
Φ	the set of roots of G
Φ^{red}	the set of reduced roots of G
Φ^+, Φ^-	the set of positive resp. negative roots in Φ
Δ	the set of simple roots in Φ^+
U_α	the root subgroup corresponding to the root α
Θ	a subset of the set Δ of simple roots
$<\Theta>$	the subset $\{\alpha \in \Phi : \alpha$ is a integral linear combination of roots from $\Theta\}$
S_Θ	the connected component of $\bigcap_{\alpha \in \Theta} \ker(\alpha)$
M_Θ	the centralizer of S_Θ in G, i.e., the Levi subgroup corresponding to Θ
U_Θ	the unipotent subgroup of G generated by all root subgroups U_α for $\alpha \in \Phi^+ \setminus <\Theta>$
$P_\Theta = M_\Theta U_\Theta$	the parabolic subgroup of type Θ with respect to the choices S, Φ^+
δ_Θ	the modulus character of the parabolic subgroup P_Θ
X	the Bruhat-Tits building corresponding to G

d	the distance function on the metric space X
$X_{(q)}$ (X_q)	the set of oriented (nonoriented) q-dimensional polysimplices of X
$X^{(q)}$	the q-skeleton of X
A	a fixed (basic) apartment of X, a d-dimensional affine space
D	a fundamental Weyl chamber in the apartment A
F	a polysimplex of X
$St(F)$	the star of the facet F
$\overline{X}$	the Borel-Serre compactification of X
X_∞	the boundary of X in $\overline{X}$
P_Ω	the pointwise stabilizer in G of a subset $\Omega \subseteq \overline{X}$
P_Ω^+	the stabilizer in G of a subset $\Omega \subseteq X$
$U_F^{(e)}$	for any integer $e \geq 0$, the e-th filtration subgroup of P_F^+
$U_{\alpha,r}$	subgroup of U_α of ℓ-value $\geq r$
$T = G/U_x^{(e)}$	for a special vertex $x \in A$, the basic homogeneous G-set
G_{ell}	the subset of all regular elliptic elements of G
C_{ell}	the set of conjugacy classes of regular elliptic elements of G

V	a smooth representation of G
$\mathrm{Alg}(G)$	the category of smooth G-modules
$\mathrm{Alg}^U(G)$	for U a compact open subgroup, the subcategory of $\mathrm{Alg}(G)$ of those G-modules which are generated by the subspace of their U-fixed vectors V^U
$\mathrm{Alg}_\chi(G)$	for χ a character on the center C of G the full subcategory of $\mathrm{Alg}(G)$ of those G-modules on which C acts by the character χ
$\mathrm{Alg}_{\chi,\Theta}^{fl}(G)$	the full subcategory of $\mathrm{Alg}_\chi(G)$ of those G-modules which are of finite length and whose irreducible subquotients are all of type Θ
$R(G)$	the Grothendieck group of representations of finite length in $\mathrm{Alg}(G)$ tensorized by $\mathbb{C}$
$\mathrm{Ind}_{P_\Theta}^G(E)$	the (unnormalized) induction to G of a smooth P_Θ-module E

$\mathrm{Ind}(E)$	the normalized induction to G of a smooth P_Θ-module E
$\mathcal{H}_\chi$	for a character χ of the center C, the Hecke algebra of locally constant functions on G, compactly supported modulo the center C and transforming with respect to the action of C by the character χ
$EP(V, V')$	the Euler-Poincaré characteristic of the smooth G-modules V, V', V finite length, V' admissible
$d^V g$	the Euler-Poincaré measure for V
vol_V	the volume corresponding to $d^V g$
$\mathcal{E}^{d-\#\Theta}$	the involution functor on $\mathrm{Alg}^{fl}_{\chi,\Theta}(G)$

$\mathrm{Coeff}(X)$	the category of coefficient systems on X
$\gamma_e(V)$	the coefficient system associated to a smooth G-module V for a fixed integer $e \geq 0$
γ_e	the corresponding functor from $\mathrm{Alg}(G)$ to $\mathrm{Coeff}(X)$
$C^{or}_c(X_{(\cdot)}, \gamma_e(V))$	the oriented chain complex associated to a coefficient system $\gamma_e(V)$
$\underset{\approx}{V}$	the sheaf on the Bruhat-Tits building associated to the smooth representation V
$j_{*,\infty}\underset{\approx}{V}$	the smooth extension of $\underset{\approx}{V}$ to the compactification $\overline{X}$
$\underset{=}{V}$	the sheaf on the boundary X_∞ of X in $\overline{X}$ corresponding to the smooth representation V

References

[Au1] Aubert, A.-M.: Théorie de Mackey généralisée: Un théorème de décomposition. Preprint 1992

[Au2] Aubert, A.-M.: Dualité dans le groupe de Grothendieck de la catégorie des représentations lisse de longeur finie d'un groupe réductif p-adique. Transact. AMS 347, 2179-2189 (1995). Erratum. Transact. AMS, to appear

[Ber] Bernstein, J.: Le "centre" de Bernstein. In Bernstein, Deligne, Kazhdan, Vigneras, Représentations des groupes réductifs sur un corps local. Paris: Hermann 1984

[Be2] Bernstein, J.: On the support of Plancherel measure. J. Geom. Physics 5, 663-710 (1988)

[BDK] Bernstein, J., Deligne, P., Kazhdan, D.: Trace Paley-Wiener theorem for reductive p-adic groups. J. d'analyse math. 47, 180-192 (1986)

[BZ] Bernstein, J., Zelevinsky, A.: Induced representations of reductive $\wp$-adic groups I. Ann. sci. ENS 10, 441-472 (1977)

[Bla] Blanc, P.: Projectifs dans la catégorie des G-modules topologiques. C.R. Acad. Sci. Paris 289, 161-163 (1979)

[Bor] Borel, A.: Linear Algebraic Groups. 2nd Enlarged Edition. Berlin-Heidelberg-New York: Springer 1991

[BH] Borel, A., Harder, G.: Existence of discrete cocompact subgroups of reductive groups over local fields. J. reine angew. Math. 298, 53-64 (1978)

[BS] Borel, A., Serre, J.-P.: Cohomologie d'immeubles et de groupes S-arithmétiques. Topology 15, 211-232 (1976)

[BoT] Borel, A., Tits, J.: Groupes réductifs. Publ. Math. IHES 27, 55-152 (1965)

[BW] Borel, A., Wallach, N.: Continuous cohomology, discrete subgroups, and representations of reductive groups. Ann. Math. Studies 94. Princeton Univ. Press 1980

[Bou] Bourbaki, N.: Groupes et algèbres de Lie, Chap. 4-6. Paris: Masson 1981

[Bo2] Bourbaki, N.: Topologie générale, Chap. 5-10. Paris: Hermann 1974

[Bro] Brown, K.S.: Buildings. Berlin-Heidelberg-New York: Springer 1989

[BT] Bruhat, F., Tits, J.: Groupes réductifs sur un corps local I. Données radicielles valuées. Publ. Math. IHES 41 (1972). II. Schémas en groupes. Existence d'une donnée radicielle valuée. Publ. Math. IHES 60 (1984)

[Car] Cartier, P.: Representations of $\wp$-adic groups: a survey. In Automorphic Forms, Representations and L-Functions. Proc. Symp. Pure Math. 33 (1), pp. 111-155. American Math. Soc. 1979

[Cas] Casselman, W.: Introduction to the theory of admissible representations of $\wp$-adic reductive groups. Preprint

[Ca2] Casselman, W.: A new non-unitary argument for p-adic representations. J. Fac. Sci. Univ. Tokyo 28, 907-928 (1981)

[Clo] Clozel, L.: Invariant harmonic analysis on the Schwartz space of a reductive p-adic group. In Harmonic Analysis on Reductive Groups (Eds. Barker, Sally), Progress in Math. 101, pp. 101-121. Boston-Basel-Berlin: Birkhäuser 1991

[CLT] Curtis, C.W., Lehrer, G.I., Tits, J.: Spherical Buildings and the Character of the Steinberg Representation. Invent. Math. 58, 201-210 (1980)

[CR] Curtis, C., Reiner, I.: Representation Theory of Finite Groups and Associative Algebras. New York-London: J. Wiley 1962

[DL] Deligne, P., Lusztig, G.: Duality for Representations of a Reductive Group over a Finite Field I. J. Algebra 74, 284-291 (1982)

[Dix] Dixmier, J.: C^*-Algebras. Amsterdam: North-Holland 1982

[Dol] Dold, A.: Lectures on Algebraic Topology. Berlin-Heidelberg-New York: Springer 1980

[GZ] Gabriel, P., Zisman, M.: Calculus of Fractions and Homotopy Theory. Berlin-Heidelberg-New York: Springer 1967

[God] Godement, R.: Topologie algébrique et théorie des faisceaux. Paris: Hermann 1964

[HCD] Harish-Chandra, van Dijk, G.: Harmonic Analysis on Reductive p-adic Groups. Lect. Notes Math., vol. 162. Berlin-Heidelberg-New York: Springer 1970

[Har] Hartshorne, R.: Residues and duality. Lect. Notes Math., vol. 20. Berlin-Heidelberg-New York: Springer 1966

[Hat] Hattori, A.: Rank element of a projective module. Nagoya J. Math. 25, 113-120 (1965)

[KS] Kashiwara, M., Schapira, P.: Sheaves on Manifolds. Berlin-Heidelberg-New York: Springer 1990

[Kat] Kato, S.: Duality for representations of a Hecke algebra. Proc. AMS 119, 941-946 (1993)

[Ka1] Kazhdan, D.: Cuspidal geometry of p-adic groups. J. d'analyse math. 47, 1-36 (1986)

[Ka2] Kazhdan, D.: Representations of groups over close local fields. J. d'-analyse math. 47, 175-179 (1986)

[Kot] Kottwitz, R.: Tamagawa numbers. Ann. Math. 127, 629-646 (1988)

[MP] Moy, A., Prasad, G.: Unrefined minimal K-types for p-adic groups. Invent. math. 116, 393-408 (1994)

[PR] Prasad, G., Raghunatan, M.S.: Topological central extensions of semi-simple groups over local fields. Ann. Math. 119, 143-201 (1984)

[Pro] Procter, K.: The Zelevinsky Duality Conjecture for GL_N. Thesis, King's College, London 1994

[Rog] Rogawski, J.: An application of the building to orbital integrals. Compositio Math. 42, 417-423 (1981)

[SS] Schneider, P., Stuhler, U.: Resolutions for smooth representations of the general linear group over a local field. J. reine angew. Math. 436, 19-32 (1993)

[Ser] Serre, J.-P.: Cohomologie des groupes discrets. In Prospects in Mathematics, Ann. Math. Studies 70, pp. 77-169. Princeton Univ. Press 1971

[Sil] Silberger, A.: Introduction to harmonic analysis on reductive p-adic groups. Princeton Univ. Press 1979

[Tit] Tits, J.; Reductive groups over local fields. In Automorphic Forms, Representations and L-Functions. Proc. Symp. Pure Math. 33 (1), pp. 29-69. American Math. Soc. 1979

[Vig] Vigneras, M.-F.: On formal dimensions for reductive p-adic groups. In Festschrift in honor of I.I. Piatetski-Shapiro (Eds. Gelbart, Howe, Sarnak), Part I, pp. 225-266. Israel Math. Conf. Proc. 2. Jerusalem: Weizmann Science Press 1990

[Zel] Zelevinsky, A.: Induced representations of reductive p-adic groups II. Ann. sci. ENS 13, 165-210 (1980)

Prof. Dr. P. Schneider
Mathematisches Institut
der Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. U. Stuhler
Mathematisches Institut
der Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen