HOCHSCHILD AND CYCLIC HOMOLOGY OF FINITE TYPE ALGEBRAS

DAVID KAZHDAN, VICTOR NISTOR, AND PETER SCHNEIDER

Abstract. We study Hochschild and cyclic homology of finite type algebras using abelian stratifications of their primitive ideal spectrum. Hochschild homology turns out to have a quite complicated behavior, but cyclic homology can be related directly to the singular cohomology of the strata. We also briefly discuss some connections with the representation theory of reductive p–adic groups.

Contents

Introduction

Finite type algebras (Definition 1) appear in the study of reductive p–adic groups as commuting algebras of Bernstein components [5]. If A is a finite type algebra, then its primitive ideal spectrum, the space $\mathrm{Prim}(A)$, parametrizes the irreducible representations of A. In particular, if $A = A_D$ is the commuting algebra of a Bernstein component D, then $\mathrm{Prim}(A_D)$ parametrizes the irreducible admissible representations in D. Motivated, in part, by the existence of this parametrization, we study $\mathrm{Prim}(A)$ and its topology from an algebraic point of view using finite type algebras and two homology theories for algebras: Hochschild and periodic cyclic homology [8] (see also [31]).

Let us examine first what are the results in the particular case of a *commutative* finite type algebra A, i.e., that of a finitely generated commutative algebra. If A is a finitely generated commutative algebra, then the periodic cyclic homology groups of A, denoted $\mathrm{HP}_*(A)$, are known to be isomorphic to the de Rham cohomology of $\mathrm{Prim}(A)$, see [10, 15, 26]; moreover, if A is reduced and regular, then its Hochschild homology groups $\mathrm{HH}_*(A)$ are naturally isomorphic to the space of algebraic forms on $\mathrm{Prim}(A)$, see [14, 20]. These results suggest a close connection between the geometry and topology of $\mathrm{Prim}(A)$ and the groups $\mathrm{HH}_*(A)$ and $\mathrm{HP}_*(A)$.

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In this paper we generalize these results on Hochschild and periodic cyclic homology, from finitely generated commutative algebras, to the larger class of finite type algebras. Our main result, roughly stated, relates $\mathrm{HP}_*(A)$ and the topology of $\mathrm{Prim}(A)$, showing in particular that periodic cyclic homology groups of a finite type algebra A are a good substitute for de Rham cohomology groups of $\mathrm{Prim}(A)$. Otherwise, it is not known how to define classically the de Rham cohomology of $\mathrm{Prim}(A)$ because of its "multiple points." In periodic cyclic homology the "multiple points" are dealt with by considering certain filtrations of $\mathrm{Prim}(A)$.

Before stating our main result, let us explain how they relate to p–adic groups. Let $\mathcal{H}(G)$ be the Hecke algebra of G, i.e., the algebra of compactly supported, locally constant functions on G with the convolution product being defined using a fixed Haar measure. Then the groups $\mathrm{HP}_*(\mathcal{H}(G))$ have two different descriptions: one using the algebraic structure of G and another one using the representation theory of G.

The first, algebraic, description of the groups $\mathrm{HP}_*(\mathcal{H}(G))$ was obtained independently in [13, 28] using the action of G on a building, thus without relying on the representation theory of G.

In the present paper we adopt a point of view different from that of [13, 28], and we obtain, as a consequence of our results, a second description, of representation theoretic nature, of the groups $\mathrm{HP}_*(\mathcal{H}(G))$. This representation theoretic description is based on the relation between $\mathrm{HP}_*(A)$ and $\mathrm{Prim}(A)$ that we prove for a finite type algebra A. In particular, for commuting algebras, this description takes into account the multiplicities of induced representations. To see more precisely how finite type algebras can be used to obtain a representation theoretic description of $\mathrm{HP}_*(\mathcal{H}(G))$, we need to recall some results on reductive p–adic groups.

If $\hat{G}$ is the admissible dual of a reductive p–adic group G, then the Bernstein components of $\hat{G}$ are its connected components. Moreover, as a topological space, each Bernstein component D is homeomorphic to the primitive ideal spectrum of its associated commuting algebra A_D, and thus we have the following disjoint union decomposition of $\hat{G}$

$$\mathrm{Prim}(\mathcal{H}(G)) = \hat{G} = \cup_D D = \cup_D \mathrm{Prim}(A_D). \tag{1}$$

In terms of periodic cyclic homology, this means that

$$\mathrm{HP}_*(\mathcal{H}(G)) \simeq \oplus_D \mathrm{HP}_*(A_D). \tag{2}$$

The strategy of relating $\mathrm{HP}_*(\mathcal{H}(G))$ to the topology of $\hat{G}$ is now as follows. First, using (2), we reduce the computation of $\mathrm{HP}_*(\mathcal{H}(G))$ to that of the computation of $\mathrm{HP}_*(A_D)$. Next, for the finite type algebras A_D, their periodic cyclic homology groups are related to the topology of theeir primitive ideal spectrum by our main result (Theorem 1, below). Then, finally, we determine the topology of $\hat{G}$ from that of the individual Bernstein components using (1).

We now address some more specific aspects of the paper. The starting idea is that that for the homology theories we use–Hochschild and cyclic–a semiprimitive finite type algebra with only irreducible representations of dimension k, for some fixed k, behaves like its center. Our approach attempts to generalize this behavior to arbitrary finite type algebras A. More precisely, the technique to compare to commutative algebras is based on a certain class of filtrations of A by ideals, called *abelian filtrations* (Definition 3). Thus, let us consider a decreasing filtration $(\mathfrak{I}_k)$

by ideals,

$$A = \mathfrak{I}_0 \supset \mathfrak{I}_1 \supset \ldots \supset \mathfrak{I}_n = \mathrm{Jac}(A) \supset (0),$$

of a finite type algebra A. Theorem 1 below states the existence of a spectral sequence associated to an abelian filtration $(\mathfrak{I}_k)$ of A convergent to $\mathrm{HP}_*(A)$. Before we can state our main result, Theorem 1, we need to introduce some notations.

Consider an abelian filtration $(\mathfrak{I}_k)$ of a finite type algebra A. Then, for each k, the center Z_k of $A/\mathfrak{I}_k$ is a finitely generated complex algebra, and hence it is isomorphic to the ring of regular functions on an affine, complex algebraic variety X_k. Let $Y_k \subset X_k$ be the subvariety defined by $Z_k \cap (\mathfrak{I}_{k-1}/\mathfrak{I}_k)$ and X_k^{an} and Y_k^{an}, where $k = 1, \ldots, n$, be the analytic varieties underlying X_k and Y_k. We shall refer to X_k^{an} and Y_k^{an} as the *analytic spaces* associated to the filtration $(\mathfrak{I}_k)$ of A. Let

$$\mathrm{H}^{[j]}(A, B) = \prod_{k \in \mathbb{Z}} \mathrm{H}^{j+2k}(A, B; \mathbb{C})$$

be the "$\mathbb{Z}/2\mathbb{Z}$–periodic" Čech cohomology groups with complex coefficients of a pair (A, B) of *paracompact Hausdorff* topological spaces. Here is our main result.

Theorem 1. *If $Y_p^{\mathrm{an}} \subset X_p^{\mathrm{an}}$ are the analytic spaces associated to an abelian filtration of a finite type algebra A, then there exists a natural spectral sequence with*

$$E^1_{-p,q} = \mathrm{H}^{[q-p]}(X_p^{\mathrm{an}}, Y_p^{\mathrm{an}})$$

convergent to $\mathrm{HP}_{q-p}(A)$.

If the algebra A in the above theorem is commutative, then any decreasing filtration of A (by radical ideals) is abelian, which explains the terminology "abelian filtration," and our spectral sequence reduces to the spectral sequence in Čech cohomology associated to the filtration of a space by closed subsets.

Our theorem is general enough to apply to all finite type algebras, because each finite type algebra has abelian filtrations. An example is obtained by letting $\mathfrak{I}_k$ to be the intersection of the kernels of all irreducible representations of dimension at most k (this filtration is called the *standard filtration*, see equation (5) and Proposition 1). The excision theorem of Cuntz and Quillen plays an important part in the proof of our results.

We now proceed to describe the contents of the paper. We begin Section 1 with a few general properties of finite type algebras that extend well known properties of finitely generated commutative algebras. We then introduce abelian filtrations and prove that the standard filtration is abelian. We have tried to rely as little as possible on the general theory of algebras with polynomial identities (or PI–algebras), and the material included in this section reflects this desire.

In Section 2 we prove that the Hochschild homology of finite type algebras behaves well with respect to I–adic completions. This is done by comparing ordinary and topological resolutions. Topological resolutions require the framework of "relative homological algebra" [21], in which all resolutions must have continuous (or bounded) contractions. It turns out that this condition is automatically satisfied for finite type algebras and finitely generated resolutions, and this gives Theorem 3, after a sequence of Lemmata.

In Section 3 we use central localization to study the relation between abelian filtrations and Hochschild homology. An illustration of this relation is a certain analog of the Trace Paley-Wiener property [4], valid for semiprimitive finite type algebras. We conclude this section with a remark attesting to the complications one has to deal with when using Hochschild homology.

Section 4 contains the most significant results of this paper. We begin this section with a brief review of the needed results from cyclic homology. Then, in Theorem 8, we give a powerful criterion for two algebras to have the same periodic cyclic homology. Theorem 9 identifies the periodic cyclic homology of ideals of finitely generated commutative algebras with a relative cohomology group, generalizing results of [10, 15]. Our main result, Theorem 1, stated above, follows then by identifying the E^1–term of a general spectral sequence in periodic cyclic homology. (We actually prove a more general result, Theorem 10, which contains the result of Theorem 1 as a particular case.)

We hope that the results we have obtained can be used to obtain a better understanding of the admissible dual of a reductive p–adic group G. We also plan to use the results of this paper to obtain a description of the "K-theory" of $\hat{G}$ [18].

1. Filtrations of finite type algebras

In this section we introduce finite type algebras and abelian filtrations, and study their most basic properties. We shall also use this opportunity to fix notation and to recall a few facts and definitions.

Finite type algebras form a subclass of the class of algebras with polynomial identities (or PI–algebras). Although we tried to make this section as self–contained as possible, we have found it difficult, and not reasonable, to avoid two results: the existence of central polynomials and the Artin–Procesi's Theorem. A good reference to these results and to some other properties of PI–algebras is [27]. Some of the facts we present below have generalizations to PI–algebras, but they are usually more difficult to prove.

In what follows, $Z(\mathfrak{A})$ will denote the center of an algebra $\mathfrak{A}$. An algebra with unit will be called a *unital* algebra. All algebras considered in this paper will be complex algebras and all ideals will be two-sided ideals, unless otherwise stated. By $\mathbf{k}$ we shall denote a finitely generated commutative unital complex algebra; a $\mathbf{k}$–algebra is a (not necessarily unital) algebra together with a unital morphism $\mathbf{k} \to Z(M(A))$ to the center of $M(A)$, the multiplier algebra of A.

Definition 1. *A* finite type $\mathbf{k}$*–algebra is a (not necessarily unital)* $\mathbf{k}$*–algebra A that is a finitely generated* $\mathbf{k}$*–module.*

A *finite type* algebra is an algebra that is a finite type $\mathbf{k}$–algebra for some commutative unital finitely generated algebra $\mathbf{k}$. Clearly, the center Z of a unital finite type algebra A is a finitely generated algebra and A is a finite type Z–algebra, as well. Also, it follows from the definition that, if $J \subset A$ is a two-sided ideal of a finite type algebra, then A/J is also a finite type algebra.

Recall that a *polynomial identity algebra* (or a *PI–algebra*) is an algebra A for which there exists a non-zero complex polynomial $P(X_1, X_2, \ldots, X_m)$, in noncommuting variables $X_1, X_2, \ldots, X_m$, without constant term such that

$$P(a_1, a_2, \ldots, a_m) = 0,$$

for any $a_1, a_2, \cdots, a_m \in A$. The polynomial P is called an *identity* of A. A finite type $\mathbf{k}$–algebra A satisfies an alternating multilinear identity of degree $q+1$, where q is the number of elements in a system of generators of A as a $\mathbf{k}$–module, so finite type algebras are PI-algebras.

Recall that a primitive ideal of A is a two-sided ideal $\mathfrak{P} \subset A$ which is the kernel of a nonzero irreducible complex representation of A. Denote by $\mathrm{Prim}(A)$ the primitive ideal spectrum of A endowed with the Jacobson topology; the closed sets of the Jacobson topology are the sets of the form $V(S_0)$,

$$V(S_0) = \{\mathfrak{P} \in \mathrm{Prim}(A), S_0 \subset \mathfrak{P}\},$$

$S_0 \subset A$ arbitrary. The *Jacobson radical* of A, $\mathrm{Jac}(A)$, is the intersection of all primitive ideals of A.

Wedderburn's Theorem (see also [27, 1.5.16]) gives that, if $\mathfrak{Q}$ is a primitive ideal of a PI–algebra $\mathfrak{A}$, then there exists a division algebra $\mathbb{D}$ such that $\mathfrak{A}/\mathfrak{Q} \simeq M_n(\mathbb{D})$. Moreover, if $\mathfrak{A}$ satisfies a non-trivial identity of degree d and if $[\mathbb{D} : Z(\mathbb{D})] = r^2$, then $nr \leq d/2$. It follows that all irreducible representations of A will have dimensions uniformly bounded from above.

Lemma 1. *Let A be a unital finite type algebra A with center $Z = Z(A)$. Then*

(i) If $\mathfrak{P}$ is a primitive ideal of A, then $\mathfrak{P} \cap Z$ is a maximal ideal of Z and $A/\mathfrak{P} \simeq M_n(\mathbb{C})$, for some n; moreover, $n \leq C$, where C is a constant that depends only on A and not on $\mathfrak{P}$.

(ii) The "infinitesimal character" map,

$$\Theta : \mathrm{Prim}(A) \ni \mathfrak{P} \longrightarrow \mathfrak{p} = Z(A) \cap \mathfrak{P} \in \mathrm{Prim}(Z(A)), \tag{3}$$

satisfies $\Theta^{-1}(V(I)) = V(IA)$ for any ideal $I \subset Z$, and hence Θ is continuous and surjective.

(iii) $\mathrm{Jac}(A)$ *is nilpotent.*

(iv) If $J_1 \subset J_2$ are two–sided ideals of A and $V(J_1) = V(J_2)$, then J_2/J_1 is nilpotent.

Proof. (i) We observe that Z has finite or countable dimension, and hence A and any simple left A–module V also have finite or countable dimensions. The center Z_0 of $\mathrm{End}_A(V)$ is a field extension of $\mathbb{C}$, because V is irreducible. Since the only field extension of finite or countable dimension of $\mathbb{C}$ is $\mathbb{C}$ itself, we obtain that $Z \to Z_0$ is surjective, and hence $\mathfrak{P} \cap Z$, the kernel of $Z \to \mathrm{End}_A(V)$, is a maximal ideal of Z.

Hilbert's Nullstellensatz implies that $Z/Z \cap \mathfrak{P} \simeq \mathbb{C}$. Because $\mathbb{C}$ is the only finite dimensional complex division algebra, we obtain that $A/\mathfrak{P} \simeq M_n(\mathbb{C})$. Moreover, if A is generated as a Z–module by q elements, then $n \leq (q+1)/2$.

(ii) Let $I \subset Z$ be an ideal. We then have

$$\mathfrak{P} \in \Theta^{-1}(V(I)) \Leftrightarrow Z(A) \cap \mathfrak{P} \supset I \Leftrightarrow \mathfrak{P} \supset I \Leftrightarrow \mathfrak{P} \supset IA \Leftrightarrow \mathfrak{P} \in V(IA).$$

This proves that $\Theta^{-1}(V(I)) = V(IA)$. Since the closed subsets of the Jacobson topology on $\mathrm{Prim}(A)$ are the sets of the form $V(S_0)$, $S_0 \subset A$, the continuity of Θ follows.

We now prove the surjectivity of Θ. If $\mathfrak{p}$ is a maximal ideal of $Z(A)$, then $\mathfrak{p}A$ is a proper two–sided ideal of A and hence contained in a maximal *left* ideal $L \subset A$. The quotient A/L is consequently a simple left module of A and, if we let $\mathfrak{P}$ denote the annihilator of A/L, then $\mathfrak{P} \in V(\mathfrak{p}A)$, which hence is nonempty. This proves that

$$\Theta^{-1}(\mathfrak{p}) = \Theta^{-1}(V(\mathfrak{p})) = V(\mathfrak{p}A) \neq \emptyset,$$

so the map Θ is surjective.

(iii) The result is well known if A is commutative, so we shall assume the contrary.

Recall that the nilradical of an algebra is the sum of all nilpotent ideals. Since Z is noetherian and A is finitely generated, it follows that the nilradical of A, denoted $\mathrm{Nil}(A)$, is nilpotent. We claim that it is enough to show that Z contains non-zero nilpotent elements if $\mathrm{Jac}(A) \neq (0)$. Indeed, then $\mathrm{Jac}(A/\mathrm{Nil}(A)) = \mathrm{Jac}(A)/\mathrm{Nil}(A)$ and, if $\mathrm{Jac}(A) \neq \mathrm{Nil}(A)$, then $A/\mathrm{Nil}(A)$ contains non-trivial nilpotent ideals, contradicting the definition of $\mathrm{Nil}(A)$. It follows that $\mathrm{Jac}(A) = \mathrm{Nil}(A)$, and hence $\mathrm{Jac}(A)$ is nilpotent.

We now check that Z contains non-zero nilpotent elements if $\mathrm{Jac}(A) \neq (0)$. Define $A^{(0)} = A$ and $A^{(j+1)} = [A, A^{(j)}]$. Then $A^{(j+1)} = \{0\}$ for large j; in fact, this holds if A has a system of generators, as a Z–module, with less than j elements. Choose j to be the largest integer such that $A^{(j)} \neq 0$. If follows then that $A^{(j)}$ is a subset of Z, the center of A. Choose $a \in A^{(j)}$ different from 0. We shall prove that a is nilpotent. Indeed, it is enough to check that a belongs to each maximal ideal of Z. Choose then a maximal ideal $\mathfrak{p} \subset Z$ and a primitive ideal $\mathfrak{P}$ of A containing $\mathfrak{p}$, which exists using (ii) above. Using also (i), we obtain that $\mathfrak{P}$ intersects Z in a proper ideal containing $\mathfrak{p}$, and hence that $\mathfrak{p} = \mathfrak{P} \cap Z$, because $\mathfrak{p}$ is maximal. It will be then enough–in order to check that a belongs to $\mathfrak{p}$–to prove that $\pi(a) = 0$ for any irreducible representation π with kernel $\mathfrak{P}$.

Now, $\pi(A) \simeq M_n(\mathbb{C})$, and $\pi(a)$ is a scalar matrix. Because A is not commutative, $[A, A] \neq 0$, hence $j \geq 1$ and a is a linear combination of commutators. Consequently, $\pi(a)$ has trace zero. This proves that $\pi(a) = 0$, as desired.

(iv) follows from (iii) because J_2/J_1 is contained in the Jacobson radical of the finite type algebra A/J_1. □

If A is a finite type algebra we shall denote by $\mathrm{Prim}_n(A) \subset \mathrm{Prim}(A)$ the set of primitive ideals $\mathfrak{P} \subset A$ that are kernels of irreducible representations of A of dimension n.

The *PI–class* of a finite type algebra A is, by definition, the smallest integer n such that any identity of $M_n(\mathbb{C})$ is also an identity of A. The PI–class of A has representation theoretic significance because the largest integer n such that $\mathrm{Prim}_n(A)$ is not empty is the PI–class of $A/\mathrm{Jac}(A)$; this is proved using central polynomials.

A complex[1] polynomial $g_n(X_1, \ldots, X_m)$ with no constant term in the noncommutative variables $X_1, X_2, \ldots, X_m$ is called an *M_n–central polynomial* if and only if $g_n(a_1, \ldots, a_m)$ is a scalar matrix for any $a_1, \ldots, a_m \in M_n(\mathbb{C})$, but g_n is not identically 0 on $M_n(\mathbb{C})$. It is known that central polynomials exist for each n [27, 1.4.15] (this is a nontrivial fact). Fix an M_n-central polynomial g_n for each n.

Lemma 2. *Let A be a unital semiprimitive finite type algebra such that every irreducible representation of A has dimension $\leq n$. Then A has PI–class $\leq n$ and $g_n(A)$ is contained in the center of A.*

Proof. The kernel of the morphism $j : A \to \prod_{\mathfrak{P}} A/\mathfrak{P}$ is by definition the Jacobson radical of A, so it vanishes, by assumption. It follows that the algebra A embeds into the direct product $\prod_{\mathfrak{P}} A/\mathfrak{P}$. By assumption, any identity of $M_n(\mathbb{C})$ is satisfied by each $A/\mathfrak{P}$. In particular, g_n is either a central polynomial or an identity of each factor $A/\mathfrak{P}$; in both cases, $g_n(A/\mathfrak{P})$ consists of scalar matrices.

It follows that A has PI–class $\leq n$ and $g_n(A)$ is in the center of A. □

[1] For general algebras one has to consider polynomials with integer coefficients.

Let A be a finite type algebra and $Ag_n(A)A$ be the two sided ideal of A generated by $g_n(a_1,\dots,a_m)$ for all $a_1,\dots,a_m \in A$. Note that g_n is an identity for $M_k(\mathbb{C})$ for all $k < n$. If $\mathfrak{P}$ is a primitive ideal of A that is the kernel of an irreducible representation π, we denote by $d_{\mathfrak{P}}$ the dimension of the space on which π acts. If $\mathfrak{P}$ is a primitive ideal such that $d_{\mathfrak{P}} < n$, then g_n vanishes identically on $\pi(A)$, and hence $Ag_n(A)A \subset \mathfrak{P}$. Conversely, if $Ag_n(A)A \subset \mathfrak{P}$, for a primitive ideal $\mathfrak{P}$, then $A/\mathfrak{P} \simeq M_k(\mathbb{C})$, for some k, and we must have $k < n$, because g_n is not an identity of $M_n(\mathbb{C})$. We finally obtain that

$$X_k = \mathrm{Prim}_1(A) \cup \mathrm{Prim}_2(A) \cup \ldots \cup \mathrm{Prim}_k(A) = V(Ag_{k+1}(A)A), \tag{4}$$

and, consequently, X_k are closed subset of $\mathrm{Prim}(A)$ defining an increasing filtration of $\mathrm{Prim}(A)$. This filtration then defines a *decreasing* filtration by ideals of the PI–algebra A, called *the standard filtration* of A:

$$\begin{gathered} A = \mathfrak{I}_0^{\mathrm{st}} \supset \mathfrak{I}_1^{\mathrm{st}} \supset \ldots \supset \mathfrak{I}_n^{\mathrm{st}} = \mathrm{Jac}(A) \supset (0), \\ \mathfrak{I}_k^{\mathrm{st}} = \mathrm{Jac}(A/Ag_{k+1}(A)A) = \bigcap_{d_{\mathfrak{P}} \le k} \mathfrak{P}, \text{ and } \quad X_k = V(\mathfrak{I}_k^{\mathrm{st}}). \end{gathered} \tag{5}$$

Thus $\mathfrak{I}_k^{\mathrm{st}}$ is the intersection of the kernels of all irreducible representations of dimension $\le k$. Note that the ideals $\mathfrak{I}_k^{\mathrm{st}}$ are not distinct in general, but that $\mathfrak{I}_n^{\mathrm{st}} \neq \mathfrak{I}_{n-1}^{\mathrm{st}}$, if the PI–class of $A/\mathrm{Jac}(A)$ is n.

Definition 2. *An* Azumaya algebra *of rank k over the commutative algebra Z is a unital Z–algebra A which is a finitely generated projective module of rank k over Z such that*

$$A \otimes_Z A^{\mathrm{op}} \simeq \mathrm{End}_Z(A).$$

It follows that Z coincides with the center of A. The dimensions $[A_{\mathfrak{p}} : Z_{\mathfrak{p}}]$ of the localizations $A_{\mathfrak{p}} = (Z \setminus \mathfrak{p})^{-1}A$ are equal to k, for all maximal ideals $\mathfrak{p} \subset A$, and $k = n^2$, where n is the PI–class of A. Moreover, the central character defines a homeomorphism $\Theta : \mathrm{Prim}(A) \to \mathrm{Prim}(Z)$, see [23]. Azumaya algebras will play an important rôle in what follows, because of the following basic result.

Theorem 2 (Artin-Procesi). *Suppose that $\mathfrak{A}$ is a unital PI–algebra of PI–class n and that $\mathrm{Prim}(\mathfrak{A}) = \mathrm{Prim}_n(\mathfrak{A})$, then $\mathfrak{A}$ is an Azumaya algebra of rank n^2.*

See [27, 1.8.48] or the original papers [1, 24] for a proof.

We now introduce abelian filtrations and prove, as a consequence of the Artin-Procesi theorem, that the standard filtration, equation (5), is abelian.

Definition 3. *A finite decreasing sequence*

$$A = \mathfrak{I}_0 \supset \mathfrak{I}_1 \supset \ldots \supset \mathfrak{I}_{n-1} \supset \mathfrak{I}_n = \mathrm{Jac}(A)$$

of two-sided ideals of a unital finite type algebra A is an abelian filtration *if and only if the following three conditions are satisfied for each k:*

(i) The quotient $A/\mathfrak{I}_k$ is semiprimitive.

(ii) For each maximal ideal $\mathfrak{p} \subset Z_k = Z(A/\mathfrak{I}_k)$ not containing $I_{k-1} = Z_k \cap (\mathfrak{I}_{k-1}/\mathfrak{I}_k)$, the localization $(A/\mathfrak{I}_k)_{\mathfrak{p}} = (Z_k \setminus \mathfrak{p})^{-1}(A/\mathfrak{I}_k)$ is an Azumaya algebra over $(Z_k)_{\mathfrak{p}}$,

(iii) The quotient $(\mathfrak{I}_{k-1}/\mathfrak{I}_k)/I_{k-1}(A/\mathfrak{I}_k)$ is nilpotent.

If $(\mathfrak{I}_k)$ is an abelian filtration of A, then the central characters define bijections

$$\mathrm{Prim}(\mathfrak{I}_{k-1}/\mathfrak{I}_k) \to \mathrm{Prim}(Z_k) \setminus V(I_{k-1}),$$

by Lemma 1. This suggests that the quotients $\mathfrak{I}_{k-1}/\mathfrak{I}_k$ behave like commutative algebras, and, in a certain sense, we will show that this is indeed true.

Any filtration of a commutative algebra by radical ideals is an abelian filtration, and any refinement of an abelian filtration is again an abelian filtration, provided (i) is satisfied.

Proposition 1. *The standard filtration, $(\mathfrak{I}_k^{\mathrm{st}})$, equation (5), is an abelian filtration.*

Proof. We need to check that the three conditions of Definition 3 are satisfied. Condition (i) is satisfied because $\mathfrak{I}_k^{\mathrm{st}}$ is defined as an intersection of primitive ideals.

By replacing A with $A/\mathfrak{I}_k^{\mathrm{st}}$, we may assume that $\mathfrak{I}_k^{\mathrm{st}} = (0)$ and that $Z_k = Z$, the center of A. We then obtain that $A_{\mathfrak{p}}$ has center $Z_{\mathfrak{p}}$ (see [27, 1.7.11] for a proof of this easy fact) and that

$$\mathrm{Prim}_k(A_{\mathfrak{p}}) = \mathrm{Prim}_k(A) \cap \Theta^{-1}(\mathfrak{p}) = \mathrm{Prim}(A) \cap \Theta^{-1}(\mathfrak{p}) = \mathrm{Prim}(A_{\mathfrak{p}}).$$

Condition (ii) then follows from Artin-Procesi's theorem.

The last condition, (iii), is obtained as follows. First, we have that $g_k(A) \subset \mathfrak{I}_{k-1}^{\mathrm{st}}$, so $g_k(A) \subset I_{k-1}$ and hence $V(Ag_k(A)A) \supset V(I_{k-1}A)$. On the other hand, $I_{k-1}A \subset \mathfrak{I}_{k-1}$ and, consequently,

$$V(Ag_k(A)A) = V(\mathfrak{I}_{k-1}) \subset V(I_{k-1}A).$$

This proves that $V(\mathfrak{I}_{k-1}) = V(I_{k-1}A)$, and hence that $\mathfrak{I}_{k-1}/I_{k-1}A$ is the Jacobson radical of $A/I_{k-1}A$. Lemma 1 (iv) finally gives the desired result. $\square$

We now introduce a class of examples that will be used several times in the paper to illustrate various results.

Let $k \geq 1$ and let $A_k \subset M_2(\mathbb{C}[X])$ be the algebra of matrices $P = [p_{ij}]$ such that X^k divides p_{21}. Then A_k is a finite type algebra of PI–class 2. The standard filtration of A_k is

$$\mathfrak{I}_1^{\mathrm{st}} = \{P = [p_{ij}] \in A, X|p_{11}, \text{ and } X|p_{22}\},$$

$\mathfrak{I}_2^{\mathrm{st}} = (0)$. Also, using the notation of Definition 3, $Z_1 \simeq \mathbb{C} \oplus \mathbb{C}$, $Z_2 \simeq \mathbb{C}[X]$ and $I_1 = X\mathbb{C}[X] = \mathfrak{I}_1^{\mathrm{st}} \cap Z_2$.

2. Hochschild homology and completion

Fix throughout this section a unital finite type $\mathbf{k}$–algebra A with center Z and an ideal $I \subset \mathbf{k}$. For any $\mathbf{k}$–module the products $I^n M$ define a filtration, the I–adic filtration, of M. In this section we shall study the effect of I–adic completion on Hochschild homology. These results are used in the last section.

Consider a vector space V endowed with a decreasing filtration

$$V = F_0 V \supset F_1 V \supset \ldots \supset F_n V \supset \ldots ,$$

and completion $\hat{V} = \lim\limits_{\leftarrow} V/F_n V$. A linear map $\phi : V \to W$ between two filtered vector spaces is called *bounded* if there is an integer k such that $\phi(F_n V) \subset F_{n-k} W$ for all n. A bounded map ϕ will extend to a linear map $\hat{\phi} : \hat{V} \to \hat{W}$ between completions. We shall use a few standard results on filtered modules and their completions (such as Artin-Rees' Lemma, the exactness of $-\otimes_{\mathbf{k}}\hat{\mathbf{k}}$, and $\hat{M} \simeq M\otimes_{\mathbf{k}}\hat{\mathbf{k}}$, if M is finitely generated) that can be found in [2, 6].

We agree that all modules considered below are *left* modules, unless otherwise specified.

Lemma 3. *Let M be a finitely generated A–module. Then M has a resolution $(A\otimes V_i, d_i)$ by finitely generated free A–modules. Moreover, for any such resolution, the I–adically complete complex $(\hat{A}\otimes V_i, \hat{d}_i)$ has a contraction $s = (s_i)$ consisting of bounded maps $s_i : \hat{A}\otimes V_i \to \hat{A}\otimes V_{i+1}$.*

Proof. Since M is finitely generated, we can find a finite dimensional space V_0 and a surjective A–linear morphism $d_0 : A\otimes V_0 \to M$.

Suppose now that we have constructed finite dimensional vector spaces $V_0, \dots, V_n$ and A–linear maps $d_0 : A\otimes V_0 \to M$, and $d_i : A\otimes V_i \to A\otimes V_{i-1}$ satisfying $d_{i-1}d_i = 0$ for all $i \leq n$, and such that

$$M \xleftarrow{d_0} A\otimes V_0 \xleftarrow{d_1} A\otimes V_1 \xleftarrow{d_2} \dots \xleftarrow{d_n} A\otimes V_n$$

is exact. Then $\ker(d_n)$ is a $\mathbf{k}$–submodule of a finitely generated $\mathbf{k}$–module, and hence $\ker(d_n)$ is itself a finitely generated $\mathbf{k}$–module because $\mathbf{k}$ is noetherian. Any set $\xi_1, \dots, \xi_m$ of generators of $\ker(d_n)$ as a $\mathbf{k}$–module will generate $\ker(d_n)$ as an A–module, so, if we define $V_{n+1} = \mathbb{C}^m$, with basis $e_1, \dots, e_m$ and $d_{n+1}(a\otimes e_k) = a\xi_k$, then $\ker(d_n) = \mathrm{Im}(d_{n+1})$. Continuing this procedure, we obtain the desired resolution.

The existence of the contraction s is an application of Artin-Rees' Lemma. Since the contraction s will be defined as an infinite sum, we replace the complex $(A\otimes V_i, d_i)$ with its completion, $(\hat{A}\otimes V_i, \hat{d}_i)$. We need to define $\mathbb{C}$–linear maps $s_{-1} : \hat{M} \to \hat{A}\otimes V_0$ and $s_n : \hat{A}\otimes V_n \to \hat{A}\otimes V_{n+1}$ satisfying $d_0\circ s_{-1} = 1$ and $d_{i+1}s_i + s_{i-1}d_i = 1$ for all $i \geq 0$.

Now, for each i, we have two filtrations on $\ker(d_i) = \mathrm{Im}(d_{i+1})$: the one induced from $\hat{A}\otimes V_i$,

$$F_n \ker(d_i) = (I^n\hat{A}\otimes V_i)\cap\ker(d_i),$$

and the one induced from $\hat{A}\otimes V_{i+1}$,

$$F_n' \ker(d_i) = d_{i+1}(I^n\hat{A}\otimes V_{i+1}) = I^n d_{i+1}(\hat{A}\otimes V_{i+1}).$$

¿From the $\hat{A}$–linearity of d_{i+1}, we see that $F_n'\ker(d_i) \subset F_n\ker(d_i)$. All modules in sight are finitely generated $\mathbf{k}$–modules, so Artin-Rees' Lemma [2, 6] may be used to find p such that $F_n\ker(d_i) \subset F_{n-p}'\ker(d_i)$. We now define a bounded contraction as follows.

First choose for each $n \geq 0$ elements $(v_l^{(n)})_{l\geq 0}$ of $F_n\ker(d_i)$ which project onto a basis of $F_n\ker(d_i)/F_{n+1}\ker(d_i)$. Then choose lifts $(\overline{v}_l^{(n)})_l \in I^{n-p}(\hat{A}\otimes V_{i+1})$, $d_{i+1}(\overline{v}_l^{(n)}) = v_l^{(n)}$.

By the exactness of the completion functor for finitely generated A–modules, we have that the kernel of $\hat{d}_i$ is the completion of the kernel of d_i. Using this fact, we obtain that each element $x \in \hat{A}\otimes V_i$ that is in the kernel of $\hat{d}_i$ can uniquely be expressed as a convergent series

$$x = \sum_{n=0}^{\infty}\sum_l \lambda_{n,l}v_l^{(n)}\ , \quad \lambda_{n,l}\in\mathbb{C},$$

where $\lambda_{n,l}$ are zero, except finitely many, for each fixed n. Then define $s_i(x)$, for $x \in \ker(\hat{d}_i)$ as above, by the series

$$s_i(x) = \sum_{n=0}^{\infty} \sum_{l} \lambda_{n,l} \overline{v}_l^{(n)},$$

which is still convergent (the definition of s_i as an infinite sum explains why we needed to replace the complex $(A \otimes V_i, d_i)$ by its completion). We then extend s_i to $\hat{A} \otimes V_i$ by the formula $s_i(v) = s_i(v - s_{i-1} d_i(v))$. Since, by induction, s_{i-1} is bounded, s_i will also be bounded. This completes the proof. $\square$

We need the following generalization of a well known result in commutative algebra.

Proposition 2. *Let A be a unital finite type* $\mathbf{k}$*–algebra. If N, respectively M, are a right, respectively left, finitely generated A–module, then $\mathrm{Tor}_q^A(N, M)$ is a finitely generated* $\mathbf{k}$*–module, for each q.*

Proof. Choose, using Lemma 3, a resolution $(A \otimes V_i, d_i)$ of M by free finitely generated left A–modules. Then the complex $(N \otimes V_i, 1 \otimes d_i)$ is a complex of finitely generated $\mathbf{k}$–modules, whose homology is $\mathrm{Tor}_*^A(N, M)$. Since $\mathbf{k}$ is noetherian, the result follows. $\square$

Let A^{op} be the algebra A with the opposite multiplication and $A^e = A \otimes A^{\mathrm{op}}$, so that A becomes a left A^e–module. Then $\mathrm{HH}_*(A) = \oplus_q \mathrm{HH}_q(A)$, the *Hochschild homology* groups of the unital algebra A are, by definition [21],

$$\mathrm{HH}_q(A) \simeq \mathrm{Tor}_q^{A^e}(A, A).$$

The groups $\mathrm{HH}_q(A)$ can be computed as the homology groups of the complex $(A^{\otimes n+1}, b)$, where

$$\begin{aligned} b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \qquad \text{and} \\ b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) + (-1)^n a_n a_0 \otimes \ldots \otimes a_{n-1}. \end{aligned} \tag{6}$$

If A is a $\mathbf{k}$–algebra, for some commutative ring $\mathbf{k}$, then the differential b is $\mathbf{k}$–linear, and hence each $\mathrm{HH}_q(A)$ is naturally a $\mathbf{k}$–module.

Corollary 1. *If A is a unital finite type* $\mathbf{k}$*–algebra, then $\mathrm{HH}_q(A)$ is a finitely generated* $\mathbf{k}$*–module, for each q.*

Proof. Use Proposition 2 for $M = N = A$ and for A^e in place of A. $\square$

If V and W are filtered vector spaces, we let

$$V \hat{\otimes} W = \lim_{\leftarrow} V/F_n V \otimes W/F_n W.$$

(Note that this is the completion of $V \otimes W$ by the filtration induced by the images of $\sum_{i+j=n} F_i V \otimes F_j W$, for all n.)

Similarly, if M and N are a right, respectively a left, A–module endowed with the I–adic filtration, then we define

$$M \hat{\otimes}_A N = \lim_{\leftarrow} M/I^n M \otimes_A N/I^n N.$$

The basic results of homological algebra extend to filtered vector spaces and A–modules if one is careful to use only *admissible* resolutions, that is resolutions that have *bounded* $\mathbb{C}$–linear contractions, in the spirit of relative homological algebra. By Lemma 3, the completion of every resolution by finitely generated free modules is admissible, and this is essential for our argument.

The straightforward proof of the following result is the same as in the case of modules without filtrations [21].

Lemma 4. *Let V_n and V_n' be filtered vector spaces and $(\hat{A}\hat{\otimes}V_n, d_n)$ and $(\hat{A}\hat{\otimes}V_n', d_n')$ be two admissible resolutions (i.e., with bounded contractions) of an $\hat{A}$–module. Then there exists an $\hat{A}$–linear bounded morphism of resolutions $\phi : \hat{A}\hat{\otimes}V_\bullet \to \hat{A}\hat{\otimes}V_\bullet'$, uniquely determined up to homotopy.*

If $\psi_1, \psi_2 : (M_n, d_n) \to (M_n', d_n')$ are two complexes of right A–modules that are homotopic via a bounded homotopy $s : M_\bullet \to M_{\bullet+1}'$, then

$$\psi_1\hat{\otimes}1, \psi_2\hat{\otimes}1 : (M_n\hat{\otimes}_A N, d_n\hat{\otimes}1) \to (M_n'\hat{\otimes}_A N, d_n'\hat{\otimes}1)$$

are homotopic for any left A–module N.

Let A be a $\mathbf{k}$–algebra and I be an ideal of $\mathbf{k}$. Then the *I–adic Hochschild complex* [29] is

$$\hat{A} \xleftarrow{b} A\hat{\otimes}A \xleftarrow{b} A^{\hat{\otimes}3} \xleftarrow{b} \ldots \xleftarrow{b} A^{\hat{\otimes}n+1} \xleftarrow{b} \ldots \tag{7}$$

where $A^{\hat{\otimes}q} = \varprojlim (A/I^nA)^{\otimes q}$ is the topological tensor product in the category of complete modules. Denote by $\mathrm{HH}_*^{\mathrm{top}}(\hat{A})$ the homology of the I-adic Hochschild complex. The ordinary, i.e., uncomplete, Hochschild complex maps canonically to the topological Hochschild complex, and hence we have a natural map

$$\mathrm{HH}_q(A) \to \mathrm{HH}_q^{\mathrm{top}}(\hat{A}). \tag{8}$$

The $\hat{\mathbf{k}}$–linearity of b guaranties that each $\mathrm{HH}_q^{\mathrm{top}}(\hat{A})$ is a $\hat{\mathbf{k}}$–module and that the map of equation (8) is $\mathbf{k}$–linear.

Theorem 3. *Suppose that A is a unital finite type $\mathbf{k}$–algebra and $I \subset \mathbf{k}$ is an ideal. Then the natural map $\mathrm{HH}_*(A) \to \mathrm{HH}_*^{\mathrm{top}}(\hat{A})$ and the $\mathbf{k}$–module structure on $\mathrm{HH}_*(A)$ define an isomorphism*

$$\mathrm{HH}_*(A) \otimes_{\mathbf{k}} \hat{\mathbf{k}} \simeq \mathrm{HH}_*^{\mathrm{top}}(\hat{A})$$

of $\hat{\mathbf{k}}$–modules.

In particular, using also Corollary 1, we obtain that the I–adic Hochschild homology groups of A are the I–adic completions of the Hochschild homology groups of A, if A is as above.

Proof. We can assume that $\mathbf{k}$ is the center Z of A. Consider the ideal $I_2 = I \otimes Z + Z \otimes I \subset Z \otimes Z$ and endow $Z^e = Z \otimes Z$ and A^e with the I_2–filtrations. We shall denote by $\hat{Z}^e$ and $\hat{A}^e$ their completions. The assumptions of the theorem imply that A^e is a finitely generated Z^e–module so we can use the above lemmata with Z^e in place of Z and A^e in place of A.

By Lemma 3 we can find a resolution

$$A \xleftarrow{d_0} A^e \otimes V_0 \xleftarrow{d_1} A^e \otimes V_1 \xleftarrow{d_2} \ldots \xleftarrow{d_n} A^e \otimes V_n \longleftarrow \ldots \tag{9}$$

of A with finitely generated free A^e–modules. Since Z^e is Noetherian, tensoring with $\hat{Z}^e$ is exact [2, 6], and hence we obtain an exact sequence

$$\hat{A} \xleftarrow{d_0} \hat{A}^e \otimes V_0 \xleftarrow{d_1} \hat{A}^e \otimes V_1 \xleftarrow{d_2} \ldots \xleftarrow{d_n} \hat{A}^e \otimes V_n \longleftarrow \quad . \tag{10}$$

By Lemma 3 the complex (10) has a bounded contraction. The I–adic b'–complex

$$\hat{A} \xleftarrow{b'} A \hat{\otimes} A \xleftarrow{b'} A^{\hat{\otimes} 3} \xleftarrow{b'} \ldots \xleftarrow{b'} A^{\hat{\otimes} n+1} \xleftarrow{b'} \ldots \tag{11}$$

also has a bounded contraction s, $s(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = 1 \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n$.

¿From Lemma 4 we obtain that the b'–complex and the complex (10) are homotopic equivalent via a $\hat{A}^e$–linear homotopy $\phi = (\phi_n)$.

$$\phi_n : (\hat{A}^e \otimes V_n, d_n) \longrightarrow (A^{\hat{\otimes} n+1}, b'). \tag{12}$$

Tensoring with $\hat{A}$ over $\hat{A}^e$ we obtain, again from Lemma 4, a homotopy equivalence

$$\phi \hat{\otimes} 1 : (\hat{A} \otimes V_n, d_n \hat{\otimes} 1) \longrightarrow (A^{\hat{\otimes} n+1}, b). \tag{13}$$

We claim the homology groups of the two complexes in (13) are exactly the groups appearing in the statement of the theorem.

Indeed, the homology groups of the complex $(A \otimes V_n, d_n \otimes 1)$ are isomorphic to $\mathrm{HH}_*(A)$, since HH_* is a derived functor (it is a Tor–group)). From this, using the exactness of the functor $\hat{Z} \otimes_Z -$, we obtain that the homology groups of

$$(\hat{A} \otimes V_n, d \hat{\otimes} 1) = \hat{Z} \hat{\otimes}_Z (A \otimes V_n, d_n \otimes 1)$$

are isomorphic to $\mathrm{HH}_*(A) \otimes_Z \hat{Z}$. This takes care of the first complex.

The second complex is the I–adic Hochschild complex whose homology groups are $\mathrm{HH}_*^{\mathrm{top}}(\hat{A})$, by definition. The desired isomorphism is obtained then from (13). The proof of the theorem is now complete. □

As a corollary, we obtain the following result of [15]. Let $\mathcal{O}(X)$ be the ring of regular functions on an affine algebraic variety X. Denote by $\Omega^l(X)$ the space of l–forms on X, if X is smooth. If I is an ideal of $\mathcal{O}(X)$ defining the subvariety $Y \subset X$, let

$$\hat{\Omega}^l(\hat{Y}) = \lim_{\leftarrow} \Omega^l(X)/I^n \Omega^l(X), \quad n \to \infty.$$

We shall denote by χ the Hochschild-Kostant-Rosenberg map

$$\chi(a_0 \otimes \ldots \otimes a_k) \to \frac{1}{k!} a_0 da_1 \ldots da_k. \tag{14}$$

Corollary 2. *Let $Z = \mathcal{O}(X)$ be the ring of regular functions on a affine smooth complex algebraic variety X and $I \subset Z$ be an ideal defining a subvariety $Y \subset X$. Then the Hochschild-Kostant-Rosenberg map χ induces an isomorphism*

$$\mathrm{HH}_l^{\mathrm{top}}(\hat{Z}) = \hat{\Omega}^l(\hat{Y}).$$

Proof. The classical Hochschild-Kostant-Rosenberg result [14] states that χ establishes an isomorphism $\mathrm{HH}_l(Z) = \Omega^l(X)$. Then, using Theorem 3 for $A = Z$, we obtain

$$\mathrm{HH}_q^{\mathrm{top}}(\hat{Z}) \simeq \mathrm{HH}_l(Z) \otimes_Z \hat{Z} \simeq \Omega^l(X) \otimes_Z \hat{Z} \simeq \lim_{\leftarrow} \Omega^l(X)/I^n \Omega^l(X),$$

because $\Omega^l(X)$ is a finitely generated Z–module. □

3. Hochschild homology and localization

In this section we use central localization techniques to establish a Trace Paley-Wiener Property for general finite type algebras. Central localization will play an important rôle in the last section.

Definition 4. *If A is a unital finite type algebra with an abelian filtration $(\mathfrak{I}_k)$, then the commutative algebras $Z_k = Z(A/\mathfrak{I}_k)$ will be called the* subcenters *of the the filtration, and the ideals $I_{k-1} = Z_k \cap (\mathfrak{I}_{k-1}/\mathfrak{I}_k)$ will be called the* reducibility *ideals of the filtration.*

We fix *throughout* this section a unital finite type algebra A with an abelian filtration, $(\mathfrak{I}_k)$, and we shall use the notation of Definition 4. The first integer n such that $\mathfrak{I}_n = \mathrm{Jac}(A)$ will be called the *length of the filtration* $(\mathfrak{I}_k)$.

In the proof of the following theorem we shall use the Künneth theorem for Hochschild homology [21] to conclude that the diagonal embedding $a \to a \otimes E_q$ of A into $M_q(A)$ induces an isomorphism $\mathrm{HH}_*(A) \simeq \mathrm{HH}_*(M_q(A))$ (cf. [19], corollary 1.2.14). Here E_q denotes the multiplicative unit of $M_q(\mathbb{C})$, and we regard $M_q(A)$ as $A \otimes M_q(\mathbb{C})$.

Proposition 3. *Let A be an Azumaya algebra over Z. Then the inclusion $j : Z \to A$ defines an isomorphism $j_* : \mathrm{HH}_*(Z) \to \mathrm{HH}_*(A)$.*

Proof. Let V be a projective Z–module such that $A \otimes_Z V \simeq Z^q$ is free. The existence of V follows from [3], page 476. Left multiplication defines a morphism of A into Z–linear endomorphisms of $A \otimes_Z V$ which then gives a morphism

$$\phi : A \ni a \longrightarrow a \otimes 1 \in A \otimes_Z (A^{\mathrm{op}} \otimes_Z \mathrm{End}_Z(V)) = \mathrm{End}_Z(A \otimes_Z V) \simeq M_q(Z) \tag{15}$$

where we have identified $A \otimes_Z A^{\mathrm{op}}$ with $\mathrm{End}_Z(A)$.

We will show that ϕ induces a map, ϕ_*, inverse to j_*, if we identify $\mathrm{HH}_*(M_q(Z))$ with $\mathrm{HH}_*(Z)$. This will be obtained by studying the compositions $\phi \circ j$ and $j \circ \phi$. Observe first that both j and ϕ are Z–linear.

The composition $\phi \circ j : Z \to M_q(Z)$ is Z–linear and unital, so $\phi \circ j(z) = z \otimes E_q$, where E_q is the unit of $M_q(Z)$. It follows that the map $\phi \circ j$ induces an isomorphism $\phi_* \circ j_* : \mathrm{HH}_*(Z) \simeq \mathrm{HH}_*(M_q(Z))$.

The other composition, $j \circ \phi : A \to M_q(A) = M_q(Z) \otimes_Z A$, can be written, up to an automorphism of $M_q(Z)$, as

$$A \ni a \longrightarrow a \otimes 1 \otimes 1 \in A \otimes_Z (A^{\mathrm{op}} \otimes_Z \mathrm{End}_Z(V)) \otimes_Z A \simeq A \otimes_Z M_q(Z). \tag{16}$$

We obtain that the map $j \circ \phi$ is given, up to an automorphism of the algebra $M_q(A) \simeq A \otimes_Z M_q(Z)$, by $j \circ \phi(a) = aE_q$. This, in turn, implies that $j_* \circ \phi_*$ is also an isomorphism. Since $\phi_* \circ j_*$ is the identity (up to a canonical identification) the result follows. $\square$

In the proof of the next proposition we shall use the following result of [7], which can also be used to give an alternative proof of the previous proposition.

Proposition 4 (Brylinski)**.** *Let S be a multiplicative subset of the center Z of the algebra A. Then $\mathrm{HH}_*(S^{-1}A) \simeq S^{-1}\,\mathrm{HH}_*(A)$.*

The following proposition relates the Hochschild homology of a finite type algebra to the structure of an abelian filtration.

Proposition 5. *If $(\mathfrak{I}_k)$ is an abelian filtration of a unital finite type algebra A, $Z_k = Z(A/\mathfrak{I}_k)$, $I_{k-1} = Z_k \cap (\mathfrak{I}_{k-1}/\mathfrak{I}_k)$, and $\mathfrak{p} \subset Z_k$ is a maximal ideal not containing I_{k-1}, then the inclusion $Z_k \to A/\mathfrak{I}_k$ induces an isomorphism*

$$\mathrm{HH}_*(Z_k)_\mathfrak{p} \to \mathrm{HH}_*(A/\mathfrak{I}_k)_\mathfrak{p}.$$

Proof. By replacing A by $A/\mathfrak{I}_k$, we may assume that $\mathfrak{I}_k = (0)$ and that $Z_k = Z$, the center of A. From the definition of abelian filtrations, Definition 3, we know that $A_\mathfrak{p}$ is an Azumaya with center $Z_\mathfrak{p}$. Combining this with Brylinski's result, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{HH}_*(Z)_\mathfrak{p} & \longrightarrow & \mathrm{HH}_*(A)_\mathfrak{p} \\ \downarrow & & \downarrow \\ \mathrm{HH}_*(Z_\mathfrak{p}) & \longrightarrow & \mathrm{HH}_*(A_\mathfrak{p}) \end{array}$$

in which the bottom and the vertical arrows are isomorphisms (Propositions 3 and 4). It follows that the top arrow is also an isomorphism, as stated. □

As an application of abelian filtrations, using localization techniques, we now discuss a generalization of the Trace Paley-Wiener property to the setting of general finite type algebras.

To any abelian filtration $(\mathfrak{I}_k)$ of lenght n of a semiprimitive finite type algebra A we associate a filtration of $\mathrm{HH}_0(A)$ by $Z = Z(A)$–modules,

$$0 = M_n \subset L_n \subset \ldots \subset M_1 \subset L_1 = \mathrm{HH}_0(A) = A/[A,A], \tag{17}$$

defined by induction on the length n of the filtration $(\mathfrak{I}_k)$. If $n = 1$, that is if A is an Azumaya algebra over Z, we let $M_1 = 0$ and $L_1 = A = \mathrm{HH}_0(A) = Z$. Assume now that the modules of (17) were defined for all semiprimitive finite type algebras and all abelian filtrations of length $n-1$, and proceed to define (17) for a filtration of length n, $(\mathfrak{I}_k)$, of A. The projection $\pi : A \to A/\mathfrak{I}_{n-1}$ induces a filtration $(\pi(\mathfrak{I}_k))$ of length $n-1$ of the algebra $A/\mathfrak{I}_{n-1}$, therefore, by induction, we have obtained Z–modules

$$0 = M'_{n-1} \subset L'_{n-1} \subset \ldots \subset M'_1 \subset L'_1 = \mathrm{HH}_0(A/\mathfrak{I}_{n-1}).$$

Then, we let $M_l = \pi_*^{-1}(M'_l)$ and $L_l = \pi_*^{-1}(L'_l)$, for $l \leq n-1$, where $\pi_* : \mathrm{HH}_0(A) \to \mathrm{HH}_0(A/\mathfrak{I}_{n-1})$ is the induced morphism. Also we let $M_n = 0$. The only group not yet defined is L_n, and we let L_n to be the image of $Z \cap \mathfrak{I}_{n-1}$ in $\mathrm{HH}_0(A)$ induced by the inclusion $Z \to A$.

We shall analyse the filtration (17) with the help of the following lemma, valid for not necessarily semiprimitive algebras A.

Lemma 5. *Let $(\mathfrak{I}_k)$ be an abelian filtration of length n of the unital finite type algebra A. If $j : Z \to A$ is the inclusion of the center and $I = \mathfrak{I}_{n-1} \cap Z$, then the map $j_* : \mathrm{HH}_l(Z) \to \mathrm{HH}_l(A)$ satisfies*

(i) $I^k \ker(j_*) = I^k \operatorname{coker}(j_*) = 0$, for some k that depends on l.

(ii) The kernel of the map $j_* : Z = \mathrm{HH}_0(Z) \to \mathrm{HH}_0(A)$ consists of nilpotent elements.

(iii) The support of $\mathrm{HH}_0(A)/j_*(\mathrm{HH}_0(Z))$ is nowhere dense in $\mathrm{Prim}(Z)$.

Proof. Both $\mathrm{HH}_l(Z)$ and $\mathrm{HH}_l(A)$ are finitely generated Z–modules by Corollary 1, and $\mathrm{HH}_l(Z)_\mathfrak{p} \simeq \mathrm{HH}_l(A)_\mathfrak{p}$ for maximal ideals $\mathfrak{p} \subset Z$ not containing I (Proposition 5). Since Z is noetherian, this is enough to conclude (i).

It follows from definition that $\mathrm{HH}_0(Z) = Z$. Assume first that Z is a domain. Then, applying (i) to the standard filtration, equation (5), and using that the annihilator of I^k vanishes, we obtain (ii).

In general, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ be the minimal prime ideals of Z so that $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_r = \mathrm{Nil}(Z)$, the radical of A. Since each $Z/\mathfrak{q}_j$ is a domain, the map $Z/\mathfrak{q}_j \to \mathrm{HH}_0(A/\mathfrak{q}_j A)$ is injective, and hence the kernel of the composite map

$$Z \to \mathrm{HH}_0(A) \to \oplus_j \mathrm{HH}_0(A/\mathfrak{q}_j A)$$

is $\mathrm{Nil}(Z)$.From this (ii) follows.

Again (iii) follows from (i) if Z is a domain. This shows that the support of $\mathrm{HH}_0(A)/j_*(\mathrm{HH}_0(Z))$ contains no irreducible component of $\mathrm{Prim}(Z)$. Since $\mathrm{Prim}(Z)$ is a noetherian space, the result follows. □

Lemma 5 can be used to identify some subquotients of the filtration (17). We shall use the notation of Definition 4.

Corollary 3. *If $(\mathfrak{I}_k)$ is an abelian filtration, then the subquotients L_k/M_k of (17) are isomorphic to the reducibility ideals $L_k/M_k \simeq I_{k-1} = Z_k \cap (\mathfrak{I}_{k-1}/\mathfrak{I}_k)$.*

Proof. For each k the map $\mathrm{HH}_0(A) \to \mathrm{HH}_0(A/\mathfrak{I}_k)$ is surjective; then apply Lemma 5 (ii) to the center Z_k of the quotient $A/\mathfrak{I}_k$. □

In particular, if A is a unital semiprimitive finite type algebra and the inclusion $Z \to A$ induces an isomorphism $\mathrm{HH}_0(Z) \simeq \mathrm{HH}_0(A)$, then A is an Azumaya algebra, and consequently the central character $\Theta : \mathrm{Prim}(A) \to \mathrm{Prim}(Z)$, equation (3), is a homeomorphism.

Suppose now that the quotients $A/\mathfrak{I}_k$, for $k = 1, \ldots, n$, are *algebras with trace*, in the sense that there exist Z_k–linear maps $\mathrm{Tr}_k : A/\mathfrak{I}_k \to Z_k$ that vanish on commutators and are normalized such that $\mathrm{Tr}_k(1) = 1$. See [25]. By Lemma 5 (iii), Tr_k, if it exists, is uniquely determined and is the identity on Z_k. By abuse of notation, we shall denote by Tr_k the induced map $A \to Z_k$, as well. Define then the *total trace* map,

$$\mathrm{TR} : \mathrm{HH}_0(A) \to \bigoplus_{k=1}^{n} Z_k, \tag{18}$$

by $\mathrm{TR} = \oplus \mathrm{Tr}_k$. We want to describe the image and kernel of the total trace TR. To this end we need the following lemma.

We shall use the notation of Definition 4.

Lemma 6. *We have that $\mathrm{Tr}_k(\mathfrak{I}_{k-1}/\mathfrak{I}_k) \subset I_{k-1}$.*

Proof. By replacing A with $A/\mathfrak{I}_k$, we may assume that $\mathfrak{I}_k = (0)$ and that $Z_k = Z$, the center of A. Let $\mathfrak{p} \subset Z$ be a maximal ideal containing $I_{k-1} = \mathfrak{I}_{k-1} \cap Z$. Since A is a finite type algebra, the quotient $A/\mathfrak{p}A$ is a finite dimensional $Z/\mathfrak{p}$–vector space. Denote by $\tau_{\mathfrak{p}} : A/\mathfrak{p}A \to Z/\mathfrak{p} \simeq \mathbb{C}$ the map induced by Tr_k.

Since $\tau_{\mathfrak{p}}$ is a trace, it vanishes on all nilpotent elements of $A/\mathfrak{p}A$. In particular, $\tau_{\mathfrak{p}}$ vanishes on the image of $\mathfrak{I}_{k-1}$ in $A/\mathfrak{p}A$, because $\mathfrak{I}_{k-1}/I_k A$ is nilpotent (Definition 3). We finally obtain that $\mathrm{Tr}_k(\mathfrak{I}_{k-1})$ is contained in all maximal ideals $\mathfrak{p}$ that contain I_{k-1}. Since

$$I_{k-1} = Z_k \cap \mathfrak{I}_{k-1} = Z_k \cap \Big(\bigcap_{\mathfrak{I}_{k-1} \subset \mathfrak{P}} \mathfrak{P} \Big) = \bigcap_{\mathfrak{I}_{k-1} \subset \mathfrak{P}} (Z_k \cap \mathfrak{P}),$$

we obtain that I_{k-1} is the intersection of the maximal ideals of Z_k that contain it. This proves the result. □

A consequence of the previous lemma is that the trace Tr_k descends to a map $\overline{\mathrm{Tr}}_k : \mathrm{HH}_0(A/\mathfrak{I}_{k-1}) \to Z_k/I_{k-1}$ satisfying $\overline{\mathrm{Tr}}_k(a + \mathfrak{I}_{k-1}) = \mathrm{Tr}_k(a) + I_{k-1}$. Denote by t_{k-1} the composition

$$Z_{k-1} \longrightarrow \mathrm{HH}_0(A/\mathfrak{I}_{k-1}) \xrightarrow{\overline{\mathrm{Tr}}_k} Z_k/I_{k-1}.$$

Lemma 7. *We have that* $\mathrm{Tr}_k(a) + I_{k-1} = t_{k-1}(\mathrm{Tr}_{k-1}(a))$, *for all* $a \in A$.

Proof. To begin with, the difference $t_{k-1} \circ \mathrm{Tr}_{k-1} - \overline{\mathrm{Tr}}_k$ is a Z_k–linear map

$$\mathrm{HH}_0(A/\mathfrak{I}_{k-1}) \to Z_k/I_{k-1}$$

that vanishes on Z_{k-1} by the definition of t_{k-1} and because Tr_{k-1} is the identity map on Z_{k-1}.

Then, by Lemma 5, the support of the quotient $\mathrm{HH}_0(A/\mathfrak{I}_{k-1})/Z_{k-1}$ is nowhere dense in the maximal ideal spectrum of Z_{k-1}. Since Z_{k-1} is integral over Z_k/I_{k-1}, it follows that the support of $\mathrm{HH}_0(A/\mathfrak{I}_{k-1})/Z_{k-1}$, as a Z_k/I_{k-1}–module, is nowhere dense as well. From this we obtain that any Z_k/I_{k-1}–linear map

$$\mathrm{HH}_0(A/\mathfrak{I}_{k-1})/Z_{k-1} \to Z_k/I_{k-1}$$

vanishes. In particular $t_{k-1} \circ \mathrm{Tr}_{k-1} - \overline{\mathrm{Tr}}_{k-1} = 0$. □

Denote by

$$T_A = \{(z_1, z_2, \ldots, z_n), z_j \in Z_j, t_{k-1}(z_{k-1}) = z_k + I_{k-1} \text{ for each } k\}.$$

We have the following analog of the 'Trace Paley Wiener' Theorem [4].

Theorem 4. *Let* $(\mathfrak{I}_k)$ *be an abelian filtration of length* n *of a unital finite type algebra* A *such that all quotients* $A/\mathfrak{I}_k$ *are algebras with trace. Assume that* A *is semiprimitive, then the range of the total trace*

$$\mathrm{TR} : \mathrm{HH}_0(A) \to \oplus_{k=1}^n Z_k$$

is T_A *and its kernel has a composition series with subquotients isomorphic to* M_{k-1}/L_k, $k = 2, \ldots, n$, *in this order.*

Proof. Observe first that the map $t_{n-1} : Z_{n-1} \to Z_n/I_{n-1}$ is surjective because it is Z_n–linear and $t_{n-1}(1) = 1$. So, if we let $\tilde{T}_A = T_{A/\mathfrak{I}_{n-1}}$, then the kernel of the projection map $T_A \to \tilde{T}_A$ is naturally isomorphic to I_{n-1}. By Lemma 7, we obtain that $\mathrm{TR}(A) \subset T_A$. It remains to prove that TR is surjective.

The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{n-1} & \longrightarrow & \mathrm{HH}_0(A) & \longrightarrow & \mathrm{HH}_0(A/\mathfrak{I}_{n-1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_{n-1} & \longrightarrow & T_A & \longrightarrow & \tilde{T}_A & \longrightarrow & 0 \end{array}$$

commutes and has exact lines, by definition. Now proceed by induction on the length n of the abelian filtration $(\mathfrak{I}_k)$, and assume that

$$\widetilde{\mathrm{TR}} : \mathrm{HH}_0(A/\mathfrak{I}_{n-1}) \to \tilde{T}_A$$

is surjective. Since the morphism $M_{n-1} \to I_{n-1} \simeq L_n$ induced by TR is surjective, by Corollary 3, it follows that $\mathrm{TR} : \mathrm{HH}_0(A) \to T_A$ is also surjective.

The "ker-coker" Lemma gives an exact sequence,

$$0 \longrightarrow M_{n-1}/L_n \longrightarrow \ker(\mathrm{TR}) \longrightarrow \ker(\widetilde{\mathrm{TR}}) \longrightarrow 0,$$

from which the rest of the proof follows. □

For commuting algebras of p–adic groups it follows directly from results of Bernstein that the map TR is injective.

We conclude this section with a remark on the Hochschild homology of the algebras A_k introduced in the first section.

Remark. We observe that TR is not an isomorphism for the algebras $A_k \subset M_2(\mathbb{C}[X])$ considered at the end of Section 1, if $k \geq 2$. We shall use the notation introduced there.

If $k \geq 2$, then the map $f(P) = p'_{11}(0) - p'_{22}(0)$ is a trace of A_k that vanishes on $L_2 \simeq X\mathbb{C}[X]$, but does not descend to a trace on A_k/J_1, so $M_1 \neq L_2$ and TR is not injective. However, for $k = 1$, the algebra A_1 is a crossed-product: $A_1 \simeq \mathbb{C}[Y] \rtimes \mathbb{Z}/2\mathbb{Z}$, where α maps Y to $-Y$, so we can use the results of [22] to prove that TR is an isomorphism

$$\mathrm{TR} : \mathrm{HH}_0(A_1) \simeq T_{A_1} \simeq \mathbb{C}[X] \oplus \mathbb{C},$$

where $X = Y^2$. We also obtain that $\mathrm{HH}_1(A_1) \simeq \mathbb{C}[Y]^{\mathbb{Z}/2\mathbb{Z}} \simeq \mathbb{C}[X]$ and that $\mathrm{HH}_k(A_1) \simeq 0$ for $k \geq 2$.

4. Periodic cyclic homology of finite type algebras

In this section we shall prove two theorems establishing a very close connection between the periodic cyclic homology groups of finite type algebras and their spectra. This will lead to a proof of Theorem 1 stated in the introduction. Several results of this section will be devoted to proving that various algebras have the same periodic cyclic homology groups.

We begin with a short review of the needed definitions and results from cyclic homology.

Recall [8, 20, 31] that the *cyclic homology* groups of a unital algebra A, denoted $\mathrm{HC}_n(A)$, are the homology groups of the cyclic complex $(\mathcal{C}(A), b + B)$, where

$$\mathcal{C}_n(A) = \bigoplus_{k \geq 0} A \otimes (A/\mathbb{C}1)^{\otimes n-2k},$$

b is the Hochschild homology boundary map (6) and B is defined by

$$B(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = s \sum_{k=0}^{n} t^k (a_0 \otimes a_1 \otimes \ldots \otimes a_n). \tag{19}$$

Here we have used the notation of [8], that $s(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = 1 \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n$ and $t(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}$.

More generally, cyclic homology groups can be defined for "mixed complexes," [17]. A *mixed complex* $(\mathcal{X}, b, B)$ is a $\mathbb{Z}_+$–graded complex vector space $\mathcal{X} = (\mathcal{X}_n)_{n \geq 0}$ endowed with two differentials b and B, $b : \mathcal{X}_n \to \mathcal{X}_{n-1}$ and $B : \mathcal{X}_n \to \mathcal{X}_{n+1}$, satisfying $b^2 = B^2 = bB + Bb = 0$. The *cyclic complex*, $\mathcal{C}(\mathcal{X})$, associated to a mixed complex $(\mathcal{X}, b, B)$ is the complex

$$\mathcal{C}_n(\mathcal{X}) = \mathcal{X}_n \oplus \mathcal{X}_{n-2} \oplus \mathcal{X}_{n-4} \ldots = \oplus_k \mathcal{X}_{n-2k}$$

with differential $b + B$. It is the total complex associated to the bicomplex shown in Figure 1. The cyclic homology groups of the mixed complex $\mathcal{X}$ are the homology groups of the cyclic complex of $\mathcal{X}$,

$$\mathrm{HC}_n(\mathcal{X}) = \mathrm{H}_n(\mathcal{C}(\mathcal{X}), b + B).$$

$$\begin{tikzcd}
{} \arrow[d, "b"'] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
\mathcal{X}_2 \arrow[d, "b"'] & \mathcal{X}_1 \arrow[l, "B"'] \arrow[d, "b"] & \mathcal{X}_0 \arrow[l, "B"'] \\
\mathcal{X}_1 \arrow[d, "b"'] & \mathcal{X}_0 \arrow[l, "B"'] & \\
\mathcal{X}_0 & &
\end{tikzcd}$$

FIGURE 1. The bicomplex associated to a mixed complex.

We review below a few necessary results on mixed complexes. The interested reader can find more information in [16].

The homology of the complex $(\mathcal{X}_n, b)$ is called the Hochschild homology of the mixed complex $(\mathcal{X}, b, B)$ and is denoted $\mathrm{HH}_*(\mathcal{X})$. If A is an algebra with unit and if we let $\mathcal{X}_n = A \otimes (A/\mathbb{C})^{\otimes n}$, with the differentials b and B given in the equations (6) and (19) above (note that b passes to the quotient), then we recover the usual definitions of the Hochschild and cyclic homology groups of A. If A is an arbitrary algebra, denote by $A^+ = A + \mathbb{C}$ the algebra with an adjoined unit and let $\mathcal{X}_n(A) = \ker(\mathcal{X}_n(A^+) \to \mathcal{X}_n(\mathbb{C}))$. The cyclic and Hochschild homology groups of the resulting mixed complex $(\mathcal{X}_n(A), b, B)$ are, by definition, the cyclic and Hochschild homology groups of the not necessarily unital algebra A. This recovers the previous definitions if A already had a unit. The cyclic and Hochschild homology groups are covariant functors on the category of mixed complexes: a morphism $f : A \to B$ of algebras induces a morphism of the corresponding mixed complexes and hence it induces morphisms $f_* : \mathrm{HC}_*(A) \to \mathrm{HC}_*(B)$ and $f_* : \mathrm{HH}_*(A) \to \mathrm{HH}_*(B)$.

The cyclic complex of a mixed complex $(\mathcal{X}, b, B)$ has a periodicity endomorphism $S : \mathcal{C}_n(\mathcal{X}) \to \mathcal{C}_{n-2}(\mathcal{X})$ which maps $\mathcal{X}_n$ to 0 and is the identity on the other factors. This defines a map $\mathrm{HC}_n(\mathcal{X}) \longrightarrow \mathrm{HC}_{n-2}(\mathcal{X})$, also denoted by S. The homology groups $\mathrm{HP}_i(\mathcal{X})$ of the projective limit complex

$$\mathcal{C}_n^{\mathrm{per}}(\mathcal{X}) = \lim_{\leftarrow} \mathcal{C}_{n+2k}(\mathcal{X}) \tag{20}$$

with respect to the periodicity operator S are called the *periodic cyclic homology* groups of A:

$$\mathrm{HP}_i(\mathcal{X}) = \mathrm{H}_i(\mathcal{C}_n^{\mathrm{per}}(\mathcal{X}))\,. \tag{21}$$

Periodic cyclic homology is a $\mathbb{Z}/2\mathbb{Z}$–graded homology theory, i.e., $i \in \mathbb{Z}/2\mathbb{Z}$. Standard homological algebra shows that cyclic and periodic cyclic homology fit into the following $\lim^1$ exact sequence

$$0 \longrightarrow \lim_{\leftarrow}{}^1\, \mathrm{HC}_{m+1}(\mathcal{X}) \longrightarrow \mathrm{HP}_m(\mathcal{X}) \longrightarrow \lim_{\leftarrow} \mathrm{HC}_m(\mathcal{X}) \longrightarrow 0. \tag{22}$$

Lemma 8. *If $0 \to \mathcal{X}' \to \mathcal{X} \to \mathcal{X}'' \to 0$ is a short exact sequence of mixed complexes, then there exists a six-term periodic long exact sequence*

$$\ldots \to \mathrm{HP}_n(\mathcal{X}') \to \mathrm{HP}_n(\mathcal{X}) \to \mathrm{HP}_n(\mathcal{X}'') \to \mathrm{HP}_{n-1}(\mathcal{X}') \to \mathrm{HP}_{n-1}(\mathcal{X}) \to \ldots \tag{23}$$

of periodic cyclic homology groups. Analogous exact sequences exist for Hochschild and cyclic homology.

Proof. Since the periodicity operator is surjective, we obtain a short exact sequence of periodic cyclic complexes

$$0 \longrightarrow \mathcal{C}_n^{\mathrm{per}}(\mathcal{X}') \longrightarrow \mathcal{C}_n^{\mathrm{per}}(\mathcal{X}) \longrightarrow \mathcal{C}_n^{\mathrm{per}}(\mathcal{X}'') \longrightarrow 0\,. \tag{24}$$

A short exact sequence of complexes gives rise to a long exact sequence of homology groups (the snake lemma). This exact sequence gives the statement of the lemma. The same proof gives similar long exact sequences for Hochschild and cyclic homology. □

Mixed complexes present the advantage that they form an abelian category, unlike algebras. This is useful in the following situation. If $J \subset A$ is a two sided ideal then we denote by $\mathcal{X}_n(A, J)$ the kernel of the canonical projection $\mathcal{X}_n(A) \to X_n(A/J)$, to obtain a mixed complex $(\mathcal{X}(A, J), b, B)$ whose Hochschild, cyclic and periodic cyclic homology groups are denoted $\mathrm{HH}_*(A, J)$, $\mathrm{HC}_*(A, J)$ and $\mathrm{HP}_*(A, J)$, respectively. The above Lemma shows that they fit into long exact sequences

(25)

$$.. \to \mathrm{HH}_n(A, J) \to \mathrm{HH}_n(A) \to \mathrm{HH}_n(A/J) \xrightarrow{\partial} \mathrm{HH}_{n-1}(A, J) \to \mathrm{HH}_{n-1}(A) \to ..$$

$$.. \to \mathrm{HP}_n(A, J) \to \mathrm{HP}_n(A) \to \mathrm{HP}_n(A/J) \xrightarrow{\partial} \mathrm{HP}_{n-1}(A, J) \to \mathrm{HP}_{n-1}(A) \to ..$$

Cyclic and Hochschild homology of a mixed complex $(\mathcal{X}, b, B)$ are related by the 'SBI'–exact sequence, a natural long exact sequence due to Connes in cohomology [8] and to Loday and Quillen in homology [20]

$$\ldots \to \mathrm{HH}_n(\mathcal{X}) \xrightarrow{I} \mathrm{HC}_n(\mathcal{X}) \xrightarrow{S} \mathrm{HC}_{n-2}(\mathcal{X}) \xrightarrow{B} \mathrm{HH}_{n-1}(\mathcal{X}) \xrightarrow{I} \mathrm{HC}_{n-1}(\mathcal{X}) \to \ldots \tag{26}$$

where S is the periodicity operator defined above.

A consequence of the 'SBI'–exact sequence is that, if a morphism $f : \mathcal{X} \to \mathcal{X}'$ of mixed complexes is such that $f_* : \mathrm{HH}_*(\mathcal{X}) \to \mathrm{HH}_*(\mathcal{X}')$ is an isomorphism, then f also induces an isomorphism $f_* : \mathrm{HC}_*(\mathcal{X}) \to \mathrm{HC}_*(\mathcal{X}')$ of cyclic homology groups. Using the $\lim^1$–exact sequence (22) we also obtain that f induces an isomorphism $f_* : \mathrm{HP}_*(\mathcal{X}) \to \mathrm{HP}_*(\mathcal{X}')$ of periodic cyclic homology groups, as well.

The basic property of periodic cyclic homology leading to all the other theorems is the following result of Goodwillie [11].

Theorem 5 (Goodwillie)**.** *If $I \subset A$ is a nilpotent ideal then the quotient morphism $A \to A/I$ induces an isomorphism $\mathrm{HP}_*(A) \to \mathrm{HP}_*(A/I)$.*

Let J be a two–sided ideal of A and denote by $\mathrm{HP}_*^{\mathrm{top}}(A)$ the periodic cyclic homology of the mixed complex

$$\hat{\mathcal{X}}(A) = (A^{\hat{\otimes} n+1}, b, B), \tag{27}$$

where $A^{\hat{\otimes} n+1} = \varprojlim (A/J)^{\otimes n+1}$. The Hochschild homology of this mixed complex, $\mathrm{HH}_*^{\mathrm{top}}(\hat{A})$, was considered also in the previous section. The periodic cyclic complex $\mathcal{C}^{\mathrm{per}}(\hat{\mathcal{X}}(A))$, associated to $\hat{\mathcal{X}}(A)$, will be also denoted by $\hat{\mathcal{C}}^{\mathrm{per}}(\hat{A})$.

A first consequence of Goodwillie's theorem is the following result of [29].

Theorem 6 (Seibt). *Let J be a two–sided ideal of the algebra A. Then the quotient morphism $A \to A/J$ induces an isomorphism $\mathrm{HP}_*^{\mathrm{top}}(\hat{A}) \to \mathrm{HP}_*(A/J)$.*

An equivalent form of Seibt's theorem is given in the following proposition.

Proposition 6. *The kernel $\hat{\mathcal{X}}(A,J)$ of the morphism $\hat{\mathcal{X}}(A) \to \mathcal{X}(A/J)$ satisfies $\mathrm{HP}_*(\hat{\mathcal{X}}(A,J)) \simeq 0$.*

Proof. This follows from Theorem 6 and Lemma 8. □

A deep result is the Excision Theorem in periodic cyclic homology [9].

Theorem 7 (Cuntz-Quillen). *Any two-sided ideal J of an algebra A over a characteristic 0 field gives rise to a periodic six-term exact sequence*

$$\begin{array}{ccccc} \mathrm{HP}_0(J) & \longrightarrow & \mathrm{HP}_0(A) & \longrightarrow & \mathrm{HP}_0(A/J) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ \mathrm{HP}_1(A/J) & \longleftarrow & \mathrm{HP}_1(A) & \longleftarrow & \mathrm{HP}_1(J). \end{array} \tag{28}$$

An equivalent way of stating the excision theorem is the following result.

Proposition 7. *Let $\mathrm{HP}_*(A,J)$ be the periodic cyclic homology of the mixed complex $\mathcal{X}(A,J)$. Then the inclusion $\mathcal{X}(J) \to \mathcal{X}(A,J)$ gives $\mathrm{HP}_*(J) \simeq \mathrm{HP}_*(A,J)$.*

Proof. This follows directly by comparing the exact sequences of Theorem 7 and Lemma 8, and by using the Five Lemma. □

An immediate consequence of excision and of Goodwillie's theorem is the following corollary.

Corollary 4. *For any not necessarily unital complex algebra B, the inclusion $B^n \to B$ induces a natural isomorphism $\mathrm{HP}_*(B^n) \to \mathrm{HP}_*(B)$.*

We now begin the proof of one of the main results of this section, Theorem 8. We start with some lemmata.

Lemma 9. *Let L be a finite type $\mathbf{k}$–algebra and $I \subset \mathbf{k}$ be an ideal such that $L^k \subset IL$ for some large k. Endow $\mathbf{k}$ and $B = \mathbf{k} + L$, the $\mathbf{k}$–algebra with adjoined unit, with the I-adic filtrations. Then the projection $B \to \mathbf{k}$ gives an isomorphism $\mathrm{HP}_*^{\mathrm{top}}(\hat{B}) \to \mathrm{HP}_*^{\mathrm{top}}(\hat{\mathbf{k}})$.*

Proof. For each q there exists a commutative diagram

$$\begin{array}{ccc} \mathrm{HP}_q^{\mathrm{top}}(\hat{B}) & \longrightarrow & \mathrm{HP}_q^{\mathrm{top}}(\hat{\mathbf{k}}) \\ \downarrow & & \downarrow \\ \mathrm{HP}_q(\mathbf{k}/I + L/IL) = \mathrm{HP}_q(B/IB) & \longrightarrow & \mathrm{HP}_q(\mathbf{k}/I) \end{array}$$

whose vertical maps are induced by the projections $\hat{B} \to B/IB$ and $\hat{\mathbf{k}} \to \mathbf{k}/I$. It follows from Theorem 6 that these vertical arrows are isomorphisms. The bottom map is also an isomorphism since L/IL is a nilpotent ideal. We finally obtain that the top arrow is an isomorphism, as stated. □

The following lemma applies to $\mathbf{k}$–algebras that are not necessarily finite type $\mathbf{k}$–algebras, and to arbitrary commutative rings $\mathbf{k}$, not necessarily noetherian.

Lemma 10. *Let L be a $\mathbf{k}$–algebra and $\mathfrak{p} \subset \mathbf{k}$ be a maximal ideal such that $L_\mathfrak{p}$ is unital. If $B = \mathbf{k} + L$ is the $\mathbf{k}$–algebra with adjoined unit, then $\mathrm{HH}_*(B)_\mathfrak{p} \simeq \mathrm{HH}_*(\mathbf{k}_\mathfrak{p}) \oplus \mathrm{HH}_*(L_\mathfrak{p})$.*

Proof. We have the following isomorphisms

$$\begin{aligned} \mathrm{HH}_*(B)_\mathfrak{p} &\simeq \mathrm{HH}_*(\mathbf{k}_\mathfrak{p} + L_\mathfrak{p}) && \text{by Proposition 4} \\ &\simeq \mathrm{HH}_*(\mathbf{k}_\mathfrak{p} \oplus L_\mathfrak{p}) && \text{because } 1 \in L_\mathfrak{p} \\ &\simeq \mathrm{HH}_*(\mathbf{k}_\mathfrak{p}) \oplus \mathrm{HH}_*(L_\mathfrak{p}) \end{aligned}$$

where the last isomorphism is due to the fact that Hochschild homology and direct sums of unital algebras commute. □

We are ready now to prove a powerful criterion for two algebras to have the same periodic cyclic homology. The idea is to compare two $\mathbf{k}$–algebras that have a similar behavior with respect to an ideal $I \subset \mathbf{k}$; slightly more precisely, their periodic cyclic homology vanishes "at the points in $V(I)$," and their Hochschild homology groups are isomorphic "at the points outside of $V(I)$."

Theorem 8. *Let $I \subset \mathbf{k}$ be an ideal and let $L \subset J$ be finite type $\mathbf{k}$–algebras such that L/IL and J/IJ are nilpotent. Suppose that, for all maximal ideals $\mathfrak{p} \subset \mathbf{k}$ such that $I \not\subset \mathfrak{p}$, the localizations $L_\mathfrak{p}$ and $J_\mathfrak{p}$ are unital and $\mathrm{HH}_q(L_\mathfrak{p}) \to \mathrm{HH}_q(J_\mathfrak{p})$ is an isomorphism. Then the inclusion $L \to J$ gives an isomorphism*

$$\mathrm{HP}_q(L) \simeq \mathrm{HP}_q(J).$$

Proof. Let $B = \mathbf{k} + J$ and $C = \mathbf{k} + L$ be the "$\mathbf{k}$–algebras with adjoined unit," endowed with the filtrations defined by the powers of I. Denote

$$I_B = I + J \subset B \quad \text{and} \quad I_C = I + L \subset C.$$

We divide the proof into several steps, the first being a reduction of the proposition to the proof that $\mathrm{HP}_*(I_C) \simeq \mathrm{HP}_*(I_B)$.

Step 1: $\mathrm{HP}_*(L) \simeq \mathrm{HP}_*(J) \Leftrightarrow \mathrm{HP}_*(I_C) \simeq \mathrm{HP}_*(I_B)$.

The split exact sequence $0 \longrightarrow L \longrightarrow I_C \longrightarrow I \longrightarrow 0$ gives rise to a natural exact sequence

$$0 \longrightarrow \mathrm{HP}_q(L) \longrightarrow \mathrm{HP}_q(I_C) \longrightarrow \mathrm{HP}_q(I) \longrightarrow 0.$$

By replacing I_C by I_B and L by J, we obtain a similar exact sequence

$$0 \longrightarrow \mathrm{HP}_q(J) \longrightarrow \mathrm{HP}_q(I_B) \longrightarrow \mathrm{HP}_q(I) \longrightarrow 0.$$

By definition, the inclusion $I_C \to I_B$ induces a morphism of these exact sequences. Then, using the Five Lemma, we obtain that $\mathrm{HP}_*(I_C) \simeq \mathrm{HP}_*(I_B)$ if and only if $\mathrm{HP}_*(L) \simeq \mathrm{HP}_*(J)$.

We continue by further reducing the proof of the theorem to proving the vanishing of the periodic cyclic homology of the mixed complex $X = \mathcal{X}(B, I_B)/\mathcal{X}(C, I_C)$.

Step 2: $\mathrm{HP}_*(L) \simeq \mathrm{HP}_*(J) \Leftrightarrow \mathrm{HP}_*(X) = 0$.

The excision property (more precisely Proposition 7) implies that we have natural isomorphisms $\mathrm{HP}_*(I_C) \simeq \mathrm{HP}_*(C, I_C)$ and $\mathrm{HP}_q(I_B) \simeq \mathrm{HP}_q(B, I_B)$, for $q \in \mathbb{Z}/2\mathbb{Z}$. It is then enough to prove that the morphism $\mathcal{X}(C, I_C) \to \mathcal{X}(B, I_B)$ of mixed complexes gives isomorphisms $\mathrm{HP}_*(C, I_C) \simeq \mathrm{HP}_*(B, I_B)$. Thus for the rest of the proof we shall work with the mixed complexes $\mathcal{X}(C, I_C)$ and $\mathcal{X}(B, I_B)$. Let

$j : I_C \to I_B$ be the inclusion and $X = \mathcal{X}(B, I_B)/\mathcal{X}(C, I_C)$, as above, then Lemma 8 gives a periodic exact sequence

$$\ldots \to \mathrm{HP}_q(X) \to \mathrm{HP}_{q-1}(C, I_C) \xrightarrow{j_*} \mathrm{HP}_{q-1}(B, I_B) \to \mathrm{HP}_{q-1}(X) \to \ldots ,$$

from which we see that, in order to prove that j_* is an isomorphism, it is enough to check that $\mathrm{HP}_*(X) = 0$. The rest follows from Step 1.

We now show that the condition $\mathrm{HP}_*(\hat{X}) = 0$ is satisfied if we consider the 'I–adically complete' analog $\hat{X} = \hat{\mathcal{X}}(B, I_B)/\hat{\mathcal{X}}(C, I_C)$ of X. Thus, to define $\hat{X}$ we shall consider 'I–adically complete' versions of the previous constructions.

Step 3: $\mathrm{HP}_*(\hat{X}) = 0$.

Recall that $\hat{\mathcal{X}}(B)$ was defined in equation (27) using complete tensor products. Because $J^k \subset IJ$ and $L^k \subset IL$, by assumption, the requirements of Lemma 9 are satisfied, giving that the morphism $\hat{\mathcal{X}}(C) \to \mathcal{X}(C/I_C) = \mathcal{X}(\mathbf{k}/I)$ induces an isomorphism in periodic cyclic homology, and hence that its kernel, namely $\hat{\mathcal{X}}(C, I_C)$, has vanishing periodic cyclic homology. Similarly, $\hat{\mathcal{X}}(B, I_B)$, the kernel of $\hat{\mathcal{X}}(B) \to \mathcal{X}(B/I_B) = \mathcal{X}(\mathbf{k}/I)$, also has vanishing periodic cyclic homology (Lemma 8). The 'I–adically complete version' of the mixed complex X is the quotient $\hat{X} = \hat{\mathcal{X}}(B, I_B)/\hat{\mathcal{X}}(C, I_C)$. Using once again Lemma 8, we obtain that $\mathrm{HP}_*(\hat{X}) \simeq 0$. The proof of Step 3 is complete.

The crucial idea of the proof now is to verify that the morphism $X \to \hat{X}$ induces an isomorphism of Hochschild homology groups, which would imply, by a standard argument reviewed earlier, that $\mathrm{HP}_*(X) \to \mathrm{HP}_*(\hat{X})$ is also an isomorphism, and hence, finally, that $\mathrm{HP}_*(X) \simeq 0$, thereby proving the theorem. Therefore, in the following steps we concentrate on the Hochschild homologies of X and $\hat{X}$.

We begin by proving that we can obtain the Hochschild homology groups of $\hat{X}$ from those of X by 'completion.' We will use Theorem 3 repeatedly.

Step 4: $\mathrm{HH}_q(\hat{X}) \simeq \mathrm{HH}_q(X) \otimes_{\mathbf{k}} \hat{\mathbf{k}}$.

Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_{q+1}(C) \arrow[r] \arrow[d] & \mathrm{HH}_{q+1}(C/I_C) \arrow[r] \arrow[d, equal] & \mathrm{HH}_q(C, I_C) \arrow[r] \arrow[d] & \mathrm{HH}_q(C) \arrow[r] \arrow[d] & \mathrm{HH}_q(C/I_C) \arrow[d, equal] \\
\mathrm{HH}^{\mathrm{top}}_{q+1}(\hat{C}) \arrow[r] & \mathrm{HH}_{q+1}(C/I_C) \arrow[r] & \mathrm{HH}_q(\hat{\mathcal{X}}(C, I_C)) \arrow[r] & \mathrm{HH}^{\mathrm{top}}_q(\hat{C}) \arrow[r] & \mathrm{HH}_q(C/I_C)
\end{tikzcd}$$

whose lines are exact due the exactness of the sequences in (25). Tensoring with $\hat{\mathbf{k}}$ over $\mathbf{k}$ preserves exactness (since $\hat{\mathbf{k}}$ is a flat $\mathbf{k}$–module), so, denoting by $M\hat{} = M \otimes_{\mathbf{k}} \hat{\mathbf{k}}$ for any $\mathbf{k}$–module M, we obtain a new commutative diagram

(29)

$$\begin{tikzcd}
\mathrm{HH}_{q+1}(C)\hat{} \arrow[r] \arrow[d] & \mathrm{HH}_{q+1}(C/I_C)\hat{} \arrow[r] \arrow[d] & \mathrm{HH}_q(C, I_C)\hat{} \arrow[r] \arrow[d] & \mathrm{HH}_q(C)\hat{} \arrow[r] \arrow[d] & \mathrm{HH}_q(C/I_C)\hat{}\,, \arrow[d] \\
\mathrm{HH}^{\mathrm{top}}_{q+1}(C) \arrow[r] & \mathrm{HH}_{q+1}(C/I_C) \arrow[r] & \mathrm{HH}_q(\hat{\mathcal{X}}(C, I_C)) \arrow[r] & \mathrm{HH}^{\mathrm{top}}_q(C) \arrow[r] & \mathrm{HH}_q(C/I_C)
\end{tikzcd}$$

whose lines are still exact. A commutative diagram with exact lines similar to (29) is obtained if we replace C by B and I_C by I_B.

We claim that the complex (29) satisfies the assumptions of the Five Lemma.

Indeed, from Theorem 3 we know that the inclusion $C \to \hat{C}$ determines natural isomorphisms

$$\mathrm{HH}_*(C) \otimes_{\mathbf{k}} \hat{\mathbf{k}} \simeq \mathrm{HH}_*^{\mathrm{top}}(\hat{C}).$$

Also, since $I\,\mathrm{HH}_*(C/I_C) = 0$, we obtain a straightforward isomorphism

$$\mathrm{HH}_q(C/I_C) \otimes_{\mathbf{k}} \hat{\mathbf{k}} \simeq \mathrm{HH}_q(C/I_C)$$

for each q. It follows then from the Five Lemma that the middle arrow of (29) is an isomorphism.

The same argument works if we replace C with B, and we conclude that there exist isomorphisms

$$\begin{aligned} \mathrm{HH}_*(C, I) \otimes_{\mathbf{k}} \hat{\mathbf{k}} \simeq \mathrm{HH}_*(\hat{\mathcal{X}}(C, I_C)) =: \mathrm{HH}_*^{\mathrm{top}}(\hat{C}, \hat{I}_C) \quad \text{and} \\ \mathrm{HH}_*(B, I_B) \otimes_{\mathbf{k}} \hat{\mathbf{k}} \simeq \mathrm{HH}_*(\hat{\mathcal{X}}(B, I_B)) =: \mathrm{HH}_*^{\mathrm{top}}(\hat{B}, \hat{I}_B). \end{aligned} \tag{30}$$

We now fit the isomorphisms (30) into the following commutative diagram

$$\begin{CD} \mathrm{HH}_q(C, I_C)\hat{} @>>> \mathrm{HH}_q(B, I_B)\hat{} @>>> \mathrm{HH}_q(X)\hat{} @>>> \mathrm{HH}_{q-1}(C, I_C)\hat{} @>>> \mathrm{HH}_{q-1}(B, I_B)\hat{} \\ @VVV @VVV @VVV @VVV @VVV \\ \mathrm{HH}_q^{\mathrm{top}}(\hat{C}, \hat{I}_C) @>>> \mathrm{HH}_q^{\mathrm{top}}(\hat{B}, \hat{I}_B) @>>> \mathrm{HH}_q(\hat{X}) @>>> \mathrm{HH}_{q-1}^{\mathrm{top}}(\hat{C}, \hat{I}_C) @>>> \mathrm{HH}_{q-1}^{\mathrm{top}}(\hat{B}, \hat{I}_B), \end{CD}$$

in which the bottom line is the exact sequence in Hochschild homology, (25), and the top line is exact since it is obtained from the exact sequence (25) by tensoring with $\hat{\mathbf{k}}$. Moreover, the first two and the last two vertical arrows are the isomorphisms of (30). Then, from the Five lemma we conclude that the natural morphism $X \to \hat{X}$ of mixed complexes gives the desired isomorphism

$$\mathrm{HH}_*(X) \otimes_{\mathbf{k}} \hat{\mathbf{k}} \simeq \mathrm{HH}_*(\hat{X}). \tag{31}$$

We now use a commutative algebra argument to complete the last step of the proof.

Step 5: $I^k\,\mathrm{HH}_q(X) = 0$, for some k depending on q, and hence

$$\mathrm{HH}_q(X) = \mathrm{HH}_q(X) \otimes_{\mathbf{k}} \hat{\mathbf{k}} \simeq \mathrm{HH}_q(\hat{X})$$

for all q.

¿From Corollary 1 and the exact sequence (25), we know that $\mathrm{HH}_q(C, I_C)$ and $\mathrm{HH}_q(B, I_B)$ are finitely generated $\mathbf{k}$–modules, for each q. It follows that $\mathrm{HH}_q(X)$ is also a finitely generated $\mathbf{k}$–module for each q. Fix now an arbitrary maximal ideal $\mathfrak{p} \subset \mathbf{k}$ not containing I. Since $\mathrm{HH}_q(C/I_C)_{\mathfrak{p}} = \mathrm{HH}_q(\mathbf{k}/I)_{\mathfrak{p}} = 0$, we get $\mathrm{HH}_q(C, I_C)_{\mathfrak{p}} \simeq \mathrm{HH}_q(C)_{\mathfrak{p}}$, which gives, using Lemma 10, that

$$\mathrm{HH}_q(C, I_C)_{\mathfrak{p}} \simeq \mathrm{HH}_q(C)_{\mathfrak{p}} = \mathrm{HH}_q(\mathbf{k} + L)_{\mathfrak{p}} \simeq \mathrm{HH}_q(\mathbf{k}_{\mathfrak{p}}) \oplus \mathrm{HH}_q(L_{\mathfrak{p}}),$$

where the last isomorphism is based on the fact that $L_{\mathfrak{p}}$ is unital. Similarly,

$$\mathrm{HH}_q(B, I_B)_{\mathfrak{p}} \simeq \mathrm{HH}_q(B)_{\mathfrak{p}} \simeq \mathrm{HH}_q(\mathbf{k}_{\mathfrak{p}}) \oplus \mathrm{HH}_q(J_{\mathfrak{p}})$$

for $I \not\subset \mathfrak{p}$. The assumption that $\mathrm{HH}_q(L_{\mathfrak{p}}) \simeq \mathrm{HH}_q(J_{\mathfrak{p}})$ then shows that the natural map $\mathrm{HH}_q(C, I_C)_{\mathfrak{p}} \to \mathrm{HH}_q(B, I_B)_{\mathfrak{p}}$ is an isomorphism. From this, using Lemma 8 for Hochschild homology, we finally obtain that $\mathrm{HH}_q(X)_{\mathfrak{p}} = 0$, for $I \not\subset \mathfrak{p}$. This is enough to conclude that $I^k\,\mathrm{HH}_q(X) = 0$, for some large k that depends on q.

¿From Step 4 we obtain that $\mathrm{HH}_q(X) = \mathrm{HH}_q(X) \otimes_{\mathbf{k}} \hat{\mathbf{k}} \simeq \mathrm{HH}_q(\hat{X})$ for all q. The proof of Step 5 is now complete.

Summing up, we have obtained, as desired, that $\mathrm{HH}_*(X) \simeq \mathrm{HH}_*(\hat{X})$, and hence also that $\mathrm{HP}_*(X) \simeq \mathrm{HP}_*(\hat{X}) \simeq 0$, due to Step 3 and to the fact that a morphism of mixed complexes that induces an isomorphism in Hochschild homology also induces an isomorphism in periodic cyclic homology. In view of Step 2, this completes the proof of the Theorem. ☐

Lemma 11. *Let $L \subset A$ be a $\mathbf{k}$–subalgebra of a unital finite type $\mathbf{k}$–algebra A such that $L \subset IA$ for some ideal $I \subset \mathbf{k}$. Then $L^k \subset IL$ for some large k, and hence L/IL is nilpotent.*

Proof. The Artin-Rees Lemma states that we can find k such that

$$L \cap I^k A = I(L \cap I^{k-1} A).$$

In particular, we obtain that $L \cap I^k A \subset IL$. Then $L^k \subset (IA)^k \cap L = I^k A \cap L \subset IL$. ☐

Corollary 5. *Let A be a unital finite type $\mathbf{k}$–algebra and $L \subset J \subset IA$ be inclusions of finite type $\mathbf{k}$–algebras, where $I \subset \mathbf{k}$ is an ideal. Suppose that, for all maximal ideals $\mathfrak{p}$ such that $I \not\subset \mathfrak{p}$, the localizations $L_{\mathfrak{p}}$ and $J_{\mathfrak{p}}$ are unital and $\mathrm{HH}_q(L_{\mathfrak{p}}) \to \mathrm{HH}_q(J_{\mathfrak{p}})$ is an isomorphism. Then the inclusion $L \to J$ gives an isomorphism*

$$\mathrm{HP}_q(L) \simeq \mathrm{HP}_q(J).$$

Proof. Lemma 11 shows that the assumptions of Theorem 8 are satisfied. ☐

We can drop the assumption that $L \to J$ is an inclusion in the above Corollary, by using mapping cone complexes in the proof of Theorem 8.

An important situation when the previous theorem applies is given in the next proposition. We shall use the notation and terminology from Definition 4; thus $Z_k = Z(A/\mathfrak{I}_k)$ are the subcenters and $I_{k-1} = Z_k \cap (\mathfrak{I}_{k-1}/\mathfrak{I}_k)$ are the reducibility ideals associated to an abelian filtration $(\mathfrak{I}_k)$ of A.

Proposition 8. *Let $(\mathfrak{I}_k)$ be an abelian filtration of a unital finite type algebra A. Let Z_k and I_k be the subcenters and the reducibility ideals of the filtration $(\mathfrak{I}_k)$. Then, for each k and for each two–sided ideal $\mathfrak{a}$ of A, the inclusion*

$$I_{k-1}^{\mathfrak{a}} := ((\mathfrak{a} + \mathfrak{I}_k)/\mathfrak{I}_k) \cap I_{k-1} \hookrightarrow ((\mathfrak{a} + \mathfrak{I}_k) \cap \mathfrak{I}_{k-1})/\mathfrak{I}_k = (\mathfrak{a} \cap \mathfrak{I}_{k-1})/(\mathfrak{a} \cap \mathfrak{I}_k)$$

induces an isomorphism in periodic cyclic homology. In particular, we have isomorphisms $\mathrm{HP}_*(I_{k-1}) \to \mathrm{HP}_*(\mathfrak{I}_{k-1}/\mathfrak{I}_k)$, for all $k \geq 1$.

Proof. By replacing A by $A/\mathfrak{I}_k$, we may assume that $\mathfrak{I}_k = (0)$ and that Z_k is the center Z of A. We claim that all assumptions of Corollary 5 are satisfied for $I = L = \mathfrak{a} \cap I_{k-1} =: I_{k-1}^{\mathfrak{a}}$ and $J = IA$, and hence that we can conclude from that theorem the existence of a natural isomorphism $\mathrm{HP}_*(I) \simeq \mathrm{HP}_*(J)$, induced by inclusion.

The conditions $L, J \subset IA$ of Corollary 5 are satisfied by definition. By Proposition 5, we have an isomorphism $\mathrm{HH}_*(Z_{\mathfrak{p}}) \to \mathrm{HH}_*(A_{\mathfrak{p}})$, for any maximal ideal $\mathfrak{p}$ not containing $I_{k-1} = \mathfrak{I}_{k-1} \cap Z$. If $\mathfrak{p}$ does not contain $\mathfrak{a}$ either, then $I_{\mathfrak{p}} \simeq Z_{\mathfrak{p}}$ and $J_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$, which shows that $I_{\mathfrak{p}}$ and $J_{\mathfrak{p}}$ are unital and that $\mathrm{HH}_*(I_{\mathfrak{p}}) \simeq \mathrm{HH}_*(J_{\mathfrak{p}})$.

Now, in order to conclude the proof, it is enough to verify that

$$I_{k-1}^{\mathfrak{a}}/J = (\mathfrak{a} \cap \mathfrak{I}_{k-1})/J$$

is nilpotent. To this end, using Lemma 1 (iv), we see that it is enough to check that $V(\mathfrak{a} \cap \mathfrak{I}_{k-1}) = V(J)$. Thus, let us consider a primitive ideal $\mathfrak{P} \subset A$ containing J. Consider the maximal ideal $\mathfrak{p} = \mathfrak{P} \cap Z$ of Z.

First, if $I_{k-1} \subset \mathfrak{p}$, then $I_{k-1}A \subset \mathfrak{P}$, and hence $\mathfrak{I}_{k-1} \subset \mathfrak{P}$, because $\mathfrak{I}_{k-1}/I_{k-1}A$ is nilpotent by the definition of an abelian filtration (Definition 3).

Next, if $I_{k-1} \not\subset \mathfrak{p}$, then $A_\mathfrak{p}$ is an Azumaya algebra over $Z_\mathfrak{p}$, also by Definition 3, and hence

$$\mathfrak{a}_\mathfrak{p} = (\mathfrak{a}_\mathfrak{p} \cap Z_\mathfrak{p})A_\mathfrak{p} = (\mathfrak{a}_\mathfrak{p} \cap (I_{k-1})_\mathfrak{p})A_\mathfrak{p} = J_\mathfrak{p} \subset \mathfrak{P}_\mathfrak{p}.$$

Since $A/\mathfrak{P} = (A/\mathfrak{P})_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{P}_\mathfrak{p}$, the inclusion $\mathfrak{a}_\mathfrak{p} \subset \mathfrak{P}_\mathfrak{p}$ implies $\mathfrak{a} \subset \mathfrak{P}$. Thus, whether $\mathfrak{p} = \mathfrak{P} \cap Z$ contains I_{k-1} or not, we obtain that $\mathfrak{a} \cap \mathfrak{I}_{k-1} \subset \mathfrak{P}$, if $J \subset \mathfrak{P}$. The proof is now complete. □

We now identify the periodic cyclic homology of ideals in finitely generated algebras. This is the last significant result that we need before we can deal with the main theorem stated in the introduction. We first review mapping fiber complexes.

If $f = (f_n) : A \to B$ is a morphism of two complexes $A = (A_n, d_n)$ and $B = (B_n, d'_n)$, then we shall denote by $\mathcal{M}(f)$ the *mapping fiber* complex associated to A, B, and f. Recall that $\mathcal{M}_n(f) = A_n \oplus B_{n+1}$ is the complex with differential $\delta(a, b) = (d_n a, -d'_{n+1} b - f_n(a))$.

The following lemmata will be used in the proof of Theorem 9. A morphism of complexes is a *quasi-isomorphism* if the map induced on homology is an isomorphism.

Lemma 12. *Let $\phi : A \to A'$, $\psi : B \to B'$, $f : A \to B$, and $f' : A' \to B'$ be morphisms of complexes such that $f' \circ \phi = \psi \circ f$ and such that ϕ and ψ are quasi-isomorphisms. Then the mapping fiber morphism $(\phi, \psi) : \mathcal{M}(f) \to \mathcal{M}'(f')$ is also a quasi-isomorphism.*

Proof. The mapping fiber complex $\mathcal{M}(f)$ contains B as a subcomplex, up to a degree shift, such that $\mathcal{M}(f)/B \simeq A$. This isomorphism gives rise to a natural long exact sequence relating the homology groups of A, B and $\mathcal{M}(f)$. A similar exact sequence is satisfied by $\mathcal{M}'(f')$. The morphism $(\phi, \psi) : \mathcal{M}(f) \to \mathcal{M}'(f')$ is part of a morphism of the corresponding long exact sequences, the two other morphisms being induced by ϕ and ψ. The Lemma is then a consequence of the Five Lemma. □

We are interested in mixed complexes because they compute *relative* cohomology groups.

Denote, in what follows, by $\mathbb{A}^m_\mathbb{C}$ the m–dimensional complex affine space (whose coordinate ring is $R := \mathbb{C}[X_1, X_2, \dots, X_m]$). Let X^{an} be the analytic variety associated to a closed subvariety $X \subset \mathbb{A}^m_\mathbb{C}$, and let $\hat{\Omega}^q_X$ be the sheaf (on $\hat{X}$) of algebraic q–forms defined in an infinitesimal neighborhood of X. If X is defined by an ideal $\mathfrak{a} \subset R$, then its global sections are given by

$$\hat{\Omega}^q_X(\hat{X}) = \lim_{\leftarrow} \Omega^q(\mathbb{A}^m_\mathbb{C})/\mathfrak{a}^n \Omega^q(\mathbb{A}^m_\mathbb{C}).$$

Correspondingly let $\hat{\Omega}^{q,\mathrm{an}}_X$ be the sheaf (on $\hat{X}^{an}$) of holomorphic q–forms defined in an infinitesimal neighbourhood of X^{an}.

We have the following *relative* de Rham theorem for formally smooth varieties.

Proposition 9. *Let $Y \subset X \subset \mathbb{A}^m_{\mathbb{C}}$ be closed subvarieties and let $j^* : \hat{\Omega}^q_X(\hat{X}) \to \hat{\Omega}^q_Y(\hat{Y})$ be the restriction. Then there exists a natural isomorphism*

$$\mathrm{H}^*(\mathcal{M}(j^*)) \simeq \mathrm{H}^*(X^{\mathrm{an}}, Y^{\mathrm{an}}; \mathbb{C}).$$

Proof. It is known, see [12], that the canonical homomorphism of complexes of sheaves from $\hat{\Omega}^*_X$ to the total direct image $\mathrm{R}(an)_* \hat{\Omega}^{*,\mathrm{an}}_X$ under the natural map $an : \hat{X}^{\mathrm{an}} \to \hat{X}$ is a quasi-isomorphism, i.e., induces an isomorphism of (hyper) cohomology groups

$$\mathbb{H}^*(\hat{X}^{\mathrm{an}}; \hat{\Omega}^{*,\mathrm{an}}_X) \simeq \mathbb{H}^*(\hat{X}; \hat{\Omega}^*_X). \tag{32}$$

Let $\iota : Y \to X$ denote the inclusion map and let $\underline{j}^* : \hat{\Omega}^q_X \to \iota_* \hat{\Omega}^q_Y$ (respectively, $\underline{j}^*_{\mathrm{an}} : \hat{\Omega}^{q,\mathrm{an}}_X \to \iota_* \hat{\Omega}^{q,\mathrm{an}}_Y$) be the restriction of algebraic (respectively holomorphic) forms. Using Lemma 12 and the isomorphism (32), we obtain that

$$\mathbb{H}^*(\hat{X}^{\mathrm{an}}; \mathcal{M}(\underline{j}^*_{\mathrm{an}})) \simeq \mathbb{H}^*(\hat{X}; \mathcal{M}(\underline{j}^*)) \simeq \mathrm{H}^*(\mathcal{M}(j^*)). \tag{33}$$

The analytic Poincaré lemma, see [12], states that the complex of sheaves $\hat{\Omega}^{*,\mathrm{an}}_X$ (respectively $\hat{\Omega}^{*,\mathrm{an}}_Y$) is a resolution of the constant sheaf $\underline{\mathbb{C}}_X$ (respectively $\underline{\mathbb{C}}_Y$). Let $r : \underline{\mathbb{C}}_X \to \iota_* \underline{\mathbb{C}}_Y$ be the natural morphism. Then

$$\mathrm{H}^*(X^{\mathrm{an}}, Y^{\mathrm{an}}; \mathbb{C}) \simeq \mathbb{H}^*(X^{\mathrm{an}}; \mathcal{M}(r)) \simeq \mathbb{H}^*(\hat{X}^{\mathrm{an}}; \mathcal{M}(\underline{j}^*_{\mathrm{an}})),$$

by the analytic Poincaré lemma, Lemma 12 and (33). □

We shall continue to use the notation of the above proof. Let $\mathfrak{a} \subset \mathfrak{b} \subset R = \mathbb{C}[X_1, X_2, \dots, X_m]$ be ideals defining the varieties $Y \subset X \subset \mathbb{A}^m_{\mathbb{C}}$. Denote then by $\hat{R}_X$ be the completion of R with respect to the $\mathfrak{a}$–adic topology and by $\hat{R}_Y$ the completion of R with respect to the $\mathfrak{b}$–adic topology. Then the natural morphism $\hat{f} : \hat{R}_X \to \hat{R}_Y$, the Hochschild-Kostant-Rosenberg map χ, equation (14), and the restriction morphism j^* fit into the following commutative diagram

$$\begin{CD} \mathrm{HH}^{\mathrm{top}}_q(\hat{R}_X) @>{\hat{f}_*}>> \mathrm{HH}^{\mathrm{top}}_q(\hat{R}_Y) \\ @V{\chi}VV @VV{\chi}V \\ \Omega^q(\hat{X}) @>{j^*}>> \Omega^q(\hat{Y}). \end{CD}$$

By Corollary 2, the vertical morphisms (i.e., the Hochschild-Kostant-Rosenberg maps χ) are isomorphisms.

In the following we shall use the notation $\mathrm{H}^{[j]}(A, B) = \prod_{k \in \mathbb{Z}} \mathrm{H}^{j+2k}(A, B; \mathbb{C})$ to denote the $\mathbb{Z}/2\mathbb{Z}$–periodic cohomology groups of a pair (A, B) of Hausdorff locally compact spaces.

We will also need to identify the relation between χ and the differential B. This relation is best expressed by the commutativity of the following diagram

$$\begin{CD} \mathrm{HH}^{\mathrm{top}}_q(\hat{R}_X) @>{B}>> \mathrm{HH}^{\mathrm{top}}_{q+1}(\hat{R}_X) \\ @V{\chi}VV @VV{\chi}V \\ \Omega^q(\hat{X}) @>{d_{DR}}>> \Omega^{q+1}(\hat{X}). \end{CD}$$

Since $\chi \circ b = 0$, we get that the map χ gives a morphism, denoted also χ, of mixed complexes

$$\chi : (\hat{\mathcal{X}}(\hat{R}_X), b, B) \longrightarrow (\Omega^*(\hat{X}), 0, d_{DR}), \tag{34}$$

which induces an isomorphism of the Hochschild homology groups, by Corollary 2. We finally obtain than that χ induces an isomorphism of periodic cyclic homology groups as well. The morphism (34) will be used in the proof of Theorem 9.

Because the periodic cyclic complex of $(\Omega^*(\hat{X}), 0, d_{DR})$ is a direct product of copies of the de Rham complex of $\hat{X}$, shifted by even integers (see Figure 1), we obtain that

$$\mathrm{HP}_q^{\mathrm{top}}(\hat{R}_X) \simeq \mathrm{H}_{DR}^{[q]}(\hat{X}) = \prod_k \mathrm{H}_{DR}^{q+2k}(\hat{X}).$$

Because $\hat{R}_X/\mathfrak{a}\hat{R}_X = R/\mathfrak{a} \simeq \mathcal{O}(X)$, Seibt's theorem gives an isomorphism

$$\mathrm{HP}_*^{\mathrm{top}}(\hat{R}_X) \simeq \mathrm{HP}_*(\mathcal{O}(X)).$$

Combining this with the isomorphism $\mathrm{H}_{DR}^*(\hat{X}) \simeq \mathrm{H}^*(X^{\mathrm{an}}; \mathbb{C})$ between de Rham and Čech cohomology (see [12]), we recover the isomorphism

$$\mathrm{HP}_q(\mathcal{O}(X)) \simeq \mathrm{H}^{[q]}(X^{\mathrm{an}}; \mathbb{C}) \tag{35}$$

of [10, 15]. This result will be extended in the next theorem to ideals of $\mathcal{O}(X)$, by generalizing the previous argument and using mapping fiber complexes.

Theorem 9. *Let I be an ideal of the ring of regular functions on a complex affine algebraic variety X, and let $Y \subset X$ be the subvariety defined by I. Then we have a natural isomorphism*

$$\mathrm{HP}_q(I) \simeq \mathrm{H}^{[q]}(X^{\mathrm{an}}, Y^{\mathrm{an}}).$$

Proof. Let I_0 be the radical of I. Because $\mathcal{O}(X)$ is noetherian, we have that $I_0^N \subset I \subset I_0$, for some N, and hence $\mathrm{HP}_*(I_0) \simeq \mathrm{HP}_*(I)$, by Corollary 4. Since I and I_0 define the same subvariety, namely Y, we may assume that $I = I_0$, that is, that I is reduced, and hence $\mathcal{O}(X)/I = \mathcal{O}(Y)$.

Let $f : \mathcal{O}(X) \to \mathcal{O}(Y)$ be the restriction morphism. Denote by $\mathcal{X}(\mathcal{O}(X), I)$ the kernel of the morphism of mixed complexes, $\mathcal{X}(\mathcal{O}(X)) \to \mathcal{X}(\mathcal{O}(Y))$, associated to f. The excision property (Proposition 7) implies that $\mathrm{HP}_*(I) \simeq \mathrm{HP}_*(\mathcal{X}(\mathcal{O}(X), I))$.

Let $f_* : \mathcal{C}^{\mathrm{per}}(\mathcal{O}(X)) \to \mathcal{C}^{\mathrm{per}}(\mathcal{O}(Y))$ be the morphism of periodic cyclic complexes induced by f. Because f_* is surjective, we obtain that the inclusion $\ker(f_*) \to \mathcal{M}(f_*)$ induces an isomorphism of homology groups. Thus $\mathcal{M}(f_*)$ is quasi-isomorphic to $\mathcal{C}^{\mathrm{per}}(\mathcal{O}(X), I)$, and hence

$$\mathrm{HP}_*(I) \simeq \mathrm{H}_*(\mathcal{M}(f_*)). \tag{36}$$

Fix an embedding $X \subset \mathbb{A}_{\mathbb{C}}^m$ of X into an affine space. We shall use the notation we have used before: $\mathfrak{a} \subset \mathfrak{b} \subset R = \mathbb{C}[X_1, X_2, \ldots, X_q]$ are the ideals defining X and Y, $\mathcal{O}(X) = R/\mathfrak{a}$, and $\mathcal{O}(Y) = R/\mathfrak{b}$. We shall also denote by $\hat{R}_X$ the completion of R with respect to the $\mathfrak{a}$–adic topology, by $\hat{R}_Y$ the completion of R with respect to the $\mathfrak{b}$–adic topology, and by $\hat{f} : \hat{R}_X \to \hat{R}_Y$ the induced continuous morphism.

Theorem 6 implies that the natural projections $\pi_X : \hat{R}_X \to \mathcal{O}(X)$ and $\pi_Y : \hat{R}_X \to \mathcal{O}(X)$ induce isomorphisms

$$\pi_{X*} : \mathrm{HP}_*^{\mathrm{top}}(\hat{R}_X) \simeq \mathrm{HP}_*(\mathcal{O}(X)) \quad \text{and} \quad \pi_{Y*} : \mathrm{HP}_*^{\mathrm{top}}(\hat{R}_Y) \simeq \mathrm{HP}_*(\mathcal{O}(Y)). \tag{37}$$

The restriction morphism $f : \mathcal{O}(X) \to \mathcal{O}(Y)$ defines a continuous morphism $\hat{f} : \hat{R}_X \to \hat{R}_Y$. We now repeat, using $\hat{f}$, some of the constructions done using f.

First, the map $\hat{f}$ induces a morphism $\hat{f}_* : \hat{\mathcal{C}}^{\rm per}(\hat{R}_X) \to \hat{\mathcal{C}}^{\rm per}(\hat{R}_Y)$. Second, the projections π_X and π_Y combine to define a morphism $(\pi_{X*}, \pi_{Y*}) : \mathcal{M}(\hat{f}_*) \to \mathcal{M}(f_*)$. And last, Lemma 12 and the isomorphisms of equation (37) together then give the isomorphism

$$\mathrm{H}_*(\mathcal{M}(\hat{f}_*)) \simeq \mathrm{H}_*(\mathcal{M}(f_*)). \tag{38}$$

Consider now the mixed complexes

$$\Omega_{\hat{X}} = (\Omega^*(\hat{X}), 0, d_{DR}) \quad \text{and} \quad \Omega_{\hat{Y}} = (\Omega^*(\hat{Y}), 0, d_{DR})$$

used in equation (34). The restriction morphism $j^* : \Omega^*(\hat{X}) \to \Omega^*(\hat{Y})$ then gives a morphism

$$j^*_{[2]} : \mathcal{C}^{\rm per}(\Omega_{\hat{X}}) \longrightarrow \mathcal{C}^{\rm per}(\Omega_{\hat{Y}})$$

of the periodic cyclic complexes associated to the $\mathbb{Z}/2\mathbb{Z}$–periodic de Rham complexes of $\hat{X}$ and $\hat{Y}$. The morphism χ of equation (34) extends to a morphism of mapping fiber complexes and defines an isomorphism

$$\mathrm{H}_*(\mathcal{M}(\hat{f}_*)) \simeq \mathrm{H}_*(\mathcal{M}(j^*_{[2]})). \tag{39}$$

Now the periodic cyclic complex of $\Omega_{\hat{X}}$ (respectively $\Omega_{\hat{Y}}$) splits as the direct product of copies of the de Rham complex $(\Omega^*(\hat{X}), d_{DR})$ of $\hat{X}$ (respectively as the direct product of copies of the de Rham complex $(\Omega^*(\hat{Y}), d_{DR})$ of $\hat{Y}$) *naturally.* We then obtain a similar decomposition of $\mathcal{M}(j^*_{[2]})$. Precisely, the complex $\mathcal{M}(j^*_{[2]})$ is the direct product of copies of the complex $\mathcal{M}(j^*)$, shifted by even integers (see Figure 1)

$$\mathcal{M}_n(j^*_{[2]}) \simeq \prod_k \mathcal{M}_{n+2k}(j^*).$$

We finally obtain that

$$\mathrm{H}_q(\mathcal{M}(j^*_{[2]})) \simeq \prod_k \mathrm{H}_{q+2k}(\mathcal{M}(j^*). \tag{40}$$

The isomorphism of the theorem follows by putting together the isomorphisms (36), (38), (39), (40), and the isomorphism $\mathrm{H}_q(\mathcal{M}(j^*)) \simeq \mathrm{H}^q(X^{\rm an}, Y^{\rm an}; \mathbb{C})$ of Proposition 9. ☐

Proposition 10. *Let $\mathfrak{A}$ be a (not necessarily unital) complex algebra endowed with a filtration,*

$$(0) = J_n \subset J_{n-1} \subset \ldots \subset J_1 \subset J_0 = \mathfrak{A},$$

by ideals. Then there exists a spectral sequence with $E^1_{-p,q} = \mathrm{HP}_{q-p}(J_p/J_{p+1})$ convergent to $\mathrm{HP}_{q-p}(\mathfrak{A})$.

Proof. This spectral sequence is obtained from the filtration of the periodic cyclic complex $\mathcal{C}^{\rm per}(\mathfrak{A})$ by the kernels $(F_p)_{p\in\mathbb{Z}}$ of the natural projections:

$$F_{-p} = \ker(\mathcal{C}^{\rm per}(\mathfrak{A}) \longrightarrow \mathcal{C}^{\rm per}(\mathfrak{A}/J_p)).$$

Since

$$F_{-p}/F_{-p+1} \simeq \ker(\mathcal{C}^{\rm per}(\mathfrak{A}/J_{p-1}) \to \mathcal{C}^{\rm per}(\mathfrak{A}/J_p)),$$

and the kernel complex computes the periodic cyclic homology of J_{p-1}/J_p, by excision, the result follows. ☐

The proof of our main result, Theorem 1, is now a direct consequence of the facts that we have established. Actually, we shall establish a more general result, valid for two–sided ideals $\mathfrak{a} \subset A$ of unital finite type algebras.

Before stating this more general result, we need to introduce the necessary notation. Consider, as in the statement of Theorem 1, an abelian filtration $(\mathfrak{I}_k)$ of a unital finite type algebra A. Let X_k be the complex affine algebraic variety associated to the center Z_k of $A/\mathfrak{I}_k$. Let $\mathfrak{a}$ be a two–sided ideal of A, and let $Y_k(\mathfrak{a}) \subset X_k$ be the subvariety defined by

$$I_{k-1}^{\mathfrak{a}} = ((\mathfrak{a} + \mathfrak{I}_k)/\mathfrak{I}_k) \cap I_{k-1} = ((\mathfrak{a} + \mathfrak{I}_k)/\mathfrak{I}_k) \cap Z_k$$

and X_k^{an} and $Y_k^{\mathrm{an}}(\mathfrak{a})$, for $k = 1, \dots, n$, be the analytic varieties underlying X_k and $Y_k(\mathfrak{a})$. We shall refer to X_k^{an} and $Y_k^{\mathrm{an}}(\mathfrak{a})$ as the *analytic spaces* associated to the filtration $(\mathfrak{I}_k)$ of A and the two–sided ideal $\mathfrak{a}$.

Denote, as before, by

$$\mathrm{H}^{[j]}(A, B) = \prod_{k \in \mathbb{Z}} \mathrm{H}^{j+2k}(A, B; \mathbb{C})$$

the "$\mathbb{Z}/2\mathbb{Z}$–periodic" Čech cohomology groups of a pair (A, B) of *paracompact Hausdorff* topological spaces.

Theorem 10. *If $Y_p^{\mathrm{an}}(\mathfrak{a}) \subset X_p^{\mathrm{an}}$ are the analytic spaces associated to an abelian filtration of a unital finite type algebra A and the ideal $\mathfrak{a}$, then there exists a natural spectral sequence with*

$$E^1_{-p,q} = \mathrm{H}^{[q-p]}(X_p^{\mathrm{an}}, Y_p^{\mathrm{an}}(\mathfrak{a})),$$

convergent to $\mathrm{HP}_{q-p}(\mathfrak{a})$.

Proof. The abelian filtration, $\mathfrak{I}_k$, of A gives rise to a filtration of $\mathfrak{a}$,

$$\mathfrak{a} = \mathfrak{a} \cap \mathfrak{I}_0 \supset \mathfrak{a} \cap \mathfrak{I}_1 \supset \dots \supset \mathfrak{a} \cap \mathfrak{I}_n = \mathfrak{a} \cap \mathrm{Jac}(A).$$

Using Proposition 10, we obtain a spectral sequence with

$$E^1_{-p,q} = \mathrm{HP}_{q-p}(\mathfrak{a} \cap \mathfrak{I}_p / \mathfrak{a} \cap \mathfrak{I}_{p+1})$$

convergent to $\mathrm{HP}_{q-p}(\mathfrak{a}/\mathfrak{a} \cap \mathrm{Jac}(A)) = Hp_{q-p}(\mathfrak{a})$. Proposition 8 then gives that

$$\mathrm{HP}_{q-p}(\mathfrak{a} \cap \mathfrak{I}_p / \mathfrak{a} \cap \mathfrak{I}_{p+1}) = \mathrm{HP}_{q-p}(I_p^{\mathfrak{a}}).$$

¿From Theorem 9, we finally obtain that

$$\mathrm{HP}_{q-p}(I_p^{\mathfrak{a}}) \simeq \mathrm{H}^{[q-p]}(X_p^{\mathrm{an}}, Y_p^{\mathrm{an}}(\mathfrak{a})).$$

The proof is now complete. □

We continue with some applications, the first one being an easy computation.

Remark. Let $A_k \subset M_2(\mathbb{C}[X])$, for $k \geq 1$, be the algebras considered at the end of Section 1. Then X_1 consists of two points, so that $Z_1 \simeq \mathbb{C} \oplus \mathbb{C}$. Also, $X_2 = \mathbb{A}^1_{\mathbb{C}}$ is the one dimensional affine complex space, and $Y_2 = \{0\}$. We then obtain that $\mathrm{H}^*(X_2^{\mathrm{an}}, Y_2^{\mathrm{an}}) = 0$, so the spectral sequence degenerates at E^1, and the only nonzero terms are

$$E^1_{-1,q} = \mathrm{H}^{[q+1]}(X_1^{\mathrm{an}}), \quad q \in \mathbb{Z},$$

because Y_1^{an} is empty. We obtain that $\mathrm{HP}_q(A_k) \simeq \mathbb{C} \oplus \mathbb{C}$, if q is even, and that $\mathrm{HP}_q(A_k) \simeq 0$, otherwise.

The point of this remark is that, unlike Hochschild homology, periodic cyclic homology gives the same result for all the algebras A_k, where $k \geq 1$. Because the algebras A_k have the same spectrum, it follows that the periodic cyclic homology of an algebra is more closely related to the spectrum of that algebra than the Hochschild homology is.

Theorem 11. *Suppose $L \subset M_N(\mathbf{k})$ is a $\mathbf{k}$–subalgebra such that $L \subset M_N(I)$, for some ideal I of $\mathbf{k}$. Assume that, for any maximal ideal $\mathfrak{p}$ not containing I, the algebra $L/L \cap M_N(\mathfrak{p})$ has a unique simple quotient. Then the inclusion $L \to M_N(I)$ induces an isomorphism $\mathrm{HP}_*(L) \to \mathrm{HP}_*(M_N(I)) \simeq \mathrm{HP}_*(I)$.*

We agree that simple algebras have units and $1 \neq 0$.

Proof. We denote for any finite dimensional algebra B by B_{ss} the semisimple quotient $B/\operatorname{Jac}(B)$. Thus our hypothesis is that for each maximal ideal $\mathfrak{p} \subset \mathbf{k}$ the algebra $(L/L \cap M_N(\mathfrak{p}))_{ss}$ is either simple or 0. The proof will consist of two steps.

Step 1. The theorem is true if $\dim(L/L \cap M_N(\mathfrak{p}))_{ss} = k^2$ *is constant for all maximal ideals $\mathfrak{p} \subset \mathbf{k}$ such that $I \not\subset \mathfrak{p}$.*

Since the radical of $\mathbf{k}$ is nilpotent, we may assume that $\mathbf{k}$ is reduced. By replacing $M_N(\mathbf{k})$ with $M_{2N+1}(\mathbf{k})$ and L by $M_2(L)$, if necessary, we may also assume that $L \cap \mathbf{k} = \{0\}$ and that L has no irreducible representations of dimension 1 (here we identified $\mathbf{k}$ with the center of $M_N(\mathbf{k})$). Then using Lemma 11 we see that the standard filtration of the algebra $B = L + \mathbf{k}$ is

$$(0) \subset \operatorname{Jac}(L) \subset L \subset B.$$

Let $K_0 = Z(B/\operatorname{Jac}(L)) \cap (L/\operatorname{Jac}(L))$ and $K_0' \subset L$ its preimage in B. Using Proposition 8 and excision, we obtain

$$\mathrm{HP}_*(K_0') \simeq \mathrm{HP}_*(K_0) \simeq \mathrm{HP}_*(L/\operatorname{Jac}(L)) \simeq \mathrm{HP}_*(L).$$

It is then enough to prove that $\mathrm{HP}_*(K_0') \simeq \mathrm{HP}_*(I)$.

The trace $Tr : M_N(\mathbf{k}) \to \mathbf{k}$ maps K_0' to an ideal $K_L \subset I$, by $\mathbf{k}$–linearity. We claim that $I^m \subset K_L$, for some large m. Indeed, it is enough to check that, if $\mathfrak{p} \subset \mathbf{k}$ is a maximal ideal containing K_L, then $I \subset \mathfrak{p}$. So let $K_L \subset \mathfrak{p}$, $\mathfrak{p}$ maximal, and $Tr_{\mathfrak{p}} : L/L \cap M_N(\mathfrak{p}) \to \mathbf{k}/\mathfrak{p} = \mathbb{C}$ be the factor morphism induced by the trace. Then

$$K_L \subset \mathfrak{p} \Rightarrow Tr_{\mathfrak{p}} = 0 \Rightarrow L/L \cap M_N(\mathfrak{p}) \text{ is nilpotent} \Rightarrow I \subset \mathfrak{p},$$

where the last implication is obtained from the assumptions of the theorem. We obtain in this way that $\mathrm{HP}_*(I) \simeq \mathrm{HP}_*(K_L)$, and hence it suffices to prove that $\mathrm{HP}_*(K_0') \simeq \mathrm{HP}_*(K_L)$.

We consider again the trace, more precisely, its restriction $Tr : K_0' \to K_L$ which descends to a linear isomorphism $f : K_0 \to K_L$. If $I \not\subset \mathfrak{p}$, then the composition $K_0 \to K_L/K_L \cap \mathfrak{p}$ descends to a trace $K_0/\mathfrak{p}K_0 \to \mathbb{C} = \mathbf{k}/\mathfrak{p}$ that takes 1 to k. We obtain in this way that f/k is a multiplicative linear isomorphism, i.e., an algebra isomorphism. Moreover, the morphism induced by trace $\mathrm{HP}_*(K_0') \to \mathrm{HP}_*(K_L)$, factors as

$$\mathrm{HP}_*(K_0') \longrightarrow \mathrm{HP}_*(K_0) \xrightarrow{kf_*} \mathrm{HP}_*(K_L),$$

where the first morphism is the isomorphism induced by the surjective morphism $K_0' \to K_0$ with nilpotent kernel. Since f is an isomorphism, it follows that $\mathrm{HP}_*(K_0') \simeq \mathrm{HP}_*(K_L)$, and this is enough to prove the claim, as explained above.

Step 2. Induction: Suppose that the theorem is true for all subalgebras L such that $\dim(L/L \cap M_N(\mathfrak{p}))_{ss} < k$ *and we prove it for all subalgebras such that* $\dim(L/L \cap$

$M_N(\mathfrak{p}))_{ss} \leq k$. Here $\mathfrak{p}$ is an arbitrary maximal ideal of $\mathbf{k}$.

For any central polynomial g_k, the set $g_k(L)$ is a $\mathbf{k}$–submodule of L and hence $\mathrm{Tr}(g_k(L))$ is going to be an ideal of $\mathbf{k}$ with radical K_1. Let $L_1 = L \cap M_N(K_1)$. Then L_1 satisfies the assumptions of the Theorem and of the first step, and hence $L_1 \to M_N(K_1)$ gives an isomorphism $\mathrm{HP}_*(L_1) \simeq \mathrm{HP}_*(K_1)$. Also let $L_2 = L/L_1 \subset M_N(I/K_1)$, then all quotients $L_2/L_2 \cap M_N(\mathfrak{p})$ have dimensions $< k$, for all maximal ideals $\mathfrak{p}$ of $\mathbf{k}_2 = \mathbf{k}/K_1$ that do not contain $K_2 = I/K_1$. If follows that L_2 satisfies the assumptions of the inductive hypothesis with K_2 in place of I, and we obtain that $\mathrm{HP}_*(L_2) \simeq \mathrm{HP}_*(K_2)$. The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & L_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_N(I_1) \arrow[r] & M_N(I) \arrow[r] & M_N(I_2) \arrow[r] & 0
\end{tikzcd}$$

with exact lines gives rise to a natural transformation of the corresponding six term periodic exact sequences in periodic cyclic homology ($A = M_N(\mathbf{k})$):

$$\begin{tikzcd}
\mathrm{HP}_n(L_2) \arrow[r] \arrow[d] & \mathrm{HP}_{n-1}(L_1) \arrow[r] \arrow[d] & \mathrm{HP}_{n-1}(L) \arrow[r] \arrow[d] & \mathrm{HP}_{n-1}(L_2) \arrow[r] \arrow[d] & \mathrm{HP}_n(L_1) \arrow[d] \\
\mathrm{HP}_n(I_2A) \arrow[r] & \mathrm{HP}_{n-1}(I_1A) \arrow[r] & \mathrm{HP}_{n-1}(IA) \arrow[r] & \mathrm{HP}_{n-1}(I_2A) \arrow[r] & \mathrm{HP}_n(I_1A)
\end{tikzcd}$$

in which the first two and last two vertical arrows are isomorphisms, as argued above. The Five Lemma then gives that the middle vertical arrow is also an isomorphism, and the proof is complete. $\square$

Corollary 6. *Suppose $A \subset M_n(Z)$ is a unital finite type algebra with center Z and $J \subset A$ is a two–sided ideal. If the infinitesimal character $\Theta : \mathrm{Prim}(A) \to \mathrm{Prim}(Z)$ induces a bijection between $\mathrm{Prim}(J)$ and $\mathrm{Prim}(Z)$, then the inclusion $J \subset M_n(Z)$ gives an isomorphism*

$$\mathrm{HP}_*(J) \to \mathrm{HP}_*(M_n(Z)) \simeq \mathrm{HP}_*(Z).$$

Proof. This follows from Theorem 11 for $L = J$ and $I = Z = \mathbf{k}$. $\square$

References

[1] M. Artin, *On Azumaya algebras and finite dimensional representations of rings*, J. Algebra **11** (1969), 532–563.

[2] M.F. Atiyah and MacDonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, Mass.-London, 1969.

[3] Hyman Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[4] J. Bernstein, P. Deligne, and D. Kazhdan, *Trace Paley-Wiener theorem for reductive p–adic groups*, J. Analyse Math. **47** (1986), 180–192.

[5] J. N. Bernstein, *Le "centre" de Bernstein*, Representations of reductive groups over a local field (Paris) (P. Deligne, ed.), Hermann, Paris, 1984, pp. 1–32.

[6] N. Bourbaki, *Algèbre Commutative*, Hermann, Paris, 1961.

[7] J.-L. Brylinski, *Central localization in Hochschild homology*, J. Pure Appl. Algebra **57** (1989), 1–4.

[8] A. Connes, *Non commutative differential geometry*, Publ. Math. IHES **62** (1985), 41–144.

[9] J. Cuntz and D. Quillen, *Excision in bivariant periodic cyclic cohomology*, Invent. Math. **127** (1997), 67–98.

[10] B. Feigin and B. L. Tsygan, *Additive K–Theory and cristaline cohomology*, Funct. Anal. Appl. **19** (1985), 52–62.

[11] T. G. Goodwillie, *Cyclic homology, derivations, and the free loopspace*, Topology **24** (1985), 187–215.

[12] R. Hartshorne, *On the de Rham cohomology of algebraic varieties*, Publ. Math. IHES **45** (1976), 5–99.

[13] N. Higson and V. Nistor, *Cyclic cohomology of groups acting on buildings*, J. Funct.Anal. **141** (1996), 466–495.

[14] G. Hochschild, B. Kostant, and A. Rosenberg, *Differential forms on regular affine algebras*, Trans. Amer. Math. Soc. **102** (1962), 383–408.

[15] R. Hübl, *A note on the Hochschild homology and cyclic homology of a topological algebra*, Manuscripta Math. **77** (1992), 63–70.

[16] John D. Jones and C. Kassel, *Bivariant cyclic theory*, K-Theory **3** (1989), 339–365.

[17] C. Kassel, *Cyclic homology, comodules, and mixed complexes*, J. Algebra **107** (1987), 195–216.

[18] D. Kazhdan, V. Nistor, and P. Schneider, *On the Baum–Connes conjecture for reductive p–adic groups*, work in progress.

[19] J.-L. Loday, *Cyclic Homology*, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[20] J.-L. Loday and D. Quillen, *Cyclic homology and the Lie homology of matrices*, Comment. Math. Helv. **59** (1984), 565–591.

[21] S. Mac Lane, *Homology*, Springer-Verlag, Berlin-Heidelberg-New York, 1995.

[22] V. Nistor, *Group cohomology and the cyclic cohomology of crossed products*, Invent. Math. **99** (1990), 411–424.

[23] R. S. Peirce, *Associative algebras*, Springer-Verlag, 1982.

[24] C. Procesi, *On a theorem of M. Artin*, J. Algebra **22** (1972), 309–315.

[25] ———, *Trace identities and standard diagrams*, Ring Theory (New York), Lecture Notes in Pure and Appl. Math., vol. 51, Dekker, 1978, pp. 309–315.

[26] G. Rinehart, *Differential forms for general commutative algebras*, Trans. Amer. Math. Soc. **108** (1963), 195–222.

[27] L. H. Rowen, *Polynomial identities in ring theory*, Academic Press, New York-London-Toronto, 1980.

[28] P. Schneider, *The cyclic homology of p-adic reductive groups*, J. Reine Angew. Math. **475** (1996), 39–54.

[29] P. Seibt, *Local cyclic homology*, K-Theory **4** (1990), 143–155.

[30] J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier **6** (1956), 1–42.

[31] B. L. Tsygan, *Homology of matrix Lie algebras over rings and Hochschild homology*, Uspekhi Math. Nauk. **38** (1983), 217–218.

Harvard University, Math. Dept., Cambridge, MA 02138, USA
E-mail address: kazhdan@math.harvard.edu

Pennsylvania State University, Math. Dept., University Park, PA 16802, USA
E-mail address: nistor@math.psu.edu

WWU Münster, Math. Institut, Einsteinstr. 62, 48149 Münster, Germany
E-mail address: pschnei@math.uni-muenster.de