Equivariant homology for totally disconnected groups

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One of the far-reaching problems about continuous group actions is the Baum-Connes conjecture ([BCH]). Although it is an assertion about certain K-theoretic invariants in the equivariant setting the understanding of the corresponding equivariant (co)homological invariants should be most useful. The literature contains various proposals how to construct in various situations, mostly for discrete groups or real Lie groups acting on manifolds, the delocalized equivariant (co)homology. Now that we have the beautiful work of Bernstein/Lunts ([BL]) on the equivariant derived category of a real Lie group action this undoubtedly provides the correct framework in that case.

In this paper we consider the case of a locally compact and totally disconnected group G acting continuously on a locally compact space X. We develop the basic homological algebra of G-equivariant sheaves on X including Verdier duality. It turns out that certain functors, e.g. the direct image, do not correspond to their nonequivariant counterparts under the forgetful map. That this was to be expected was pointed out to me by Bernstein. We give a derived functor definition of delocalized equivariant homology and show that it unifies and generalizes the corresponding concepts in [BC] and [BCH]. Finally we relate the delocalized equivariant homology of the point to the cyclic homology of the Hecke algebra of G. This generalizes the main result of [HN] and [Sch].

I want to thank P. Baum for several enlightening discussions about the paper [BC] and J. Bernstein for his helpful comments.

1. The construction

Let G be a locally compact and totally disconnected group. A **smooth** G-module V is a $\mathbb{C}$-vector space V together with a linear G-action such that $\{g \in G : gv = v\}$ is open in G for any $v \in V$. (Working with $\mathbb{C}$ as coefficient field is just for convenience; it could be any field of characteristic 0.) The abelian category $\mathrm{Alg}(G)$ of smooth G-modules contains enough projective objects ([Bla]). Hence the group homology

$$\begin{aligned} H_*(G,\cdot) := {} & \text{left derived functor of the functor} \\ & (\cdot)_G \text{ of taking } G\text{-coinvariants} \end{aligned}$$

is defined.

Now let X be a locally compact space with a continuous G-action

$$\begin{aligned} \mu : G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx \quad . \end{aligned}$$

To define the notion of a G-equivariant sheaf on X we need the maps

$$G \times G \times X \;\overset{\xrightarrow{\partial_0}}{\underset{\xrightarrow{\partial_2}}{\xrightarrow{\partial_1}}}\; G \times X \;\overset{\xrightarrow{\pi}}{\underset{}{\xrightarrow{\mu}}}\; X$$

given by

$$\pi(g,x) := x,$$
$$\partial_0(g,h,x) := (h,x),\ \partial_1(g,h,x) := (gh,x),\ \partial_2(g,h,x) := (g,hx) \quad .$$

Definition:
A G-equivariant sheaf (of $\mathbb{C}$-vector spaces) F on X is a sheaf F (of $\mathbb{C}$-vector spaces) on X together with a sheaf isomorphism

$$\alpha = \alpha_F : \pi^* F \overset{\cong}{\longrightarrow} \mu^* F$$

which satisfies the cocycle condition

(1) $$\partial_1^*(\alpha) \;=\; \partial_2^*(\alpha) \circ \partial_0^*(\alpha) \quad .$$

Let us unravel this definition a little bit. Pulling α back via the map

$$X \longrightarrow G \times X \ ,$$
$$x \longmapsto (g,x)$$

for a given $g \in G$, results in a sheaf homomorphism

$$\alpha_g : F \longrightarrow g^* F \quad .$$

Moreover pulling back the identity (1) via the map

$$X \longrightarrow G \times G \times X$$
$$x \longmapsto (g,h,x)$$

gives the identity

(2) $$\alpha_1 = id_F \text{ and } \alpha_{gh} = h^*(\alpha_g) \circ \alpha_h \text{ for any } g,h \in G \quad .$$

By adjunction α induces a sheaf homomorphism $F \longrightarrow \pi_* \mu^* F$. Let $U \subseteq X$ be an open subset. Writing out the definition of $(\mu^* F)(G \times U)$ we obtain the following continuity condition:

For any section $s \in F(U)$ there is an open covering
$G \times U = \bigcup_{i \in I} \Omega_i \times U_i$ and sections $s_i \in F(\Omega_i U_i)$
such that
$$\alpha_g(s|U_i) = s_i|gU_i$$
for any $g \in \Omega_i$ and any $i \in I$.

A nicer way to write down that condition is as follows: First of all we call an open subset $U \subseteq X$ G-good if its stabilizer $P_U^\dagger := \{g \in G : gU = U\}$ is open in G. Since the G-good open subsets form a basis for the topology of X the above continuity condition is equivalent to:

(3) For any section $s \in F(U)$ there is an open covering $U = \bigcup_{i \in I} U_i$ by G-good subsets U_i and there are open subgroups $H_i \subseteq P_{U_i}^\dagger$ such that $s|U_i$ is H_i-invariant for any $i \in I$.

It is not difficult to check that giving an α which satisfies (1) is equivalent to giving the α_g for $g \in G$ such that (2) and (3) are fulfilled.
Let $\mathrm{Sh}_G(X)$ denote the category of G-equivariant sheaves on X. It is abelian since π^* and μ^* are exact functors. Evidently $\mathrm{Alg}(G) = \mathrm{Sh}_G(pt)$. In the following we develop only the more elementary theory of the category $\mathrm{Sh}_G(X)$. In particular we do not discuss the problem how to construct the equivariant derived category (compare [BL]).
Let Y be a second locally compact space with a continuous G-action (a G-space for short) and let $f : X \longrightarrow Y$ be a G-equivariant continuous map (a G-map for short). We then have the exact inverse image functor

$$\begin{array}{rccc} f^* : & \mathrm{Sh}_G(Y) & \longrightarrow & \mathrm{Sh}_G(X) \\ & (F, \alpha_F) & \longmapsto & (f^* F, (id \times f)^*(\alpha_F)) \end{array} .$$

Using base change ([KS] 2.6.7) we obtain the higher direct images with proper support

$$\begin{array}{rccc} R^i f_! : & \mathrm{Sh}_G(X) & \longrightarrow & \mathrm{Sh}_G(Y) \\ & (F, \alpha_F) & \longmapsto & (R^i f_! F,\ R^i (id \times f)_!\ (\alpha_F)) \end{array} .$$

In particular, for any G-equivariant sheaf F on X, its cohomology with compact support $H_c^*(X, F)$ in a natural way is a smooth G-module. We need a more precise version of this latter fact.
If G^{dis} denotes the group G equipped with the discrete topology then we have the fully faithful and exact embedding

$$\mathrm{Sh}_G(X) \subseteq \mathrm{Sh}_{G^{\mathrm{dis}}}(X) \quad .$$

A G^{dis}-equivariant sheaf on X is a sheaf on X together with homomorphisms α_g for $g \in G$ which satisfy (2). For any such sheaf S we may define a subsheaf $S^{\mathrm{smooth}} \subseteq S$ by

$$S^{\mathrm{smooth}}(U) := \{s \in S(U) : s \text{ satisfies (3) } \} \quad .$$

One checks that the α_g respect this subsheaf S^{smooth}. This means that the G^{dis}-equivariant structure on S induces a G-equivariant structure on S^{smooth}. In this way we obtain a left exact functor

$$\begin{aligned} \mathrm{Sh}_{G^{\mathrm{dis}}}(X) &\longrightarrow \mathrm{Sh}_G(X) \\ S &\longmapsto S^{\mathrm{smooth}} \end{aligned}$$

which obviously is right adjoint to the above embedding. Hence it preserves injective objects. If F is G-equivariant then $F^{\mathrm{smooth}} = F$.

Lemma 1:
The category $\mathrm{Sh}_G(X)$ has enough injective objects.
Proof: For $\mathrm{Sh}_{G^{\mathrm{dis}}}(X)$ this is shown in [Gro] 5.1.1. Using the properties of the functor $(\cdot)^{\mathrm{smooth}}$ which we have listed above it then follows for $\mathrm{Sh}_G(X)$.

Let $\mathrm{Sh}(X)$ be the abelian category of sheaves of $\mathbb{C}$-vector spaces on X. There is the forgetful functor

$$\begin{array}{rccc} \mathrm{For}: & \mathrm{Sh}_G(X) & \longrightarrow & \mathrm{Sh}(X) \\ & (F, \alpha_F) & \longmapsto & F \quad . \end{array}$$

I don't know whether this functor always respects injective objects. Instead we will use the following fact.

Proposition 2:
For any sheaf F in $\mathrm{Sh}_G(X)$ there is an embedding $F \hookrightarrow S$ in $\mathrm{Sh}_G(X)$ such that $\mathrm{For}(S)$ *is c-soft.*
Proof: (I learnt the subsequent argument in a different context from V. Berkovich.) Let $\tilde{F}$ denote the Godement resolution of F, i.e., the sheaf given by

$$\tilde{F}(U) = \prod_{x \in U} F_x \quad .$$

This sheaf obviously is $G^{\rm dis}$-equivariant. Hence the embedding

$$F \hookrightarrow S := \tilde{F}^{\rm smooth}$$

lies in $\mathrm{Sh}_G(X)$. We claim that $\mathrm{For}(S)$ is c-soft. Consider a section s_0 of S over some compact subset $K \subset X$. It extends to a section $s' \in S(U)$ for some open neighbourhood U of K. Since K is compact we may assume that U as well as s' are fixed by some open subgroup $H \subseteq G$. We extend s' by zero to a section $s \in \tilde{F}(X)$. Since it is fixed by H, too, it actually lies in $S(X)$.

Corollary 3:
The functor For transforms injective objects in $\mathrm{Sh}_G(X)$ *into c-soft sheaves.*
Proof: This is a standard argument (compare [Ive] proof of II.3.5) which we briefly recall for the convenience of the reader. The point to observe is that any direct summand of a c-soft sheaf is c-soft, too. Let now F be an injective object in $\mathrm{Sh}_G(X)$. According to the previous result we find an embedding $F \hookrightarrow S$ in $\mathrm{Sh}_G(X)$ such that $\mathrm{For}(S)$ is c-soft. But because of its injectivity F actually is a direct summand of S.

In particular we obtain that the functors $R^i f_!$ can be computed by injective resolutions in $\mathrm{Sh}_G(X)$. We now assume that the group G has finite homological dimension. Then the group hyperhomology of unbounded complexes of smooth G-modules is defined. The **equivariant homology** of a sheaf F in $\mathrm{Sh}_G(X)$ is defined to be

$$H_*(X, G; F) := H_*(G, \Gamma_c(X, I^{-\cdot}))$$

where $F \xrightarrow{\sim} I^\cdot$ is an injective resolution in $\mathrm{Sh}_G(X)$. By construction we have the spectral sequence

$$E_2^{r,s} = H_r(G, H_c^s(X, F)) \;\Rightarrow\; H_{r-s}(X, G; F) \enspace .$$

If $F = \mathbb{C}$ is the constant sheaf then the groups $H_*(X, G; \mathbb{C})$ might be called the equivariant homology of the space X. A delocalized version of these groups can be introduced as follows. An element $g \in G$ is called compact if g is contained in a compact subgroup of G; put

$$G_0 := \{g \in G : \; g \text{ is compact } \} \enspace .$$

We always make the assumption that G_0 is locally closed in G. Then the subspace

$$\hat{X} := \{(g, x) \in G \times X : \; g \text{ compact and } gx = x\}$$

of $G \times X$ makes the diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\text{pr}} & X \\ \subseteq\downarrow & & \downarrow \text{diag} \\ G_0 \times X & \xrightarrow{(\mu,id)} & X \times X \end{array}$$

cartesian and hence is locally closed and therefore locally compact. It is a G-space via

$$\begin{array}{ccc} G \times \hat{X} & \longrightarrow & \hat{X} \\ (g,(h,x)) & \longmapsto & (ghg^{-1}, gx) \end{array} \quad .$$

The **delocalized equivariant homology** of the space X is defined to be

$$H_*^G(X) := H_*(\hat{X}, G; \mathbb{C}) \quad .$$

This generalizes the corresponding concept introduced in [BC] for discrete groups. As we will see in the next section it also comprises the cosheaf homology of the building in [BCH].

2. The building

We now take G to be the group of K-rational points of a connected reductive group over a nonarchimedean locally compact field K. The assumptions which we made in the previous section are satisfied:

– Homological dimension of $G \leq K$-rank of G.

– G_0 is open and closed in G.

Let $\mathcal{B}$ denote the Bruhat-Tits building of G. It is a G-space which carries a natural G-equivariant polysimplicial structure. Those (open) polysimplices are called facets. For any such facet $F \subseteq \mathcal{B}$ let $P_F \subseteq G$ be its pointwise stabilizer; this is an open subgroup in G.

In this section we always assume that G is semisimple and simply connected. Then the subgroups P_F are compact open and

$$G_0 = \bigcup_F P_F \quad .$$

This allows us to introduce a natural open covering of $\hat{X}$ in order to compute $H_*^G(X)$ for any G-space X. For any facet F in $\mathcal{B}$ we put

$$\hat{X}_F := \{(g,x) \in \hat{X} \ : \ g \in P_F\} \quad .$$

These are open and closed subsets in $\hat{X}$ such that

$$\hat{X} = \bigcup_F \hat{X}_F \quad .$$

Let us fix once and for all a G-invariant orientation of $\mathcal{B}$. (Nothing will really depend on this choice. At the cost of a more complicated notation it could be avoided by working with oriented chain complexes in the following.) For any sheaf $S \in \mathrm{Sh}_G(\hat{X})$ we then have the augmented complex

$$(*) \qquad \begin{array}{ccc} \bigoplus_{\dim F=d} \Gamma_c(\hat{X}_F, S) \xrightarrow{\partial} \ldots \xrightarrow{\partial} & \bigoplus_{\dim F=0} \Gamma_c(\hat{X}_F, S) \\ & \downarrow \\ & \Gamma_c(\hat{X}, S) \ . \end{array}$$

Here d denotes the dimension of $\mathcal{B}$ which is equal to the K-rank of G. The boundary maps ∂ as well as the augmentation are induced by the extension by zero of sections with compact support. Obviously G acts on that complex. Since each $\Gamma_c(\hat{X}_F, S)$ is a smooth P_F-module $(*)$ actually lies in $\mathrm{Alg}(G)$. It follows from [Sch] 2.1 that each term of $(*)$ is a projective object in $\mathrm{Alg}(G)$.

Proposition 1:
If For(S) is c-soft then $(*)$ is an exact projective resolution of $\Gamma_c(\hat{X}, S)$ in $\mathrm{Alg}(G)$.
Proof: This is a straightforward generalization of the arguments in the proof of [Sch] 2.2. $(*)$ is the complex of global sections with compact support of the complex of sheaves

$$0 \longrightarrow \bigoplus_{\dim F=d} i_{F*}i_F^*S \longrightarrow \ldots \longrightarrow \bigoplus_{\dim F=0} i_{F*}i_F^*S \longrightarrow S \longrightarrow 0$$

on $\hat{X}$; here $i_F : \hat{X}_F \xrightarrow{\subseteq} \hat{X}$ denotes the inclusion map. All sheaves in this complex are c-soft. Hence it suffices to check that this complex of sheaves is exact which can be done stalkwise. But the complex of stalks in the point $(g, x) \in \hat{X}$ computes the singular homology of the nonempty subcomplex

$$\mathcal{B}^{(g)} := \text{ all facets which are fixed pointwise by } g$$

of the building $\mathcal{B}$ with coefficients in the abelian group $S_{(g,x)}$. Under our assumption on the group G that subcomplex $\mathcal{B}^{(g)}$ coincides with the fixed point set $\mathcal{B}^g$ of g in $\mathcal{B}$. The latter is contractible since the G-action on $\mathcal{B}$ respects geodesics.

Now choose an injective resolution $S \xrightarrow{\sim} I^\cdot$ in $\mathrm{Sh}_G(\hat{X})$. Combining the above result and Corollary 1.3 we see that the equivariant homology $H_*(\hat{X}, G; S)$ can be computed as the homology of the double complex

$$\left[\bigoplus_{\dim F=*} \Gamma_c(\hat{X}_F, I^{-\cdot})\right]_G \quad .$$

It is an easy exercise to deduce from this description that in the case $X = pt$ the point and $S = \mathbb{C}$ the constant sheaf our delocalized equivariant homology $H_*^G(pt)$ coincides with what in [BCH] (6.9) is called the equivariant homology for the building.

3. Equivariant Verdier duality

Let G again be a general locally compact and totally disconnected group and let $f : X \to Y$ be a G-map. It is obvious that the direct image $f_* S$ of a G-equivariant sheaf S on X at least is a G^{dis}-equivariant sheaf on Y. Hence we may define the smooth direct image functor $f_{*,\infty}$ to be the composite

$$\mathrm{Sh}_G(X) \xrightarrow{f_*} \mathrm{Sh}_{G^{\mathrm{dis}}}(Y) \xrightarrow{(.)^{\mathrm{smooth}}} \mathrm{Sh}_G(Y) \ .$$

It is right adjoint to f^* and, in particular, respects injective objects. Its right derived functors $R^i f_{*,\infty}$ are called the higher smooth direct images. One should always keep in mind that under the forgetful functor they do not correspond to the usual higher direct images. As a special case, for the map from X to the point, we obtain the smooth cohomology groups

$$H^i_\infty(X,.) : \mathrm{Sh}_G(X) \longrightarrow \mathrm{Alg}(G) \ .$$

On the other hand the right derived functors of the functor

$$\begin{array}{ccc} \mathrm{Sh}_G(X) & \longrightarrow & \mathbb{C}-\ \text{vector spaces} \\ S & \longmapsto & S(X)^G \end{array}$$

of taking the G-invariant global sections are called the **equivariant cohomology** groups $H^*(X, G; .)$. We have the spectral sequence

$$E_2^{r,s} = H^r(G; H^s_\infty(X,S)) \ \Rightarrow \ H^{r+s}(X, G; S)$$

where $H^*(G,.)$ is the usual group cohomology on $\mathrm{Alg}(G)$.

Next we introduce the functor of “compact induction”. Let $H \subseteq G$ be an open subgroup and let S be a sheaf in $\mathrm{Sh}_H(X)$. We define a presheaf $c\text{-}\mathrm{Ind}(S)$ on X by setting, for $U \subseteq X$ open,

$$
\begin{aligned}
c\text{-Ind}(S)(U) := \ & \text{all maps } \varphi : G \to \dot{\bigcup_{g\in G}} S(g^{-1}U) \\
& \text{such that} \\
& - \varphi(g) \in S(g^{-1}U) \text{ for any } g \in G, \\
& - \text{for any compact subset } A \subseteq U \text{ the map} \\
& \qquad\qquad g \longmapsto \varphi(g)|g^{-1}A \\
& \quad \text{vanishes off finitely many right cosets of} \\
& \quad H \text{ in } G, \text{ and} \\
& - \varphi(gh) = \alpha_{h^{-1}}(\varphi(g)) \text{ for any } g \in G,\ h \in H\ .
\end{aligned}
$$

This clearly is a sheaf on X on which G^{dis} acts by left translations.

Lemma 1:
The sheaf c-Ind(S) is G-equivariant.
Proof: The claim is that the G-equivariant subsheaf $c\text{-Ind}(S)^{\text{smooth}}$ actually coincides with c-Ind(S). If we replace the second condition in the definition of c-Ind(S) by the stronger requirement that φ itself is supported on finitely many right cosets of H then we obtain a sub-presheaf whose sheafification is c-Ind(S). All we therefore have to show is that any section $\varphi \in c\text{-Ind}(S)(U)$ which is supported on H satisfies the condition (3) of the first paragraph. Since S is H-equivariant we find an open covering $U = \bigcup_{i\in I} U_i$ by H-good and hence G-good subsets U_i and open subgroups $H_i \subseteq H \cap P^{\dagger}_{U_i}$ such that $\varphi(1)|U_i$ is H_i-invariant. It immediately follows that $\varphi|U_i$ is H_i-invariant.

The resulting functor

$$
c\text{-Ind} \ : \ \mathrm{Sh}_H(X) \longrightarrow \mathrm{Sh}_G(X)
$$

is called compact induction.

Lemma 2:
The functor c-Ind$(\,.\,)$ is exact and left adjoint to the forgetful functor For.
Proof: Since X is locally compact the stalk of c-Ind(S) in a point $x \in X$ is equal

to

$$c\text{-Ind}(S)_x = \text{all maps } \psi : G \longrightarrow \dot{\bigcup_{g\in G}} S_{g^{-1}x}$$
$$\begin{aligned}&\text{such that}\\ &- \psi(g) \in S_{g^{-1}x} \text{ for any } g \in G,\\ &- \psi \text{ vanishes off finitely many right}\\ &\quad \text{cosets of } H \text{ in } G, \text{ and}\\ &- \psi(gh) = \alpha_{h^{-1}}(\psi(g)) \text{ for any } g \in G, h \in H\ .\end{aligned}$$

From this the exactness assertion is immediate. The two adjunction homomorphisms are given by

$$\begin{array}{rcl} S(U) & \longrightarrow & \mathrm{For}(c\text{-Ind}(S))(U) \\ s & \longmapsto & \left[g \longmapsto \left\{\begin{array}{ll} \alpha_{g^{-1}}(s) & \text{if } g \in H \\ 0 & \text{otherwise} \end{array}\right\}\right] \end{array}$$

and

$$\begin{array}{rcl} c\text{-Ind}(\mathrm{For}(T))(U) & \longrightarrow & T(U) \\ \varphi & \longmapsto & \displaystyle\sum_{g\in G/U} \alpha_g(\varphi(g)) \quad . \end{array}$$

The sum in the second definition although infinite is well defined due to the finiteness property of φ on compact subsets of U.

Corollary 3:
For any open subgroup $H \subseteq G$ the forgetful functor $\mathrm{For} : \mathrm{Sh}_G(X) \longrightarrow \mathrm{Sh}_H(X)$ *respects injective objects.*

This has the following useful consequence. For any two sheaves $S, T \in \mathrm{Sh}_{G^{\mathrm{dis}}}(X)$ the homomorphisms $\mathrm{Hom}_{\mathrm{Sh}(X)}(S,T)$ form, of course, a G^{dis}-module. So we may define

$$\mathrm{Hom}_{X,\infty}(S,T) := \mathrm{Hom}_{\mathrm{Sh}(X)}(S,T)^{\mathrm{smooth}} \quad .$$

It is evident that, for $S \in \mathrm{Sh}_G(X)$, we have

$$\mathrm{Hom}_{X,\infty}(S,T) = \mathrm{Hom}_{X,\infty}(S,T^{\mathrm{smooth}}) \quad .$$

Corollary 4:
Let T be an injective object in $\mathrm{Sh}_G(X)$; then the functor

$$\mathrm{Hom}_{X,\infty}(\,.\,,T) : \mathrm{Sh}_G(X) \longrightarrow \mathrm{Alg}(G)$$

is exact.
Proof: We have

$$\mathrm{Hom}_{X,\infty}(S,T) \;=\; \bigcup_H \mathrm{Hom}_{\mathrm{Sh}_H(X)}(S,T)$$

where H runs over the open subgroups of G.

Remark 5:
If the functor $\mathrm{Hom}_{X,\infty}(\,.\,,T)$ is exact then T is injective in $\mathrm{Sh}_H(X)$ for some open subgroup $H \subseteq G$.
Proof: Consider an embedding $T \hookrightarrow I$ in $\mathrm{Sh}_G(X)$ into an injective object I. By assumption it has to have a smooth section, i.e., a section which is invariant under some open subgroup $H \subseteq G$. Using Corollary 3 we see that T is injective in $\mathrm{Sh}_H(X)$.

We similarly define the G-equivariant hom-sheaf by

$$\underline{\mathrm{Hom}}_{X,\infty}(S,T) := \underline{\mathrm{Hom}}_{\mathrm{Sh}(X)}(S,T)^{\mathrm{smooth}} \quad .$$

The relation to the hom-module is given by

$$H^0_\infty(X, \underline{\mathrm{Hom}}_{X,\infty}(S,T)) = \mathrm{Hom}_{X,\infty}(S,T) \quad .$$

Again, for $S \in \mathrm{Sh}_G(X)$, we have

$$\underline{\mathrm{Hom}}_{X,\infty}(S,T) = \underline{\mathrm{Hom}}_{X,\infty}(S,T^{\mathrm{smooth}}) \quad .$$

For smooth G-modules V, V' we usually will use the more traditional notation

$$\mathrm{Hom}^\infty_{\mathbb{C}}(V,V') := \mathrm{Hom}_{\mathbb{C}}(V,V')^{\mathrm{smooth}} \quad .$$

This latter functor is exact in both variables.
Clearly with two G-equivariant sheaves S and T also their tensor product $S \underset{\mathbb{C}}{\otimes} T$ is a G-equivariant sheaf on X. The adjunction formula

$$\mathrm{Hom}_{X,\infty}(S \underset{\mathbb{C}}{\otimes} T, T') = \mathrm{Hom}_{X,\infty}(S, \underline{\mathrm{Hom}}_{X,\infty}(T,T'))$$

holds.

Lemma 6:
If $T \in \mathrm{Sh}_G(X)$ *is arbitrary and* $T' \in \mathrm{Sh}_G(X)$ *is an injective object then* $\underline{\mathrm{Hom}}_{X,\infty}(T,T')$ *is injective in* $\mathrm{Sh}_G(X)$.
Proof: We also have the adjunction formula

$$\mathrm{Hom}_{\mathrm{Sh}_G(X)}(S \underset{\mathbf{C}}{\otimes} T, T') = \mathrm{Hom}_{\mathrm{Sh}_G(X)}(S, \underline{\mathrm{Hom}}_{X,\infty}(T,T')) \quad .$$

Because of the exactness of the tensor product the left hand side is an exact functor in S.

As a first application we see that we have the local-to-global spectral sequence

$$E_2^{r,s} = H^r_\infty(X, \underline{\mathrm{Ext}}^s_{X,\infty}(S,T)) \;\Rightarrow\; \mathrm{Ext}^{r+s}_{X,\infty}(S,T)$$

for $S,T \in \mathrm{Sh}_G(X)$ where $\underline{\mathrm{Ext}}^*_{X,\infty}$, resp. $\mathrm{Ext}^*_{X,\infty}$, are the right derived functors of $\underline{\mathrm{Hom}}_{X,\infty}$, resp. $\mathrm{Hom}_{X,\infty}$, on $\mathrm{Sh}_G(X)$.

We now assume that X has finite c-cohomological dimension. For any smooth G-module V and any sheaf $I \in \mathrm{Sh}_G(X)$ the presheaf $\Gamma^!_I(V)$ on X is defined by

$$\Gamma^!_I(V)(U) := \mathrm{Hom}_{\mathbf{C}}(\Gamma_c(U,I),V) \quad \text{for } \; U \subseteq X \; \text{ open} \; .$$

It is G^{dis}-equivariant in an obvious sense. We recall the following facts.

Lemma 7:
If $\mathrm{For}(I)$ *is c-soft then we have:*

i. The functor

$$\begin{array}{rcl} \mathrm{Sh}_G(X) & \longrightarrow & \mathrm{Alg}(G) \\ S & \longmapsto & \Gamma_c(X, S \underset{\mathbf{C}}{\otimes} I) \end{array}$$

is exact;

ii. $\Gamma^!_I(V)$ *is a sheaf ;*

iii. there is the natural isomorphism of functors

$$\mathrm{Hom}_{\mathbf{C}}(\Gamma_c(X, \,.\, \underset{\mathbf{C}}{\otimes} I), V) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Sh}(X)}(\,.\,, \Gamma^!_I(V))$$

on $\mathrm{Sh}_G(X)$.
Proof: [KS] 3.1.2 and 3.1.3.

Assume $\mathrm{For}(I)$ to be c-soft. According to the second assertion in the previous lemma the functor

$$\begin{array}{rcl} \Gamma_I^{!,\infty} \ : \ \mathrm{Alg}(G) & \longrightarrow & \mathrm{Sh}_G(X) \\ V & \longmapsto & \Gamma_I^{!}(V)^{\mathrm{smooth}} \end{array}$$

is defined. The third assertion gives the natural isomorphism

$$\mathrm{Hom}_{\mathrm{Alg}(G)}(\Gamma_c(X, \,.\, \underset{\mathbb{C}}{\otimes} I), V) \ \xrightarrow{\cong} \ \mathrm{Hom}_{\mathrm{Sh}_G(X)}(\,.\,, \Gamma_I^{!,\infty}(V))$$

on $\mathrm{Sh}_G(X)$ which together with the first assertion implies that the functor $\Gamma_I^{!,\infty}$ respects injective objects.

Our assumption on X together with Corollary 1.3 ensures that there is a bounded resolution $\mathbb{C} \xrightarrow{\sim} I^\cdot$ in $\mathrm{Sh}_G(X)$ of the constant sheaf such that each $\mathrm{For}(I^j)$ is c-soft. Also let $V \xrightarrow{\sim} J^\cdot$ be an injective resolution in $\mathrm{Alg}(G)$. Then the total complex

$$\Gamma^{!,\infty}(V) := \mathrm{Tot}(\Gamma_{I^{-\cdot}}^{!,\infty}(J^\cdot))$$

consists of injective objects in $\mathrm{Sh}_G(X)$, is bounded below, and as an object in the derived category $D^+(\mathrm{Sh}_G(X))$ only depends on V. We obtain a functor

$$\Gamma^{!,\infty} \ : \ D^+(\mathrm{Alg}(G)) \ \longrightarrow \ D^+(\mathrm{Sh}_G(X))$$

which is right adjoint to $R\Gamma_c(X, \,.\,)$ (compare [KS] 3.1.5). Following the usual terminology we call the complex

$$\omega_{X,G} \ := \Gamma^{!,\infty}(\mathbb{C})$$

for the trivial G-module $\mathbb{C}$ the **equivariant dualizing complex** of sheaves on X. Again we point out that $\mathrm{For}(\omega_{X,G})$ in general does not coincide with the usual dualizing complex ω_X on X. We will come back to this problem at the end of this section. For the moment being we just make the easy observation that $\omega_{pt,G} = \mathbb{C}$ in $D^+(\mathrm{Alg}(G))$.

Proposition 8: (Smooth Verdier duality)
For any $S \in \mathrm{Sh}_G(X)$ *and any* $V \in \mathrm{Alg}(G)$ *we have*

$$\mathrm{Hom}^{\infty}_{\mathbb{C}}(H_c^{-*}(X,S),V) \;=\; H^*_{\infty}(X,\underline{\mathrm{Hom}}_{X,\infty}(S,\Gamma^{!,\infty}(V))) \;\; .$$

Proof: With the previous notations we compute

$$\begin{aligned}
\mathrm{Hom}^{\infty}_{\mathbb{C}}(H_c^{-*}(X,S),V) &= \mathrm{Hom}^{\infty}_{\mathbb{C}}(h^*(\Gamma_c(X,S\underset{\mathbb{C}}{\otimes} I^{-\cdot})),V) \\
&= h^*(\mathrm{Hom}^{\infty}_{\mathbb{C}}(\Gamma_c(X,S\underset{\mathbb{C}}{\otimes} I^{-\cdot}),V)) \\
&= h^*(\mathrm{Tot}\,\mathrm{Hom}^{\infty}_{\mathbb{C}}(\Gamma_c(X,S\underset{\mathbb{C}}{\otimes} I^{-\cdot}),J^{\cdot})) \\
&= h^*(\mathrm{Tot}\,\mathrm{Hom}_{X,\infty}(S,\Gamma^{!,\infty}_{I^{-\cdot}}(J^{\cdot}))) \\
&= h^*(\mathrm{Tot}\,H^0_{\infty}(X,\underline{\mathrm{Hom}}_{X,\infty}(S,\Gamma^{!,\infty}_{I^{-\cdot}}(J^{\cdot})))) \\
&= H^*_{\infty}(X,\underline{\mathrm{Hom}}_{X,\infty}(S,\Gamma^{!,\infty}(V))) \;\; .
\end{aligned}$$

In the first step we use the fact that with I^j also $S\underset{\mathbb{C}}{\otimes} I^j$ is c-soft, the fourth step comes from Lemma 7iii, and the last step is a consequence of Lemma 6.

The equivariant duality functor is given by

$$\begin{aligned}
D_{X,G} : D^b(\mathrm{Sh}_G(X)) &\longrightarrow D^+(\mathrm{Sh}_G(X)) \\
S^{\cdot} &\longmapsto R\,\underline{\mathrm{Hom}}_{X,\infty}(S^{\cdot},\omega_{X,G}) \, .
\end{aligned}$$

As a consequence of Proposition 8 it satisfies

$$(*) \qquad D_{pt,G}\circ R\Gamma_c(X,\,.\,) \;=\; RH^0_{\infty}(X,\,.\,)\circ D_{X,G} \;\; .$$

It is clear how to generalize all this (following [KS] 3.1.5) to the relative situation of a G-map $f : X \to Y$ (with $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ of finite cohomological dimension) in producing a functor

$$f^{!,\infty} \;:\; D^+(\mathrm{Sh}_G(Y)) \longrightarrow D^+(\mathrm{Sh}_G(X))$$

which is right adjoint to $Rf_!$. Given sheaves F in $\mathrm{Sh}_G(Y)$ and I in $\mathrm{Sh}_G(X)$ one has the G^{dis}-equivariant presheaf on X defined by

$$(f^!_I F)(U) := \mathrm{Hom}_{\mathrm{Sh(Y)}}((f|U)_!(I|U),F) \;\; .$$

One shows that, whenever $\mathrm{For}(I)$ is f-soft ([KS] 3.1.1), the G-equivariant sheaf

$$f_I^{!,\infty}(F) := (f_I^! F)^{\mathrm{smooth}}$$

on X is well defined. We now fix a bounded resolution $\mathbb{C} \xrightarrow{\sim} I^{\cdot}$ in $\mathrm{Sh}_G(X)$ of the constant sheaf such that each $\mathrm{For}(I^j)$ is f-soft. If $F^{\cdot}$ is a bounded below complex in $\mathrm{Sh}_G(Y)$ then let $F^{\cdot} \xrightarrow{\sim} J^{\cdot}$ be an injective resolution in $\mathrm{Sh}_G(Y)$ and define

$$f^{!,\infty}(F^{\cdot}) := \begin{array}{l} \text{class in } D^+(\mathrm{Sh}_G(X)) \text{ of the} \\ \text{total complex } \mathrm{Tot}(f_{I^{-\cdot}}^{!,\infty}(J^{\cdot})) \end{array} .$$

We leave it as an exercise to the reader to work out the details.

Corollary 9: (Equivariant Verdier duality)
For any $S \in \mathrm{Sh}_G(X)$ *we have*

$$\mathrm{Hom}_{\mathbb{C}}(H_*(X, G; S), \mathbb{C}) \ = \ H^*(X, G; D_{X,G}(S)) \ .$$

Proof: Combine $(*)$ and the identity

$$\mathrm{Hom}_{\mathbb{C}}(LH_0(G, \,.\,), \mathbb{C}) \ = \ RH^0(G, \,.\,) \circ D_{pt,G} \ .$$

We now come back to the problem how $\omega_{X,G}$ is related to the usual (nonequivariant) dualizing complex ω_X on X. As before let $\mathbb{C} \xrightarrow{\sim} I^{\cdot}$ be a bounded resolution in $\mathrm{Sh}_G(X)$ such that each $\mathrm{For}(I^j)$ is c-soft and let $\mathbb{C} \xrightarrow{\sim} J^{\cdot}$ be an injective resolution in $\mathrm{Alg}(G)$. Recall that $\Gamma_{I^{-\cdot}}^!(\mathbb{C})$ represents the dualizing complex ω_X.

Proposition 10:
The natural map

$$\Gamma_{I^{-\cdot}}^!(\mathbb{C})^{\mathrm{smooth}} \xrightarrow{\sim} \mathrm{Tot}\ (\Gamma_{I^{-\cdot}}^{!,\infty}(J^{\cdot}))$$

is a quasi-isomorphism.
Proof: Let $I \in \mathrm{Sh}_G(X)$ be a sheaf such that $\mathrm{For}(I)$ is c-soft. We show that the complex

$$0 \to \Gamma_I^!(\mathbb{C})^{\mathrm{smooth}} \longrightarrow \Gamma_I^!(J^0)^{\mathrm{smooth}} \to \ldots \to \Gamma_I^!(J^i)^{\mathrm{smooth}} \to \ldots$$

is exact. First of all we note that the complex

$$0 \to \Gamma_I^!(\mathbb{C}) \to \Gamma_I^!(J^0) \to \ldots \to \Gamma_I^!(J^i) \to \ldots$$

is exact even in the sense of presheaves. For a G-good open subset $U \subseteq X$ the sections with compact support $\Gamma_c(U, I)$ form a smooth $P_U^\dagger$-module. Hence

$$\mathrm{Hom}^\infty_{\mathbb{C}}(\Gamma_c(U, I), \mathbb{C}) \xrightarrow{\sim} \mathrm{Hom}^\infty_{\mathbb{C}}(\Gamma_c(U, I), J^\cdot)$$

is an exact resolution. Because of (3) this implies the exactness of the first complex of sheaves above.

We see that the complex $\Gamma^!_{I^{-\cdot}}(\mathbb{C})$ represents ω_X whereas $\Gamma^!_{I^{-\cdot}}(\mathbb{C})^{\mathrm{smooth}}$ represents $\omega_{X,G}$. More can be said if we impose an additional condition on the G-space X. An open subset $U \subseteq X$ is called G-special if its pointwise stabilizer $P_U := \{g \in G : gx = x \text{ for any } x \in U\}$ is open in G. The G-space X is called **special** if any point in X has a fundamental system of G-special open neighbourhoods. A typical example of a special G-space is the building $\mathcal{B}$ which we have considered in the second section. A G-equivariant sheaf F on a special G-space X is called **special** if there is an open covering $X = \bigcup\limits_{i \in I} U_i$ by G-special subsets U_i together with open subgroups $H_i \subseteq P_{U_i}$ such that the induced action of H_i on $F|U_i$ is trivial.

Lemma 11:
If X is a special G-space and $F \in \mathrm{Sh}_G(X)$ is a special sheaf then there is an embedding $F \hookrightarrow S$ in $\mathrm{Sh}_G(X)$ such that S is special, too, and $\mathrm{For}(S)$ is c-soft.
Proof: The embedding constructed in the proof of Proposition 1.2 satisfies the requirements.

On a special G-space X we therefore can choose the resolution $\mathbb{C} \xrightarrow{\sim} I^\cdot$ in such a way that each I^j in addition is a special sheaf. It is rather obvious that then $\Gamma^!_{I^{-j}}(\mathbb{C})$ is a special, in particular G-equivariant, sheaf on X, too. Hence in this case $\Gamma^!_{I^{-\cdot}}(\mathbb{C})$ represents ω_X as well as $\omega_{X,G}$.

Corollary 12:
If X is a special G-space then $\mathrm{For}(\omega_{X,G}) = \omega_X$.

Actually we can say more insofar as we can relate the duality functors

$$D_{X,G} = R\,\underline{\mathrm{Hom}}_{X,\infty}(\,.\,, \omega_{X,G}) : D^b(\mathrm{Sh}_G(X)) \longrightarrow D^+(\mathrm{Sh}_G(X))$$

and

$$D_X = R\,\underline{\mathrm{Hom}}_X(\,.\,, \omega_X) : D^b(\mathrm{Sh}(X)) \longrightarrow D^b(\mathrm{Sh}(X)) \quad .$$

Proposition 13:
Let X be a special G-space; for any bounded complex $S^{\cdot}$ in $\mathrm{Sh}_G(X)$ consisting of special sheaves we have

$$\mathrm{For}(D_{X,G}(S^{\cdot})) = D_X(S^{\cdot}) \quad .$$

Proof: Using [Bor] V.7.8 (ii) we have

$$\begin{aligned} D_{X,G}(S^{\cdot}) = &\ \mathrm{Tot}\, \underline{\mathrm{Hom}}_{X,\infty}(S^{\cdot}, \Gamma^{!;\infty}_{I^{-\cdot}}(J^{\cdot})) \\ = &\ \mathrm{Tot}\, \underline{\mathrm{Hom}}_X(S^{\cdot}, \Gamma^{!}_{I^{-\cdot}}(J^{\cdot}))^{\mathrm{smooth}} \\ = &\ \mathrm{Tot}\, \mathrm{Hom}_{\mathbb{C}}(\Gamma_c(\,.\,, S^{\cdot} \underset{\mathbb{C}}{\otimes} I^{-\cdot}), J^{\cdot})^{\mathrm{smooth}} \end{aligned}$$

and

$$\begin{aligned} D_X(S^{\cdot}) = &\ \mathrm{Tot}\, \underline{\mathrm{Hom}}_X(S^{\cdot}, \Gamma^{!}_{I^{-\cdot}}(\mathbb{C})) \\ = &\ \mathrm{Tot}\, \mathrm{Hom}_{\mathbb{C}}(\Gamma_c(\,.\,, S^{\cdot} \underset{\mathbb{C}}{\otimes} I^{-\cdot}), \mathbb{C}) \quad . \end{aligned}$$

Since any $S^i \underset{\mathbb{C}}{\otimes} I^j$ is a special and c-soft sheaf the same argument as for Proposition 10 and Corollary 12 shows that the two complexes in question are naturally quasi-isomorphic.

4. Cyclic homology

In this paragraph we will relate the delocalized equivariant homology of the point to cyclic homology. This generalizes corresponding results for discrete groups in [Con] III.2.δ and for p-adic reductive groups in [HN] and [Sch]. We will follow very closely the method developped in [Sch]. The role of the building is taken over by an appropriate version of the space $\underline{E}G$ from [BCH]. Throughout the following assumption is made:
– G_0 is equal to the union of all compact open subgroups of G.
Since $\hat{pt} = G_0$ has c-cohomological dimension 0 the groups $H^G_*(pt)$ are defined even if G does not have finite homological dimension.
Let

$$\mathcal{E} := \dot{\bigcup_H} G/H$$

denote the disjoint union of all coset spaces G/H for H a compact open subgroup of G and define

$$\mathcal{E}. := \text{ simplicial set of finite sequences in } \mathcal{E} \quad .$$

The group G acts on $\mathcal{E}_.$ through the left translation action on the set $\mathcal{E}$. The stabilizer $P_\sigma^\dagger$ of a simplex $\sigma = (\sigma_0, \ldots, \sigma_q) = (g_0H_0, \ldots, g_qH_q) \in \mathcal{E}_q$ is a compact open subgroup in G which contains with finite index the pointwise stabilizer $P_\sigma = g_0H_0g_0^{-1} \cap \ldots \cap g_qH_qg_q^{-1}$. Let X be a G-space such that $\hat{X}$ has finite c-cohomological dimension. The subset

$$\hat{X}_\sigma := \{(g, x) \in \hat{X} : g \in P_\sigma\}$$

is open and closed (and possibly empty) in $\hat{X}$; let $j_\sigma : \hat{X}_\sigma \overset{\subseteq}{\hookrightarrow} \hat{X}$ denote the inclusion map. We have

– $\hat{X}_\sigma = \hat{X}_{\sigma_0} \cap \ldots \cap \hat{X}_{\sigma_q}$, and

– $g\hat{X}_\sigma = \hat{X}_{g\sigma}$ for $g \in G$.

By assumption the $\hat{X}_\sigma$ for $\sigma \in \mathcal{E}_0$ form an open covering

$$\hat{X} = \bigcup_{\sigma \in \mathcal{E}_0} \hat{X}_\sigma \quad .$$

It follows that we have, for any sheaf $S \in \mathrm{Sh}_G(\hat{X})$, the exact augmented complex

$$\ldots \longrightarrow \bigoplus_{\sigma \in \mathcal{E}_q} j_{\sigma!}j_\sigma^* S \longrightarrow \ldots \longrightarrow \bigoplus_{\sigma \in \mathcal{E}_0} j_{\sigma!}j_\sigma^* S \longrightarrow S \longrightarrow 0$$

in $\mathrm{Sh}_G(\hat{X})$ (compare [KS] 2.8). The corresponding complex of global sections with compact support reads

$$\ldots \longrightarrow \bigoplus_{\sigma \in \mathcal{E}_q} \Gamma_c(\hat{X}_\sigma, S) \longrightarrow \ldots \longrightarrow \bigoplus_{\sigma \in \mathcal{E}_0} \Gamma_c(\hat{X}_\sigma, S) \longrightarrow \Gamma_c(\hat{X}, S) \longrightarrow 0 \, .$$

By our assumption on $\hat{X}$ it still is exact provided the sheaf S is c-soft. The important point is that each term in degree ≥ 0 of that complex is a direct sum of smooth G-modules which are compactly induced from the stabilizer groups $P_\sigma^\dagger$ and hence is a projective object in $\mathrm{Alg}(G)$ ([Sch] 2.1).
Let $S \xrightarrow{\sim} I^\cdot$ be a bounded resolution in $\mathrm{Sh}_G(\hat{X})$ such that each sheaf $\mathrm{For}(I^j)$ is c-soft. The above discussion shows that the equivariant homology $H_*(\hat{X}, G; S)$ is computable as the homology of the double complex

$$\left[\bigoplus_{\sigma \in \mathcal{E}_.} \Gamma_c(\hat{X}_\sigma, I^{-\cdot})\right]_G \quad .$$

We now specialize to the case $X = pt$. The constant sheaf $\mathbb{C}$ on $\hat{pt} = G_0$ is c-soft. We therefore obtain

$$(*) \qquad H_*^G(pt) = h_*\left(\left[\bigoplus_{\sigma \in \mathcal{E}_.} \Gamma(P_\sigma, \mathbb{C})\right]_G\right) \quad .$$

Let $\mathcal{H}$ be the Hecke algebra of G, i.e., the convolution algebra of $\mathbb{C}$-valued locally constant functions with compact support on G. We recall that the cyclic homology of $\mathcal{H}$ can be localized in any conjugation-invariant open subset $\Omega \subseteq G$ giving rise to the groups $HC_*(\mathcal{H})_\Omega$ ([BB] §3).

Theorem:
$HC_*(\mathcal{H})_{G_0} = \bigoplus\limits_{i \geq 0} H^G_{*-2i}(pt)$.

Proof: Starting from the formula $(*)$ the argument is exactly the same as in [Sch] §3 if one replaces everywhere (in a purely formal way) the building by the simplicial set $\mathcal{E}$. .

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