CONTINUOUS AND LOCALLY ANALYTIC REPRESENTATION THEORY

Lectures at Hangzhou, August 2004

by

Peter Schneider and Jeremy Teitelbaum

Lecture I: Basic functional analysis

§1 Nonarchimedean fields

Let K be a field. A nonarchimedean absolute value on K is a function $|\ | : K \longrightarrow \mathbb{R}$ such that, for any $a, b \in K$ we have

(i) $|a| \geq 0$,

(ii) $|a| = 0$ if and only if $a = 0$,

(iii) $|ab| = |a| \cdot |b|$,

(iv) $|a + b| \leq \max(|a|, |b|)$.

The condition (iv) is called the strict triangle inequality. Because of (iii) the map $|\ | : K^\times \longrightarrow \mathbb{R}_+^\times$ is a homomorphism of groups. In particular we have $|1| = |-1| = 1$. We always will assume in addition that $|\ |$ is non-trivial, i.e., that

(v) there is an $a_0 \in K$ such that $|a_0| \neq 0, 1$.

It follows immediately that $|n \cdot 1| \leq 1$ for any $n \in \mathbb{Z}$. Moreover, if $|a| \neq |b|$ for some $a, b \in K$ then the strict triangle inequality actually can be sharpened into the equality

$$|a + b| = \max(|a|, |b|) .$$

To see this we may assume that $|a| < |b|$. Then $|a| < |b| = |b + a - a| \leq \max(|b+a|, |a|)$, hence $|a| < |a+b|$ and therefore $|b| \leq |a+b| \leq \max(|a|, |b|) = |b|$.

Via the distance function $d(a, b) := |b - a|$ the set K is a metric and hence topological space. For any $a \in K$ and any real number $\epsilon > 0$ the subsets

$$B_\epsilon(a) := \{b \in K : |b - a| \leq \epsilon\} \qquad \text{and} \qquad B_\epsilon^-(a) := \{b \in K : |b - a| < \epsilon\}$$

are called *closed balls* and *open balls*, respectively. Both systems, for varying ϵ, form a fundamental system of neighborhoods of a in the metric space K. One checks that addition $+ : K \times K \longrightarrow K$ and multiplication $\cdot : K \times K \longrightarrow K$ are continuous maps. So K is a topological field.

Lemma 1.1: *i. $B_\epsilon(a)$ is open and closed in K;*

ii. if $B_\epsilon(a) \cap B_\epsilon(a') \neq \emptyset$ then $B_\epsilon(a) = B_\epsilon(a')$;

iii. If B and B' are any two closed balls in K with $B \cap B' \neq \emptyset$ then either $B \subseteq B'$ or $B' \subseteq B$;

iv. K is totally disconnected.

Proof: The assertions i. and ii. are immediate consequences of the strict triangle inequality. The assertion iii. follows from ii. To see iv. let $M \subseteq K$ be a nonempty connected subset. Pick a point $a \in M$. By i. the intersection $M \cap B_\epsilon(a)$ is open and closed in M. It follows that M is contained in any ball around a and therefore must be equal to $\{a\}$.

The assertions i.-iii. hold similarly for open balls. Talking about open and closed balls in particular does not refer to a topological distinction but only to the nature of the inequality sign in the definition. The assertion ii. says that any point of a ball can serve as its midpoint. On the other hand the real number ϵ is not uniquely determined by the set $B_\epsilon(a)$ and therefore cannot be considered as the "radius" of this ball.

Another consequence of the strict triangle inequality is the fact that a sequence $(a_n)_{n\in\mathbb{N}}$ in K is a Cauchy sequence if and only if the consecutive distances $|a_{n+1} - a_n|$ converge to zero if n goes to infinity.

Definition: *The field K is called nonarchimedean if it is equipped with a nonarchimedean absolute value such that the corresponding metric space K is complete (i.e., every Cauchy sequence in K converges).*

From now on throughout the course K always denotes a nonarchimedean field with absolute value $|\ |$.

Lemma 1.2: *i. $o := \{a \in K : |a| \leq 1\}$ is an integral domain with quotient field K;*

ii. $\mathfrak{m} := \{a \in K : |a| < 1\}$ is the unique maximal ideal of o;

iii. $o^\times = o \backslash \mathfrak{m}$;

iv. every finitely generated ideal in o is principal.

Proof: The assertions i.-iii. again are simple consequences of the strict triangle inequality. For iv. consider an ideal $\mathfrak{a} \subseteq o$ generated by the finitely many elements $a_1, \ldots, a_m$. Among the generators choose one, say a, of maximal absolute value. Then $\mathfrak{a} = oa$.

The ring o, resp. the field $o/\mathfrak{m}$, is called the *ring of integers* of K, resp. the *residue class field* of K.

Exercise: For any $a \in o$ and any $\epsilon \leq 1$, the ball $B_\epsilon(a)$ is an additive coset $a + \mathfrak{b}$ for an appropriate ideal $\mathfrak{b} \subseteq o$.

Examples: 1) The completion $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the p-adic absolute value $|a|_p := p^{-r}$ if $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p. The field $\mathbb{Q}_p$ is locally compact.

2) The p-adic absolute value $|\ |_p$ extends uniquely to any given finite field extension K of $\mathbb{Q}_p$. Any such K again is locally compact.

3) The completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$. This field is not locally compact since the set of absolute values $|\mathbb{C}_p|$ is dense in $\mathbb{R}_+$.

4) The field of formal Laurent series $\mathbb{C}\{\{T\}\}$ in one variable over $\mathbb{C}$ with the absolute value $|\sum_{n\in\mathbb{Z}} a_n T^n| := e^{-\min\{n: a_n \neq 0\}}$. The ring of integers of this field is the ring of formal power series $\mathbb{C}[[T]]$ over $\mathbb{C}$. Since $\mathbb{C}[[T]]$ is the infinite disjoint union of the open subsets $a + T \cdot \mathbb{C}[[T]]$ with a running over the complex numbers the field $\mathbb{C}\{\{T\}\}$ is not locally compact.

The above examples show that the topological properties of the field K can be quite different. In particular there is an important stronger notion of completeness.

Definition: *The field K is called spherically complete if for any decreasing sequence of closed balls $B_1 \supseteq B_2 \supseteq \ldots$ in K the intersection $\bigcap_{n\in\mathbb{N}} B_n$ is nonempty.*

Any finite extension K of $\mathbb{Q}_p$ is locally compact and hence spherically complete. On the other hand the field $\mathbb{C}_p$ is not spherically complete.

We mention as a fact that the value group $|K^\times|$ either is a discrete or a dense subset of $\mathbb{R}_+^\times$. In the former case the field is called *discretely valued*. Examples of discretely valued fields K are finite extensions of $\mathbb{Q}_p$ and the field of Laurent series $\mathbb{C}\{\{T\}\}$. The field $\mathbb{C}_p$ on the other hand is not discretely valued.

Exercise: Any discretely valued field is spherically complete.

§2 Seminorms and lattices

Let V be a K-vector space throughout this section. A (nonarchimedean) seminorm q on V is a function $q : V \longrightarrow \mathbb{R}$ such that

(i) $q(av) = |a| \cdot q(v)$ for any $a \in K$ and $v \in V$,

(ii) $q(v + w) \leq \max(q(v), q(w))$ for any $v, w \in V$.

Note that as an immediate consequence of (i) and (ii) one has:

- $q(0) = |0| \cdot q(0) = 0$,

- $q(v) = \max(q(v), q(-v)) \geq q(v - v) = q(0) = 0$ for any $v \in V$,

Moreover, with the same proof as before, one has

- $q(v + w) = \max(q(v), q(w))$ for any $v, w \in V$ such that $q(v) \neq q(w)$.

The vector space V in particular is an o-module so that we can speak about o-submodules of V.

Definition: *A lattice L in V is an o-submodule which satisfies the condition that for any vector $v \in V$ there is a nonzero scalar $a \in K^\times$ such that $av \in L$.*

Exercises: 1. For a lattice $L \subseteq V$ the natural map

$$\begin{array}{ccc} K \underset{o}{\otimes} L & \xrightarrow{\cong} & V \\ a \otimes v & \longmapsto & av \end{array}$$

is a bijection.
2. The preimage of a lattice under a K-linear map again is a lattice.
3. The intersection $L \cap L'$ of two lattices $L, L' \subseteq V$ again is a lattice.

For any lattice $L \subseteq V$ we define its *gauge* p_L by

$$\begin{array}{rcl} p_L : V & \longrightarrow & \mathbb{R} \\ v & \longmapsto & \inf\limits_{v \in aL} |a| \; . \end{array}$$

We claim that p_L is a seminorm on V. First of all, for any $b \in K^\times$ and any $v \in V$, we compute

$$p_L(bv) = \inf_{bv \in aL} |a| = \inf_{v \in b^{-1}aL} |a| = \inf_{v \in aL} |ba| = |b| \cdot \inf_{v \in aL} |a| = |b| \cdot p_L(v) \; .$$

Secondly, the inequality $p_L(v + w) \leq \max(p_L(v), p_L(w))$ is an immediate consequence of the following observation: For $a, b \in K$ such that $|b| \leq |a|$ we have $aL + bL = aL$.

On the other hand for any given seminorm q on V we define the o-submodules

$$L(q) := \{v \in V : q(v) \leq 1\} \quad \text{and} \quad L^-(q) := \{v \in V : q(v) < 1\} \; .$$

We claim that $L^-(q) \subseteq L(q)$ are lattices in V. But, since we assumed the absolute value $|\ |$ to be non-trivial, we find an $a \in K^\times$ such that $|a^n|$ converges to zero if $n \in \mathbb{N}$ goes to infinity. This means that for any given vector $v \in V$ we find an $n \in \mathbb{N}$ such that $q(a^n v) = |a^n| \cdot q(v) < 1$.

Lemma 2.1: *i. For any lattice $L \subseteq V$ we have $L^-(p_L) \subseteq L \subseteq L(p_L)$;*

ii. for any seminorm q on V we have $c_o \cdot p_{L(q)} \leq q \leq p_{L(q)}$ where $c_o := \sup\limits_{|b|<1} |b|$.

§3 Locally convex vector spaces

Let $(L_j)_{j \in J}$ be a nonempty family of lattices in the K-vector space V such that we have

(lc1) for any $j \in J$ and any $a \in K^\times$ there exists a $k \in J$ such that $L_k \subseteq aL_j$, and

(lc2) for any two $i, j \in J$ there exists a $k \in J$ such that $L_k \subseteq L_i \cap L_j$.

The second condition implies that the intersection of two subsets $v + L_i$ and $v' + L_j$ either is empty or contains a subset of the form $w + L_k$. This means that the subsets $v + L_j$ for $v \in V$ and $j \in J$ form the basis of a topology on V which will be called the *locally convex topology on V defined by the family* (L_j). For any vector $v \in V$ the subsets $v + L_j$, for $j \in J$, form a fundamental system of open and closed neighborhoods of v in this topology.

Definition: *A locally convex K-vector space is a K-vector space equipped with a locally convex topology.*

Exercise: If V is locally convex then addition $V \times V \xrightarrow{+} V$ and scalar multiplication $K \times V \xrightarrow{\cdot} V$ are continuous maps.

Since on a nonzero K-vector space the scalar multiplication cannot be continuous for the discrete topology we see that the discrete topology is not locally convex. There is an alternative way to describe locally convex topologies with the help of seminorms.

Let $(q_i)_{i \in I}$ be a family of seminorms on the K-vector space V. The topology on V defined by this family $(q_i)_{i \in I}$, by definition, is the coarsest topology on V such that

- all $q_i : V \longrightarrow \mathbb{R}$, for $i \in I$, are continuous, and

- all translation maps $v + . : V \longrightarrow V$, for $v \in V$, are continuous.

For any finitely many norms $q_{i_1}, \ldots, q_{i_r}$ in the given family and any real number $\epsilon > 0$ we set

$$V(q_{i_1}, \ldots, q_{i_r}; \epsilon) := \{v \in V : q_{i_1}, \ldots, q_{i_r}(v) \leq \epsilon\} .$$

Lemma 3.1: *$V(q_{i_1}, \ldots, q_{i_r}; \epsilon)$ is a lattice in V.*

Proof: Since $V(q_{i_1}, \ldots, q_{i_r}; \epsilon) = V(q_{i_1}; \epsilon) \cap \ldots \cap V(q_{i_r}; \epsilon)$ and since the intersection of two lattices again is a lattice it suffices to consider a single $V(q_i; \epsilon)$. It is obviously an o-submodule. Choose an $a \in K^\times$ such that $|a| \leq \epsilon$. Then $V(q_i; \epsilon)$ contains the lattice $aL(q_i)$ and therefore must also be a lattice.

Clearly the family of lattices $V(q_{i_1},\ldots,q_{i_r};\epsilon)$ in V has the properties (lc1) and (lc2) and hence defines a locally convex topology on V.

Proposition 3.2: *i. The topology on V defined by the family of seminorms $(q_i)_{i\in I}$ coincides with the locally convex topology defined by the family of lattices $\{V(q_{i_1},\ldots,q_{i_r};\epsilon) : i_1,\ldots,i_r \in I, \epsilon > 0\}$.*

ii. A locally convex topology on V defined by the family of lattices $(L_j)_{j\in J}$ can also be defined by the family of gauges $(p_{L_j})_{j\in J}$.

This means that the concept of a locally convex topology is the same as the concept of a topology defined by a family of seminorms. For the rest of this section we let V be a locally convex K-vector space.

Exercise: Show that the following assertions are equivalent:
i. V is Hausdorff;
ii. for any nonzero vector $v \in V$ there is a $j \in J$ such that $v \notin L_j$;
iii. for any nonzero vector $v \in V$ there is an $i \in I$ such that $q_i(v) \neq 0$.

Definition: *A subset $B \subseteq V$ is called bounded if for any open lattice $L \subseteq V$ there is an $a \in K$ such that $B \subseteq aL$.*

It is almost immediate that any finite set is bounded, and that any finite union of bounded subsets is bounded.

Exercise: If the topology on V is defined by the family of seminorms $(q_i)_{i\in I}$ then a subset $B \subseteq V$ is bounded if and only if $\sup\limits_{v\in B} q_i(v) < \infty$ for any $i \in I$.

Lemma 3.3: *Let $B \subseteq V$ be a bounded subset; then the closure of the o-submodule of V generated by B is bounded.*

Proof: Let $L \subseteq V$ be an open lattice and $a \in K$ such that $B \subseteq aL$. Since aL is a closed o-submodule it necessarily contains the closed o-submodule generated by B.

§4 Banach spaces

Definition: *A seminorm q on V is called a norm if $q(v) = 0$ implies that $v = 0$. A K-vector space equipped with a norm is called a normed K-vector space.*

It is the usual convention to denote norms by $\|\ \|$ (and not by q). A normed vector space $(V, \|\ \|)$ will always be considered as a metric space with respect to the metric $d(v,w) := \|v-w\|$. It is therefore in particular a Hausdorff locally convex vector space.

Definition: *A normed K-vector space is called a K-Banach space if the corresponding metric space is complete.*

We mention without proof the following two facts.

Proposition 4.1: *Assume V to be Hausdorff; then the topology of V can be defined by a norm if and only if there is a bounded open lattice in V.*

Proposition 4.2: *The only locally convex and Hausdorff topology on a finite dimensional vector space K^n is the one defined by the norm $\|(a_1,\ldots,a_n)\| := \max\limits_{1\le i\le n} |a_i|$.*

Further examples: 1) Let X be any set; then

$$\ell^\infty(X) := \text{all bounded functions } \phi : X \to K$$

with pointwise addition and scalar multiplication and the norm

$$\|\phi\|_\infty := \sup_{x\in X} |\phi(x)|$$

is a K-Banach space. The following vector subspaces are closed and therefore Banach spaces in their own right:

- $c_{\mathrm{o}}(X) := \{\phi \in \ell^\infty(X) : \text{for any } \epsilon > 0 \text{ there are at most finitely many } x \in X \text{ such that } |\phi(x)| \ge \epsilon\}$; e.g., $c_{\mathrm{o}}(\mathbb{N})$ is the space of all zero sequences in K.

- $C(X) := \{\text{all continuous functions } \phi : X \to K\}$ provided X is a compact topological space.

2) Let $L \subseteq K$ be a complete subfield, $r \in |L^\times|$, and $a \in L^n$ a fixed point. Let $B := B_r(a) := \{x \in L^n : \|x-a\| \le r\}$ denote the closed polydisk of radius r around a. By the first example the K-vector space $\mathcal{A}_K(B)$ of all power series

$$f(x) = f(x_1,\ldots,x_n) = \sum_{\underline{i}} c_{\underline{i}}(x-a)^{\underline{i}} \quad \text{with } c_{\underline{i}} \in K \text{ and } \lim_{|\underline{i}|\to\infty} |c_{\underline{i}}| r^{|\underline{i}|} = 0$$

is a Banach space with respect to the norm $\|f\| = \max\limits_{\underline{i}} |c_{\underline{i}}| r^{|\underline{i}|}$. Here $\underline{i} = (i_1,\ldots,i_n)$ is a multi-index, $|\underline{i}| := i_1 + \ldots + i_n$, and $(x-a)^{\underline{i}} := (x_1-a_1)^{i_1} \cdot \ldots \cdot (x_n-a_n)^{i_n}$. In fact, $\mathcal{A}_K(B)$ is the algebra of all K-valued *rigid-analytic* functions on B, i.e., all power series which converge for any point of B with coordinates in an algebraic closure of L.

§5 Fréchet spaces

The next general class of locally convex vector spaces is formed by the *metrizable* ones, i.e., those whose topology can be defined by a metric.

Proposition 5.1: *For a Hausdorff locally convex K-vector space V the following assertions are equivalent:*

i. V is metrizable;

ii. the topology of V can be defined by a countable family of lattices;

iii. the topology of V can be defined by a countable family of seminorms.

Proof: The implications i. $\Rightarrow$ ii. $\Rightarrow$ iii. are clear. It remains to show that iii. implies i. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of seminorms which define the topology of V. By replacing p_n by $\max(p_1,\ldots,p_n)$ we may assume that $p_1(v) \leq p_2(v) \leq \ldots$ for any $v \in V$. We define

$$\|v\|_F := \sup_{n\in\mathbb{N}} \frac{1}{2^n} \cdot \frac{p_n(v)}{1+p_n(v)}$$

for $v \in V$. One checks that $d(v,w) := \|v-w\|_F$ is a metric on V satisfying the strict triangle inequality. We claim that d defines the topology of V. Because of $p_1 \leq p_2 \leq \ldots$ the lattices

$$V(n) := \{v \in V : p_n(v) \leq 2^{-n}\}$$

for $n \in \mathbb{N}$ form a fundamental system of neighborhoods of the zero vector. Note that for any real number $a \geq 0$ and any $m \in \mathbb{N}$ one has $a \leq 2^{-(m-1)}$ provided $a/(1+a) \leq 2^{-m}$. This implies that

$$\{v \in V : \|v\|_F \leq 2^{-(2m+1)}\} \subseteq V(m) \ .$$

On the other hand, using that $2^{-n} \cdot p_n(v)/(1+p_n(v)) \leq 2^{-m}$ for $n \geq m$ and $2^{-n} \cdot p_n(v)/(1+p_n(v)) \leq p_m(v)/(1+p_m(v)) \leq p_m(v)$ for $n \leq m$ we obtain

$$V(m) \subseteq \{v \in V : \|v\|_F \leq 2^{-m}\} \ .$$

Definition: *A locally convex K-vector space is called a K-Fréchet space if it is metrizable and complete.*

Any Banach space of course is a Fréchet space. More generally, any countable projective limit of Banach spaces is a Fréchet space.

Example: As in Example 2) of the last section we let $L \subseteq K$ be a complete subfield, $r \in |L^\times|$, and $a \in L^n$ be a fixed point. But this time we consider the open polydisk $B^- := B_r^-(a) := \{x \in L^n : \|x - a\| < r\}$ of radius r around a. We define $\mathcal{A}_K(B^-)$ to be the K-vector space of all power series

$$f(x) = \sum_{\underline{i}} c_{\underline{i}}(x-a)^{\underline{i}} \quad \text{with } c_{\underline{i}} \in K \text{ and } \lim_{|\underline{i}| \to \infty} |c_{\underline{i}}|\epsilon^{|\underline{i}|} = 0 \text{ for any } 0 < \epsilon < r \ .$$

With respect to the family of norms

$$\|f\|_\epsilon := \max_{\underline{i}} |c_{\underline{i}}|\epsilon^{|\underline{i}|}$$

$\mathcal{A}_K(B^-)$ is a Fréchet space. It is the algebra of all K-valued *rigid-analytic* functions on B^-, i.e., all power series which converge for any point of B^- with coordinates in an algebraic closure of L.

A very basic fact about Fréchet spaces is the following so called *open mapping theorem.*

Proposition 5.2: *Every surjective continuous linear map $f : V \longrightarrow W$ between two Fréchet spaces is open.*

Corollary 5.3: *i. Let V be a Fréchet (resp. Banach) space, and let $U \subseteq V$ be a closed vector subspace; then V/U with the quotient topology is a Fréchet (resp. Banach) space as well.*

ii. Any continuous linear bijection between two K-Fréchet spaces is a topological isomorphism.

§6 Vector spaces of compact type

In this section we introduce a much more complicated type of locally convex vector space but which is of basic importance in representation theory. To simplify the presentation we assume that our field K is locally compact.

Definition: *A continuous linear map $f : V \longrightarrow W$ between two locally convex K-vector spaces V and W is called compact if there is an open lattice $L \subseteq V$ such that the closure of the image $f(L)$ in W is compact.*

We consider now a sequence

$$V_1 \longrightarrow \ldots \longrightarrow V_n \xrightarrow{i_n} V_{n+1} \longrightarrow \ldots$$

of locally convex K-vector spaces V_n with continuous linear transition maps i_n. The vector space inductive limit $V := \varinjlim V_n$ equipped with the finest locally convex topology such that all the natural maps $j_n : V_n \longrightarrow V$ are continuous is called the *locally convex inductive limit* of this sequence.

Exercise: A lattice $L \subseteq V$ is open if and only if its preimage $j_n^{-1}(L)$ is open in V_n for any n.

Definition: *A locally convex K-vector space V is called of compact type if it is the locally convex inductive limit of a sequence*

$$V_1 \longrightarrow \ldots \longrightarrow V_n \xrightarrow{i_n} V_{n+1} \longrightarrow \ldots$$

of K-Banach spaces with injective and compact transition maps.

Lemma 6.1: *Any vector space of compact type is Hausdorff.*

Basic example: As before we consider a closed polydisk $B = B_r(a)$ in L^n. For convenience we fix a decreasing sequence $\{r_m\}_{m\in\mathbb{N}}$ in $|L^\times| \cap (0, r]$ which converges to zero. A function $f : B \longrightarrow K$ is called *locally L-analytic* if for any point $b \in B$ there is an $m \in \mathbb{N}$ such that $f|B_{r_m}(b)$ lies in $\mathcal{A}_K(B_{r_m}(b))$. Let $C^{an}(B, K)$ denote the K-vector space of all K-valued locally analytic functions on B. In $C^{an}(B, K)$ we have the increasing sequence of vector subspaces $V_1 \subseteq V_2 \subseteq \ldots$ defined by

$$V_m := \{f \in C^{an}(B, K) : f|B_{r_m}(b) \in \mathcal{A}_K(B_{r_m}(b)) \text{ for any } b \in B\} .$$

Since B is compact we have

$$C^{an}(B, K) = \bigcup_{m\in\mathbb{N}} V_m .$$

Moreover, for each $m \in \mathbb{N}$, there are finitely many points $b_1, \ldots, b_{n_m}$ such that the linear map

$$\begin{array}{ccc} V_m & \xrightarrow{\cong} & \bigoplus_{1\le i\le n_m} \mathcal{A}_K(B_{r_m}(b_i)) \\ f & \longmapsto & \sum_i f|B_{r_m}(b_i) \end{array}$$

is a bijection. Hence V_m with the norm

$$\|f\| := \max_i \|f|B_{r_m}(b_i)\|$$

is a Banach space. It is straightforward to see that the inclusion maps $V_m \hookrightarrow V_{m+1}$ are continuous. We equip $C^{an}(B, K)$ with the corresponding locally convex inductive limit topology. It then is a vector space of compact type since the inclusion maps $V_m \hookrightarrow V_{m+1}$ are compact. This latter fact is an elaborate version of the observation that, for any $a \in K$ such that $0 < |a| < 1$, the continuous linear endomorphism

$$\begin{array}{rcl} c_o(\mathbb{N}) & \longrightarrow & c_o(\mathbb{N}) \\ \phi & \longmapsto & [n \mapsto a^{n-1}\phi(n)] \end{array}$$

is compact. The image of the open lattice $\{\phi \in c_o(\mathbb{N}) : \|\phi\|_\infty \leq 1\}$ under this endomorphism is the o-submodule

$$A := \{\phi \in c_o(\mathbb{N}) : |\phi(n)| \leq |a|^{n-1} \text{ for any } n \in \mathbb{N}\} .$$

One checks that the map

$$\begin{array}{rcl} \prod\limits_{n \in \mathbb{N}} o & \xrightarrow{\cong} & A \\ (a_n)_n & \longmapsto & [n \mapsto a^{n-1}a_n] \end{array}$$

is a homeomorphism for the direct product topology on the left hand side which is compact by the compactness of o.

Lecture II: Duality theory

§7 Vector spaces of linear maps

In this Lecture V and W will denote two locally convex K-vector spaces. It is straightforward to see that

$$\mathcal{L}(V, W) := \{f : V \longrightarrow W \text{ continuous and linear}\}$$

again is a K-vector space. We describe a general technique to construct locally convex topologies on $\mathcal{L}(V, W)$. For this we choose a family $\mathcal{B}$ of bounded subsets of V which is closed under finite unions. For any $B \in \mathcal{B}$ and any open lattice $M \subseteq W$ the subset

$$\mathcal{L}(B, M) := \{f \in \mathcal{L}(V, W) : f(B) \subseteq M\}$$

is a lattice in $\mathcal{L}(V, W)$: It is clear that $\mathcal{L}(B, M)$ is an o-submodule. If $f \in \mathcal{L}(V, W)$ is any continuous linear map then, by the boundedness of B, there has to be an $a \in K^\times$ such that $B \subseteq af^{-1}(M)$. This means that $f(B) \subseteq aM$ or equivalently that $a^{-1}f \in \mathcal{L}(B, M)$.

The family of all these lattices $\mathcal{L}(B, M)$ is nonempty and satisfies (lc1) and (lc2). The corresponding locally convex topology is called the $\mathcal{B}$-topology. We write

$$\mathcal{L}_{\mathcal{B}}(V, W) := \mathcal{L}(V, W) \ \text{ equipped with the } \mathcal{B}\text{-topology.}$$

Examples: 1) Let $\mathcal{B}$ be the family of all finite subsets of V. The corresponding $\mathcal{B}$-topology is called the *weak topology* or the topology of pointwise convergence. We write $\mathcal{L}_s(V, W) := \mathcal{L}_{\mathcal{B}}(V, W)$.

2) Let $\mathcal{B}$ be the family of all bounded subsets in V. The corresponding $\mathcal{B}$-topology is called the *strong topology* or the topology of bounded convergence. We write $\mathcal{L}_b(V, W) := \mathcal{L}_{\mathcal{B}}(V, W)$.

If W is Hausdorff both locally convex vector spaces $\mathcal{L}_s(V, W)$ and $\mathcal{L}_b(V, W)$ are Hausdorff.

Exercise: *If V and W are normed vector spaces then the topology on $\mathcal{L}_b(V, W)$ is defined by the operator norm*

$$\|f\| := \sup\{\frac{\|f(v)\|}{\|v\|} : v \in V \setminus \{0\}\}.$$

It is technically important to know what the bounded subsets in $\mathcal{L}_{\mathcal{B}}(V,W)$ are. There always are some obvious ones.

Definition: *A subset $H \subseteq \mathcal{L}(V,W)$ is called equicontinuous if for any open lattice $M \subseteq W$ there is an open lattice $L \subseteq V$ such that $f(L) \subseteq M$ for every $f \in H$.*

Lemma 7.1: *Every equicontinuous subset $H \subseteq \mathcal{L}(V,W)$ is bounded in every $\mathcal{B}$-topology.*

Proof: Let $\mathcal{L} \subseteq \mathcal{L}_{\mathcal{B}}(V,W)$ be any open lattice. There is an open lattice $M \subseteq W$ and a $B \in \mathcal{B}$ such that $\mathcal{L} \supseteq \mathcal{L}(B,M)$. Since H is equicontinuous we find an open lattice $L \subseteq V$ such that $f(L) \subseteq M$ for any $f \in H$. Furthermore, there is an $a \in K^\times$ such that $B \subseteq aL$. Hence $f(B) \subseteq aM$ for any $f \in H$ which shows that $H \subseteq a\mathcal{L}(B,M) \subseteq a\mathcal{L}$.

To obtain a complete answer we have to impose a mild additional condition on the topology of V. To understand it we first note that any open lattice necessarily is also closed.

Definition: *A locally convex vector space V is called barrelled if every closed lattice in V is open.*

Proposition 7.2: (Banach-Steinhaus) *Suppose that V is barrelled; then in $\mathcal{L}_{\mathcal{B}}(V,W)$, for any $\mathcal{B}$-topology which is finer than the weak topology, the bounded subsets coincide with the equicontinuous subsets.*

Examples: Any Fréchet space and any vector space of compact type is barrelled.

§8 Dual spaces

We now specialize the content of the previous section to the case $W = K$. The vector space $V' := \mathcal{L}(V,K)$ is called the *dual space* of V; more precisely, we call the locally convex vector spaces $V'_s := \mathcal{L}_s(V,K)$ and $V'_b := \mathcal{L}_b(V,K)$ the *weak* and *strong dual*, respectively.

Unfortunately it turns out that even if V is Hausdorff (and nonzero) it can have a zero dual space $V' = 0$. This phenomenon is directly related to the field K having the property of being spherically complete or not.

Proposition 8.1: (Hahn-Banach) *Suppose that the field K is spherically complete, and let U be a K-vector space, q a seminorm on U, and $U_\circ \subseteq U$ a vector* *subspace; for any linear form $\ell_o : U_o \longrightarrow K$ such that $|\ell_o(v)| \leq q(v)$ for any $v \in U_o$ there is a linear form $\ell : U \longrightarrow K$ such that $\ell|U_o = \ell_o$ and $|\ell(v)| \leq q(v)$ for any $v \in U$.*

Proof: By a simple application of Zorn's lemma we may reduce to the case where $U = U_o + Kv_1$ for some vector $v_1 \in U$. Consider, for any $v \in U_o$, the subset

$$B(v) := \{a \in K : |a - \ell_o(v)| \leq q(v - v_1)\}$$

of K. For any two vectors $v, v' \in U_o$ we have

$$|\ell_o(v) - \ell_o(v')| \leq |\ell_o(v - v')| \leq q(v - v') \leq \max(q(v - v_1), q(v' - v_1))$$

and hence $\ell_o(v) \in B(v')$ or $\ell_o(v') \in B(v)$. This means that always

$$B(v) \cap B(v') \neq \emptyset \ .$$

If there is a vector $v_0 \in U_o$ such that $q(v_0 - v_1) = 0$ then $B(v_0)$ consists of one point and hence

$$\bigcap_{v \in U_o} B(v) = B(v_0) = \{\ell_o(v_0)\} \ .$$

Otherwise each $B(v)$ is a closed ball. Since K is assumed to be spherically complete the intersection

$$\bigcap_{v \in U_o} B(v) \neq \emptyset$$

is nonempty as well. We therefore find, in any case, a scalar

$$b \in \bigcap_{v \in U_o} B(v) \ .$$

We now extend ℓ_o to a linear form ℓ on U by $\ell(v_1) := b$. We have $|\ell| \leq q$ since

$$|\ell(v + av_1)| = |a| \cdot |\ell_o(a^{-1}v) + b| \leq |a| \cdot q(-a^{-1}v - v_1) = q(v + av_1)$$

for any $a \in K^\times$.

One of the important applications is the following result.

Proposition 8.2: *Suppose that K is spherically complete and that V is Hausdorff; the linear map*

$$\begin{array}{rcl} \delta : V & \longrightarrow & (V'_s)'_s \\ v & \longmapsto & \delta_v(\ell) := \ell(v) \end{array}$$

is a continuous bijection.

The map δ in the above proposition is called a *duality map* for V. We emphasize that it seldom is a topological isomorphism. In general the strong dual V_b' is the more interesting dual space. For example, if V is a Banach space then we know from the last exercise that V_b' again is a Banach space. One therefore is also interested in the following variant

$$\begin{array}{rcl}\delta : V & \longrightarrow & (V_b')_b' \\ v & \longmapsto & \delta_v(\ell) := \ell(v)\end{array}$$

of the duality map. It is not continuous in general!

Lemma 8.3: *Suppose that K is spherically complete and that V is Hausdorff and barrelled; then $\delta : V \longrightarrow (V_b')_b'$ is a topological isomorphism onto its image.*

This leads to the following important definition.

Definition: *A Hausdorff locally convex K-vector space V is called reflexive if the duality map $\delta : V \longrightarrow (V_b')_b'$ is a topological isomorphism.*

A reflexive vector space V can be completely recovered from its strong dual V_b'. Unfortunately, over a spherically complete field K, infinite dimensional Banach spaces never are reflexive.

But assuming for the rest of this section (and for simplicity) again that K is locally compact a complete characterization of reflexivity in terms of a compactness property can be given.

Proposition 8.4: *A Hausdorff locally convex vector space V is reflexive if and only if it is barrelled and any closed and bounded o-submodule of V is compact.*

In this light it seems promising to examine vector spaces $V = \varinjlim V_n$ of compact type. Almost by definition the dual space $V' = \varprojlim V_n'$ is the projective limit of the dual spaces V_n'. Using the open mapping theorem one shows that this even holds true topologically, i.e., V_b' is the topological projective limit of the dual Banach spaces $(V_n)_b'$. Moreover, as a countable projective limit of Banach spaces V_b' is a Fréchet space. Finally, from the compactness of the transition maps in the inductive system $\{V_n\}_n$ one deduces, using Prop. 8.4, that V and V_b' both are reflexive. This raises the question which Fréchet spaces arise as the strong dual of a vector space of compact type. The answer needs the property of being *nuclear* which to explain would be beyond the scope of these Lectures. But at least we want to formulate the precise result.

Proposition 8.5: *The functor $V \longmapsto V_b'$ is an anti-equivalence between the category of vector spaces of compact type and the category of nuclear Fréchet spaces.*

Despite the fact mentioned above that the concept of reflexivity is useless for Banach spaces there nevertheless is a counterpart of the last proposition for Banach spaces. This is made possible by the following compactness result.

Lemma 8.6: *Let V be a Banach space; then the o-submodule $V^d := \{\ell \in V' : \|\ell\| \leq 1\}$ is compact in V_s'.*

Proof: One checks that

$$\begin{array}{rcl} V^d & \hookrightarrow & \prod\limits_{\|v\|\leq 1} o \\ \ell & \longmapsto & (\ell(v))_v \end{array}$$

is a topological and closed embedding. As a direct product of compact spaces the right hand side is compact.

Obviously V^d is a linear-topological and torsionfree o-module. On the other hand, if we start with a linear-topological compact and torsionfree o-module M then

$$M^d := \text{all continuous } o\text{-linear maps } \ell : M \longrightarrow K$$

equipped with the norm

$$\|\ell\| := \max_{m \in M} |\ell(m)|$$

is a K-Banach space.

Proposition 8.7: *The functors $V \longmapsto V^d$ and $M \longmapsto M^d$ are quasi-inverse anti-equivalences between the category of K-Banach spaces $(V, \|\ \|)$ such that $\|V\| \subseteq |K|$ with norm decreasing linear maps and the category of linear-topological compact and torsionfree o-modules.*

Lecture III: Continuous and locally analytic functions and distributions

§9 p-adic analytic manifolds

We fix L, a locally compact nonarchimedean field of characteristic zero. Let M be a paracompact topological space.

A chart for M is an open set M_i together with a map

$$\phi_i : M_i \to B_{r(i)} \subseteq L^d$$

where $B_{r(i)}$ is a closed polydisk, as defined in Lecture I. We say that two charts (M_i, ϕ_i) and (M_j, ϕ_j) are compatible if the map

$$\phi_i \circ \phi_j^{-1} : B_{r(j)} \to B_{r(i)}$$

is locally L-analytic – meaning given by a collection of convergent power series.

(i) A collection of compatible charts that covers M is called an atlas for M.

(ii) Compatible charts have the same d; this is called the dimension of the atlas.

(iii) Given an atlas, one can enlarge it by adding all charts compatible with each element of the given family, giving a maximal atlas.

(iv) The space M together with such a maximal atlas is called a locally L-analytic manifold. If all charts have dimension d, then M is said to be d-dimensional.

(v) An function f on M is called locally L-analytic if $f \circ \phi_i^{-1}$ is locally L-analytic for any chart (M_i, ϕ_i). We write $C^{an}(M, K)$ for the space of K-valued locally L-analytic functions on M, if $K \supseteq L$ is a complete field.

(vi) We also have the continuous functions $C(M, K) \supseteq C^{an}(M, K)$.

Lemma 9.1: *Any locally L-analytic manifold is strictly paracompact, meaning: every open covering has a pairwise disjoint refinement.*

All of the locally L-analytic manifolds we consider have this property.

§10 Vector valued functions

Let V be a locally convex, Hausdorff topological vector space over $K \supseteq L$.

We are interested primarily in three classes of V-valued functions on M: continuous; locally analytic; and locally constant.

The continuous functions are what you might expect.

Definition 10.1: *If M is a topological space then we let $C(M, V)$ be the space of V-valued continuous functions, with the topology of uniform convergence on compact sets.*

If M is compact and V is a Banach space then so is $C(M, V)$, with the norm given by

$$\|f\| = \max_{x \in M} \|f(x)\|_V.$$

Exercise: Show that, if M is compact and V is a Fréchet space with its topology defined by a countable family of seminorms $\{q_i\}_{i \in \mathbb{N}}$, then $C(M, V)$ is a Fréchet space with seminorms $q'_i(f) = \sup_{x \in M} q_i(f(x))$.

The locally analytic functions are more complicated to define, if we wish to allow general spaces V.

Definition 10.2: *A BH-space for V is a continuous injection $f : U \to V$ where U is a Banach space.*

It is a consequence of the Open Mapping Theorem (Prop. 5.2) that if $h : W \to V$ is another BH-space such that $h(W) \subseteq f(U)$ then the inclusion $W \subseteq U$ is continuous; thus if U is a subspace of V that admits a Banach topology such that the inclusion is continuous, then that topology is unique.

Let M be a locally L-analytic manifold and let (M_i, ϕ_i) be a chart of M. The chart ϕ_i identifies M_i with a ball $B = B_r(a) \subseteq L^n$. We let $\mathcal{A}_K(M_i, \phi_i)$ be the K-Banach space of convergent power series on B, viewed as functions on M_i via pullback with ϕ_i.

Let U be another K-Banach space. The U-valued analytic functions on M_i is the Banach space

$$\mathcal{A}_K(M_i, \phi_i, U) := \mathcal{A}_K(M_i, \phi_i) \hat{\otimes}_K U.$$

This is the space of convergent power series on the ball $\phi_i(M_i)$ with coefficients from U. Assuming for simplicity that this ball in L^n is centered at zero and has radius $r \in |L^\times|$, we may write explicitly

$$\mathcal{A}_K(M_i, \phi_i, U) = \{\sum_I a_I \mathbf{x}^I : a_i \in U\}$$

with multi-indices $I = (i_1, \ldots, i_d)$ of non-negative integers. Here the convergence condition is

$$\|a_I\|_U \ r^{|I|} \to 0 \text{ as } |I| = \sum_{j=1}^{d} i_j \to \infty.$$

Now let V be a Hausdorff locally convex K-vector space. We wish to define V-valued, locally analytic functions on M. Choose a disjoint covering of M by charts $(M_i, \phi_i)_{i \in I}$, and for each $i \in I$ choose a BH-space U_i for V. The analytic functions on M relative to these choices is the space

$$C^{an}(\{(M_i, \phi_i)\}_i, U_i) := \prod_{i \in I} \mathcal{A}_K(M_i, \phi_i, U_i).$$

A choice of covering $\mathcal{M}_j = \{M_{ij}, \phi_{ij}\}$ with associated BH-spaces U_{ij} is called an *index* for M. The indices are ordered by refinement of coverings and enlarging the BH-spaces. The space $C^{an}(M, K)$ is the direct limit over the indices:

$$C^{an}(M, V) := \varinjlim C^{an}(\{(M_{ij}, \phi_{ij})\}_i, U_{ij}).$$

Proposition 10.3: *Suppose that M is compact and that V is of compact type. Then $C^{an}(M, V)$ is of compact type.*

Proof: Since M is compact, we may assume that the direct limit defining $C^{an}(M, V)$ is over disjoint finite open coverings of M. Then, we may assume that for a given covering, all of the Banach spaces in the index are the same. Let $\{U_j\}$ be a sequence of Banach spaces defining V as a compact inductive limit. In other words, we may choose a cofinal system of indices $(\{M_{ij}, \phi_{ij}\}, U_j)$. Now consider one map in the direct limit:

$$C^{an}(\{(M_{ij}, \phi_{ij})\}_i, U_j) \to C^{an}(\{(M_{i'j'}, \phi_{i'j'})\}_{i'}, U_{j'})$$

Both sides of this map are Banach spaces. By using charts, we can reduce to considering the case where M_i is a ball in L^n and is partitioned into sub-balls: $B_r(a) = \cup_j B_{r(i)}(b_i)$:

$$\mathcal{A}_K(B_r(a)) \otimes U_j \to \prod_i \mathcal{A}_K(B_{r(i)}(b_i)) \hat{\otimes} U_{j'}.$$

Using the explicit description of these Banach spaces as power series, it is not hard to extend the results of the example at the end of Lecture I to see that this map is compact.

Remark 10.4: One may show that, when M is compact and V is of compact type, one has an isomorphism

$$C^{an}(M,V) = C^{an}(M,K)\,\hat{\otimes}_{K,\pi}\,V$$

where the tensor product has the projective tensor product topology.

If M is not compact, we have the following result.

Proposition 10.5: *If V is Hausdorff, and if $M = \bigcup M_i$ is a partition of M into pairwise disjoint open subsets, then*

$$C^{an}(M,V) = \prod_i C^{an}(M_i,V).$$

Corollary 10.6: *If V is of compact type, then $C^{an}(M,V)$ is complete, barrelled and reflexive.*

Proof: All these properties are preserved by products (See [NFA] Prop.s 9.10, 9.11, for reflexivity, 14.3 for barrelled).

Finally we introduce the locally constant functions $C^\infty(M,V)$.

Definition 10.7: *The space $C^\infty(M,V)$ is the subspace of $C^{an}(M,V)$ consisting of locally constant functions.*

The elements of $C^\infty(M,V)$ are the *smooth* functions of Langlands theory.

§11 Distributions

If M is a topological space, we let $D^c(M,K)$ denote the dual to the space $C(M,K)$ of continuous K-valued functions on M. If M is compact, then $D^c(M,K)$ is a Banach space. We will discuss various topologies on $D^c(M,K)$ later.

If M is a locally L-analytic manifold and V a locally convex vector space as in §10, we let

$$D(M,V) = C^{an}(M,V)'_b$$

be the strong dual space to the space of V-valued locally analytic functions. The elements of $D(M,V)$ are called V-valued locally analytic distributions. When V is of compact type, then by reflexivity (Cor. 10.6) we have

$$C^{an}(M,V) = D(M,V)'_b.$$

In particular, when $V = K$, a finite extension field of L, then $C^{an}(M,K)$ is of compact type. From Lecture II we may conclude that $D(M,K)$ is a Fréchet space.

The Dirac distributions δ_x, for $x \in M$, defined by $\delta_x(f) = f(x)$ are elements of $D(M,K)$.

Theorem 11.1: *If V is a countable union of BH-spaces, then the map*

$$\begin{aligned} I^{-1} : \mathcal{L}(D(M,K),V) &\stackrel{\cong}{\longrightarrow} C^{an}(M,V) \\ A &\longmapsto [x \longmapsto A(\delta_x)] \end{aligned}$$

is a well defined K-linear isomorphism.

Proof. Clearly the map in the assertion is compatible with disjoint open coverings of M. We therefore may assume that M is compact so that $D(M,K)$ is a Fréchet space. On the other hand, since the sum of two BH-spaces again is a BH-space we find, by our assumption on V, an increasing sequence $V_1 \subseteq V_2 \subseteq \ldots$ of BH-spaces of V such that $V = \bigcup_{n\in\mathbb{N}} V_n$. By the open mapping theorem, (Prop. 5.2) any continuous linear map from the Fréchet space $D(M,K)$ into V factors through some V_n. In other words we have

$$\mathcal{L}(D(M,K),V) = \varinjlim_{n\in\mathbb{N}} \mathcal{L}(D(M,K),V_n) \ .$$

Moreover, again by the open mapping theorem, any BH-space of V is contained in some V_n. Since M is compact (so that in the definition of $C^{an}(M,V)$ we need to consider only V-indices whose underlying covering of M is finite) this means that

$$C^{an}(M,V) = \varinjlim_{n\in\mathbb{N}} C^{an}(M,V_n) \ .$$

Hence we are further reduced to the case that V is a Banach space. The assertion that, for a Banach space V, we have an isomorphism

$$\begin{aligned} C : \varinjlim C^{an}(\{(M_i,\phi_i)\},K) \hat{\otimes} V &\stackrel{\cong}{\longrightarrow} \mathcal{L}(D(M,K),V) \\ C(f\otimes v)(\ell) &\longmapsto \ell(f)v. \end{aligned}$$

is ([Dist], Prop. 1.5.). The only additional observation to make is that this map satisfies $I(f \otimes v)(\delta_x) = f(x) \cdot v$ for $x \in M$.

Since any locally analytic function on a compact manifold factors through a BH-space the arguments in the above proof show that for an arbitrary V we still have a natural map

$$I : C^{an}(M, V) \longrightarrow \mathcal{L}(D(M, K), V)$$

such that $I(f)(\delta_x) = f(x)$ for any $f \in C^{an}(M, K)$ and $x \in M$. This map I should be considered as integration: Given a locally analytic function $f : M \longrightarrow V$ and a distribution λ on M we may formally write

$$I(f)(\lambda) = \int\limits_M f(x) d\lambda(x).$$

We now consider briefly the case of continuous functions. Here we assume for simplicity that M is compact. The strong dual space $D^c(M, K)_b$ to $C(M, K)$ is a Banach space with the usual dual norm

$$\|\ell\| = \sup_{0 \neq f \in C(M,K)} \frac{|\ell(f)|}{\|f\|}.$$

However, we will require another topology on $D^c(M, K)$. Let $o[[M]]$ be the unit ball in $D^c(M, K)_b$. (The notation will make more sense later). Alaogolu's theorem (Lemma 8.6) says that $o[[M]]$ is compact in the weak topology.

Definition 11.2: *We let $D^c(M, K)$ be the continuous dual to $C(M, K)$ equipped with the "bounded-weak" topology. This is by definition the finest locally convex topology such that the inclusion of the unit ball $o[[M]]$, with its weak topology, is continuous.*

We have the following version of Thm. 11.1 for continuous functions.

Theorem 11.3: *Let V be quasi-complete, Hausdorff, and locally convex. Then evaluation on Dirac distributions gives a K-linear isomorphism*

$$\mathcal{L}(D^c(M, K), V) \to C(M, V).$$

Proof: See [Iwasawa] Cor. 2.2.

§12 Distribution Algebras

In the particular case where the manifold $M = G$ is a locally L-analytic group the spaces of distributions on G become topological algebras.

Another construction of $D^c(G,K)$ for compact G makes its ring structure apparent. As a compact locally L-analytic group, G is profinite (see [DDMS]). Therefore we have the completed group ring

$$o[[G]] = \varprojlim o[G/H]$$

where the limit is over open normal subgroups of G. It carries the projective limit topology.

(i) $o[[G]]$ is a torsionfree compact o-module, and a topological ring.

(ii) Choose a cofinal sequence H_n of open normal subgroups of G. Any element μ of $o[[G]]$ is a projective limit of $\mu_n = \sum_{G/H_n} a_g g \in o[G/H_n]$; and any continuous function f on G may be uniformly approximated by a sequence f_n of locally constant functions that are right H_n-invariant. There is a well-defined integration pairing

$$\int_G f d\mu = \lim_{n\to\infty} \sum_{g\in G/H_n} a_g f_n(g).$$

This pairing gives a map $o[[G]] \to D^c(G,K)$, and its image can be identified with the unit ball in $D^c(G,K)$. In fact $o[[G]]$ is naturally the unit ball in $D^c(G,K)$ equipped with its weak topology.

(iii) $D^c(G,K)$ is a completion of $K[[G]] = K \otimes o[[G]]$ which sits inside it as a dense subspace. (When K is finite over L, then $K[[G]] = D^c(G,K)$.) As stated in Definition 3.2, we give $D^c(G,K)$ the finest locally convex topology such that the inclusion of $o[[G]]$ is continuous.

Proposition 12.1: *The ring $D^c(G,K)$ is a K-algebra with a separately continuous multiplication. (Separately continuous means that the map $y \to x \times y$ is continuous for fixed x, and $x \to x \times y$ is continuous for fixed y).*

Next we turn to the locally analytic case. Here the main result is due originally to Féaux de Lacroix, but one may find a proof in [Duality], Appendix to Section 3).

Theorem 12.2: *The ring $D(G,K)$ has a separately continuous multiplication, with the Dirac distribution δ_1 as identity element. When G is compact, $D(G,K)$ is a Fréchet algebra – meaning that the separately continuous multiplication is continuous as a bilinear map $D(G,K) \times D(G,K) \to D(G,K)$.*

The proof of this result is too technical for these Lectures, but we make a few remarks. First of all, it is not hard to show that for any compact locally L-analytic manifolds M and N we have

$$C^{an}(M \times N, K) = C^{an}(M, C^{an}(N, K)).$$

Since $C^{an}(N, K)$ is of compact type, using Remark 10.4 we have

$$C^{an}(M \times N, K) = C^{an}(M, K) \hat{\otimes}_\pi C^{an}(N, K).$$

Using reflexivity and [NFA] 20.13 and 20.14 gives

$$D(M \times N, K) = D(M, K) \hat{\otimes}_\pi D(N, K) = D(M, K) \hat{\otimes}_\iota D(N, K)$$

since by [NFA] 17.6 the inductive and projective topologies coincide for Fréchet spaces. To pass from the compact to the general case one uses a covering argument and the compatibility of the inductive topology with locally convex direct sums. Then for groups we have

$$D(G, K) \hat{\otimes}_\iota D(G, K) = D(G \times G, K).$$

The convolution product is then given by the sequence of maps

$$D(G, K) \times D(G, K) \to D(G, K) \hat{\otimes}_\iota D(G, K) = D(G \times G, K) \to D(G, K)$$

with the last map induced by multiplication $m : G \times G \to G$.

§13 Examples

We consider the explicit case $G = \mathbb{Z}_p$.

We take advantage of theorems of Mahler and Amice. As usual we let

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-(n-1))}{n!}.$$

We first consider continuous functions.

Theorem 13.1: *Let f be a continuous function on $\mathbb{Z}_p$. Then any $f \in C(\mathbb{Z}_p, K)$ has a unique representation*

$$f = \sum_{n=0}^{\infty} T_n(f) \binom{x}{n}$$

where the coefficients $T_n(f) \in K$ go to zero as $n \to \infty$. Conversely any such series converges uniformly to a continuous function. The norm $\|f\| = \max_n |T_n(f)|$.

Thm. 13.1 gives an explicit isomorphism between $C(\mathbb{Z}_p, K)$ and $c_0(\mathbb{N})$. Consequently the dual $D^c(\mathbb{Z}_p, K)$, as a vector space, is the space $\ell^\infty(\mathbb{N})$ of bounded sequences, and the elements of the unit ball $o[[\mathbb{Z}_p]]$ may be represented as sums

$$\mu = \sum b_n T_n \text{ with } |b_n| \text{ bounded.}$$

The linear maps T_n give the coefficients of f in the expansion of Thm. 13.1. In fact the T_n may be computed explicitly as finite differences.

To compute the ring structure on $D^c(\mathbb{Z}_p, K)$ we use some Fourier theory. Any distribution is determined by its values on the dense subspace of locally constant functions in $C(\mathbb{Z}_p, K)$, and the characters of finite order

$$\chi_\zeta(x) = \zeta^x \text{ for } \zeta \text{ a } p\text{-power root of } 1$$

span the locally constant functions. We may compute on the one hand that

$$T_n(\zeta^x) = T_n(((\zeta - 1) + 1)^x) = T_n(\sum \binom{x}{i} (\zeta - 1)^i) = (\zeta - 1)^n$$

and

$$T_n * T_m(\zeta^x) = T_n^{(x)} T_m^{(y)}(\zeta^{x+y}) = T_n(\zeta^x) T_m(\zeta^y) = (\zeta - 1)^{m+n} = T_{n+m}(\zeta^x).$$

It follows inductively that $T_n = T_1^n$ and so, writing $T_1 = T$,

$$o[[\mathbb{Z}_p]] = o[[T]].$$

For the locally analytic case we use the generalization of Thm. 13.1 due to Amice.

Theorem 13.2: *An element $f \in C(\mathbb{Z}_p, K)$ is locally analytic if there is an $r > 1$ such that*

$$\lim_{n \to \infty} |T_n(f)| r^n \to 0$$

as $n \to \infty$. The dual space $D(\mathbb{Z}_p, K)$ is given by all series

$$\mu = \sum_{n=0}^{\infty} b_n T_n$$

such that, for all $r < 1$ in $p^{\mathbf{Q}}$ we have $|b_n|r^n \to 0$. The Fréchet topology on $D(\mathbb{Z}_p, K)$ is defined by the family of seminorms q_r for $r \in p^{\mathbf{Q}}$, $r < 1$, with

$$q_r(f) = \max_n |b_n| r^n.$$

The computation of the ring structure is similar to that for the continuous case, except that the locally constant characters are not dense in the locally analytic functions. Instead one must consider *all* locally analytic characters

$$\chi_z = z^x$$

where z is any element of K with $|z - 1| < 1$. These are dense in $C^{an}(\mathbb{Z}_p, K)$ and a similar computation shows again that $T_n = T^n$. It follows that

$$D(\mathbb{Z}_p, K) = \{\text{power series over } K \text{ converging on the open unit disk in } \hat{\overline{K}}\}.$$

Lecture IV: Continuous and locally analytic representations

In this Lecture we turn to the consideration of continuous and locally analytic representations.

§14 The Lie algebra of G

We assume that G is a locally L-analytic group.

The tangent space $\mathfrak{g}$ at the identity of a locally L-analytic group G has the structure of a Lie algebra. The Campbell-Baker-Hausdorff formula converges p-adically in a sufficiently small neighborhood U of zero in $\mathfrak{g}$ and gives an analytic map

$$\exp : U \to G.$$

In fact, there is an open subalgebra U and subgroup V of $\mathfrak{g}$ and G respectively so that there are inverse analytic isomorphisms exp and log:

$$\begin{array}{rcl} \exp : \mathfrak{g} \supseteq U & \to & V \subseteq G \\ \log : G \supseteq V & \to & U \subseteq \mathfrak{g} \end{array}$$

In the special case $G = L^*$, the exponential and logarithm maps are given by the usual power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and

$$\log(z) = \sum_{n=1}^{\infty} \frac{(1-z)^n}{n}.$$

Exercise: Show that $\exp(x)$ converges to an analytic function for $|x| < p^{1/(p-1)}$ and that $\log(z)$ converges to an analytic function for $|z-1| < 1$.

In general, given a finite dimensional Lie algebra $\mathfrak{g}$ over L with a faithful finite dimensional representation, the Campbell-Baker-Hausdorff formula gives a map from a small neighborhood of zero in $\mathfrak{g}$ to GL_n for some n; introducing a group operation on this neighborhood by the formula

$$\mathfrak{x}\mathfrak{y} = \log(\exp(\mathfrak{x})\exp(\mathfrak{y}))$$

constructs a locally L-analytic group with Lie algebra $\mathfrak{g}$. This construction is functorial in the sense that the category of "sufficiently small locally L-analytic groups" is equivalent to the category of Lie algebras over L.

We recall also that the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ over a field K is a (canonical) associative K-algebra with a structural map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ such that:

(i) Viewing $U(\mathfrak{g})$ as a Lie algebra via the commutator $[a, b] = ab - ba$, the structure map is a Lie algebra homomorphism;

(ii) If A is any other associative algebra (similarly viewed as a Lie algebra), and $f : \mathfrak{g} \to A$ is any Lie algebra homomorphism, then f lifts to a map f' from $U(\mathfrak{g}) \to A$ such that $f' \circ \iota = f$.

The Poincaré-Birkhoff-Witt theorem says that, if $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ is an (ordered) basis for $\mathfrak{g}$, then $U(\mathfrak{g})$ has the monomials

$$\prod_{i=1}^{n} \mathfrak{x}_i^{k_i}$$

as basis.

As in classical differential geometry, the Lie algebra of such a group acts on $C^{an}(G, K)$ as an algebra of differential operators. For $\mathfrak{x} \in \mathfrak{g}$ this action is given by

$$\mathfrak{x}(f)(g) = \frac{d}{dt} f(\exp(-t\mathfrak{x})g)|_{t=0}.$$

In particular, evaluating at 1 gives a linear form on $C^{an}(G, K)$. That this form is continuous follows from the Banach-Steinhaus theorem which asserts that a pointwise convergent limit of linear forms on a barrelled locally convex space is continuous. Consequently we have a map $\mathfrak{g} \to D(G, K)$.

Proposition 14.1: *There is an injection $t : U(\mathfrak{g}) \otimes_L K \to D(G, K)$ where, for $\mathfrak{x} \in \mathfrak{g}$, $t(\mathfrak{x})(f) = (-\mathfrak{x})(f)(1)$.*

In light of Prop. 14.1 we view $U(\mathfrak{g})$ as a subalgebra of $D(G, K)$ in the future without comment.

We also point out that this subalgebra doesn't change if we shrink G to any open subgroup, because the Lie algebra depends only on the group near the identity.

§15 Locally analytic and continuous representations

Let V be a K-Banach space, and G a locally L-analytic group. Particularly important examples of such groups are

(i) Abelian groups such as the additive group of L or the multiplicative group L^*. Particularly important are the maximal compact subgroups o_L and o_L^*.

(ii) The group $\mathrm{GL}_n(L)$ of invertible $n \times n$ matrices over the field L, along with its subgroup $\mathrm{SL}_n(L)$ of determinant one matrices and quotient group by its center $\mathrm{PGL}_n(K)$. The compact subgroup $\mathrm{GL}_n(o_L)$ is of particular importance. More generally, the group of L-valued points of a connected algebraic group over L is a locally L-analytic group.

Definition 15.1: *A K-Banach space representation of G (on V) is a G-action by continuous linear automorphisms such that the map $G \times V \to V$ giving the action is continuous.*

Example 15.2: Let G be a compact group and let V be $C(G, K)$. Then the (left) translation action $(y \cdot (f))(x) = f(y^{-1}x)$ is such a representation. More generally, the translation action of a non-compact G on the bounded functions is another example.

Remark 15.3: For any barrelled topological K-vector space and any locally compact topological group, the requirement that the action $G \times V \to V$ be continuous is equivalent to it being "separately continuous": in other words that each g act as a continuous linear endomorphism of V and that, for each fixed $v \in V$, the map $g \mapsto gv$ is continuous on G. This is a consequence of the Banach-Steinhaus theorem. (II.7.2)

Now we define locally analytic representations of G. We assume that G is a d-dimensional locally L-analytic group.

Definition 15.4: *A locally analytic G representation is an action of G on a locally convex barrelled K-vector space such that, for each $v \in V$, the map $g \mapsto gv$ belongs to $C^{an}(G, V)$.*

By Remark 15.3, a locally analytic representation is continuous as a map $G \times V \to V$.

The maps $\rho_v(g) = gv$ are called "orbit maps." One may obtain some special classes of locally analytic representations by further restricting the properties of these maps. The following special case is particularly important.

Definition 15.5: *A locally analytic representation is* smooth *if the orbit maps are* locally constant.

§16 Examples of continuous and locally analytic representations

Example 16.1: Let $G = \mathbb{Z}_p$. This group is topologically cyclic and generated by 1, so the continuous K-valued characters $\chi : \mathbb{Z}_p \to K^*$ are determined by the value $z = \chi(1)$. Since the image of G must be compact and the powers of z go to zero, we must have $|z - 1| < 1$. The function

$$\chi(a) = z^a = \sum_{n=0}^{\infty} (z-1)^n \binom{a}{n}$$

is in fact locally analytic as follows for example from Amice's Theorem (Thm. 13.2). If z is sufficiently close to 1, then

$$\chi(a) = \exp(a \log(z)).$$

Example 16.2: Let $G = o_L$ where L is finite over K. As a topological group, or as a $\mathbb{Q}_p$-analytic group, o_L is just $\mathbb{Z}_p^d$ where $d = [L : \mathbb{Q}_p]$. The continuous characters of G are therefore of the form

$$\chi(a) = z_1^{a_1} \cdots z_d^{a_d}$$

where $a = \sum_{i=1}^{d} a_i e_i$ for some basis for o_L over $\mathbb{Z}_p$, and all such characters are $\mathbb{Q}_p$-analytic and give (one-dimensional) $\mathbb{Q}_p$-analytic representations.

Proposition 16.3: *A character χ as above is locally L-analytic if and only if its derivative $d\chi$, which is a priori a $\mathbb{Q}_p$-linear map $L \to K$, is in fact L-linear.*

Example 16.4: Let $G = L^*$ be the multiplicative group. The valuation allows one to split G as

$$G = \mathbb{Z} \times o_L^*.$$

A character that is trivial on o_L^* is called *unramified.* There is a subgroup of o_L^* of finite index that is isomorphic, via the exponential and logarithm maps, to o_L. Consequently, given a K-valued locally L-analytic character χ, there is an element $c(\chi) \in K$ so that

$$\chi(a) = \exp(c(\chi) \log(a))$$

for $a \in o_L^*$ sufficiently close to one.

Example 16.5: Let V be a representation of G such that the subgroup of G stabilizing any vector $v \in V$ is open in G. Then V is a locally analytic representation, because the orbit maps are locally constant and therefore locally analytic. In fact, this is a *smooth* representation. The notion of smoothness can be defined without reference to a topology on V. Given an abstract vector space V with a smooth G-action, one may give V a topology by viewing it as the direct limit of its finite dimensional subspace. This makes V a locally analytic representation.

Example 16.6: Let G be the group of L-points of an algebraic group over L, and let V be a finite dimensional algebraic representation of G. Then V is locally analytic, because the orbit maps are polynomial functions on G, which are locally analytic.

Example 16.7: Let V be as in Example 16.5, and W be as in Example 16.6. Then $V \otimes W$, with the diagonal action, is locally analytic, because the orbit maps are "locally polynomial" functions.

Example 16.8: Let G be a locally analytic group and let P be a locally analytic subgroup such that the homogeneous space G/P is compact. Let V be a locally analytic representation of P. The analytic induction $\mathrm{Ind}_P^G(V)$ is the representation

$$\mathrm{Ind}_P^G(V) = \{f \in C^{an}(G, V) : f(gb) = b^{-1}(f(g)) \text{ for any } b \in P\}$$

where G acts on the left by $g \cdot f(h) = f(g^{-1}h)$. By [Feaux] Satz 4.1.5, $\mathrm{Ind}_P^G(V)$ is locally analytic. For example, if G is the group of L-points of a connected reductive algebraic group (e.g. GL_n), P is a parabolic subgroup with Levi decomposition $P = MU$, and V is a locally analytic representation of M extended to P by making it trivial on U, then $\mathrm{Ind}_P^G(V)$ is locally analytic.

Example 16.9: A particularly important case for us is when $G = \mathrm{GL}_n(L)$, P is the lower triangular Borel subgroup, and V is given by a character χ of the maximal split torus $T \subseteq B$. The family of representations $\mathrm{Ind}_P^G(\chi)$ is called the locally analytic principal series of G.

To give a concrete example, suppose that $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and let χ be the trivial character $\mathbf{1}$. Then G/P is the projective line $\mathbf{P}^1$ over $\mathbb{Q}_p$ and $\mathrm{Ind}_P^G(\mathbf{1})$ is the space of locally analytic functions on $\mathbf{P}^1$.

Example 16.10: The construction in Example 16.8 has a continuous version. Suppose that G is compact, P is a closed subgroup, and V is a Banach space representation of P. Then one may form ${}^c\mathrm{Ind}_P^G(V)$ as a subspace of the Banach space of V-valued functions on G equipped with the supremum norm.

If G is locally analytic but not compact, but G/P is compact, then one may show ([Feaux] 4.1.1) that the map $G \to G/P$ has a section and one may find an analytic splitting $\iota : G/P \times P \to G$ of the projection map as manifolds. Then, given a Banach space representation of P, one may define

$$^c\mathrm{Ind}_P^G(V) = \{f : G \to V : \ f \text{ continuous}, f(gb) = b^{-1}(f(g)) \text{ for any } b \in P\}$$

and give this the norm

$$\|f\| = \sup_g \|f(\iota(gP, 1))\|.$$

Different choices of splittings give equivalent norms.

If $G = \mathrm{GL}_n(L)$, P is the lower triangular Borel, χ is a continuous character of the maximal torus T, and one chooses the splitting corresponding to the Iwasawa decomposition $G = \mathrm{GL}_n(o_L)P$, then one obtains the continuous principal series.

Example 16.11: Let $\mathcal{X}$ denote the p-adic upper half plane over L. This is the rigid space whose K-points are the elements of $K \backslash L$. It has a natural action of $G = \mathrm{GL}_2(L)$. Let Ω^1 be the nuclear Fréchet space of 1-forms on $\mathcal{X}$, with its associated G-action. Then the strong dual V of Ω^1 is a locally analytic representation. In fact, a theorem of Morita says that, if P is a Borel subgroup of G, then V is isomorphic to

$$\mathrm{Ind}_P^G(\mathbf{1})/K = C^{an}(\mathbf{P}^1, K)/K.$$

We conclude with a definition of another special class of locally analytic representations.

Definition 16.12: *Let G be the group of L-points of an algebraic group, and let V be a K-vector space on which G acts through a representation π. We say that this representation is locally algebraic if*

(i) The restriction of π to any compact subgroup H of G is a sum of finite dimensional H-representations;

(ii) For any vector $v \in V$, there is a compact open subgroup H of G so that Hv is contained in a finite dimensional space U and the action of H on U comes via restriction to H of a finite dimensional algebraic representation of G.

Representations as in Example 16.7 are locally algebraic.

§17 Modules over distribution algebras

The main tool for studying Banach space and locally analytic representations is to make the problem algebraic by viewing the representations as modules over the distribution algebras.

Proposition 17.1: *Let G be a compact locally analytic group, and let V be a Banach space representation of G on V. The G-action on V extends to a separately continuous $D^c(G, K)$-module structure on V. Moreover, G-equivariant continuous linear maps extend to module homomorphisms.*

Proposition 17.2: *Let G be a compact locally analytic group, and let V be a locally analytic representation of G on V. Then the G-action extends to a separately continuous $D(G, K)$-module structure on V. Moreover, G-equivariant continuous linear maps extend to module homomorphisms.*

These results are applications of the integration results of Lecture III. In the locally analytic case, the discussion of §11 gave an integration map

$$I : C^{an}(G, V) \to \mathcal{L}(D(G, K), V).$$

This allows us to define a module structure by setting $\mu * v = I(\rho_v)(\mu)$ where ρ_v is the orbit map for $v \in V$ belonging to $C(G, V)$ or $C^{an}(G, V)$. The map $I(\rho_v)$ is continuous as a function of μ. Since $D(G, K)$ is metrizable (it is a Fréchet space), any μ is the limit of a sequence μ_i of finite linear combinations of Dirac distributions. Therefore, for any fixed v, we have $I(\rho_v)(\mu)$ is the limit of $I(\rho_v)(\mu_i)$. Since each $I(\cdot)(\mu_i)$ is continuous as a function of v – it essentially is the G-action – and V is barrelled, by Banach-Steinhaus $I(\cdot)(\mu)$ is continuous as a function of v. The module property

$$(\mu * \nu) * v = \mu * (\nu * v)$$

also follows by continuity, since it holds for Dirac distributions.

In the continuous case, we may use the fact that $\mathcal{L}_s(V, V)$ is quasi-complete (which follows from Banach-Steinhaus and the Ascoli theorem) and then we may apply Thm. 11.3 directly.

In fact it is more useful to consider, not the spaces V, but their dual spaces V'. Via the transpose action these, too, are modules for the distribution algebras.

In the locally analytic case, we assume that V is of compact type, and therefore is reflexive. It follows that we may recover the original space V from the strong dual Fréchet space V'_b. Even more we may characterize those Fréchet spaces that are duals of compact type spaces by the nuclearity property (Prop. 8.5) We have the following equivariant version of Prop. 8.5:

Theorem 17.3: *Assuming G compact, the functor $V \to V_b'$ is an (anti-)equivalence of categories between the category of locally analytic G-representations on K-vector spaces of compact type, with continuous linear G-maps, and the category of continuous $D(G,K)$-modules on nuclear Fréchet spaces, with continuous $D(G,K)$-module homomorphisms.*

Remark: If G is not compact, one obtains a similar result except that one obtains only a separately continuous module structure.

In the Banach space case, we are somewhat hampered by the lack of reflexivity. It is certainly true that, if V is a Banach space representation of G, then V' is a $D^c(G,K)$-module. However, to identify the essential image of this functor we must use the ideas at the end of §8. The unit ball in the dual V' is the linear-topological torsionfree o-module denoted V^d in Lemma 8.6 and Prop. 8.7. One can show that, taking into account the $D^c(G,K)$-action on V, one may construct such an o-module with an action of the ring $o[[G]]$.

Theorem 17.4: *Assuming G compact let $\mathcal{M}(o[[G]])_{\mathbb{Q}}$ be the category of linear-topological compact o-torsionfree $o[[G]]$-modules, localized at $\mathbb{Q}$. Then the functors $V \to V^d$ and $M \to M^d$ are inverse equivalences of categories between $\mathcal{M}(o[[G]])_{\mathbb{Q}}$ and the category of K-Banach space representations with G-invariant continuous linear maps.*

We will discuss this further in Lecture V.

Lecture V: Admissible continuous representations

§18 The duality functor

In this Lecture K will be assumed to be a finite extension of $\mathbb{Q}_p$. Let $\mathrm{Ban}(K)$ denote the category of K-Banach space with continuous linear maps and $\mathrm{Ban}(K)^{\leq 1}$ the category of normed K-Banach spaces $(V, \| \ \|)$ satisfying $\|V\| \subseteq |K|$ with norm decreasing linear maps. We remark that $\mathrm{Ban}(K)$ can be reconstructed from $\mathrm{Ban}(K)^{\leq 1}$ by "localization in $\mathbb{Q}$"; this simply means that, for any two Banach spaces V and W, one has

$$\mathrm{Hom}_{\mathrm{Ban}(K)}(V, W) = \mathrm{Hom}_{\mathrm{Ban}(K)^{\leq 1}}(V, W) \otimes \mathbb{Q} \ .$$

At the end of Lecture I (Prop. 8.7) we had seen that the functor

$$\begin{array}{rcl} \mathrm{Ban}(K)^{\leq 1} & \longrightarrow & \mathcal{M}(o) \\ V & \longmapsto & V^d = \{\ell \in V' : \|\ell\| \leq 1\} \end{array}$$

is an anti-equivalence of categories where $\mathcal{M}(o)$ denotes the category of linear-topological compact and torsionfree o-modules.

Let now G be a compact locally $\mathbb{Q}_p$-analytic group. In §11 of Lecture III the identification

$$D^c(G, K) = K[[G]] = K \otimes_o o[[G]]$$

was explained. It is an important point that the completed group ring $o[[G]]$ as an o-module is linear-topological, compact, and torsionfree, i.e., lies in the category $\mathcal{M}(o)$.

Another important technical point (as already indicated in Def. 11.2) is to consider on $\mathcal{L}(V, W)$, for two Banach spaces V and W, not the natural Banach space topology but the bounded weak topology which is the finest locally convex topology which restricts to the weak topology on some open lattice in $\mathcal{L}(V, W)$. We write $\mathcal{L}_{bs}(V, W)$ in this case. Observe that by the above equivalence of categories we have a natural linear isomorphism

$$\mathcal{L}(V, W) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{M}(o)}(W^d, V^d) \otimes \mathbb{Q} \ .$$

On the other hand, for two modules M and N in $\mathcal{M}(o)$, the natural topology to consider on $\mathrm{Hom}_{\mathcal{M}(o)}(M, N)$ is the topology of compact convergence.

Proposition 18.1: *The bounded weak topology on $\mathcal{L}(V, W)$ induces the topology of compact convergence on $\mathrm{Hom}_{\mathcal{M}(o)}(W^d, V^d)$.*

Let now $\mathrm{Ban}_G(K)$ denote the category of K-Banach space representations of G with continuous linear and G-equivariant maps. By Prop. 17.1 (which was based on Thm. 11.3), to give a continuous representation of G on the Banach space V is the same as to give a continuous algebra homomorphism

$$K[[G]] \longrightarrow \mathcal{L}_s(V,V) .$$

Because $o[[G]]$ is compact this furthermore is the same as to give a continuous algebra homomorphism

$$K[[G]] \longrightarrow \mathcal{L}_{bs}(V,V) .$$

Remark 18.2: *On any Banach space representation V of G there is a G-invariant defining norm.*

Proof: Let $L \subseteq V$ be any bounded open lattice. By the continuity of the G-action there is an open subgroup $H \subseteq G$ and an open lattice $L_0 \subseteq L$ such that $H \cdot L_0 \subseteq L$. Then $L_1 := \bigcap_{h \in H} hL$ contains L_0 and hence is an open lattice as well. The G-invariant intersection

$$L_2 := \bigcap_{g \in G} gL = \bigcap_{g \in G/H} gL_1$$

therefore in fact is a finite intersection of open lattices. The corresponding gauge p_{L_2} is a G-invariant norm on V which defines the topology.

Going back to the above discussion we therefore see, by using Prop. 18.1, that to give a continuous representation of G on the Banach space V up to isomorphism is the same as to give a continuous module structure

$$o[[G]] \times V^d \longrightarrow V^d .$$

If we let $\mathcal{M}(o[[G]])$ denote the category of all continuous (left) $o[[G]]$-modules such that the underlying o-module lies in $\mathcal{M}(o)$ then we have established the following result.

Proposition 18.3: *The functor*

$$\begin{array}{rcl} \mathrm{Ban}_G(K) & \longrightarrow & \mathcal{M}(o[[G]])_{\mathbb{Q}} \\ V & \longmapsto & V' = V^d \otimes \mathbb{Q} \end{array}$$

is an anti-equivalence of categories.

§19 Admissibility

In order to motivate what is to come we want to mention the following two pathologies of Banach space representations.

1. In general there exist non-isomorphic topologically irreducible Banach space representations V and W of G for which nevertheless there is a nonzero G-equivariant continuous linear map $V \longrightarrow W$.

2. Even such a simple commutative group like $G = \mathbb{Z}_p$ has infinite dimensional topologically irreducible Banach space representations. We mention the following examples constructed by Diarra. It was explained in Example 16.1 that any $z \in \mathbb{C}_p$ such that $|z| < 1$ gives rise to the continuous character $(1+z)^a$ on $\mathbb{Z}_p$. Let $K \subseteq V_z \subseteq \mathbb{C}_p$ denote the smallest complete field containing z. We may let $\mathbb{Z}_p$ continuously act on V_z by $a \cdot v := (1+z)^a v$. Then V_z is a Banach space representation of $\mathbb{Z}_p$. Diarra proves that V_z is infinite dimensional and topologically irreducible provided z is transcendental over $\mathbb{Q}_p$, and that V_{z_1} and V_{z_2} are non-isomorphic if $|z_1| \neq |z_2|$.

It is clear that in order to avoid such pathologies we have to impose an additional finiteness condition on our Banach space representations. For the moment we keep assuming that the group G is compact and locally $\mathbb{Q}_p$-analytic. We have the following fundamental result by Lazard (Publ. Math. IHES 26, 1965).

Theorem 19.1: *The rings $o[[G]]$ and $K[[G]]$ are noetherian.*

The most natural finiteness condition one can impose on modules over a ring is to be finitely generated. If the ring is noetherian then the finitely generated modules over it form a nice abelian category. In view of Prop. 18.3 we therefore propose the following definition.

Definition: *A K-Banach space representation V of G is called admissible if the dual V' as a $K[[G]]$-module is finitely generated.*

We let $\mathrm{Ban}^a_G(K)$ denote the full subcategory in $\mathrm{Ban}_G(K)$ of all admissible Banach space representations. We also let $\mathcal{M}_{fg}(o[[G]])$, resp. $\mathcal{M}_{fg}(K[[G]])$, denote the category of finitely generated and o-torsionfree $o[[G]]$-modules, resp. finitely generated $K[[G]]$-modules. We remark that because of the anti-involution $g \longmapsto g^{-1}$ there is no need to distinguish between left and right modules.

Since the ring $o[[G]]$ is compact and noetherian the following facts are more or less exercises.

Lemma 19.2: *i. A finitely generated* $o[[G]]$*-module* M *carries a unique Hausdorff topology - its canonical topology - such that the action* $o[[G]] \times M \longrightarrow M$ *is continuous;*

ii. any submodule of a finitely generated $o[[G]]$*-module is closed in the canonical topology;*

iii. any $o[[G]]$*-linear map between two finitely generated* $o[[G]]$*-modules is continuous for the canonical topologies.*

It follows that equipping a module in $\mathcal{M}_{fg}(o[[G]])$ with its canonical topology induces a fully faithful embedding

$$\mathcal{M}_{fg}(o[[G]]) \longrightarrow \mathcal{M}(o[[G]]) \ .$$

This then in turn induces a fully faithful embedding

$$\mathcal{M}_{fg}(K[[G]]) \longrightarrow \mathcal{M}(o[[G]])_{\mathbb{Q}} \ .$$

Together with Prop. 18.3 we obtain the following.

Theorem 19.3: *The functor*

$$\begin{array}{rcl} \mathrm{Ban}^a_G(K) & \longrightarrow & \mathcal{M}_{fg}(K[[G]]) \\ V & \longmapsto & V' \end{array}$$

is an anti-equivalence of categories.

In particular the category of admissible K-Banach space representations of G is abelian and is completely algebraic in nature. One also deduces easily the following consequences.

Corollary 19.4: *i. The functor* $V \longmapsto V'$ *induces a bijection*

$$\begin{array}{ccc} \text{set of isomorphism classes} & & \\ \text{of topologically irreducible} & \xrightarrow{\sim} & \text{set of isomorphism classes} \\ \text{admissible } K\text{-Banach space} & & \text{of simple } K[[G]]\text{-modules;} \\ \text{representations of } G & & \end{array}$$

ii. any nonzero G*-equivariant continuous linear map between two topologically irreducible admissible* K*-Banach space representations of* G *is an isomorphism.*

Our category $\mathrm{Ban}_G^a(K)$ in particular avoids the pathology 1. But also the pathology 2 disappears. We claim that for the group $G = \mathbb{Z}_p$ any topologically irreducible admissible Banach space representation is finite dimensional over K. By Cor. 19.4.i this reduces to the assertion that any simple $K[[\mathbb{Z}_p]]$-module is finite dimensional which is equivalent to any maximal ideal in $K[[\mathbb{Z}_p]]$ being of finite codimension. But we know from §13 that $K[[\mathbb{Z}_p]] = K \otimes_o o[[T]]$ is a power series ring in one variable. By Weierstrass preparation any ideal in this ring is generated by a polynomial.

Another indication that admissibility is the correct concept is the fact that it can be characterized in an intrinsic way. To formulate the result we recall that an o-module N is called of cofinite type if its Pontrjagin dual $\mathrm{Hom}_o(N, K/o)$ is a finitely generated o-module.

Proposition 19.5: *A K-Banach space representation V of G is admissible if and only if there is a G-invariant bounded open o-submodule $L \subseteq V$ such that, for any open normal subgroup $H \subseteq G$, the o-submodule $(V/L)^H$ of H-invariant elements in the quotient V/L is of cofinite type.*

Proof: Let us first assume that V' is finitely generated over $K[[G]]$. There is then a finitely generated $o[[G]]$-submodule $M \subseteq V'$ such that $V' = K \otimes_o M$. After equipping M with its canonical topology we have $V = M^d$. Moreover $L := \mathrm{Hom}_o^{cont}(M, o)$ is a G-invariant bounded open o-submodule in V. One checks that $V/L = \mathrm{Hom}_o^{cont}(M, K/o)$ and hence that

$$(*) \qquad (V/L)^H = \mathrm{Hom}_o^{cont}(M, K/o)^H = \mathrm{Hom}_o^{cont}(M/I_H M, K/o)$$

where I_H denotes the kernel of the projection map $o[[G]] \longrightarrow o[G/H]$. Hence $(V/L)^H$ is of cofinite type.

On the other hand fix now an open normal subgroup $H \subseteq G$ which is pro-p (there is a fundamental system of such) and let $L \subseteq V$ be a G-invariant bounded open o-submodule such that $(V/L)^H$ is of cofinite type. One checks that the G-invariant o-submodule $M := \{\ell \in V' : |\ell(v)| \leq 1 \text{ for any } v \in L\}$ in V_s' is compact. Since L is bounded we have $V' = K \otimes_o M$. So the identities $(*)$ apply correspondingly and we obtain that $\mathrm{Hom}_o^{cont}(M/I_H M, K/o)$ is of cofinite type. But since I_H is finitely generated as a right ideal the submodule $I_H M$ is the image of finitely many copies $M \times \ldots \times M$ under a continuous map and hence is closed in M. By Pontrjagin duality and the Nakayama lemma over o applied to the compact o-module $M/I_H M$ the latter therefore is finitely generated over o. The Nakayama lemma over $o[[G]]$ finally says that M is finitely generated over $o[[G]]$ and hence that V' is finitely generated over $K[[G]]$.

The above proof shows that the condition of this proposition in fact only needs to be checked for a single open normal subgroup $H \subseteq G$ which is pro-p. We also point out that the condition in this proposition is rather similar in spirit to Harish Chandra's admissibility condition for smooth representations (see the next Lecture). In fact, it implies that $L/\mathfrak{m}L$ is an admissible smooth representation of G over the residue class field of K.

We no drop the condition that the group G is compact. A Banach space representation of an arbitrary locally $\mathbb{Q}_p$-analytic group G is called admissible if it is admissible as a representation of every compact open subgroup $H \subseteq G$. Obviously this again gives rise to an abelian category.

Exercise: Admissibility can be tested on a single compact open subgroup $H \subseteq G$.

Example: The continuous principal series representations ${}^c\mathrm{Ind}_P^G(\chi)$ of Example 16.10 are admissible. It was explained there that using the Iwasawa decomposition $G = HP$ with $H := GL_n(o_L)$ we have ${}^c\mathrm{Ind}_P^G(\chi) = {}^c\mathrm{Ind}_{H\cap P}^H(\chi) \subseteq C(H,K)$. The latter inclusion dualizes into a surjection $K[[H]] = D^c(H,K) \longrightarrow\rightarrow {}^c\mathrm{Ind}_P^G(\chi)'$.

Lecture VI: Locally analytic admissibility

§20 Overview of main features

In this Lecture $L \subseteq K$ are finite extensions of $\mathbb{Q}_p$ and G is a locally L-analytic group. Since admissibility in all its forms is a concept which only depends on the action of a compact open subgroup we lose nothing by assuming, as we do throughout this Lecture, that G in fact is compact.

In the last Lecture we have constructed the category $\mathrm{Ban}_G^a(K)$ of admissible Banach space representations as a full subcategory of $\mathrm{Ban}_G(K)$. The point was to avoid all kinds of pathologies which occur in $\mathrm{Ban}_G(K)$. In particular:
– The category $\mathrm{Ban}_G^a(K)$ is abelian;
– all maps in $\mathrm{Ban}_G^a(K)$ behave nicely from a topological point of view: they are strict and have closed image.

The basic principle was to pass from a representation V in $\mathrm{Ban}_G(K)$ to its continuous dual V', view it as a module over the distribution algebra $D^c(G, K)$, and impose an appropriate finiteness condition on this module. The latter was made possible by the fact that the algebra $D^c(G, K) = K[[G]]$ is noetherian. In addition $o[[G]]$ is compact which meant that for an admissible V the topology on the dual V' is entirely rigid.

In this Lecture we start with the category $\mathrm{Rep}_G(K)$ of locally analytic G-representations on K-vector spaces of compact type with continuous linear and G-equivariant maps. We again want to single out a full subcategory $\mathrm{Rep}_G^a(K)$ which:
– is abelian,
– contains only strict maps,
– and still is "rich" enough.
Naturally we try to do this by the same principle as above. And indeed we have seen in Lecture IV that for any V in $\mathrm{Rep}_G(K)$ the strong dual V_b' carries a natural continuous $D(G, K)$-module structure from which V can be completely reconstructed. But then we run into a basic difficulty: The algebra $D(G, K)$ is not noetherian! It will require a major theory, to be described in Lectures VII and VIII, to construct a reasonable abelian module category over $D(G, K)$, to be called the category $\mathcal{C}_G$ of coadmissible modules. In the next section all of this will be illustrated in the case $G = \mathbb{Z}_p$. The algebra $D(G, K)$ also has no compactness properties. Therefore a coadmissible $D(G, K)$-module will have a natural but not unique topology. This forces us to make the following slightly modified definition.

Definition: *An admissible G-representation over K is a locally analytic G-representation on a K-vector space of compact type V such that the strong dual V_b' is a coadmissible $D(G, K)$-module equipped with its canonical topology.*

We let $\mathrm{Rep}_G^a(K)$ denote the full subcategory in $\mathrm{Rep}_G(K)$ of all admissible representations. As a rather straightforward consequence of the theory of coadmissible $D(G,K)$-modules one then obtains the following facts.

Theorem 20.1: *i. The functor*

$$\begin{array}{rcl} \mathrm{Rep}_G^a(K) & \xrightarrow{\sim} & \mathcal{C}_G \\ V & \longmapsto & V' \end{array}$$

is an anti-equivalence of categories.

ii. $\mathrm{Rep}_G^a(K)$ *is an abelian category; kernel and image of a morphism in* $\mathrm{Rep}_G^a(K)$ *are the algebraic kernel and image with the subspace topology;*

iii. any map in $\mathrm{Rep}_G^a(K)$ *is strict and has closed image;*

iv. the category $\mathrm{Rep}_G^a(K)$ *is closed with respect to the passage to closed G-invariant subspaces.*

So we have achieved our goal except we still have to justify the claim that this category $\mathrm{Rep}_G^a(K)$ is "rich" enough. For this we will relate this category to our previous category $\mathrm{Ban}_G^a(K)$ as well as to the "classical" smooth representation theory.

1) Relation to smooth representations.

In Example 16.5 the category $\mathrm{Rep}_G^\infty(K)$ of smooth G-representations was defined. We also recall that a smooth representation carries no topology. Neglecting the topology a locally analytic G-representation V is smooth if and only if the derived action of the Lie algebra $\mathfrak{g}$ on V is trivial, i.e., $\mathfrak{g}V = 0$. Harish Chandra introduced the following notion.

Definition: *A smooth G-representation V is called admissible-smooth if for any open subgroup $H \subseteq G$ the subspace V^H of H-fixed vectors is finite dimensional.*

Let $\mathrm{Rep}_G^{\infty,a}(K)$ denote the full subcategory in $\mathrm{Rep}_G^\infty(K)$ of admissible-smooth G-representations. Suppose that V is admissible-smooth. We equip V with the finest locally convex topology. On the other hand we find a decreasing fundamental sequence of open subgroups $G \supseteq H_1 \supseteq \ldots \supseteq H_n \supseteq \ldots$ Then V is the increasing union

$$V = \bigcup_n V^{H_n}$$

of the finite dimensional spaces V^{H_n} and therefore is of compact type. This means we have the fully faithful embedding of categories

$$\mathrm{Rep}_G^{\infty,a}(K) \longrightarrow \mathrm{Rep}_G(K) \ .$$

Theorem 20.2: $\mathrm{Rep}_G^{\infty,a}(K)$ *is the full subcategory in* $\mathrm{Rep}_G^a(K)$ *of all admissible locally analytic representations* V *such that* $\mathfrak{g}V = 0$*.*

The reason for this fact is the following. Let $I(\mathfrak{g})$ be the 2-sided ideal in $D(G,K)$ generated by $\mathfrak{g}$ and put

$$D^\infty(G,K) := D(G,K)/I(\mathfrak{g})$$

(the algebra of locally constant distributions on G). Then admissible-smooth G-representations correspond to coadmissible $D^\infty(G,K)$-modules.

2) Relation to Banach space representations.

Let V be a continuous Banach space representation of G. A vector $v \in V$ is called locally analytic if the V-valued function $g \longmapsto gv$ on G is locally analytic. We denote by V_{an} the vector subspace of all analytic vectors in V. It is clearly G-invariant. Moreover the G-equivariant linear map

$$\begin{array}{ccc} V_{an} & \longrightarrow & C^{an}(G,V) \\ v & \longmapsto & [g \mapsto g^{-1}v] \end{array}$$

is injective. We always equip V_{an} with the subspace topology with respect to this embedding. One checks that V_{an} is closed. Let us now suppose that V is admissible. The finite generation of V' over $D^c(G,K)$ then implies the existence of finitely many continuous linear forms $l_1, \ldots, l_m$ that the map

$$\begin{array}{ccc} V_{an} & \longrightarrow & C^{an}(G,K)^m \\ v & \longmapsto & ([g \mapsto l_i(g^{-1}v)])_{1 \le i \le m} \end{array}$$

also is a closed embedding. Since the right hand side is a vector space of compact type by Prop. 10.3 the left hand side is of compact type as well and we obtain the functor

$$\begin{array}{ccc} \mathrm{Ban}_G^a(K) & \longrightarrow & \mathrm{Rep}_G^a(K) \\ V & \longmapsto & V_{an} \ . \end{array}$$

Proposition 20.3: *For any* V *in* $\mathrm{Ban}_G^a(K)$ *we have*

$$V_{an} = (D(G,K) \underset{D^c(G,K)}{\otimes} V')'_b \ .$$

Theorem 20.4: *Suppose that* $L = \mathbb{Q}_p$*; we then have:*

i. V_{an} *is dense in* V *for any* V *in* $\mathrm{Ban}_G^a(K)$*;*

ii. the functor $V \longmapsto V_{an}$ *on* $\mathrm{Ban}_G^a(K)$ *is exact.*

In view of the proposition it is rather plausible that the theorem reduces to the claim that, in case $L = \mathbb{Q}_p$, the ring extension

$$D^c(G, K) \longrightarrow D(G, K)$$

is faithfully flat.

§21 The case $G = \mathbb{Z}_p$

Let $L = \mathbb{Q}_p$ and $G = \mathbb{Z}_p$. From §13 we know that

$$\mathcal{O}^b := D^c(G, K) = \text{bounded power series over } K$$

and

$$\mathcal{O} := D(G, K) = \text{power series over } K \text{ converging in } \mathcal{X}$$

where $\mathcal{X} := \{|z| < 1\}$ is the open unit disk. As a consequence of Weierstrass division $\mathcal{O}^b$ is a principal ideal domain all of whose nonzero ideals are of finite codimension. The elementary divisor theorem then implies that any V in $\mathrm{Ban}^a_{\mathbb{Z}_p}(K)$ is of the form

$$V = C(\mathbb{Z}_p, K)^r \oplus V_0$$

where V_0 is a finite dimensional continuous $\mathbb{Z}_p$-representation.

Surely $\mathcal{O}$ is an integral domain and therefore is flat over $\mathcal{O}^b$. But $\mathcal{O}$ is not noetherian although it is hard to write down an explicit example of an ideal which is not finitely generated. From Lazard one knows that in $\mathcal{O}$ the following three classes of ideals coincide:
– finitely generated ideals,
– principal ideals,
– closed ideals.
Moreover, any nonzero closed ideal is of finite codimension.

We choose a sequence of rational numbers $0 < q_n < 1$ converging to 1 and let $\mathcal{X}_n$ denote the closed disk of radius q_n. Then

$$\mathcal{O} = \varprojlim \mathcal{O}_n$$

where $\mathcal{O}_n$ denotes the ring of power series converging on the disk $\mathcal{X}_n$. Weierstrass division again shows that the rings $\mathcal{O}_n$ are principal ideal domains.

Definition: *An $\mathcal{O}$-module M is called coadmissible if*
– $M_n := \mathcal{O}_n \otimes_{\mathcal{O}} M$ is finitely generated over $\mathcal{O}_n$ for any n and
– $M = \varprojlim M_n$.

The reason that the coadmissible $\mathcal{O}$-modules form a well behaved abelian category is that $\mathcal{X}$ is a Stein space. We list a few features of this category:

– Since the number of generators of M_n can increase with n a coadmissible module need not be finitely generated.

– Vice versa, if the ideal $I \subseteq \mathcal{O}$ is not closed then the module $M := \mathcal{O}/I$ is finitely generated but not coadmissible.

– Any finitely presented $\mathcal{O}$-module is coadmissible.

A certain weaker form of the elementary divisor theorem still holds for $\mathcal{X}$ and implies:

– Any V in $\mathrm{Rep}_G^a(K)$ such that V' is finitely presented is of the form

$$V = C^{an}(\mathbb{Z}_p, K)^r \oplus V_0$$

where V_0 is a finite dimensional continuous $\mathbb{Z}_p$-representation.

– Simple coadmissible $\mathcal{O}$-modules are finite dimensional.

Lecture VII:
Fréchet-Stein Algebras
and
Coadmissible Modules

§22 Banach algebras

In this Lecture we give the general definitions for the algebras (Fréchet-Stein algebras) and modules (coadmissible modules) that make it possible to apply reasoning similar to that used for $\mathbb{Z}_p$ to general p-adic Lie groups.

A K-Banach algebra A is a K-Banach space $(A, |\ |_A)$ with a structure of an associative unital K-algebra such that the multiplication is continuous, which means that there is a constant $c > 0$ such that

$$|ab|_A \leq c|a|_A|b|_A \quad \text{for any } a, b \in A .$$

The norm will be called *submultiplicative* if it satisfies the stronger condition that

$$|1|_A = 1 \quad \text{and} \quad |ab|_A \leq |a|_A|b|_A \quad \text{for any } a, b \in A .$$

We let $\mathcal{M}_A$ denote the category of finitely generated (left) A-modules.

Proposition 22.1: *Suppose that A is a (left) noetherian K-Banach algebra; we then have:*

i. Each module M in $\mathcal{M}_A$ carries a unique K-Banach space topology (called its canonical topology) such that the A-module structure map $A \times M \to M$ is continuous;

ii. every A-submodule of a module in $\mathcal{M}_A$ is closed in the canonical topology; in particular, every (left) ideal in A is closed;

iii. any map in $\mathcal{M}_A$ is continuous and strict for the canonical topologies.

Exercise: Starting with the fact that a finitely generated module M can be presented as a quotient $A^n \to M \to 0$, and using the open mapping theorem, prove this proposition (compare [ST] Lemma 1).

Example 22.2: The Tate algebra $K << T_1, \ldots, T_d >>$ consisting of all power series

$$f = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} a_{n_1,\ldots,n_d} T_1^{n_1} \cdots T_d^{n_d}$$

such that

$$a_{n_1,\dots,n_d} \in K, |a_{n_1,\dots,n_d}| \to 0 \text{ as } n_1 + \dots n_d \to \infty$$

is the prototypical K-Banach algebra. The norm is given by:

$$\|f\| = \max_{n_1,\dots,n_d} |a_{n_1,\dots,n_d}|.$$

The Tate algebra is noetherian, and its maximal ideals are in bijection with points of the closed unit disk in $\overline{K}$ modulo the action of the Galois group of $\overline{K}/K$.

§23 Fréchet-Stein algebras

The definition of admissibility for the $D(G,K)$-modules associated to locally analytic representations is based on special properties of the ring $D(G,K)$.

These properties are inspired by some ideas from complex analysis, in particular the notions of Stein algebra and Stein module.

The underlying principle is that, when G is a compact p-adic Lie group, the ring $D(G,K)$ is a non-commutative version of the ring of functions on an open p-adic polydisk.

Such a polydisk is an example of a (rigid analytic) Stein space. The ring of functions $\mathcal{O}$ on such a space is a projective limit of Banach algebras where the transition maps are flat. The essential properties of Stein spaces are that:

(i) (Theorem A) The stalks of coherent sheaves are always generated by global sections.

(ii) (Theorem B) Stein spaces have trivial sheaf cohomology (so that one may pass from coherent sheaves to modules over $\mathcal{O}$ and back without obstructions). Since coherent sheaves form an abelian category, so do the associated modules of global sections.

We will see how these properties, suitably reinterpreted, play a crucial role in the definition of local-analytic admissibility.

Let A be a K-Fréchet algebra.

A continuous seminorm on A induces a norm on the quotient space $A/\{a \in A : q(a) = 0\}$. The completion of the latter with respect to q is a K-Banach space which will be denoted by A_q.

A_q comes with a natural continuous linear map $A \to A_q$ with dense image.

For any two continuous seminorms $q' \leq q$ the identity on A extends to a continuous, in fact norm decreasing, linear map $\phi_q^{q'} : A_q \longrightarrow A_{q'}$ with dense image such that the diagram

$$\begin{tikzcd} & A_q \arrow[dd, "\phi_q^{q'}"] \\ A \arrow[ur] \arrow[dr] & \\ & A_{q'} \end{tikzcd}$$

commutes.

For any sequence $q_1 \leq q_2 \leq \ldots \leq q_n \leq \ldots$ of seminorms on A which define the Fréchet topology (such a sequence always exists), the map

$$A \xrightarrow{\cong} \varprojlim_{n \in \mathbb{N}} A_{q_n}$$

where on the right hand side the projective limit is formed with respect to the $\phi_{q_{n+1}}^{q_n}$ as the transition maps is an isomorphism of locally convex K-vector spaces.

A continuous seminorm q on A will be called an algebra seminorm if the multiplication on A is continuous with respect to q, i.e., if there is a constant $c > 0$ such that

$$q(ab) \leq cq(a)q(b) \quad \text{for any } a, b \in A \ .$$

In this case A_q in a natural way is a K-Banach algebra and the natural map $A \rightarrow A_q$ is a homomorphism of K-algebras.

If the sequence $q_1 \leq \ldots \leq q_n \leq \ldots$ consists of algebra seminorms then the transition maps $\phi_{q_{n+1}}^{q_n}$ are algebra homomorphisms and

$$A \xrightarrow{\cong} \varprojlim_{n \in \mathbb{N}} A_{q_n}$$

is an isomorphism of Fréchet algebras.

Definition 23.1: *The K-Fréchet algebra A is called a K-Fréchet-Stein algebra if there is a sequence $q_1 \leq \ldots \leq q_n \leq \ldots$ of continuous algebra seminorms on A which define the Fréchet topology such that*
(i) A_{q_n} is (left) noetherian, and
(ii) A_{q_n} is flat as a right $A_{q_{n+1}}$-module (via $\phi_{q_{n+1}}^{q_n}$)
for any $n \in \mathbb{N}$.

Definition 23.2: *A coherent sheaf for $(A,(q_n))$ is a family $(M_n)_{n\in\mathbb{N}}$ of modules M_n in $\mathcal{M}_{A_{q_n}}$ together with isomorphisms $A_{q_n}\otimes_{A_{q_{n+1}}} M_{n+1}\xrightarrow{\cong} M_n$ in $\mathcal{M}_{A_{q_n}}$ for any $n\in\mathbb{N}$.*

The coherent sheaves for $(A,(q_n))$ with the obvious notion of a homomorphism form a category $Coh_{(A,(q_n))}$.

The flatness hypothesis in the definition of Fréchet-Stein algebras means that, given a compatible family of maps $f_n : M_n \to N_n$, the kernels of the f_n again define a coherent sheaf.

It is the noncommutative realization of the fact in geometry that "localization is flat."

It follows that the category of coherent sheaves over a Fréchet-Stein algebra is abelian, with the obvious notions of (co)kernels and (co)images.

For any coherent sheaf $(M_n)_n$ for $(A,(q_n))$ its A-module of "global sections" is defined by

$$\Gamma(M_n) := \varprojlim_n M_n \ .$$

We may now give a crucial definition.

Definition 23.3: *A (left) A-module is called coadmissible if it is isomorphic to the module of global sections of some coherent sheaf for $(A,(q_n))$.*

We let $\mathcal{C}_A$ denote the full subcategory of coadmissible modules in the category $\mathrm{Mod}(A)$.

A cofinality argument shows that $\mathcal{C}_A$ is independent of the choice of the sequence $(q_n)_n$. Passing to global sections defines a functor

$$\Gamma : Coh_{(A,(q_n))} \longrightarrow \mathcal{C}_A \ .$$

The crucial properties (i) and (ii) for coherent sheaves on Stein spaces are captured in our case by the following theorem.

Theorem 23.4: *Let $(M_n)_n$ be a coherent sheaf for $(A,(q_n))$ and put $M := \Gamma(M_n)$; we have:*

i. (Theorem A) For any $n\in\mathbb{N}$ the natural map $M\longrightarrow M_n$ has dense image with respect to the canonical topology on the target;

ii. (Theorem B)

$$\varprojlim_n{}^{(i)} M_n = 0$$

for any natural number $i \geq 1$. Expressed differently: the projective limit functor from coherent sheaves to modules is exact.

Proof: Both of these results follow from the Mittag-Leffler property, which asserts in general terms that if one has a projective system $(M_n)_n$ of complete metrizable spaces, and if the transition maps for $f_{mn} : M_m \to M_n$ for $m > n$ are uniformly continuous and have dense image, then the two conclusions of the theorem are valid for $(M_n)_n$. See [EGA] III.0.13.2.2 and III.0.13.2.4.

Corollary 23.5: *For any coherent sheaf $(M_n)_n$ for $(A, (q_n))$ and $M := \Gamma(M_n)$ the natural map*

$$A_{q_n} \otimes_A M \xrightarrow{\cong} M_n$$

is an isomorphism for any $n \in \mathbb{N}$.

Proof: By Theorem A the A_{q_n}-submodule of M_n generated by the image of M is dense in M_n. Prop. 22.1.ii. then says that this submodule, in fact, must be equal to M_n. This establishes the surjectivity of the map in question and, more precisely, that M_n as an A_{q_n} -module is generated by finitely many elements in the image of the map $M \longrightarrow M_n$.

Corollary 23.6: *The functor $\Gamma : Coh_{(A,(q_n))} \xrightarrow{\sim} \mathcal{C}_A$ is an equivalence of categories.*

Proof: By definition the functor is essentially surjective. According to the previous corollary it is fully faithful. Both properties together amount to the functor being an equivalence of categories.

Corollary 23.7: *i. The direct sum of two coadmissible A-modules is coadmissible;*

ii. the (co)kernel and (co)image of an arbitrary A-linear map between coadmissible A-modules are coadmissible;

iii. the sum of two coadmissible submodules of a coadmissible A-module is coadmissible;

iv. any finitely generated submodule of a coadmissible A-module is coadmissible;

v. any finitely presented A-module is coadmissible.

Proof: The first assertion is obvious. The last three assertions are immediate consequences of the first two. Hence it remains to establish the second assertion. By the previous corollary any map between coadmissible modules comes from a map between coherent sheaves. But, by Theorem B, the functor Γ into $\mathrm{Mod}(A)$ commutes with the formation of (co)kernels and (co)images.

These properties amount to the essential property of admissible representations, as discussed in Lecture VI:

Corollary 23.8: *$\mathcal{C}_A$ is an abelian subcategory of* $\mathrm{Mod}(A)$.

Write

$$M = \varprojlim_n M_n \ .$$

By Prop. 22.1.i, each M_n carries its canonical Banach space topology as a finitely generated A_{q_n}-module.

We equip M with the projective limit topology of these canonical topologies. This makes M into a K-Fréchet space. Moreover, the A-module structure map $A \times M \to M$ clearly is continuous. This topology is called the *canonical topology* on M.

Lemma 23.9: *For any coadmissible A-module M and any submodule $N \subseteq M$ the following assertions are equivalent:*

i. N is coadmissible;

ii. M/N is coadmissible;

iii. N is closed in the canonical topology of M.

Moreover, we have:

iv. Any A-linear map $f : M \to N$ between coadmissible A-modules is continuous and strict for the canonical topologies;

v. in particular, any finitely generated left ideal of A is closed.

The following two propositions are useful in two special situations. If G is locally L-analytic, the locally L-analytic functions are a subspace of the locally $\mathbb{Q}_p$-analytic functions; dually, $D(G, K)$ is a quotient of $D(G_{\mathbb{Q}_p}, K)$ where $G_{\mathbb{Q}_p}$ is the restriction of scalars of G to $\mathbb{Q}_p$. This proposition makes it possible to pass from $\mathbb{Q}_p$ to L.

Proposition 23.10: *Let I be a closed two sided ideal in a K-Fréchet-Stein algebra A; then A/I is a K-Fréchet-Stein algebra as well.*

The next result makes it possible to deduce that, if the algebra $D(H, K)$ is Fréchet-Stein, and H is of finite index and open in G, then $D(G, K)$ is also Fréchet-Stein.

Proposition 23.11: *Let $A \longrightarrow B$ be a continuous unital algebra homomorphism between K-Fréchet-Stein algebras such that B is coadmissible as a (left) A-module, and let M be a (left) B-module; then M is coadmissible as a B-module if and only if it is coadmissible as an A-module.*

Example 23.12: Let $G = \mathbb{Z}_p$ viewed as a locally analytic group. The distribution algebra $D(G, K) = \mathcal{O}$, discussed in §21, is the ring of power series convergent on the open unit disk. For each $x \in \overline{K}$ with $|x| < 1$ we have the seminorm q_x defined by $q_x(f) = |f(x)|$. For each radius $0 < r < 1$, with $r \in p^{\mathbf{Q}}$ we have the supremum norm

$$q_r(f) = \max_{x \in \overline{K}, |x| \leq r} q_x(f).$$

These norms are the same as those from the discussion in Thm. 13.2, though presented differently.

If $r \in p^{\mathbf{Q}}$, then the Banach algebra $A_r = A_{q_r}$ is the so-called "affinoid algebra" of rigid analytic functions on the closed disk of radius r. In explicit terms we have that A_r consists of the power series

$$f = \sum_{n=0}^{\infty} a_n T^n$$

where $|a_n| r^n \to 0$ as $n \to \infty$. If K contains an element π so that $|\pi| = p^r$ (which can always be achieved by making a finite extension of K) then a change of variables $U = T/\pi$ identifies A_r with $K << U >>$.

The algebra $D^c(\mathbb{Z}_p, K)$ is the subalgebra of $D(G, K)$ consisting of power series with bounded coefficients. It corresponds to the bounded functions on the open unit disk.

Theorem 23.13: *$D(\mathbb{Z}_p, K)$ is a Fréchet-Stein algebra.*

Because $D(\mathbb{Z}_p, K)$ really *is* the ring of functions on a Stein space, this fact follows from the theory of rigid analytic geometry (Kiehl's Theorem A and Theorem B). Of course $D(\mathbb{Z}_p, K)$ has even more properties (for example closed ideals are finitely generated, and even principal) as discussed in Lecture VI.

In the next Lecture we will prove the following main result.

Theorem 23.14: *Let G be a compact locally L-analytic group, and assume that K is a complete discretely valued extension field of L. Then $D(G, K)$ is a Fréchet-Stein algebra.*

As a consequence of this theorem we have the definition proposed in Lecture VI:

Definition 23.15: *A locally analytic G-representation on a vector space V is called admissible if V is of compact type and the strong dual V_b' is a coadmissible $D(H,K)$-module with its canonical topology for one (or all) compact open subgroups of H of G.*

The reason one may check the condition for one compact open alone is Prop. 23.11.

The properties of coadmissible modules tell us that, as discussed in Lecture VI, we have the hoped-for properties: $\mathrm{Rep}_K^a(G)$ is an abelian category; kernels and image are the algebraic kernel and image (as subspaces). Any map in the category is strict and has closed image. If V is admissible, so is any closed G-invariant subspace.

Example: The locally analytic principal series representations $\mathrm{Ind}_P^G(\chi)$ for $G = \mathrm{GL}_n(L)$ are admissible. In fact by the Iwasawa decomposition $G = G_o P$ where $G_o = \mathrm{GL}_n(o_L)$, any function $f \in \mathrm{Ind}_P^G(\chi)$ is determined by its restriction to G_o, and in fact as a representation space for G_o we have

$$\mathrm{Ind}_P^G(\chi) = \mathrm{Ind}_{P\cap G_o}^{G_o}(\chi|_{P\cap G_o}).$$

But the right side of this inequality is a closed subspace of $C^{an}(G_o, K)$, so its dual is a (Hausdorff) quotient of $D(G_o, K)$. Consequently the dual is coadmissible and therefore the principal series is admissible.

Lecture VIII: Distribution Algebras of p-adic Lie groups are Fréchet-Stein

§24 Distribution algebras of p-adic Lie groups

In this Lecture we outline a proof of the following theorem.

Theorem 24.1: *Let G be a compact locally L-analytic group, and suppose that K is discretely valued. Then $D(G, K)$ is a Fréchet-Stein algebra.*

The proof of this result relies on two main ingredients:

(i) The theory of p-valued groups. ([Laz], [DDMS])

(ii) The theory of filtered and graded rings. ([LVO])

We briefly recall aspects of each of these areas.

§25 p-valued groups.

A *p-valuation* on a group G is a real valued function $\omega : G\backslash\{1\} \longrightarrow (1/(p-1), \infty)$ such that

$$\begin{aligned}&\omega(gh^{-1}) \geq \min(\omega(g), \omega(h)),\\&\omega(g^{-1}h^{-1}gh) \geq \omega(g) + \omega(h), \text{and}\\&\omega(g^p) = \omega(g) + 1\end{aligned}$$

for any $g, h \in G$ ([Laz] III.2.1.2).

As usual one puts $\omega(1) := \infty$.

For each real number $\nu > 0$ we define the normal subgroups

$$G_\nu := \{g \in G : \omega(g) \geq \nu\} \quad \text{and} \quad G_{\nu+} := \{g \in G : \omega(g) > \nu\}$$

of G, and we put

$$gr(G) := \bigoplus_{\nu>0} G_\nu/G_{\nu+} \ .$$

To simplify the following discussion we assume that p is *odd.* However this assumption is not necessary for the truth of the theorem.

Theorem 25.1: *Any compact locally $\mathbb{Q}_p$-analytic group has an open subgroup H with an (ordered) set $h_1, \ldots, h_d$ of topological generators such that:*

(i) The map

$$\begin{array}{rccc} \psi : & \mathbb{Z}_p^d & \to & H \\ & (x_1, \ldots, x_d) & \to & h_1^{x_1} \cdots h_d^{x_d} \end{array}$$

is a global chart for the manifold H.

(ii) The function

$$\omega(h_1^{x_1} \cdots h_d^{x_d}) = \inf_{1 \leq i \leq d} (1 + \omega_p(x_i))$$

where ω_p is the p-adic valuation on $\mathbb{Z}_p$ is a p-valuation on H.

(iii) Every element of $H_2 = \{h : \omega(h) \geq 2\}$ is a p^{th} power in H.

The coordinates given by the map ψ give us explicit coordinates for $C^{an}(H, K)$ and $C(H, K)$. In fact we have isomorphisms

$$\psi^* : C^{an}(H, K) \xrightarrow{\cong} C^{an}(\mathbb{Z}_p^d, K)$$

and

$$\psi^* : C(H, K) \xrightarrow{\cong} C(\mathbb{Z}_p^d, K).$$

The multivariate version of the results in §13 tell us that $C(\mathbb{Z}_p^d, K)$ can be viewed as the space of all series

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha \binom{x}{\alpha}$$

with $c_\alpha \in K$ and such that $|c_\alpha| \longrightarrow 0$ as $|\alpha| \longrightarrow \infty$. Here we put, as usual,

$$\binom{x}{\alpha} := \binom{x_1}{\alpha_1} \cdots \binom{x_d}{\alpha_d}$$

and

$$|\alpha| := \sum_{i=1}^{d} \alpha_i$$

for $x = (x_1, \ldots, x_d) \in \mathbb{Z}_p^d$ and multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$.

The multivariate version of Thm. 13.2 (Amice's theorem) says that the Mahler expansion of f lies in the subspace $C^{an}(\mathbb{Z}_p^d, K)$ if and only if $|c_\alpha| r^{|\alpha|} \longrightarrow 0$ for some real number $r > 1$ as $|\alpha| \longrightarrow \infty$.

For $h = \psi(x)$ we have

$$h(f) = \delta_{\psi(x)}(f) = \psi^*(f)(x)$$

for any $f \in C(H, K)$ and any $x \in \mathbb{Z}_p^d$.

Write $b_i := h_i - 1$ and $\mathbf{b}^\alpha := b_1^{\alpha_1} b_2^{\alpha_2} \cdots b_d^{\alpha_d}$, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. We have

$$\mathbf{b}^\alpha \in \mathbb{Z}_p[H] \subseteq D(H, K).$$

If c_α denote the coefficients of the Mahler expansion of $\psi^*(f)$ for some $f \in C(H, K)$ then

$$\mathbf{b}^\alpha(f) = c_\alpha \ .$$

Any distribution $\lambda \in D(H, K)$ has a unique convergent expansion

$$\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha$$

with $d_\alpha \in K$ such that, for any $0 < r < 1$, the set $\{|d_\alpha| r^{|\alpha|}\}_{\alpha \in \mathbb{N}_0^d}$ is bounded. Conversely, any such series is convergent in $D(H, K)$. The Fréchet topology on $D(H, K)$ is defined by the family of norms

$$\|\lambda\|_r := \sup_{\alpha \in \mathbb{N}_0^d} |d_\alpha| r^{|\alpha|}$$

for $0 < r < 1$.

Since the multiplication in $D(H, K)$ is jointly continuous, we obtain the expansion of the product of two distributions by multiplying their expansions, inserting the expansions

$$\mathbf{b}^\beta \mathbf{b}^\gamma = \sum_\alpha c_{\beta\gamma,\alpha} \mathbf{b}^\alpha \ ,$$

and rearranging. Here the coefficients $c_{\beta\gamma,\alpha}$ belong to $\mathbb{Z}_p$.

The inclusion $\mathbb{Z}_p[H] \subseteq D(H, K)$ extends to an embedding of topological rings

$$\mathbb{Z}_p[[H]] \hookrightarrow D(H, K).$$

A distribution $\lambda = \sum_\alpha d_\alpha \mathbf{b}^\alpha$ lies in $\mathbb{Z}_p[[H]]$ if and only if all $d_\alpha \in \mathbb{Z}_p$.

Proposition 25.2: (Lazard) *The norm $\| \ \|_{1/p}$ is multiplicative, satisfies $\|h - 1\|_{1/p} \leq p^{-\omega(h)}$, and induces the natural compact topology on $\mathbb{Z}_p[[H]]$.*

Corollary 25.3: *Each norm $\| \ \|_r$, for $1/p \leq r < 1$, is submultiplicative on $D(H, K)$.*

Proof. This is a consequence of the proposition and certain elementary estimates for the coefficients in the expansion of $\mathbf{b}^\alpha \mathbf{b}^\beta$ given above.

Our goal is to show that $D(H, K)$ is a Fréchet-Stein algebra for the structure given by the family of norms $\| \ \|_r$.

Let $D_r(H, K)$ be the completion of $D(H, K)$ in the norm $\| \ \|_r$. Explicitly we have

$$D_r(H, K) = \{f = \sum d_\alpha \mathbf{b}^\alpha : |d_\alpha| r^{|\alpha|} \to 0 \text{ as } |\alpha| \to \infty\}.$$

In addition we define

$$D_{<r}(H, K) = \{f = \sum d_\alpha \mathbf{b}^\alpha : |d_\alpha| r^{|\alpha|} \text{ is bounded as } |\alpha| \to \infty\}$$

We have

$$D_r(H, K) \subseteq D_{<r}(H, K) \subseteq D_{r'}(H, K) \subseteq \ldots \subseteq D_{1/p}(H, K)$$

when $1/p \leq r' < r < 1$. Also

$$D(H, K) = \varprojlim D_r(H, K) = \varprojlim D_{<r}(H, K).$$

§26 Filtered Rings

The second key ingredient in our proof is the theory of filtered rings.

The general principle is that a filtered ring R inherits properties from its associated graded ring.

In our situation the graded rings we will consider are commutative and thus are much simpler than the original rings.

We begin with some general definitions.

Let R be an associative unital ring. We call R *filtered* if it is equipped with a family $(F^s R)_{s \in \mathbb{R}}$ of additive subgroups $F^s R \subseteq R$ such that, for any $r, s \in \mathbb{R}$,

– $F^r R \supseteq F^s R$ if $r \leq s$,

– $F^r R \cdot F^s R \subseteq F^{r+s} R$,

– $\bigcup_{s \in \mathbb{R}} F^s R = R$ and $1 \in F^0 R$.

For any $s \in \mathbb{R}$ put

$$F^{s+} R := \bigcup_{r > s} F^r R \quad \text{ and } \quad gr^s R := F^s R / F^{s+} R\ .$$

Then

$$gr^{\cdot} R := \bigoplus_{s \in \mathbb{R}} gr^s R$$

with the obvious multiplication is called the *associated graded ring.*

The filtration is called *quasi-integral* if there exists an $n_0 \in \mathbb{N}$ such that $\{s \in \mathbb{R} : gr^s R \neq 0\} \subseteq \mathbb{Z} \cdot 1/n_0$.

We say that the filtered ring R is (separated and) *complete*, if the natural map

$$R \xrightarrow{\cong} \varprojlim_s R/F^s R$$

is bijective.

If $x \in R$, then the representative of x in $gr^\cdot R$ (denoted $\sigma(x)$) is called the principal symbol of x.

Exercise: Let K be a complete discretely valued extension field of $\mathbb{Q}_p$. Filter K by

$$F^s K = \{x : |x| < p^{-s}\}.$$

Show that the associated graded ring $gr^\cdot K$ is the ring $\mathbb{F}[\sigma(\pi), \sigma(\pi)^{-1}]$ where $\mathbb{F}$ is the residue field of K and π is a uniformizing parameter.

Any homomorphism $\phi : R \longrightarrow A$ between filtered rings which respects the filtrations induces in the obvious way a homomorphism of graded rings $gr^\cdot \phi : gr^\cdot R \longrightarrow gr^\cdot A$.

In the following we consider two complete filtered rings R and A whose filtrations are quasi-integral and a unital ring homomorphism $\phi : R \longrightarrow A$ which respects the filtrations.

Proposition 26.1: *(i) If the graded ring $gr^\cdot R$ is (left) noetherian then R is (left) noetherian as well.*

(ii) Suppose that $gr^\cdot R$ and $gr^\cdot A$ are left noetherian and that $gr^\cdot A$ as a right $gr^\cdot R$-module (via $gr^\cdot \phi$) is flat; then A is flat as a right R-module (via ϕ).

§27 The main steps of the proof

We now combine the ideas of §25 and §26 to show that $D(H, K)$ is a Fréchet-Stein algebra.

The algebras $D_r(H, K)$ for each $1/p \leq r' \leq r$, have the filtrations:

$$F^s_{r'} D_r(H, K) = \{a \in D_r(H, K) : \|a\|_{r'} \leq p^{-s}\}.$$

The same filtrations make sense for $D_{<r}(H, K)$ provided that $1/p \leq r' < r$. Also, all of these filtrations coincide on K with the natural one.

Theorem 27.1: *For $1/p < r < 1$, and $r \in p^{\mathbf{Q}}$ the ring $gr_r^{\cdot} D_r(H,K)$ is a polynomial ring over $gr^{\cdot} K$ in the principal symbols $\sigma(b_i)$ for $i = 1, \ldots, d$.*

Proof. It is easy to see, from the explicit presentation of $D_r(H,K)$, that $gr_r^{\cdot} D_r(H,K)$ is a free $gr^{\cdot} K$ module generated by the principal symbols of the monomials $\sigma(\mathbf{b}^\alpha)$. However, since the norms are multiplicative for monomials by definition, we see that

$$\sigma(\mathbf{b}^\alpha) = \prod_{i=1}^{d} \sigma(b_i)^{\alpha_i}$$

Therefore the graded ring "looks like" a polynomial ring; the key question is the ring structure. For this we use the following fact:

$$(*) \qquad \|b_i b_j - b_j b_i\|_r < \|b_i b_j\|_r.$$

This implies that the $\sigma(b_i)$ are commuting variables and this implies the claimed result.

Now we must see why $(*)$ holds. Since $i < j$ we have $\|b_i b_j\|_r = r^2$. To estimate the left hand side in the assertion we use that the properties of a p-valuation imply that $h := h_i^{-1} h_j^{-1} h_i h_j = g^p$ for some $g \in H$. Hence

$$\begin{aligned} b_i b_j - b_j b_i &= h_i h_j - h_j h_i = h_j h_i (h-1) \\ &= h_j h_i (g^p - 1) = h_j h_i (((g-1)+1)^p - 1) \\ &= h_j h_i (g-1)^p + \sum_{n=1}^{p-1} \binom{p}{n} h_j h_i (g-1)^n \end{aligned}$$

and therefore, by the submultiplicativity of $\| \ \|_r$ and the fact that $\|h_i\|_r = 1$,

$$\|b_i b_j - b_j b_i\|_r \leq \max(\|g-1\|_r^p, |p| \|g-1\|_r) \ .$$

Since $1/p \leq r < 1$ the inequality $\|g-1\|_{1/p} \leq p^{-\omega(g)}$ implies that $\|g-1\|_r \leq r^{\omega(g)}$. To see this let $0 < z \leq 1$ such that $(1/p)^z = r$ and consider the expansion $g - 1 = \sum_\alpha d_\alpha \mathbf{b}^\alpha$. We have $|d_\alpha| p^{-|\alpha|} \leq p^{-\omega(g)}$. By exponentiating this latter inequality and using that $|d_\alpha| \leq 1$ we have

$$|d_\alpha| r^{|\alpha|} \leq |d_\alpha|^z p^{-z|\alpha|} \leq p^{-z\omega(g)} = r^{\omega(g)} \ .$$

Combining this with the previous estimate we obtain

$$\|b_i b_j - b_j b_i\|_r \leq \max(r^{p\omega(g)}, p^{-1} r^{\omega(g)}) \ .$$

From the properties of a p-valuation we deduce that

$$\omega(g) = \omega(h) - 1 \geq \omega(h_i) + \omega(h_j) - 1 = 1 \ .$$

We therefore get

$$\|b_i b_j - b_j b_i\|_r \le \max(r^p, r/p) \ .$$

The inequality $r/p < r^2$ is obvious from $1/p < r$. It remains to be seen that $2 < p$. Here is where our assumption that p is odd comes in! (The result is still true when p=2, but the p-valuation is more complicated).

Corollary 27.2: *The Banach algebras $D_r(H,K)$, for $1/p < r < 1$, and $r \in p^{\mathbf{Q}}$, are Noetherian.*

Proof: This follows from Prop. 26.1 since the filtration is quasi-integral and complete.

We remark that $\mathbb{Z}_p[[H]]$ is a noetherian integral domain, but it is not commutative – this is due to Lazard. From this fact one sees that $D_{1/p}(H,K)$ is also noetherian.

The second part of the Fréchet-Stein property is the flatness of the transition maps.

Theorem 27.3: *For $1/p < r' \le r < 1$ in $p^{\mathbf{Q}}$, the map $D_r(H,K) \to D_{r'}(H,K)$ is flat.*

Proof: We attack this in stages. First we show that $D_{<r}(H,K)$ is (left and right) noetherian. Then we show that the maps

$$D_r(H,K) \to D_{<r}(H,K)$$

and

$$D_{<r}(H,K) \to D_{r'}(H,K)$$

are each flat. Each of these facts can be checked after making a faithfully flat base extension, and so we may extend K so that $r = |\pi|^m$ and $r' = |\pi|^{m'}$ are integral powers of $|\pi|$, where π is the uniformizer of K. Then a calculation using the grading shows that

$$gr_r^0 D_{<r}(H,K) = k[[u_1, \ldots, u_d]]$$

where u_i is the principal symbol of b_i/π^m. Then

$$gr_r^{\cdot} D_{<r}(H,K) = gr^{\cdot} K \otimes gr_r^0 D_{<r}(H,K)$$

is a noetherian ring. Furthermore, it is not hard to see that $gr_r^0 D_r(H,K)$ is the subring of polynomials $k[u_1, \ldots, u_d]$. Thus on the level of graded rings the map from $D_r(H,K)$ to $D_{<r}(H,K)$ is a flat base change of the map of a polynomial ring into formal power series, and this is known to be flat. Now we apply Prop. 26.1.

Next we consider the map $D_{<r}(H,K) \to D_{r'}(H,K)$. It suffices to consider the inclusion of the unit ball

$$F_r^0 D_{<r}(H,K) \to D_{r'}(H,K)$$

and to show that this is flat. The key point is that the image of $F_r^0 D_{<r}(H,K)$ in the target is compact (at least if K is locally compact; more sophisticated methods work if K is only discretely valued). Therefore the filtration $F_{r'}$ on $F_r^0 D_{<r}(H,K)$ is complete. Once again one computes explicitly the graded ring and, without giving details, one finds the map is given by "localization at $\sigma(\pi)$" and is therefore flat. One more application of Prop. 26.1 completes the proof.

References

[DDMS] Dixon J.D., du Sautoy M.P.F., Mann A., Segal D.: Analytic Pro-p-Groups. Cambridge Univ. Press 1999

[Feaux] Féaux de Lacroix C. T.: Einige Resultate über die topologischen Darstellungen p-adischer Liegruppen auf unendlich dimensionalen Vektorräumen über einem p-adischen Körper. Thesis, Köln 1997, Schriftenreihe Math. Inst. Univ. Münster, 3. Serie, Heft 23, pp. 1-111 (1999)

[EGA] Grothendieck A., Dieudonné J.: Éléments de géométrie algébrique, Chap. III. Publ. Math. IHES 11 (1961)

[Laz] Lazard M.: Groupes analytiques p-adique. Publ. Math. IHES 26, 389-603 (1965)

[LVO] Li Huishi, van Oystaeyen F.: Zariskian Filtrations. Dordrecht: Kluwer 1996

[Iwasawa] Schneider P., Teitelbaum J.:Banach space representations and Iwasawa theory. Israel J. Math. 127, 359-380 (2002)

[NFA] Schneider P.: Nonarchimedean Functional Analysis. Berlin-Heidelberg-New York: Springer 2001

[Dist] Schneider P., Teitelbaum J.: Locally analytic distributions and p-adic representation theory, with applications to GL_2. J. AMS 15, 443-468 (2002)

[Duality] Schneider P., Teitelbaum J.: Duality for admissible locally analytic representations. Representation Theory 9, 297-326 (2005)

[ST] Sinclair A.M., Tullo A.W.: Noetherian Banach algebras are finite dimensional. Math. Ann. 211, 151–153 (1974)

Peter Schneider
Mathematisches Institut
Westfälische Wilhelms-Universität Münster
Einsteinstr. 62
D-48149 Münster, Germany
pschnei@math.uni-muenster.de
http://www.uni-muenster.de/math/u/schneider

Jeremy Teitelbaum
Department of Mathematics, Statistics, and Computer Science (M/C 249)
University of Illinois at Chicago
851 S. Morgan St.

Chicago, IL 60607, USA
jeremy@uic.edu
http://www.math.uic.edu/∼jeremy