ERRATUM

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The proof of theorem 6.3 in my paper Orthogonal calculus [W] contains a gap. This is caused by an error in the preliminaries [W, 6.2] ; the offending statement is . . . and happens to be inverse to $\rho_{T(b)}$. The purpose of this note is to fill the gap.

Notation. $J$ is the category of finite dimensional real vector spaces with a positive definite inner product. Morphisms in $J$ are the linear maps respecting the inner product. $E$ is the category of continuous functors from $J$ to spaces. (The spaces in question are assumed to be compactly generated Hausdorff, homotopy equivalent to CW–spaces). A morphism $E \to F$ (natural transformation) in $E$ is an equivalence if $E(V) \to F(V)$ is a homotopy equivalence for each $V$ in $J$. An object $E$ in $E$ is polynomial of degree $\leq n$ if, for each $V$ in $J$, the canonical map

$$\rho : E(V) \longrightarrow \operatorname{holim}_{0 \neq U \subset \mathbb{R}^{n+1}} E(U \oplus V)$$

is a homotopy equivalence. The codomain of $\rho$, which we also denote by $(\tau_n E)(V)$, is a topological homotopy (inverse) limit [W, 5.1] ; more details below, in the proof of lemma e.3. To repeat, $E$ is polynomial of degree $\leq n$ if and only if $\rho : E \to \tau_n E$ is an equivalence.

6.3. Theorem. For any $n \geq 0$, there exist a functor $T_n : E \longrightarrow E$ taking equivalences to equivalences, and a natural transformation $\eta_n : 1 \longrightarrow T_n$ with the following properties:

(1) $T_n(E)$ is polynomial of degree $\leq n$, for all $E$ in $E$.

(2) if $E$ is already polynomial of degree $\leq n$, then $\eta_n : E \longrightarrow T_n E$ is an equivalence.

(3) For every $E$ in $E$, the map $T_n(\eta_n) : T_n E \longrightarrow T_n T_n E$ is an equivalence.

What we have to re–prove is (1). The remainder of the proof of 6.3 in [W] is not affected by the error in 6.2. As in [W] define $T_n E$ as the homotopy colimit (telescope in this case) of the direct system

$$(e.1) \quad E \longrightarrow \tau_n E \longrightarrow \tau_n^2 E \longrightarrow \tau_n^3 E \longrightarrow \tau_n^4 E \longrightarrow \ldots$$

It would be equally reasonable to define $T_n E$ as the homotopy colimit of

$$(e.2) \quad E \longrightarrow \tau_n E \longrightarrow \tau_n^2 E \longrightarrow \tau_n^3 E \longrightarrow \ldots$$
where the \( k \)-th map in the direct system is \( \rho : \tau_{n-1}E \to \tau_n(\tau^{k-1}E) \). It turns out that the homotopy colimits of (e.1) and (e.2) are isomorphic, even relative to \( E \). Namely, the Fubini principle for homotopy limits gives

\[
(\tau_{n+k}E)(V) \cong \lim_{0 \neq U_1, \ldots, U_k \subset \mathbb{R}^{n+1}} E(U_1 \oplus \cdots \oplus U_k \oplus V).
\]

Using this as an identification and inspecting the maps in the direct systems (e.1) and (e.2), one finds that the direct systems are isomorphic.

**e.3. Lemma.** Let \( p : G \to F \) be a morphism in \( \mathcal{E} \). Suppose that there exists an integer \( b \) such that \( p : G(W) \to F(W) \) is \(((n + 1) \dim(W) - b)\)-connected for all \( W \) in \( \mathcal{J} \). Then \( \tau_n(p) : \tau_nG(W) \to \tau_nF(W) \) is \(((n + 1) \dim(W) - b + 1)\)-connected for all \( W \).

**Proof.** We begin with a discussion of the homotopy limits involved. Suppose first that \( Z \) is any functor from the poset \( D \) of nonzero linear subspaces of \( \mathbb{R}^{n+1} \) to spaces. Ignoring the topology on \( D \), we can define \( \text{holim} Z \) as the totalization of the incomplete cosimplicial space

\[
[k] \mapsto \prod_{L : [k] \to D} Z(L(k))
\]

where \( L \) runs over the order-preserving injections from the poset \( [k] = \{1, \ldots, k\} \) to \( D \). (An incomplete cosimplicial space is a covariant functor from the category with objects \( [k] \) for \( k \geq 0 \) and monotone injections as morphisms to the category of spaces; the totalization of such a thing is the space of natural transformations to it from the functor \( [k] \mapsto \Delta^k \).)

We could make (e.4) into a complete cosimplicial space by dropping the injectivity condition on the order-preserving maps \( L \); the totalization would not change. However, totalizations of incomplete cosimplicial spaces are usually easier to understand than totalizations of complete cosimplicial spaces.— In (e.4), it is understood that a product \( \prod_{i \in S} \) with empty \( S \) is a single point \(*\); therefore the right-hand side of (e.4) is a point for \( k > n + 1 \).

Remembering the topology on \( D \) now, we note that \( D \) is a union of Grassmanianns. Let us suppose that the spaces \( Z(U) \) are the fibers of a fiber bundle \( \xi \) on \( D \) (that is, \( Z(U) \) is the fiber over \( U \in D \)), and that maps \( Z(U_1) \to Z(U_2) \) induced by inclusions \( U_1 \subset U_2 \) depend continuously on \( U_1, U_2 \). Then it is appropriate to replace the incomplete cosimplicial space (e.4) by another incomplete cosimplicial space,

\[
[k] \mapsto \Gamma(e_k^*\xi)
\]

where \( e_k \) is the evaluation map \( L \mapsto L(k) \), with domain equal to the space of monotone injections \( L : [k] \to D \), and codomain \( D \). The symbol \( \Gamma \) denotes a section space. The totalization of (e.5) is the topological homotopy limit of \( Z \). For us, the relevant examples are \( Z(U) := G(U \oplus W) \) and \( Z(U) := F(U \oplus W) \) where \( W \) is fixed; the topological homotopy limits are then \( \tau_nG(W) \) and \( \tau_nF(W) \), respectively.
The space of monotone injections $[k] \to D$ is a disjoint union of manifolds $C(\lambda)$. Here $\lambda : [k] \to [n+1]$ is a monotone injection avoiding the element $0 \in [n+1]$, and $C(\lambda)$ consists of those $L : [k] \to D$ for which $L(i)$ has dimension $\lambda(i)$. Writing $\lambda_i = \lambda(i)$ we find

$$\dim(C(\lambda)) = (n+1)\lambda_k + \sum_{i=0}^{k-1}(\lambda_{i+1} - \lambda_i)\lambda_i$$

$$= (n+1)\lambda_k + \sum_{i=0}^{k-1}\lambda_i\lambda_{i+1} - \sum_{i=0}^{k-1}\lambda_i^2$$

$$< (n+1)\lambda_k - k.$$ We see from (e.5) that the connectivity of $\tau_n(p) : \tau_n G(W) \to \tau_n F(W)$ is greater than or equal to the minimum of the numbers

$$\text{(connectivity of } p : G(L(k) \oplus W) \to F(L(k) \oplus W)) - \dim(C(\lambda)) - k$$

taken over all triples $(L, \lambda, k)$ with $L \in C(\lambda)$ and $\lambda : [k] \to [n+1]$. By our hypothesis on $p : G \to F$, the connectivity of $p : G(L(k) \oplus W) \to F(L(k) \oplus W)$ is at least equal to $(n+1)(\lambda_k + \dim(W)) - b$. By the inequality for $\dim(C(\lambda))$, the minimum in question is greater than $(n+1)\dim(W) - b$. \qed

**Remark.** The hypothesis in lemma e.3 is strongly reminiscent of what Goodwillie in his calculus calls *agreement to $n$–th order*, in [Go3] and (for $n = 1$) in [Go1, 1.13]. Goodwillie also has lemmas similar to e.3, such as [Go1, 1.17] and [Go3, 1.6].

We fix some $V$ in $\mathcal{J}$ from now on; the goal is to prove that $p$ from $T_n E(V)$ to $\tau_n(T_n E)(V)$ is a homotopy equivalence for any $E$ in $\mathcal{E}$.

For $W$ in $\mathcal{J}$ let $\text{mor}(V,W)$ be the space of morphisms $V \to W$ in $\mathcal{J}$ and let $\gamma_1(V,W)$ be the Riemannian vector bundle on $\text{mor}(V,W)$ whose total space is the set of $(f,x)$ in $\text{mor}(V,W) \times W$ with $x \perp \text{im}(f)$. Let $\gamma_{n+1}(V,W)$ be the Whitney sum of $n+1$ copies of $\gamma_1(V,W)$, and let $S\gamma_{n+1}(V,W)$ be the unit sphere bundle of $\gamma_{n+1}(V,W)$. We abbreviate

$$F(W) := \text{mor}(V,W)$$

$$G(W) := S\gamma_{n+1}(V,W)$$

and write $p : G \to F$ for the projection. By [W, 4.2, 5.2] the object $G$ in $\mathcal{E}$ co–represents the functor $E \mapsto \tau_n E(V)$ from $\mathcal{E}$ to spaces. In more detail, writing $\text{nat}(\ldots)$ for spaces of natural transformations, we have a commutative diagram,

$$
\begin{array}{ccc}
E(V) & \xrightarrow{\rho} & \tau_n E(V) \\
\approx & \approx & \\
\text{nat}(F,E) & \xrightarrow{p^*} & \text{nat}(G,E).
\end{array}
$$

(e.6)
Lemma. $T_n p : T_n G \to T_n F$ is an equivalence.

Proof. It is clear that $p : G \to F$ satisfies the hypothesis of lemma e.3 with $b$ equal to $(n + 1) \dim(V) + 1$. (Here $V$ is not a variable; we fixed it, and used it in the definition of $G$ and $F$.) Repeated application of lemma e.3 shows that the connectivity of

$$\tau^k_n(p) : \tau^k_n G(W) \to \tau^k_n F(W)$$

tends to infinity as $k$ goes to infinity, for any $W$ in $\mathcal{J}$. Therefore $T_n p$ is an equivalence. □

We shall use (e.7) to prove that the commutative square

$$(e.8) \quad E(V) \xrightarrow{c} T_n E(V) \xrightarrow{\rho} \tau_n E(V) \xrightarrow{\rho} \tau_n(T_n E)(V)$$

can be enlarged to a commutative diagram of the form

$$(e.9) \quad E(V) \xrightarrow{} X \xrightarrow{} T_n E(V) \xrightarrow{\rho} \tau_n E(V) \xrightarrow{\rho} \tau_n(T_n E)(V)$$

in which the map $g$ is a homotopy equivalence. (That is, (e.8) is obtained from (e.9) by deleting the middle column.) According to (e.6), diagram (e.8) is isomorphic to

$$(e.10) \quad \text{nat}(F, E) \xrightarrow{c} \text{nat}(F, T_n E) \xrightarrow{p^*} \text{nat}(G, E) \xrightarrow{c} \text{nat}(G, T_n E)$$

and clearly (e.10) can be enlarged to

$$(e.11) \quad \text{nat}(F, E) \xrightarrow{} \text{nat}(T_n F, T_n E) \xrightarrow{\text{res}} \text{nat}(F, T_n E) \xrightarrow{p^*} \text{nat}(G, E) \xrightarrow{} \text{nat}(T_n G, T_n E) \xrightarrow{\text{res}} \text{nat}(G, T_n E)$$

where the arrows labelled res are restriction maps. We are now very close to having constructed a diagram like (e.9). The idea is that since $T_n p : T_n G \to T_n F$ is
an equivalence by lemma e.7, the middle arrow in (e.11) ought to be a homotopy equivalence. Of course, it does not work exactly like that.

What is needed here is the notion of cofibrant object in \( E \) from the appendix of [W]. If \( v : A \to B \) is an equivalence in \( E \) where \( A \) and \( B \) are cofibrant, then \( v \) admits a homotopy inverse \( u : B \to A \), with (natural) homotopies relating \( vu \) and \( uv \) to the respective identity maps. Every object in \( E \) is the codomain of an equivalence whose domain is a so-called CW–object [W, A.4], and CW–objects are cofibrant [W, A.3]. More generally, every morphism \( w : C \to D \) in \( E \) has a factorization

\[
C \hookrightarrow D^\circ \to D
\]

where \( D^\circ \to D \) is an equivalence and \( D^\circ \) is a CW–object relative to \( D \). (I leave definition and proof to the reader.) This factorization can be constructed functorially in \( w : C \to D \), and if \( C \) is already cofibrant, then \( D^\circ \) will be cofibrant.

We apply this with \( w \) equal to the inclusion \( F \to T_n F \) or to the inclusion \( G \to T_n G \). It follows from (e.6) that \( F \) and \( G \) are cofibrant. Therefore \((T_n F)^\circ \) and \((T_n G)^\circ \) in the factorizations

\[
F \hookrightarrow (T_n F)^\circ \to T_n F, \quad G \hookrightarrow (T_n G)^\circ \to T_n G
\]

are cofibrant. Replacing \( T_n F \) and \( T_n G \) by \((T_n F)^\circ \) and \((T_n G)^\circ \) in (e.11) we obtain a commutative diagram

\[
\begin{array}{ccc}
nat(F, E) & \longrightarrow & nat((T_n F)^\circ, T_n E) \\
\downarrow \rho^* & & \downarrow \rho^* \\
nat(G, E) & \longrightarrow & nat((T_n G)^\circ, T_n E)
\end{array}
\]

and now the middle arrow is a homotopy equivalence. Diagram (e.12) is the explicit form or fulfillment of (e.9).

**Proof of (1) in 6.3.** We have to show that \( \rho : T_n E(V) \to \tau_n(T_n E)(V) \) is a homotopy equivalence. It is enough to show that the vertical arrows in the commutative diagram

\[
\begin{array}{ccccccc}
E(V) & \xrightarrow{\rho} & \tau_n E(V) & \xrightarrow{\rho} & \tau_n^2 E(V) & \xrightarrow{\rho} & \tau_n^3 E(V) & \xrightarrow{\rho} & \cdots \\
\downarrow \rho & & \downarrow \rho & & \downarrow \rho & & \downarrow \rho & & \cdots \\
\tau_n E(V) & \xrightarrow{\tau_n(\rho)} & \tau_n^2 E(V) & \xrightarrow{\tau_n(\rho)} & \tau_n^3 E(V) & \xrightarrow{\tau_n(\rho)} & \tau_n^4 E(V) & \xrightarrow{\tau_n(\rho)} & \cdots
\end{array}
\]

induce a map between the homotopy colimits of the rows which is a homotopy equivalence. It is enough because \( \tau_n \) commutes with homotopy colimits over \( \mathbb{N} \) up to homotopy equivalence, and because we can define \( T_n E \) as the homotopy colimit
of (e.2). Denote the homotopy colimits of the rows in (e.13) by $P$ and $Q$, and the map under investigation by $r : P \to Q$. For each $i \geq 0$ the commutative diagram

$$
\begin{array}{ccc}
\tau_i^* E(V) & \xrightarrow{\subset} & P \\
\downarrow^\rho & & \downarrow^r \\
\tau_{i+1}^* E(V) & \xrightarrow{\subset} & Q
\end{array}
$$

can be enlarged, as in (e.9) and (e.12), to a commutative diagram

$$
\begin{array}{ccc}
\tau_i^* E(V) & \longrightarrow & X \longrightarrow & P \\
\downarrow^\rho & & \downarrow & \downarrow^r \\
\tau_{i+1}^* E(V) & \longrightarrow & Y \longrightarrow & Q
\end{array}
$$

where the middle vertical arrow is a homotopy equivalence. It follows easily that $r : P \to Q$ is a homotopy equivalence. □

References


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