# IMMERSION THEORY FOR HOMOTOPY THEORISTS

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## 1. Introduction

These notes grew out of a lecture course on immersion theory, submersion theory and related $h$-principles given to graduate students at the University of Aberdeen in 2004. They are based on the viewpoint that immersion theory is applied fibration theory. A more specific aim of the course was to explain the *corrugation method* in immersion theory, advertised beautifully in the film *Outside In* [15] about the eversion of the 2-sphere.

The remainder of this introduction is a rather detailed summary of the paper, trying to explain

- what immersion theory is,
- how the proof of its main theorem reduces to showing that certain restriction maps are fibrations,
- what methods we have for showing that certain restriction maps are fibrations, and what these methods have to do with corrugation.

**Definition 1.1.** Suppose that $M^m$ and $N^n$ are smooth manifolds, $\partial N = \emptyset$. A smooth map $f\colon M \to N$ is an *immersion* if for all $x \in M$, the differential $df(x)\colon T_xM \longrightarrow T_{f(x)}N$ is an *injective* linear map.

**Remark 1.2.** Let $f\colon M \to N$ be an immersion, $x \in M$. By the implicit function theorem, there exist smooth local coordinates near $x \in M$ and $f(x) \in N$ such that, in these coordinates, $f$ has the form $(z_1, z_2, \dots, z_m) \mapsto (z_1, z_2, \dots, z_m, 0, 0, 0, \dots, 0)$.

Let $M^m$, $N^n$ be smooth manifolds, $m \le n$ and $\partial N = \emptyset$. In the case $m = n$, suppose that $M \smallsetminus \partial M$ has no compact components. Let $\mathrm{imm}(M, N)$ be the space of smooth immersions from $M$ to $N$ (details later). Let $\mathrm{fimm}(M, N)$ be the space of "formal" immersions, i.e. the space of pairs $(f, \delta f)$ where $f\colon M \to N$ is continuous (not necessarily smooth) and $\delta f$ is *some* vector bundle map $TM \to f^*TN$ which is injective on each fiber. (See the remark below.)

**Theorem 1.3.** (Main theorem of immersion theory; [16], [9], [8].) *In these circumstances the map* $\mathrm{imm}(M, N) \longrightarrow \mathrm{fimm}(M, N)$ *given by* $f \mapsto (f, df)$ *is a weak homotopy equivalence.*

**Remark 1.4.** The vector bundle map $\delta f\colon TM \to f^*TN$ amounts to a choice of a linear injection $T_xM \to T_{f(x)}M$ for each $x \in M$, depending continuously on $x$. Think of $\delta f$ as a "formal" total derivative for the continuous map $f$. It is not required to agree with the honest derivative $df$ of $f$ and $f$ may not even have an honest derivative. — The map $\mathrm{imm}(M, N) \longrightarrow \mathrm{fimm}(M, N)$ given by $f \mapsto (f, df)$ is also known as 1-*jet prolongation.*

**Remark 1.5.** If $M$ is compact, then the weak homotopy equivalence of theorem 1.3 is a genuine homotopy equivalence. Indeed, both $\mathrm{imm}(M,N)$ and $\mathrm{fimm}(M,N)$ are then homotopy equivalent to CW-spaces. This follows from [14].

**Example 1.6.** Take $M = S^1$ and $N = \mathbb{R}^2$. Since $\mathbb{R}^2$ is contractible, $\mathrm{fimm}(S^1,\mathbb{R}^2)$ is homotopy equivalent to the space of vector bundle embeddings from the tangent bundle $TS^1$ to a trivial vector bundle $S^1 \times \mathbb{R}^2$ on $S^1$. That in turn is homotopy equivalent to the space of maps from $S^1$ to $S^1$. So we have

$$\mathrm{imm}(S^1,\mathbb{R}^2) \simeq \mathrm{fimm}(S^1,\mathbb{R}^2) \simeq S^1 \times \mathbb{Z}\ .$$

In particular $\pi_0\mathrm{imm}(S^1,\mathbb{R}^2)$ is identified with $\mathbb{Z}$ via rotation numbers (Whitney-Graustein theorem).

**Example 1.7.** Take $M = S^{n-1}$ and $N = \mathbb{R}^n$, with $n \geq 2$. Since $\mathbb{R}^n$ is contractible, $\mathrm{fimm}(S^{n-1},\mathbb{R}^n)$ is homotopy equivalent to the space of vector bundle embeddings from the tangent bundle $TS^{n-1}$ to a trivial vector bundle $S^{n-1}\times\mathbb{R}^n$ on $S^{n-1}$. That in turn is homotopy equivalent to the space of maps from $S^{n-1}$ to $\mathrm{SO}(n)$ (using $TS^{n-1} \times \mathbb{R} \cong S^{n-1} \times \mathbb{R}^n$). So we have

$$\mathrm{imm}(S^{n-1},\mathbb{R}^n) \simeq \mathrm{SO}(n) \times \Omega^{n-1}\mathrm{SO}(n).$$

In particular $\pi_0\mathrm{imm}(S^{n-1},\mathbb{R}^n)$ is identified with $\pi_{n-1}\mathrm{SO}(n)$. If $n = 3$, this is a trivial group. Hence all immersions of $S^2$ in $\mathbb{R}^3$ are *regularly homotopic* (i.e., homotopic through a 1-parameter family of immersions). In particular the inclusion $S^2 \to \mathbb{R}^3$ is regularly homotopic to the composition of the same inclusion with any orientation-reversing linear isomorphism $\mathbb{R}^3 \to \mathbb{R}^3$. In that sense, an eversion of the 2-sphere in $\mathbb{R}^3$ is possible (and that is what the film *Outside In* illustrates).

The main theorem of immersion theory has a relatively easy and well-known (to experts) reduction to a fibration statement which we now formulate. Let $A^p \subset D^p$ be the $p$-dimensional annulus obtained by removing an open ball of radius 1/2 about the origin from the standard disk $D^p$ of radius 1.

**Proposition 1.8.** *For smooth $N^n$ without boundary and integers $p, q \geq 0$ with $p < n$ and $p + q = n$, the restriction map* $\mathrm{imm}(D^p \times D^q, N) \to \mathrm{imm}(A^p \times D^q, N)$ *is a Serre fibration.*

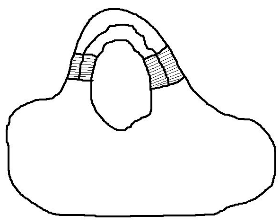
Here is a brief explanation of how this is used. First of all, the case $m < n$ in theorem 1.3 has a reduction to the case where $m = n$. The idea is to replace $M$ by the total space of a suitable disk bundle on $M$. Suppose therefore that $m = n$. Let us also assume that $M$ is compact (for the general case, see section 5). Then $M$ admits a finite handle decomposition where each handle has index $< n$. It can be set up so that handles of lower index are attached before those of higher index are attached. Then there is a filtration

$$\emptyset = M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} = M$$

where $M_p \subset M$ is the handlebody made up from all handles of index $\leq p$. We now prove by induction on $p$ that theorem 1.3 holds with $M_p$ in place of $M$. To make the induction step we set up a commutative square of restriction maps

$$\begin{array}{ccc} \mathrm{imm}(M_p, N) & \longrightarrow & \mathrm{imm}(M_{p-1}, N) \\ \downarrow & & \downarrow \\ \prod_{\lambda \in S_p} \mathrm{imm}(D^p \times D^q, N) & \longrightarrow & \prod_{\lambda \in S_p} \mathrm{imm}(A^p \times D^q, N) \end{array}$$

where $S_p$ is an indexing set for the $p$-handles, and $A^p \times D^q$ is (diffeomorphic to) the intersection of any particular $p$-handle with $M_{p-1}$. Picture of $M_p$ in the situation where $S_p$ has just one element, $p = q = 1$ and therefore $n = 2$:



[Apologies for the low quality. The shaded region is $A^p \times D^q$. Note that $M_{p-1}$ is obtained by deleting the unshaded part of the $p$-handle. It has been demoted to the status of a smooth manifold with boundary and corners in the boundary.]
It is clear from the definition of an immersion that the square is a pullback square. By proposition 1.8, the two horizontal arrows are Serre fibrations. We compare with the square of restriction maps

$$\begin{array}{ccc} \mathrm{fimm}(M_p, N) & \longrightarrow & \mathrm{fimm}(M_{p-1}, N) \\ \downarrow & & \downarrow \\ \prod_{\lambda \in S_p} \mathrm{fimm}(D^p \times D^q, N) & \longrightarrow & \prod_{\lambda \in S_p} \mathrm{fimm}(A^p \times D^q, N) \end{array}$$

which is also a pullback square in which the two horizontal maps are Serre fibrations (almost by definition). Jet prolongation gives us a (natural) map from the first square to the second. The maps

$$\begin{array}{rcl} \mathrm{imm}(M_{p-1}, N) & \longrightarrow & \mathrm{fimm}(M_{p-1}, N)\ , \\ \prod_\lambda \mathrm{imm}(D^p \times D^q, N) & \longrightarrow & \prod_\lambda \mathrm{fimm}(D^p \times D^q, N)\ , \\ \prod_\lambda \mathrm{imm}(A^p \times D^q, N) & \longrightarrow & \prod_\lambda \mathrm{fimm}(A^p \times D^q, N) \end{array}$$

are all weak homotopy equivalences. (For the first of these maps this is the inductive hypothesis. The second is a homotopy equivalence by an easy shrinking argument, lemmas 5.1 and 5.2 below. The third is again covered by our inductive hypothesis, which applies because $A^p \times D^q$ has a handle decomposition with handles of index less than $p$ only.) It follows that the jet prolongation map $\mathrm{imm}(M_p, N) \to \mathrm{fimm}(M_p, N)$ is a weak homotopy equivalence. That is the kind of conclusion which we can draw when we have a map between two pullback squares and the horizontal arrows in each pullback square are Serre fibrations. (Use for example the long exact sequence of homotopy groups of a Serre fibration and the JHC Whitehead theorem stating that a map which induces isomorphisms on all homotopy groups, for every choice of base point in the source, is a weak homotopy equivalence.)

We mention in passing that the restriction map from $\mathrm{imm}(D^p \times D^q, N)$ to $\mathrm{imm}(A^p \times D^q, N)$ can very easily be shown to be a *Serre microfibration*, i.e., has the homotopy lifting property (in the sense of Serre) for sufficiently small homotopies. See definition 2.20 and corollary 4.5 for details.

Now we turn to the much more difficult question of how proposition 1.8 can be proved. We will adopt a categorical viewpoint by regarding the elements of $\mathrm{imm}(A^p \times D^q, N)$ as objects of (something like) a category.

**Definition 1.9.** Let $f, g \in \mathrm{imm}(A^p \times D^q, N)$. A *morphism* from $f$ to $g$ is an immersion $h : S^{p-1} \times [0,3] \times D^q \to N$ such that

$$\begin{aligned} h(x,t,y) &= f(sx,y) \quad \text{for } t \in [0,1],\ s = \frac{t+1}{2}, \\ h(x,t,y) &= g(sx,y) \quad \text{for } t \in [2,3],\ s = \frac{t-1}{2}. \end{aligned}$$

It should be clear that there is some kind of composition of morphisms by concatenation. More details are given towards the end of section 4. Whatever precise formula is used to define composition, it is unlikely to be strictly associative, but it will at least be associative up to homotopy. The question of existence of identity morphisms is quite another matter. We will take that up later.

Next, we also want to regard the restriction map

$$\mathrm{imm}(D^p \times D^q, N) \to \mathrm{imm}(A^p \times D^q, N)$$

as something like a functor $\Phi$. More precisely, for each "object"

$$f \in \mathrm{imm}(A^p \times D^q, N)\ ,$$

the fiber of that restriction map over $f$ (alias space of immersions $D^p \times D^q \to N$ extending $f$) will be viewed as something like the value $\Phi(f)$ of a functor $\Phi$. Indeed, it should be clear that a morphism $h : f \to g$ induces a map $\Phi(h) : \Phi(f) \to \Phi(g)$ by concatenation with $h$.

Following is a description of the situation in an abstract setting. We assume optimistically that the so far missing "identity morphisms" can be found.

**Definition 1.10.** Let $E, B$ be spaces and let $\sigma, \tau : E \to B$ be maps. Let $E\,{}_\sigma\!\times_\tau E := \{(x,y) \in E \times E \mid \sigma(x) = \tau(y)\}$. Both $E$ and $E\,{}_\sigma\!\times_\tau E$ will be regarded as spaces over $B \times B$ using $x \mapsto (\tau(x), \sigma(x))$ for $E$, and $(x,y) \mapsto (\tau(x), \sigma(y))$ for $E\,{}_\sigma\!\times_\tau E$. A *composition structure* on $(E, B, \sigma, \tau)$ consists of

- a map $\kappa : E\,{}_\sigma\!\times_\tau E \longrightarrow E$ over $B \times B$,
- a map $\iota : B \to E$ such that $\sigma\iota = \mathrm{id}_B = \tau\iota$.

These are subject to the condition that the maps $E \to E$ given by $x \mapsto \kappa(\iota\tau(x), x)$ and $x \mapsto \kappa(x, \iota\sigma(x))$ are both homotopic to the identity, over $B \times B$.

**Remark 1.11.** The idea is that $E$ is the space of morphisms of "something like" a topological category, and $B$ is the space of objects. The maps $\sigma$ and $\tau$ are "source" and "target". The map $\iota$ is the map which to each objects assigns its identity morphism. The map $\kappa$ is "composition of morphisms". It is not required to be associative.

**Definition 1.12.** Keep the notation of definition 1.10. Let $\pi\colon Z \to B$ be some map. Let $E\,{}_\sigma{\times}_\pi\, Z = \{(x,z) \in E \times Z \mid \sigma(x) = \pi(z)\}$ and make this into a space over $B$ by $(x,z) \mapsto \tau(x)$. — An *action* of $E$ on $Z$ is a map $\alpha\colon E\,{}_\sigma{\times}_\pi\, Z \to Z$ over $B$, subject to the following conditions.

- the maps $E\,{}_\sigma{\times}_\tau\, E\,{}_\sigma{\times}_\pi\, Z \to Z$ given by $(x,y,z) \mapsto \alpha(\kappa(x,y),z)$ and $(x,y,z) \mapsto \alpha(x,\alpha(y,z))$ are fiberwise homotopic over $B$, where the reference map from $E\,{}_\sigma{\times}_\tau\, E\,{}_\sigma{\times}_\pi\, Z$ to $B$ is $(x,y,z) \mapsto \tau(x)$ ;
- the map $Z \to Z$ given by $z \mapsto \alpha(\iota(\pi(z)),z)$ for $z \in Z$ is homotopic over $B$ to the identity $\mathrm{id}_Z$.

**Remark 1.13.** The idea is that $b \mapsto Z_b = \pi^{-1}(b)$ for $b \in B$ defines something like a functor on the (almost-)category defined by $E, B, \sigma, \tau, \iota, \kappa$. In particular, for a "morphism" $e$ in $E$ with "source" $\sigma(e) = b$ and "target" $\tau(e) = c$, the "induced map" $e_* : Z_b \to Z_c$ is given by $z \mapsto \alpha(e,z)$.

The main lemma is as follows:

**Lemma 1.14.** *Keep the notation and assumptions of definition 1.12. Suppose that $(\tau,\sigma)\colon E \to B \times B$ and $\pi\colon Z \to B$ are Serre microfibrations. Then $\pi$ is actually a Serre fibration.*

For the proof of proposition 1.8, we just apply lemma 1.14, taking

- $E = \mathrm{imm}(S^{p-1} \times [0,3] \times D^q, N)$ ;
- $B = \mathrm{imm}(A^p \times D^q, N)$ ;
- $\sigma(h) = ((sx,y) \mapsto h(x,t,y))$ for $h \in E$, where $t \in [0,1]$ and $s = (t+1)/2$ ;
- $\tau(h) = ((sx,y) \mapsto h(x,t,y))$ for $h \in E$, where $t \in [2,3]$ and $s = (t-1)/2$ ;
- $Z = \mathrm{imm}(D^p \times D^q, N)$ ;
- $\pi : Z \to B$ equal to the restriction map.

The composition rule $\kappa$ and the action map $\alpha$ are both given by concatenation. They do not present a challenge. Constructing $\iota : B \to E$ is more of a challenge. But it can be met and it is at this point that corrugation (as in *corrugated cardboard*) comes in. The identity morphism $\iota(b)$ of an object $b \in B$ (an immersion from $A^p \times D^q$ to $N$) is obtained by corrugating that object. The details of that will be explained and illustrated in the proof of proposition 1.8, in section 4.

*Remark.* Revaz Kurdiani pointed out (in 2004) that definitions 1.10 and 1.12 can be formalized in the following manner. The category $\mathscr{C}$ of spaces over $B \times B$ is a *monoidal category* with monoidal operation $\square$ given by

$$E \square E' := \{(x,y) \in E \times E' \mid \sigma(x) = \tau'(y)\}$$

for spaces $E$ and $E'$ over $B \times B$, with reference maps $(\tau,\sigma)\colon E \to B \times B$ and $(\tau',\sigma')\colon E' \to B \times B$. Here $E \square E'$ is again a space over $B \times B$ with reference map $(x,y) \mapsto (\tau(x), \sigma'(y))$. There is a two-sided *unit object* $1_{\mathscr{C}}$ for the monoidal operation, given by the diagonal $B \to B \times B$, viewed as a space over $B \times B$.
Next, the category $\mathscr{C}$ with the above monoidal operation *acts* on the category $\mathscr{D}$ of spaces over $B$ by

$$E \square Z := \{(x,z) \in E \times Z \mid \sigma(x) = \pi(z)\}$$

for spaces $E$ over $B \times B$ and $Z$ over $B$, with reference maps $(\tau,\sigma)\colon E \to B \times B$ and $\pi\colon Z \to B$. Here $E \square Z$ is meant to be a new object of $\mathscr{D}$ with reference map $(x,z) \mapsto \tau(x)$.

In the categories $\mathscr{C}$ and $\mathscr{D}$, there are notions of homotopy between morphisms. (I hear the category enthusiasts shouting at me that $\mathscr{C}$ and $\mathscr{D}$ are categories enriched over simplicial sets.) What we see in definitions 1.10 and 1.12 can therefore be reformulated as follows.

- We have an object $E$ in $\mathscr{C}$ , a morphism $\kappa\colon E\,\Box\, E \to E$ and a morphism $\iota$ from $1_{\mathscr{C}}$ to $E$.
- The morphism obtained by composing
$$E \;\cong 1_{\mathscr{C}}\Box\, E \xrightarrow{\;\iota\,\Box\,\mathrm{id}_E\;} E\,\Box\, E \xrightarrow{\;\kappa\;} E$$
is homotopic to $\mathrm{id}_E$ , and the morphism obtained by composing
$$E \;\cong\; E\,\Box\, 1_{\mathscr{C}} \xrightarrow{\;\mathrm{id}_E\,\Box\,\iota\;} E\,\Box\, E \xrightarrow{\;\kappa\;} E$$
is homotopic to $\mathrm{id}_E$ .
- We have an object $Z$ in $\mathscr{D}$ and a morphism $\alpha\colon E\Box Z \to Z$ in $\mathscr{D}$.
- The morphisms from $E\Box E\Box Z$ to $Z$ given by composing, respectively,
$$E\,\Box\,(E\,\Box\, Z) \xrightarrow{\;\mathrm{id}_E\,\Box\,\alpha\;} E\,\Box\, Z \xrightarrow{\;\alpha\;} Z$$
$$(E\,\Box\, E)\,\Box\, Z \xrightarrow{\;\kappa\,\Box\,\mathrm{id}_Z\;} E\,\Box\, Z \xrightarrow{\;\alpha\;} Z$$
are homotopic.
- The morphism $Z \to Z$ given by composing
$$Z \cong 1_{\mathscr{C}} \xrightarrow{\;\iota\,\Box\,\mathrm{id}_Z\;} E\,\Box\, Z \xrightarrow{\;\alpha\;} Z$$
is homotopic to $\mathrm{id}_Z$.

## 2. Fibrations and related notions

In this section and the next, the main results from fibration theory which we will need are collected.

**Definition 2.1.** A map $p\colon E \to B$ has the *homotopy lifting property*, HLP (also known as *covering homotopy property*), if the following holds. Given any space $X$ and (continuous) maps
$$f\colon X \times [0,1] \to B, \qquad \bar{f}_0\colon X \to E$$
such that $p\bar{f}_0(x) = f(x,0)$ for all $x \in X$, there exists a map $\bar{f}\colon X \times [0,1] \to E$ such that $p\bar{f} = f$ and $\bar{f}(x,0) = \bar{f}_0(x)$ for all $x \in X$.
If $p$ has the HLP, it is called a fibration.

*More vocabulary.* It is also common to say *Hurewicz fibration* in the above circumstances. If $p$ satisfies the above whenever $X$ is a $CW$–space, then $p$ is a *Serre fibration.*

A useful extension of the HLP is called HELP (homotopy extension lifting property), respectively micro–HELP. We formulate a special case of this for Serre fibrations. Let $X$ be any CW-space, $A \subset X$ a CW-subspace.

**Proposition 2.2.** *Let $p: E \to B$ be a Serre fibration. Let $X$ be a CW-space with a CW-subspace $A$ and let $Z = X \times \{0\} \cup A \times [0,1]$, a subspace of $X \times [0,1]$. Given maps*

$$f : X \times [0,1] \to B, \qquad \bar{f}_Z : Z \to E$$

*such that $p\bar{f}_Z(x,t) = f(x,t)$ for all $(x,t) \in Z$, there exists a map $\bar{f} : X \times [0,1] \to E$ such that $p\bar{f} = f$ and $\bar{f}(x,t) = \bar{f}_Z(x,t)$ for all $x \in Z$.*

**Remark 2.3.** The following is a very useful fact: a map $p : E \to B$ is a Serre fibration if it satisfies the HLP above in all cases where $X$ is a (compact) disk $D^n$, any $n \geq 0$. Most of the proof of that can be found in or near [17, 7.2.5]. More precisely, it is shown there that if $p: E \to B$ has the HLP in all cases where the test space is a disk $D^n$, then it has the HELP in all cases where the test pair $(X, A)$ is of the form $(D^n, S^{n-1})$. From there it is easy to deduce by induction over skeleta (exercise) that $p: E \to B$ has the HELP whenever the test pair $(X, A)$ consists of a CW-space $X$ and a CW-subspace $A$. The induction is in fact carried out in [17] in those cases where $(X, A)$ is a pair of simplicial complexes.

The concept of Serre fibration has a lot to do with the notion of a *weak homotopy equivalence*, as we shall see. A map $f: Y \to Z$ is a weak homotopy equivalence if for every CW-space $X$, the induced map $f_*: [X, Y] \to [X, Z]$ is a bijection, where $[-, -]$ denotes homotopy classes of maps. According to JHC Whitehead's theorem, there is another characterization: a map $f: Y \to Z$ is a weak homotopy equivalence if and only if, for every choice of base point $y$ in $Y$ and every $n \geq 0$, the map of homotopy groups or homotopy sets $\pi_n(Y, y) \to \pi_n(Z, f(y))$ induced by $f$ is an isomorphism (bijection of pointed sets when $n = 0$).
If $Y$ and $Z$ are themselves CW-spaces, or homotopy equivalent to CW-spaces, then $f: Y \to Z$ is a weak homotopy equivalence if and only if it is a homotopy equivalence. On the other hand it is easy to give an example of maps which are weak homotopy equivalences but not homotopy equivalences.

**Exercise 2.4.** Let $Z$ be a space which is connected but not path connected. (There is a well-known subset of $\mathbb{R}^2$, with the subspace topology, which has this property.) Let $Y$ be the space which is the topological disjoint union of the path components of $Z$. Then $Y$ agrees with $Z$ as a set, but the topologies are different; more precisely the identity $Y \to Z$ is continuous but the identity $Z \to Y$ is not. Show that the identity $Y \to Z$ is a weak homotopy equivalence but not a homotopy equivalence.

**Remark 2.5.** If $p: E \to B$ is a fibre bundle and $B$ is paracompact Hausdorff, then $p$ is a fibration. This is well known, but the proof is not easy. See Spanier [17], for example. Spanier proves more (and refers to Hurewicz and others as the originators). Let $p: E \to B$ a map where $B$ is paracompact Hausdorff. Suppose that every $b \in B$ admits a neighborhood $U$ in $B$ such that the map $p^{-1}(U) \to U$ obtained by restricting $p$ is a fibration. *Then $p$ is a fibration.* This is applicable to fiber bundles because, if $p: E \to B$ is a fiber bundle, then every $b \in B$ admits a neighborhood $U$ in $B$ such that $p^{-1}(U)$ is homeomorphic, *over* $U$, to a product $U \times F$. In this situation $p^{-1}(U) \to U$ is clearly a fibration.
We will not need this but we shall need some variants and weaker forms which will be stated later.

**Exercise 2.6.** Let $p: E \to B$ be a fibration, where $B$ is path connected. Let $x, y \in B$. Show that the spaces $p^{-1}(x)$ and $p^{-1}(y)$ are homotopy equivalent.

**Example 2.7.** The map $p\colon S^1 \to S^1$ given by $z \mapsto z^2$ (in complex number notation) is a fiber bundle, therefore a fibration. The evaluation map $\mathrm{O}(n) \to S^n$ given by $A \mapsto A\mathbf{e}_1$ is a fiber bundle (where $\mathbf{e}_1$ is the first standard basis vector). The projection from the triangle $\{(x,y) \in \mathbb{R}^2 \mid x,y \geq 0, x+y \leq 1\}$ to the interval $[0,1]$ given by $(x,y) \mapsto x$ is a fibration, but not a fiber bundle.

**Example 2.8.** Let $f\colon X \to Y$ be any (continuous) map of spaces. Following Serre, we can write $f$ as a composition $X \to X^\sharp \to Y$ where the first arrow is a homotopy equivalence and the second arrow, $X^\sharp \to Y$, is a fibration. The definition of $X^\sharp$ is

$$X^\sharp = \{(x,w) \mid x \in X,\ w\colon [0,1] \to Y,\ w(0) = f(x)\ \}.$$

In words: an element of $X^\sharp$ is a pair $(x,w)$ consisting of an $x \in X$ and a path $w$ in $Y$ starting at $f(x)$. The space $X^\sharp$ is to be topologized as a subspace of $X \times Y^{[0,1]}$ where $Y^{[0,1]}$ denotes the space of continuous maps from $[0,1]$ to $Y$, with the compact-open topology. The map $X^\sharp \to Y$ in the promised factorization of $f$ is defined by

$$(x,w) \mapsto w(1)\ .$$

It was shown by Serre and is also shown for example in Spanier [17] that $X^\sharp \to Y$ is a fibration. The Serre construction is a very basic tool in homotopy theory and does, obviously, provide many examples of fibrations.

**Example 2.9.** Let $p\colon E \to B$ be a fibration and let $g\colon A \to B$ be any map. Let $g^*E$ be the subspace of $A \times E$ consisting of $(x,y) \in A \times E$ with $g(x) = p(y)$. There is a forgetful projection $g^*E \to A$. It is again a fibration. (Exercise.) We say that $g^*E \to A$ is the fibration obtained from $p\colon E \to B$ by pullback along $g\colon A \to B$.

*Remark.* The definition of the pullback as a space is obviously quite symmetric, despite the asymmetrical designation $g^*E$ chosen above. Quite generally, suppose that $u\colon X \to Z$ and $v\colon Y \to Z$ are maps of spaces. Their *pullback* is the subspace of $X \times Y$ consisting of the pairs $(x,y)$ which satisfy $u(x) = v(y)$.

**Exercise 2.10.** (Dugundji.) Let $p\colon M \to X$ be a fibration, where $M$ is a closed nonempty manifold and $X$ is any path–connected space having more than one point. Show that $p$ is not nullhomotopic. [Hint: Suppose for a contradiction that it is. Make a homotopy lifting problem out of a nullhomotopy for $p$.]

**Exercise 2.11.** An observation related to Proposition 1.8: Show that the restriction map $\mathrm{imm}(D^p \times D^q, N) \to \mathrm{imm}(A^p \times D^q, N)$ is not a fibration when $N = \mathbb{R}$, $p = 1$ and $q = 0$.

**Exercise 2.12.** Similar to the previous exercise, but harder: Show that the restriction map $\mathrm{imm}(D^p \times D^q, N) \to \mathrm{imm}(A^p \times D^q, N)$ is not a fibration when $N = \mathbb{R}^2$, $p = 2$ and $q = 0$. [Hint: It is enough to exhibit two immersions $f, g\colon A^2 \to \mathbb{R}^2$ which are in the same connected component of $\mathrm{imm}(A^2, \mathbb{R}^2)$, in such a way that $f$ extends to an immersion $D^2 \to S^2$ whereas $g$ does not. Find conditions which ensure that an immersion $A^2 \to \mathbb{R}^2$ does not extend to an immersion $D^2 \to \mathbb{R}^2$. ]

**Definition 2.13.** A map $p\colon E \to B$ is also called a *space over* $B$. Given two spaces over $B$, say $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$, a map over $B$ from $p_1$ to $p_2$ is a map $f\colon E_1 \to E_2$ such that $p_2 f = p_1$. To be honest, the standard expression is: "... a map from $E_1$ to $E_2$ over $B$ ...". Given two maps $f, g\colon E_1 \to E_2$, both over $B$, and a homotopy $h\colon E_1 \times [0,1] \to E_2$ from $f$ to $g$, we say that $h$ is a homotopy *over* $B$ if $p_2 h(x,t) = p_1(x)$ for all $x \in E_1$ and $t \in [0,1]$. In this situation we also say that $h$ is a *vertical homotopy.*

**Definition 2.14.** Let two spaces over $B$ be given, $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$. They are *fiberwise homotopy equivalent* if there exist maps $u\colon E_1 \to E_2$, $v\colon E_2 \to E_1$ and homotopies $\alpha$ from $vu$ to $\mathrm{id}_{E_1}$, and $\beta$ from $uv$ to $\mathrm{id}_{E_2}$, such that $u$, $v$ and $\alpha$, $\beta$ are all over $B$.

**Example 2.15.** Let $B = \mathbb{R}$, $E_1 = \mathbb{R}$, $E_2 = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$, $p_1 = \mathrm{id}_{\mathbb{R}}$ and let $p_2\colon E_2 \to \mathbb{R}$ be given by $(x,y) \mapsto x$. Then $p_1$ and $p_2$ are fiberwise homotopy equivalent. Namely, define $u\colon E_1 \to E_2$ by $u(x) = (x,0)$ and $v\colon E_2 \to E_1$ by $v(x,y) = x$ and $\alpha(x,t) = x$ for $x \in E_1$, and $\beta((x,y),t) = (x,(1-t)y)$ for $(x,y) \in E_2$. However: $p_1$ is a fibration and $p_2$ isn't. (The path $[0,1] \to B$ given by $t \mapsto t$ cannot be lifted to a path in $E_2$ with prescribed initial position $(0,1)$, for example.)

**Corollary 2.16.** *The HLP is not a fibrewise homotopy invariant.*

**Definition 2.17.** [3]. Let $p\colon E \to B$ be a map. We say that $p$ has the *weak homotopy lifting property*, WHLP, if for every space $X$ and maps

$$f\colon X \times [0,1] \to B, \qquad \bar{f}_0\colon X \to E$$

such that $p\bar{f}_0(x) = f(x,0)$ for all $x \in X$, there exists a map $\bar{f}\colon X \times [0,1] \to E$ such that $p\bar{f} = f$ and the map $x \mapsto \bar{f}(x,0)$ from $X$ to $E$ is *vertically* homotopic to $\bar{f}_0$. In that situation, the map $p$ is called a *weak fibration*.

**Proposition 2.18.** *Suppose that $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$ are fiberwise homotopy equivalent. If $p_1$ has the WHLP, then so does $p_2$.*

*Proof.* For each $s \in [0,1]$ let $\iota_s\colon X \to X \times [0,1]$ be given by $\iota_s(x) = (x,s)$. Let $u, v, \alpha, \beta$ be maps and homotopies as in definition 2.14. Let $f\colon X \times [0,1] \to B$ and $\bar{f}_0\colon X \to E_2$ be given such that $p_2\bar{f}_0(x) = f(x,0)$ for all $x \in X$. Together, $f$ and $\bar{f}_0$ make up a homotopy lifting problem for $p_2$. Then $f$ together with $v\bar{f}_0$ constitute a homotopy lifting problem for $p_1$. Since $p_1$ has the WHLP, there exists $\bar{f}\colon X \times [0,1] \to E_1$ such that $p_1\bar{f} = f$ and $\bar{f} \circ \iota_0$ is vertically homotopic to $v\bar{f}_0$. Then $u\bar{f}\colon X \times [0,1] \to E_2$ is a homotopy such that $p_2 \circ (u\bar{f}) = f$ and $u\bar{f} \circ \iota_0$ is vertically homotopic to $u \circ (v\bar{f}_0) = (uv) \circ \bar{f}_0$, which is vertically homotopic to $\bar{f}_0$. This homotopy solves the homotopy lifting problem for $p_2$ that we started with, in the weak sense of the WHLP. ☐

**Proposition 2.19.** *Let $p\colon E \to B$ be a Serre fibration. Let $b \in B$ and let $F = p^{-1}(b) \subset E$. Then for any choice of base point $c \in F$, and any $n \geq 0$, we have $\pi_n(E,F,c) \cong \pi_n(B,b)$ (isomorphism induced by $p$).*

This was originally proved by Serre for Serre fibrations. There is a proof in Spanier's book [17], but be warned that Spanier uses the expression "weak fibration" for a Serre fibration !

**Definition 2.20.** A map $p\colon E \to B$ is a *microfibration* (has the micro-HLP) if, for every $f\colon X \times [0,1] \to B$ and $\bar{f}_0\colon X \to E$ with $p\bar{f}_0 = f\iota_0$, there exist a neighborhood $U$ of $X \times \{0\}$ and $\bar{f}\colon U \to E$ such that $p\bar{f} = f|U$ and $\bar{f}\iota_0 = \bar{f}_0$.

There is also a notion of *Serre microfibration*: this is a map $p\colon E \to B$ which has the micro-HLP whenever the test space is a CW-space.

**Exercise 2.21.** Show that if $p\colon E \to B$ is a fibration and $V \subset E$ is open, then $p|V$ is a microfibration.

**Example 2.22.** Let $p\colon \mathbb{R}^2 \to \mathbb{R}$ be the the standard projection given by $(x,y) \mapsto x$. This is obviously a (trivial) fibration. Let $V = \mathbb{R}^2 \smallsetminus \{(0,0)\}$. This is open in $\mathbb{R}^2$, so $p|V$ is a microfibration. But $p|V$ is not a fibration, since the fiber of $p|V$ over $0 \in \mathbb{R}$ is not homotopy equivalent to the fiber of $p|V$ over $1 \in \mathbb{R}$.

**Exercise 2.23.** Let $M, N$ be smooth manifolds and let $f\colon M \to N$ be a smooth submersion (i.e., for every $x \in M$ the differential $df(x)\colon T_xM \to T_{f(x)}M$ is surjective). Show that $f$ is a Serre microfibration.

**Lemma 2.24.** A map $p\colon E \to B$ which is a weak fibration and a Serre microfibration is a Serre fibration.

*Proof.* First we discuss the case where the test space is a point. Let $w\colon [0,1] \to B$ and $\bar{w}_0 \in E$ be given, with $p(\bar{w}_0) = w(0) \in B$. Using the WHLP we obtain

$$v\colon [0,1] \to E$$

with $p \circ v = w$, and a path $u\colon [0,1] \to p^{-1}(w(0)) \subset E$ such that $u(0) = v(0)$ and $u(1) = \bar{w}_0$. For $\delta \in [0,1]$ let

$$L_\delta = \{(x,y) \in [0,1]^2 \mid xy = 0,\ x \le \delta\}\ .$$

Together $u$ and $v$ define a map

$$q\colon L_\delta \to E$$

where $q(x,y) = u(y)$ if $x = 0$ while $q(x,y) = v(x)$ if $y = 0$. Then $p \circ q$ is the map $(x,y) \mapsto w(x)$. Let

$$h\colon [0,\varepsilon_1] \times L_\delta \longrightarrow B$$

be given by $h(s,x,y) = w(x+s)$, taking $\varepsilon_1 = \delta = 1/2$, say. Together, $h$ and $q$ make a homotopy lifting problem which we can micro-solve using the micro-HLP. We obtain

$$H\colon [0,\varepsilon_2] \times L_\delta \longrightarrow E$$

so that $p \circ H = h$ where applicable, and $H$ restricted to $\{0\} \times L_\delta$ agrees with $q$. Now we make a map

$$\Phi\colon [0,\varepsilon_2] \times [0,1] \longrightarrow [0,\varepsilon_2] \times L_\delta$$

using the formula $(s,t) \mapsto (s,0,t-s)$ when $t \ge s$, and $(s,t) \mapsto (t, s-t, 0)$ when $t \le s$. Then

$$(p \circ H \circ \Phi)(x,y) = (h \circ \Phi)(x,y) = w(x) \in B$$

where applicable, and also $(H \circ \Phi)(x,y) = q(x,y) \in E$ if in addition $xy = 0$. Therefore the continuous map $W\colon [0,1] \to E$ defined by

$$W(s) = \begin{cases} H(\Phi(s, 1 - s/\varepsilon_2)) & \text{if} \quad s \in [0,\varepsilon_2] \\ q(s,0) = v(s) & \text{if} \quad s \in [\varepsilon_2, 1] \end{cases}$$

satisfies $p \circ W = w$ and $W(0) = q(0,1) = u(1) = \bar{w}_0$ . It is a solution of our path lifting problem consisting of $w$ and $\bar{w}_0$ .

The general case where the test space is a disk $D^k$ is very similar. Start with $w\colon [0,1] \times D^k \to B$ and $\bar{w}_0\colon D^k \to E$, where $p(\bar{w}_0(z)) = w(0,z)$ for all $z \in D^k$. Using the WHLP we obtain $v\colon [0,1] \times D^k \to E$ with $p \circ v = w$, and a homotopy $u\colon [0,1] \times D^k \to E$ such that $u(0,z) = v(0,z)$ and $u(1,z) = \bar{w}_0(z)$ for all $z \in D^k$, and $p(u(t,z)) = \bar{w}_0(z)$ for all $z \in D^k$. Together $u$ and $v$ define a map

$$q\colon L_\delta \times D^k \to E\ .$$

Define $h\colon [0,\varepsilon_1]\times L_\delta\times D^k \longrightarrow B$ by $h(s,x,y,z)=w(x+s,z)$, taking $\varepsilon_1=\delta=1/2$, say. Together, $h$ and $q$ make a homotopy lifting problem which we can micro-solve using the micro-HLP. We obtain $H\colon [0,\varepsilon_2]\times L_\delta\times D^k \longrightarrow E$. Then the continuous map $W\colon [0,1]\times D^k\to E$ defined by

$$W(s,z)=\begin{cases} H((\Phi(s,1-s/\varepsilon_2),z) & \text{if} \quad s\in[0,\varepsilon_2] \\ q(s,0,z)=v(s,z) & \text{if} \quad s\in[\varepsilon_2,1]\end{cases}$$

satisfies $p\circ W=w$ and $W(0,z)=q(0,1,z)=u(1,z)=\bar{w}_0(z)$. □

**Corollary 2.25.** *Let $p\colon E\to B$ be a map where $B$ is paracompact. Suppose that*

- *$p$ is locally fiber homotopy trivial, that is, every $b\in B$ admits an open neighborhood $U_b$ in $B$ such that the restriction $p^{-1}(U_b)\to U_b$ of $p$ is fiberwise homotopy equivalent to a trivial fiber bundle;*
- *$p$ is a Serre microfibration.*

*Then $p$ is actually a Serre fibration.*

*Proof.* Let $b\in B$ and choose an open neighborhood $U_b$ so that $p^{-1}(U_b)\to U_b$ is fiberwise homotopy equivalent to a trivial fiber bundle. Then $p^{-1}(U_b)\to U_b$ is a weak fibration (has the WHLP) because the WHLP is a fiberwise homotopy invariant. Since $p$ is a Serre microfibration, $p^{-1}(U_b)\to U_b$ is also a Serre microfibration. Therefore $p^{-1}(U_b)\to U_b$ is a Serre fibration by lemma 2.24. Now a map which is locally a Serre fibration is globally a Serre fibration (reference ?) . □

## 3. Composition structures and fibrations

For the following lemma, we adopt the notation and assumptions of definition 1.10. Suppose in addition that $B$ is a disk $D^i$, and that $(\tau,\sigma)\colon E\to B\times B$ is a Serre microfibration.

**Lemma 3.1.** *Then for every $b\in B$ there exists a neighborhood $U_b$ in $B$ and maps*

- *$m_{\text{out}}\colon U_b\to E$,*
- *$m_{\text{in}}\colon U_b\to E$,*
- *$h_{\text{gen}}\colon U_b\times[0,1]\to E$,*
- *$h_{\text{spec}}\colon U_b\times[0,1]\to E$*

*such that for any $c\in U_b$ ,*

- *$m_{\text{out}}(c)$ maps to $(c,b)$ under $(\tau,\sigma)\colon E\to B\times B$,*
- *$m_{\text{in}}(c)$ maps to $(b,c)$ under $(\tau,\sigma)\colon E\to B\times B$,*
- *the path $t\mapsto h_{\text{gen}}(c,t)$ begins at $\kappa(m_{\text{out}}(c),m_{\text{in}}(c))$, ends at $\iota(c)$ and runs in the fiber of $(\tau,\sigma)\colon E\to B\times B$ over $(c,c)$;*
- *the path $t\mapsto h_{\text{spec}}(c,t)$ begins at $\kappa((m_{\text{in}}(c),m_{\text{out}}(c))$, ends at $\iota(b)$ and runs in the fiber of $(\tau,\sigma)\colon E\to B\times B$ over $(b,b)$.*

*Proof.* Assume first that $b$ is in the interior of the disk $B=D^i$. Then without loss of generality it is the center $\underline{0}$ of the disk, and the disk is the cone on a sphere $S^{i-1}$. We write

$$B\cong[0,1]\times S^{i-1}/\sim$$

where $\sim$ identifies all points of the form $(0,c)$ with $\underline{0}$. Thus the identity $B\to B$ can be regarded as a homotopy from a constant map $S^{i-1}\to B$ with value $\underline{0}$ to the inclusion $S^{i-1}\to B$. This homotopy gives us *two* homotopies,

$$(g_s\colon S^{i-1}\to B\times B)_{s\in[0,1]} \quad \text{and} \quad (h_s\colon S^{i-1}\to B\times B)_{s\in[0,1]}\ ,$$

where $g_s(c) = (sc, \underline{0})$ and and $h_s(c) = (\underline{0}, sc)$. For these homotopies, we have initial lifts $\bar{g}_0\colon S^{i-1} \to E$ and $\bar{h}_0\colon S^{i-1} \to E$ respectively, which are constant with value $\iota(\underline{0})$. (*Lift* means that if we compose on the left with the map $(\tau,\sigma)\colon E \to B \times B$, we obtain $g_0$ and $h_0$, respectively.) Using the micro–HLP for $(\tau,\sigma)\colon E \to B \times B$, we therefore get the map $m_{\text{out}}$ as a micro–lift of the homotopy $(g_s)$, and $m_{\text{in}}$ as a micro–lift of the homotopy $(h_s)$. The maps $m_{\text{out}}$, $m_{\text{in}}$ are defined on a neighborhood of $\underline{0}$ in $B$. We obtain this as a neighborhood of $\{0\} \times S^{i-1}$ in $[0,1] \times S^{i-1}$, divided out by $\{0\} \times S^{i-1}$.)

The construction of the homotopies $h_{\text{gen}}$ and $h_{\text{spec}}$ is similar, except that we need the relative HLP (alias HELP). Let

$$K = B \times [0,1], \qquad L = S^{i-1} \times [0,1]\,,$$

so that we can identify $K$ with a quotient of $[0,1] \times L$. In particular the two maps from $K$ to $B \times B$ given by $(c,t) \mapsto (c,c)$ and $(c,t) \mapsto (\underline{0},\underline{0})$ can then be regarded as two homotopies

$$(G_s\colon L \longrightarrow B \times B)_{s\in[0,1]}\,, \quad (H_s\colon L \longrightarrow B \times B)_{s\in[0,1]}$$

given by $G_s(c,t) = (sc, sc)$ and $H_s(c,t) = (\underline{0},\underline{0})$, for all $c \in S^{i-1}$. We have an initial lift for both, given by the map $L \to E$ taking $(c,t) \in L = S^{i-1} \times [0,1]$ to $\omega(t)$, where $\omega$ is a path in $E$ such that $\omega(0) = \kappa(\iota(\underline{0}), \iota(\underline{0}))$ and $\omega(1) = \iota(\underline{0})$. (The path $\omega$ is meant to run in the fiber of the map $(\tau,\sigma)\colon E \to B \times B$ over the point $(\underline{0},\underline{0})$; its existence is guaranteed by definition 1.10.) We also have micro-lifts for the restricted homotopies $(G_s|S^{i-1} \times \{0,1\})$ and $(H_s|S^{i-1} \times \{0,1\})$. These lifts can be defined by the formulae

$$(c,t) \mapsto \begin{cases} \kappa(m_{\text{out}}(sc), m_{\text{in}}(sc)) & t = 0, \text{ case of } (G_s) \\ \iota(sc) & t = 1, \text{ case of } (G_s) \\ \kappa(m_{\text{in}}(sc), m_{\text{out}}(sc)) & t = 0, \text{ case of } (H_s) \\ \iota(\underline{0}) & t = 1, \text{ case of } (H_s) \end{cases}$$

where we are assuming $s \in [0, \varepsilon_1]$ , for an $\varepsilon_1 > 0$ which we have from the earlier construction of $m_{\text{out}}$ and $m_{\text{in}}$. (Note that $\iota(\underline{0}) = m_{\text{in}}(\underline{0}) = m_{\text{out}}(\underline{0})$ by construction.) From the micro–HELP of proposition 2.2, we obtain lifted (micro)–homotopies, micro–lifting $(G_s)$ and $(H_s)$. We call them $h_{\text{gen}}$ and $h_{\text{spec}}$, respectively. In more detail, we get two maps defined on $[0,\varepsilon_2] \times S^{i-1} \times [0,1]$ for some $\varepsilon_2 > 0$ which is $\le \varepsilon_1$. Their restriction to $\{0\} \times S^{i-1} \times [0,1]$ is given by $(0,c,t) \mapsto \omega(t)$, independent of $c \in S^{i-1}$. Since $B \cong [0,1] \times S^{i-1}/\sim$, we can view these maps as being defined on $U \times [0,1] \subset B \times [0,1]$, where $U$ is the closed disk of radius $\varepsilon_2$ about $\underline{0}$. In other words, our maps can be viewed as two homotopies $h_{\text{gen}}$ and $h_{\text{spec}}$ between certain maps from $U$ to $E$.

The case where $b$ is on the boundary of $B = D^i$ is very similar. We may think of $B$ as the cone on a hemisphere (closed upper half of $S^{i-1}$) and of $b = \underline{0}$ as the apex alias center of the cone. The details are left to the reader. □

*Proof of lemma 1.14.* We begin with some easy reductions. Firstly, we can reduce to the case where $B$ is a disk. Namely, suppose that $f\colon X \times [0,1] \to B$ is a homotopy (with $B$ still arbitrary) which we want to lift across $\pi\colon Z \to B$, with an initial lift $\bar{f}_0\colon X \to Z$. Since we are going for the Serre fibration property, we may assume that $X$ is a disk $D^i$. But then $B' := X \times [0,1]$ is also (homeomorphic to) a disk $D^{i+1}$. We can now use $f\colon B' \to B$ to pull the entire homotopy lifting problem and the composition structure and action data back to $B'$. Thus we replace $B$ by $B'$ and $Z$ by

$$Z' = f^*Z = \{(c, z) \in B' \times Z \mid f(c) = \pi(z)\}$$

and $E$ by $E' = \{(e, c, d) \in E \times B' \times B' \mid \tau(e) = f(c),\, \sigma(e) = f(d)\}$. The homotopy $f$ itself can be replaced by the identity $X \times [0, 1] \to B'$ (but try to forget that it is an identity map) and the initial lift $\bar{f}_0$ can be replaced by the map

$$X \longrightarrow Z' \subset B' \times Z$$

whose second coordinate is $\bar{f}_0$ and whose first coordinate is the inclusion of $X \cong X \times \{0\}$ to $X \times [0, 1]$. If this new homotopy lifting problem with base space $B'$ has a solution, then that solution determines a solution for the old homotopy lifting problem: just compose with projection $Z' \to Z$.
Secondly, it is enough to show that $\pi\colon Z \to B$ is a *weak* fibration because of exercise **??**. That is what we will do, assuming that $B$ is a disk.
Thirdly, to show that $\pi\colon Z \to B$ is a weak fibration when $B$ is a disk, we only need to show that it is locally fiber homotopy trivial, because of corollary **??**.
But this last statement is almost obvious from lemma 3.1. Given $b \in B$ we choose $U_b$ as in that lemma, along with the maps and homotopies $m_{\text{out}}$, $m_{\text{in}}$, $h_{\text{gen}}$ and $h_{\text{spec}}$. Now fix some $c \in U_b$. Then we have a preferred choice of maps

$$\pi^{-1}(b) \longrightarrow \pi^{-1}(c)\,,\ \pi^{-1}(c) \longrightarrow \pi^{-1}(b)$$

given by acting on the left with $m_{\text{out}}(c)$ and $m_{\text{in}}(c)$, respectively (using the "action" $\alpha$ described in definition 1.12). These two maps are homotopy inverses of each other. Indeed, the composite maps $\pi^{-1}(c) \longrightarrow \pi^{-1}(c)$ and $\pi^{-1}(b) \longrightarrow \pi^{-1}(b)$ are homotopic to the respective identity maps by means of the homotopies given by the actions of $h_{\text{gen}}(t, c)$ on $\pi^{-1}(c)$, and $h_{\text{spec}}(t, c)$ on $\pi^{-1}(b)$, respectively, for $t \in [0, 1]$. Allowing $c$ to vary, we have a homotopy equivalence

$$\pi^{-1}(b) \times U_b \longrightarrow \pi^{-1}(U_b)$$

which is, by construction, a fiberwise homotopy equivalence over $U_b$. This completes the proof. □

## 4. Spaces of smooth maps and spaces of immersions

Our main goal in this section is to prove proposition 1.8 by deducing it from lemma 1.14. We have to start with a lengthy discussion of spaces of smooth maps and spaces of immersions.

Let $M$ and $N$ be smooth manifolds. Assume $\partial N = \emptyset$, but let's not assume that $\partial M = \emptyset$. We denote by $C^\infty(M, N)$ the set of all smooth maps from $M$ to $N$. This comes with a preferred topology, the compact–open $C^\infty$ topology. This is described in [10].

**Example 4.1.** If $M$ is compact, then $\text{imm}(M, N) \subset C^\infty(M, N)$ is an open subset. (If $M$ is noncompact, this is usually not the case. Show that $\text{imm}(\mathbb{R}^m, \mathbb{R}^n)$ is not open in $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ if $m \leq n$.)

Next we have a well-known lemma about constructing smooth functions with prescribed higher derivatives. This is essentially due to E. Borel.

**Lemma 4.2.** *Let $L$ be a smooth compact manifold. For $i = 0, 1, 2, \ldots$ let $f_i : L \to \mathbb{R}$ be smooth functions. Then there exists a smooth $F : L \times \mathbb{R} \to \mathbb{R}$ such that the $i$–th partial derivative of $F$ in the $\mathbb{R}$ direction, evaluated along $L \times \{0\} \cong L$, equals $f_i$.*

*Proof.* (The proof is reproduced here from [5] because it proves more than the lemma states, and we need that extra information.) There is no loss of generality in assuming that $L$ is a codimension zero compact smooth submanifold of a euclidean space $\mathbb{R}^n$. (Otherwise, embed $L$ in a euclidean space, and replace it by the total space of a normal disk bundle in the euclidean space.) The advantage which we have from that is that we can use standard notation for partial derivatives.
To begin with fix a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho(t) = 1$ for $|t| \leq \frac{1}{2}$ and $\rho(t) = 0$ for $|t| \geq 1$. Set

$$(*) \qquad F(x,t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \rho(\mu_i t) f_i(x)$$

where the (large) real numbers $\mu_i \geq 1$ are yet to be determined. We want to choose them in such a way that the series

$$(**) \qquad \sum_{i=0}^{\infty} D^{\alpha}\Big( \frac{t^i}{i!} \rho(\mu_i t) f_i(x) \Big)$$

is uniformly convergent for every multi–index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$. If that can be achieved, then $F$ is well defined and $(*)$ can be differentiated term by term, and $F$ solves our problem.
To determine the numbers $\mu_i$, write the $i$–th term in $(*)$ in the form

$$(\mu_i)^{-i} f_i(x) \cdot (i!)^{-1} (\mu_i t)^i \rho(\mu_i t) = (\mu_i)^{-i} f_i(x) \cdot \psi_i(\mu_i t)$$

where $\psi_i(t) = (i!)^{-1} t^i \rho(t)$. Next let

$$M_i = \max \left\{ D^{\alpha} \left( f_i(x) \psi_i(t) \right) \;\middle|\; (x,t) \in L \times \mathbb{R},\ |\alpha| < i \right\}.$$

(Because $\psi_i$ vanishes outside $[-1, 1]$, the maximum can be taken over $(x, t)$ in the compact set $L \times [-1, 1]$ and over the finitely many $\alpha$ which satisfy $|\alpha| < i$.) Since $\mu_i > 1$, it follows for $|\alpha| < i$ that

$$|i\text{–th element in } (**)| \leq (\mu_i)^{|\alpha|} (\mu_i)^{-i} M_i \leq M_i \mu_i^{-1}.$$

Now choose $\mu_i = \max\{1, 2^i M_i\}$. Then for fixed $\alpha$ and for any $i > |\alpha|$, the $i$–th term of $(**)$ is bounded by $2^{-i}$. □

But let's not stop there. The construction of $F$ in terms of the $f_i$ is quite explicit. The only "random" choice which we made was the choice of the function $\rho$, which really should be made once and for all at the beginning. Then the construction amounts to a map of the form

$$\prod_{i=0}^{\infty} C^{\infty}(L, \mathbb{R}) \quad \longrightarrow \quad C^{\infty}(L \times \mathbb{R}, \mathbb{R}) \quad .$$

This map is *continuous* (with the product topology in the LHS). To verify this, let's first observe that the formula for each number $\mu_i = \mu_i(f_0, f_1, f_2, \ldots)$ is continuous as a function of the variables $f_0, f_1, f_2, \ldots$. In fact it depends only on $f_i$ and that dependence can be expressed in terms of the values of $f_i$ and the partial derivatives of $f_i$ , of order $< i$. (The derivatives of $\rho$ are also involved in that expression, but only those of order $< i$. Each of them has a maximum since $\rho$ vanishes outside a compact interval.) Next we need to know that (∗∗) depends continuously on $(f_0, f_1, f_2, \dots)$ for fixed $\alpha$. Write (∗∗) as a sum

$$\sum_{i=0}^{k} D^\alpha\Big(\frac{t^i}{i!}\rho(\mu_i t) f_i(x)\Big) + \sum_{i=k+1}^{\infty} D^\alpha\Big(\frac{t^i}{i!}\rho(\mu_i t) f_i(x)\Big)$$

where $k$ is larger than $|\alpha|$. The continuous dependence of each $\mu_i$ on $(f_0, f_1, f_2, \dots)$ implies that the first of the two summands depends continuously on $(f_0, f_1, f_2, \dots)$. For the other summand, we have the bound $2^{-k} + 2^{-k-1} + 2^{-k-2} + \cdots = 2^{1-k}$, which we can make as small as we like by choosing $k$ large.
We are therefore in a position to formulate the following astonishing corollary to (the proof of) E. Borel's lemma:

**Corollary 4.3.** *The map from $C^\infty(L \times \mathbb{R}, \mathbb{R})$ to $\prod_{i \geq 0} C^\infty(L, \mathbb{R})$ given by*

$$F \mapsto \left( \frac{\partial^i F}{\partial t^i}\Big|_{t=0} \right)_{i=0,1,2,\dots}$$

*has a continuous right inverse.*

Let $M$ and $N$ be smooth as before. Let $M_0 \subset M$ be a compact codimension zero submanifold. (Boundary legislation is as follows: We require that there exist a smooth map $f : M \to \mathbb{R}$ such that $f$ and $f|\partial M$ are transverse to 0, and such that $M_0 = \{x \in M \mid f(x) \leq 0\}$. This is equivalent to saying that $M_0$ looks “locally” like $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{m-2}$ in $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^{m-2}$ , where $\mathbb{R}_+ = \{z \in \mathbb{R} \mid z \geq 0\}$. In particular $M_0$ is a manifold with corners if $M_0 \cap \partial M$ is nonempty.)

**Lemma 4.4.** *The restriction map $\rho : C^\infty(M, N) \to C^\infty(M_0, N)$ is a Serre fibration. If $N = \mathbb{R}^i$ for some $i \geq 0$, it admits a (continuous) section.*

*Proof.* For the second part, the existence of a section in the case $i = 1$ follows easily from corollary 4.3 This is left as an exercise with the following organizational hints.
*Step 1.* Reduce to the case where $M = L \times \mathbb{R}$ and $M_0 = L \times [0, \infty)$ for a compact smooth manifold with boundary $L$. This reduction will not be used heavily until later in step 4.
*Step 2.* Using corollary 4.3, show that $\rho : C^\infty(L \times \mathbb{R}, \mathbb{R}^i) \to C^\infty(L \times [0, \infty), \mathbb{R}^i)$ admits a section.
*Step 3.* Show that $\rho : C^\infty(L \times \mathbb{R}, \mathbb{R}^i) \to C^\infty(L \times [0, \infty), \mathbb{R}^i)$ is a fibration, using the section from step 2 and the vector space structure on $C^\infty(L \times \mathbb{R}, \mathbb{R}^i)$. (Indeed, any continuous linear map between topological vector spaces which admits a section is a fibration.)
*Step 4.* Using step 3 show that $\rho : C^\infty(L \times \mathbb{R}, V) \to C^\infty(L \times [0, \infty), V)$ is a fibration whenever $V$ is an open subset of $\mathbb{R}^i$. (Smooth maps $L \times \mathbb{R} \to \mathbb{R}^i$ which take $L \times [0, \infty)$ to $V$ can be precomposed with smooth maps $L \times \mathbb{R} \to L \times \mathbb{R}$ which agree with the identity on $L \times [0, \infty)$, in such a way that the composition lands in $V$.)
*Step 5.* Show that $\rho : C^\infty(L \times \mathbb{R}, N) \to C^\infty(L \times [0, \infty), N)$ is a fibration for arbitrary $N$. Without loss of generality, $N$ is a smooth submanifold of $\mathbb{R}^i$ which is also a closed subset of $\mathbb{R}^i$, and it has a tubular neighborhood $V$ in $\mathbb{R}^i$. □

**Corollary 4.5.** *The restriction map* $\mathrm{imm}(M, N) \to \mathrm{imm}(M_0, N)$ *is a Serre microfibration.*

*Proof.* See example 4.1. □

*Proof of proposition 1.8.* As explained in the introduction, this will be deduced from lemma 1.14. The appropriate interpretations of $E$, $B$, $Z$ etc. in lemma 1.14 are given in the introduction. To obtain $\kappa\colon E\,{}_\sigma\!\times_\tau E \longrightarrow E$ choose an identification of the colimit (pushout) of

$$[0,3] \xleftarrow{\;t+2 \leftarrow\!\shortmid\, t\;} [0,1] \xrightarrow{\;t\mapsto t\;} [0,3]$$

with $[0,3]$ which extends the identity on the left–hand copy of $[0,1]$ and on the right–hand copy of $[2,3]$. Similarly, to obtain the action map $\alpha\colon E\,{}_\sigma\!\times_\pi Z \to Z$ choose an appropriate identification of the colimit of

$$D^p \xleftarrow{\;(t+1)x/2 \leftarrow\!\shortmid\, (x,t)\;} S^{p-1}\times[0,1] \xrightarrow{\;(x,t)\mapsto(x,t)\;} S^{p-1}\times[0,3]$$

with $D^p$. All that is straightforward.
It remains to construct $\iota\colon B \to E$. Let $U = (\mathbb{R}^p \smallsetminus 0)\times\mathbb{R}^q$, an open neighbourhood of $A^p \times D^q$ in $\mathbb{R}^p \times \mathbb{R}^q$. The restriction map

$$r : \mathrm{imm}(U,N) \to B = \mathrm{imm}(A^p \times D^q, N)$$

admits a section, $s : B \to \mathrm{imm}(U,N)$ such that $rs = \mathrm{id}_B$. (The proof is an exercise; see remark 4.6 for instructions.) For $f \in B$ we want to define $\iota(f)$ by

$$\iota(f) = s(f)\circ v$$

where

$$v\colon S^{p-1}\times[0,3]\times D^q \longrightarrow U$$

is an immersion yet to be defined, independent of $f$. We take out a common factor $S^{p-1}$ on both sides, and so we proceed to construct an immersion

$$w\colon [0,3]\times D^q \to \mathbb{R}_+ \times \mathbb{R}^q\ .$$

Because the target $\tau$ and source $\sigma$ of $\iota(f) = s(f)\circ v$ are prescribed, $v$ is prescribed on $S^{p-1}\times[0,1]\times D^q$ and on $S^{p-1}\times[2,3]\times D^q$, and so $w$ is prescribed on $[0,1]\times D^q$ and on $[2,3]\times D^q$. Therefore we must have $w(t,y) = (s,y)$ with $s = (t+1)/2$ for $t \in [0,1]$, and $w(t,y) = w(t-2,y)$ for $t \in [2,3]$. A good solution is shown in the following picture, where $q = 1$:

The cases $q > 1$ are similar. To show that $\iota$ satisfies (for example) the last condition in definition 1.12, we make the following auxiliary choices.
1. Choice of a section $\zeta : \mathrm{imm}(D^p \times D^q, N) \to \mathrm{imm}(\mathbb{R}^p \times \mathbb{R}^q, N)$ of the restriction $\mathrm{imm}(\mathbb{R}^p \times \mathbb{R}^q, N) \to \mathrm{imm}(D^p \times D^q, N)$. Constructing $\zeta$ is an exercise very similar to the construction of the section $s$, remark 4.6.
2. For every $g \in \mathrm{imm}(D^p \times D^q, N)$, with $f = g_{|A^p\times D^q}$ , a path $\gamma_g$ in $\mathrm{imm}(U,N)$ from $\zeta(g)_{|U}$ to $s(f)$ projecting to a constant path in $\mathrm{imm}(A^p \times D^q, N)$, and depending

continuously on $g$. See instructions in remark 4.7.
Now given $g \in Z = \mathrm{imm}(D^p \times D^q, N)$ with image $\pi(g) = f \in \mathrm{imm}(A^p \times D^q, N)$ we need to construct a path in $Z$ from $\alpha(\iota(f), g)$ to $g$ which projects to a constant path in $B$. In other words, we need to construct a regular homotopy from the concatenation of $\iota(f)$ and $g$ to $g$ itself which is stationary on $A^p \times D^q$. Let $R$ be the region of $D^p \times D^q$, diffeomorphic to $A^p \times D^q$ but larger than $A^p \times D^q$, where the concatenation $\alpha(\iota(f), g)$ of $\iota(f)$ and $g$ agrees by construction with $\iota(f)$. The restriction of $\alpha(\iota(f), g) \in \mathrm{imm}(D^p \times D^q, N)$ to $R$ is therefore given by

$$s(f) \circ e$$

where $e$ is a fixed codimension zero immersion $R \to U$, not dependent on $g$ or $f$. But $s(f)$ is $\gamma_g(0)$ and so we have a path in $\mathrm{imm}(R, N)$ given by

$$t \mapsto \gamma_g(t) \circ e$$

where $t \in [0, 1]$. We lift this to a path $\omega_g$ in $Z = \mathrm{imm}(D^p \times D^q, N)$ in a trivial manner, by not changing anything on the complement of $R$. The path $\omega_g$ starts therefore with $\alpha(\iota(f), g)$ and ends with an immersion which has the form

$$\zeta(g) \circ u$$

where $u\colon D^p \times D^q \to \mathbb{R}^p \times \mathbb{R}^q$ is an immersion which no longer depends on $g$, and which agrees with the standard inclusion on $A^p \times D^q$. Finally we choose a path $\omega'$ from $u \in \mathrm{imm}(D^p \times D^q, \mathbb{R}^p \times \mathbb{R}^q)$ to the standard inclusion, projecting to the constant path in $\mathrm{imm}(A^p \times D^q, N)$. (Here we need remember that $u$ is ultimately defined in terms of $v$, so we are using a good property of $v$ which, with hindsight, explains why $v$ was defined the way it was defined.) Let $\gamma_g$ be the concatenation of paths, $\omega' * \omega_g$ . This is the solution to our problem.
The verification that $\iota$ also satisfies the remaining conditions is similar. □

**Remark 4.6.** Existence of a section $s$ for the restriction map

$$r\colon \mathrm{imm}(U, N) \to \mathrm{imm}(A^p \times D^q, N)\ ;$$

instructions.
(1) Show that the restriction $C^\infty(U, N) \to C^\infty(A^p \times D^q, N)$ admits a (continuous) section, alias right inverse. This is similar to lemma 4.4. Restrict this section to obtain a map $s_1\colon \mathrm{imm}(A^p \times D^q, N) \to C^\infty(U, N)$.
(2) Construct an isotopy of smooth embeddings $v_t\colon U \to U$, where $t \in [0, \infty)$, such that $v_0 = \mathrm{id}$ and such that $v_t(U)$ for $t \geq 1$ is contained in an $\varepsilon_t$-neighborhood of $A^p \times D^q$, where $\varepsilon_t = t^{-1}$. Also ensure $v_t \equiv \mathrm{id}$ on $A^p \times D^q$ for all $t$.
(3) Let $W \subset \mathrm{imm}(A^p \times D^q, N) \times U$ be the open subset consisting of all $(f, z)$ such that the differential of the smooth map $s_1(f) : U \to N$ at $z \in U$ is injective, alias invertible. Note that $W$ contains $\mathrm{imm}(A^p \times D^q, N) \times A^p \times D^q$. Construct a (continuous) function $\tau$ from $\mathrm{imm}(A^p \times D^q, N)$ to $[1, \infty)$ so that the set of pairs $(f, z) \in \mathrm{imm}(A^p \times D^q, N) \times U$ where the distance from $z$ to $A^p \times D^q$ is $\leq 1/\tau(f)$ is contained in $W$. Define the section $s$ by $s(f) = s_1(f) \circ v_{\tau(f)}$.

**Remark 4.7.** Existence of a homotopy $\gamma$; instructions.
The problem can be generalized as follows. Let $X$ be any paracompact space and let $\alpha, \beta\colon X \to \mathrm{imm}(U, N)$ be maps such that $r\alpha = r\beta$, where $r$ is the restriction map from $\mathrm{imm}(U, N)$ to $\mathrm{imm}(A^p \times D^q, N)$. We want to show that there exists a homotopy from $\alpha$ to $\beta$ over $\mathrm{imm}(A^p \times D^q, N)$.
(1) Show that there is an open neighborhood $W$ of $X \times (A^p \times D^q)$ in $X \times U$ so that

the equations $\alpha(x)(z) = \beta(x)(z')$ for $(x,z) \in W$ can be solved simultaneously and continuously for the unknown $z'$. This defines a map $z \mapsto z'$ from $W$ to $X \times U$, over $X$.
(2) Let $(v_t)$ be the isotopy from step (2) of remark 4.6. Construct a function $\psi : X \to [1,\infty)$ such that the image of

$$X \times U \longrightarrow X \times U \ ; \ (x,z) \mapsto (x, v_{\psi(x)}(z))$$

is contained in $W$. Here the paracompactness of $X$ should be used.
(3) Evidently $\alpha$ is homotopic over $\mathrm{imm}(A^p \times D^q, N)$ to the map taking $x \in X$ to $\alpha(x) \circ v_{\psi(x)}$. By step (1), for each $x \in X$, the immersion $\alpha(x) \circ v_{\psi(x)}$ can be written in the form $\beta(x)\circ\sigma(x)$ for some smooth immersion $\sigma(x)\colon U \to U$, depending continuously on $x \in X$.
(4) Therefore it only remains to show that the space of immersions $U \to U$ which restrict to the identity on $A^p \times D^q$ is contractible. For that, use the following procedure. For $g \in \mathrm{imm}(U,U)$, choose $t(g) \in [1,\infty)$ so large that the family

$$s \mapsto ((1-s)g + s \cdot \mathrm{id}_U) \circ v_{t(g)}$$

where $0 \le s \le 1$ defines a path in $\mathrm{imm}(U,U)$. Use the vector space structure in $\mathbb{R}^p \times \mathbb{R}^q \supset U$ to make sense of $(1-s)g + s \cdot \mathrm{id}_U$. Now we have obvious paths from $g$ to $g \circ v_{t(g)}$, and from $g \circ v_{t(g)}$ to $v_{t(g)}$, and from there back to $\mathrm{id}_U$.

## 5. Reduction to codimension zero and completion of proof

**Lemma 5.1.** *The space* $\mathrm{imm}(\mathbb{R}^m,\mathbb{R}^n)$ *is homotopy equivalent to the space of injective linear maps from* $\mathbb{R}^m$ *to* $\mathbb{R}^n$*.*

*Proof.* Clearly $\mathrm{imm}(\mathbb{R}^m,\mathbb{R}^n) \simeq \mathrm{imm}_0(\mathbb{R}^m,\mathbb{R}^n)$ where $\mathrm{imm}_0(\mathbb{R}^m,\mathbb{R}^n)$ is the subspace of $\mathrm{imm}(\mathbb{R}^m,\mathbb{R}^n)$ consisting of those immersions which take 0 to 0. Let $X$ be the space of injective linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. The inclusion of $X$ in $\mathrm{imm}_0(\mathbb{R}^m,\mathbb{R}^n)$ has a left inverse $\mathrm{imm}_0(\mathbb{R}^m,\mathbb{R}^n) \to X$ given by taking the differential at 0. It remains to show that the composition

$$\mathrm{imm}_0(\mathbb{R}^m,\mathbb{R}^n) \longrightarrow X \longrightarrow \mathrm{imm}_0(\mathbb{R}^m,\mathbb{R}^n)$$

is homotopic to the identity. An explicit homotopy $(h_t)$ is as follows. For an immersion $f \in \mathrm{imm}_0(\mathbb{R}^m,\mathbb{R}^n)$ and $x \in \mathbb{R}^m$ we define $h_t(f)(x)$ by $t^{-1}f(tx)$ provided $0 < t \le 1$, and $h_0(f)(x) = df(0)(x)$. □

The next lemma is a variant of lemma 5.1 and for $m = n$ it is also the induction beginning in the handle induction strategy described in the introduction.

**Lemma 5.2.** *For any smooth* $N$ *without boundary, the inclusion*

$$\mathrm{imm}(D^m, N) \to \mathrm{fimm}(D^m, N)$$

*is a homotopy equivalence.*

*Proof.* It is clear that $\mathrm{imm}(D^m,N) \simeq \mathrm{imm}(\mathbb{R}^m,N)$ because we have an inclusion $D^m \to \mathbb{R}^m$ and an embedding $\mathbb{R}^m \to D^m$ which induce the reciprocal homotopy equivalences. Choose an exponential map $TN \to N$, more precisely, a smooth map $e\colon TN \to N$ which agrees with the identity on the zero section and such that the restriction of $e$ to each tangent space $T_xN$ is a smooth embedding $T_xN \to N$ whose differential at $0 \in T_xN$ is the identity. We now give the remaining steps, leaving the details to the reader:

(1) Let $\mathrm{imm}'(\mathbb{R}^m, N) \subset \mathrm{imm}(\mathbb{R}^m, N)$ be the subspace consisting of those immersions $g$ for which $\mathrm{im}(g) \subset e(T_{g(0)}N)$. The inclusion of $\mathrm{imm}'(\mathbb{R}^m, N)$ in $\mathrm{imm}(\mathbb{R}^m, N)$ is a homotopy equivalence. (Prove this by pre-composing immersions $\mathbb{R}^m \to N$ with embeddings $\mathbb{R}^m \to \mathbb{R}^m$ whose image is a small neighborhood of the origin.)
(2) Let $\mathrm{imm}''(\mathbb{R}^m, N) \subset \mathrm{imm}'(\mathbb{R}^m, N)$ be the subspace consisting of those immersions $g$ which have the form $e \circ v$ for some linear injection $v\colon \mathbb{R}^m \to T_xN$, and some $x \in N$. The inclusion of $\mathrm{imm}''(\mathbb{R}^m, N)$ in $\mathrm{imm}'(\mathbb{R}^m, N)$ is a homotopy equivalence. This is a key argument and it is similar to the proof of the previous lemma.
(3) Clearly $\mathrm{imm}''(\mathbb{R}^m, N)$ can be identified with the space $X_N$ of pairs $(x, v)$ where $x \in N$ and $v\colon \mathbb{R}^m \to N$ is a linear injection. The homotopy equivalence from $\mathrm{imm}(\mathbb{R}^m, N)$ to $X_N$ can be described directly as the map which to an immersion $\mathbb{R}^m \to N$ associates its value and its first derivative at 0.
(4) The jet prolongation map $\mathrm{imm}(D^m, N) \to \mathrm{fimm}(D^m, N)$ is a map over $X_N$ and the reference map $\mathrm{fimm}(D^m, N) \to X_N$ is also, more obviously, a homotopy equivalence. □

Now we fix $M$ and $N$ as in previous sections, of dimension $m$ and $n$, respectively. Let $V \to M$ be a vector bundle, of fiber dimension $n-m$, with a Riemannian metric. Let $\bar{V}$ be the associated disk bundle. (The notation suggests that $V \subset \bar{V}$, which is correct up to diffeomorphism over $M$. On the other hand and more obviously, the disk bundle $\bar{V}$ is contained in $V$.) We are interested in (codimension zero) immersions $\bar{V} \to N$. An immersion

$$f\colon \bar{V} \to N$$

determines an immersion $g\colon M \to N$ by restriction to the zero section, and an isomorphism $\iota$ of the vector bundle $V \to M$ with the "normal bundle" of $g$. The normal bundle of the immersion $g$ is $g^*TN/\mathrm{im}(dg)$, also known as the cokernel of the differential $dg\colon TM \to g^*TN$. The isomorphism $\iota$ is simply $df$, or more precisely, what we get when we "divide"

$$T\bar{V}_{|M} \xrightarrow{df} f^*TN_{|M} = g^*TN$$

by appropriate vector subbundles (namely, the tangent bundle $TM$ in the source, and $\mathrm{im}(dg)$ in the target).
For a fixed vector bundle $V$ on $M$ as above, let $\mathrm{imm}_V(M, N)$ be the space of pairs $(g, \iota)$ where $g$ is *any* immersion $M \to N$ and $\iota$ is *any* isomorphism of $V$ with the normal bundle of $g$.

**Lemma 5.3.** *The above map* $\mathrm{imm}(\bar{V}, N) \longrightarrow \mathrm{imm}_V(M, N)$ *is a homotopy equivalence.*

*Proof.* The proof is left as an exercise with the following instructions.
(1) It is enough to show that a similarly defined map $\mathrm{imm}(V, N) \to \mathrm{imm}_V(M, N)$ is a homotopy equivalence.
(2) Let $Y$ be the space of all smooth maps $f\colon V \to N$ such that the differential $df(x)$ is invertible for every $x$ in the zero section of $V$. The inclusion $\mathrm{imm}(V, N) \to Y$ is a homotopy equivalence. (Prove this by pre-composing immersions $V \to N$ with embeddings $V \to V$ whose image is a small neighborhood of the zero section of $V$.)
(3) Choose an exponential type map $e\colon TN \to N$ as in the proof of lemma 5.2. Let $Y' \subset Y$ be the subspace consisting of those $f \in Y$ with the property that $f(V_x) \subset e(T_yN)$ for all $x \in M$, where $y = f(0_x)$ is the image of the origin of the

fiber $V_x$. The inclusion $Y' \to Y$ is a homotopy equivalence.
(4) Composition with $e$ provides a map $u$ from $\mathrm{imm}_V(M,N)$ to $Y'$. The map $u$ has an obvious left inverse. An argument by convexity, using straight line segments in $T_yN$ for every $y \in N$, shows that the left inverse is also a homotopy right inverse. □

**Corollary 5.4.** *The following is a homotopy pullback square:*

$$\begin{array}{ccc} \mathrm{imm}(\bar V, N) & \xrightarrow{\text{1-jet prolong.}} & \mathrm{fimm}(\bar V, N) \\ {\scriptstyle\text{restr.}}\downarrow & & \downarrow{\scriptstyle\text{restr.}} \\ \mathrm{imm}(M,N) & \xrightarrow{\text{1-jet prolong.}} & \mathrm{fimm}(M,N)\,. \end{array}$$

*Proof.* By the lemma, we may replace $\mathrm{imm}(\bar V, N)$ by $\mathrm{imm}_V(M,N)$. By inspection, we may also replace $\mathrm{fimm}(\bar V, N)$ by the space $\mathrm{fimm}_V(M,N)$ of triples $(f, \delta f, \iota)$ where $(f,\delta f) \in \mathrm{fimm}(M,N)$ and $\iota$ is a vector bundle isomorphism from $V$ to $\mathrm{coker}(\delta f) = f^*TN/\mathrm{im}(\delta f)$. Then our diagram turns into

$$\begin{array}{ccc} \mathrm{imm}_V(M,N) & \xrightarrow{\text{1-jet prolong.}} & \mathrm{fimm}_V(M,N) \\ {\scriptstyle\text{restr.}}\downarrow & & \downarrow{\scriptstyle\text{restr.}} \\ \mathrm{imm}(M,N) & \xrightarrow{\text{1-jet prolong.}} & \mathrm{fimm}(M,N)\,. \end{array}$$

It is a strict pullback square and it is easy to verify that the vertical arrows are fibrations. Hence it is a homotopy pullback square. □

*Reduction of theorem 1.3 to the case where $m = n$.* Suppose theorem 1.3 known in the case $m = n$. For the general case $m \le n$, choose some vector bundle $V$ on $M$ of fiber dimension $n - m$. Then, by corollary 5.4, the homotopy fiber of the jet prolongation map

$$\mathrm{imm}(M,N) \longrightarrow \mathrm{fimm}(M,N)$$

over any point in the image of the forgetful map $\mathrm{fimm}_V(M,N) \to \mathrm{fimm}(M,N)$ is weakly homotopy equivalent to a point. Since $V$ was fairly arbitrary, this means that *all* homotopy fibers of $\mathrm{imm}(M,N) \longrightarrow \mathrm{fimm}(M,N)$ are weakly homotopy equivalent to a point. □

Most of the details of the proof of theorem 1.3 outlined in the introduction have now been supplied. In the case of a noncompact $M$, we need to revise the handle induction argument (*under construction ...*)

## 6. Submersion theory and Gromov's theorem

A *submersion* is a smooth map $f\colon M \to N$ with the property that, for each $x \in M$, the differential $T_xM \to T_{f(x)}M$ is surjective. Here, as before, $N$ should be without boundary, $M$ can have a nonempty boundary, but we pay no special attention to the tangent spaces $T_x\partial M$ for $x \in \partial M$. Hence the restriction of a submersion $M \to N$ to $\partial M$ need not be a submersion.
Assuming that $M$ is compact, let $\mathrm{subm}(M,N)$ be the space of smooth submersions from $M$ to $N$, an open subspace of $C^\infty(M,N)$. Also let $\mathrm{fsubm}(M,N)$ be the space of formal submersions from $M$ to $N$. An element in $\mathrm{fsubm}(M,N)$ is a pair $(f, \delta f)$

where $f\colon M \to N$ is a continuous map and $\delta f\colon TM \to f^*TN$ is a vector bundle surjection. There is a jet prolongation map

$$\mathrm{subm}(M,N) \to \mathrm{fsubm}(M,N)$$

given by $f \mapsto (f, df)$.

**Theorem 6.1.** *Let $M$ and $N$ be smooth manifolds, with $\partial N = \emptyset$. Assume that $M \smallsetminus \partial M$ has no compact component. Then the jet prolongation map*

$$\mathrm{subm}(M,N) \to \mathrm{fsubm}(M,N)$$

*is a weak homotopy equivalence.*

*Remark.* This is uninteresting if $\dim(M) < \dim(N)$, because then both $\mathrm{subm}(M,N)$ and $\mathrm{fsubm}(M,N)$ are empty. If $\dim(L) = \dim(N)$, we recover the special case $m = n$ of theorem 1.3.

The proof of theorem 6.1 is very similar to the proof of theorem 1.3. Nevertheless it is worth highlighting a few points.
First of all, if $M$ is a compact smooth manifold, then $\mathrm{subm}(M,N)$ is an open subspace of $C^\infty(M,N)$. Hence the analogue of corollary 4.5 for submersions is valid. We need this, of course, in order to have access to lemma 1.14. Then we can prove the analogue of proposition 1.8 for submersions. The statement is that the restriction map

$$\mathrm{subm}(D^p \times D^q, N) \longrightarrow \mathrm{subm}(A^p \times D^q, N)$$

is a Serre fibration. Here $q > 0$, and we can assume $p + q \geq n$. For the proof, let $Z = \mathrm{subm}(D^p \times D^q, N)$ and $B = \mathrm{subm}(A^p \times D^q, N)$, with $\pi\colon Z \to B$ equal to the restriction map, and

$$E \quad = \quad \mathrm{subm}(S^{p-1} \times [0,3] \times D^q, N)\,.$$

In short, replace *immersion* by *submersion* wherever the opportunity arises. However, one point should be emphasized. When you reach the passage ... *for $f \in B$ we want to define $\iota(f)$ by a formula of type*

$$\iota(f) = s(f) \circ v$$

*where $v\colon S^{k-1} \times [0,3] \times D^{\ell-k} \longrightarrow U$ is an immersion yet to be determined ...* then you may or may not replace the word *immersion* by *submersion*. It does not matter because we are talking about a codimension zero situation. Thus, the proof of the analogue of proposition 1.8 in the submersion setting ($p + q \geq n$) involves not only submersions where the dimension drops from $p + q$ to $n$, but also submersions/immersions of codimension zero (dimension $p + q$ mapping to dimension $p + q$). □

There is an even more general result which can be proved with exactly the same arguments. Fix a "target" manifold $N$ and integers $m, k > 0$ and a subset $\mathfrak{W}$ of the jet space $J^k(\mathbb{R}^m, N)$. Suppose that

(i) $\mathfrak{W}$ is open in $J^k(\mathbb{R}^m, N)$
(ii) $\mathfrak{W}$ is invariant under "local diffeomorphisms" of $\mathbb{R}^m$. [This means that if you have a diffeomorphism $\varphi\colon U \to V$ between open subsets of $\mathbb{R}^m$, and some jet $s \in J^k(V,N) \cap \mathfrak{W}$, then $s \circ \varphi \in J^k(U,N) \cap \mathfrak{W}$.]

Now suppose that $M$ is any smooth $m$–manifold, possibly with boundary. Let $\mathfrak{W}_M \subset J^k(M,N)$ consist of all the jets which, in local coordinate charts about their source in $M$, belong to $\mathfrak{W} \subset J^k(\mathbb{R}^m,\mathbb{R}^n)$. [Because of the diffeomorphism invariance condition, it does not matter how you choose the coordinate charts. But see the remark just below.] Let $C^\infty_{\mathfrak{W}}(M,N)$ consist of all the smooth maps from $M$ to $N$ whose $k$–jets at any point $x \in M$ belong to $\mathfrak{W}_M$. Let $\Gamma(\mathfrak{W}_M)$ be the space of continuous sections (with the compact–open $C^0$ topology) of the bundle projection

$$\mathfrak{W}_M \longrightarrow M\,.$$

*Remark.* Officially, an element of $J^k(\mathbb{R}^m,\mathbb{R}^n)$ is represented by a triple $(x,U,f)$ where $x \in \mathbb{R}^m$ and $f$ is a smooth map from a neighborhood $U$ of $x$ to $\mathbb{R}^n$. Unofficially, however, an element of $J^k(\mathbb{R}^m,\mathbb{R}^n)$ is a collection of numbers, one for each possible "mixed partial derivative" $\partial^\alpha/\partial x_\alpha$ where $|\alpha| \le k$. With the second definition, it is easy to represent elements of $J^k(\mathbb{R}^m,\mathbb{R}^n)$ by "slightly less" than the above — for example, a triple $(x,U,f)$ where $x$ belongs to a hyperplane $\mathbb{R}^{m-1}\times\{0\} \subset \mathbb{R}^m$ and $U$ is a neighborhood of $x$ in $\mathbb{R}^{m-1}\times[0,\infty[$, and $f\colon U \to \mathbb{R}^n$ is smooth. Hence a smooth map $M \to N$ has well defined $k$–jets at any point of $M$, even at points in $\partial M$.

**Theorem 6.2** (Gromov)**.** *Suppose that $M \smallsetminus \partial M$ has no compact component. Then the jet prolongation map $C^\infty_{\mathfrak{W}}(M,N) \longrightarrow \Gamma(\mathfrak{W}_M)$ is a weak homotopy equivalence.*

*Proof.* Once you have unravelled the meaning, you will see that it can be proved exactly like the submersion theorem. The following is important: Given

$$f \in C^\infty_{\mathfrak{W}}(M,N)$$

and given any codimension zero immersion/submersion $v\colon L \to M$, the composition $f \circ v$ belongs to $C^\infty_{\mathfrak{W}}(L,N)$. The reason is that codimension zero immersions/submersions are locally diffeomorphic (inverse function theorem !) and our assumptions on $\mathfrak{W}$ include some diffeomorphism invariance. ☐

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