Topology 2: CW-spaces, cohomology and many products

Michael Weiss
Mathematisches Institut, Universität Münster

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CHAPTER 1

CW-spaces

1.1. CW-Spaces: definition and examples

CW-spaces are generalizations of simplicial complexes and geometric realizations of semi-simplicial sets (see Lecture notes WS13-14). To be more precise: a simplicial complex is a topological space $|V|_{\mathcal{S}}$ which has been obtained from a vertex scheme $(V, \mathcal{S})$, and a semi-simplicial set X has a geometric realization $|X|$ which is a topological space. Both $|V|_{\mathcal{S}}$ and $|X|$ have the additional structure that they need in order to qualify as CW-spaces.
In describing a CW-space, we do not begin with combinatorial data in order to make a space out of them. We begin with a space and we put additional structure on it by specifying an increasing sequence of subspaces. The definition is a great achievement due to J.H.C. Whitehead (probably 1949).

Definition 1.1.1. A *CW-space* is a space X together with an increasing sequence of subspaces

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset X^2 \subset X^3 \subset \dots$$

subject to the following conditions.

(1) $X = \bigcup_{n \geq -1} X^n$ and a subset A of X is closed if and only if $A \cap X^n$ is closed in X^n for all n.
(2) For every $n \geq 0$ there *exists* a pushout square of spaces (see remark 1.1.2)

$$\begin{tikzcd} \coprod_{\lambda \in \Lambda_n} S^{n-1} \arrow[r, "\text{incl.}"] \arrow[d] & \coprod_{\lambda \in \Lambda_n} D^n \arrow[d] \\ X^{n-1} \arrow[r, "\text{incl.}"] & X^n \end{tikzcd}$$

where Λ_n is a set (and D^n, S^{n-1} are unit disk and unit sphere in $\mathbb{R}^n$, respectively).

Let us unravel this and derive some of the easier consequences.

– Condition (2) implies that X^{n-1} is a closed subspace of X^n.
– Using that, we can deduce from condition (1) that X^n is a closed subspace of X.
– Points are closed in X. Indeed, for $z \in X$, determine n such that $z \in X^n \smallsetminus X^{n-1}$. In the notation of condition (2), the preimage of $\{z\}$ under

$$\coprod_{\lambda \in \Lambda_n} D^n \to X^n$$

is a single point (therefore a closed subset) and the preimage of $\{z\}$ under the inclusion $X^{n-1} \hookrightarrow X^n$ is empty (therefore a closed subset). Since X^n is the topological quotient of

$$X^{n-1} \amalg \Big(\coprod_{\lambda \in \Lambda_n} D^n \Big)$$

under these maps, it follows that $\{z\}$ is closed in X^n. We know already that X^n is closed in X; therefore $\{z\}$ is closed in X.

- X is a normal space (points are closed and disjoint closed sets have disjoint open neighborhoods) and therefore also Hausdorff. Sketch proof: let A_1 and A_2 be disjoint closed subsets of X. Inductively, construct disjoint open neighborhoods $U_{1,n}$ and $U_{2,n}$ in X^n of $A_1 \cap X^n$ and $A_2 \cap X^n$, respectively. Do this in such a way that $U_{1,n-1} = U_{1,n} \cap X^{n-1}$ and $U_{2,n-1} = U_{2,n} \cap X^{n-1}$. Then by condition (1), the sets $U_1 := \bigcup_n U_{1,n}$ and $U_2 := \bigcup_n U_{1,n}$ are open in X and they are disjoint neighborhoods of A_1 and A_2, respectively.
- A subset Y of X is closed in X if and only if its intersection with every compact subset C of X is closed in C. (This property has a name: *compactly generated*.) Proof: one direction is trivial. For the other, suppose that $Y \cap C$ is closed in C for every compact subset C of X. It suffices to show that $Y \cap X^n$ is closed in X^n, for every n. We proceed by induction on n. For the induction step, assume that $Y \cap X^{n-1}$ is closed in X^{n-1}. Choose a pushout square as in condition (2). The intersection of Y with the image of each copy of D^n under the right-hand vertical arrow is closed in that image, by assumption. Therefore the preimage of $Y \cap X^n$ is closed in $\Lambda_n \times D^n$. Therefore $Y \cap X^n$ is closed in X^n by the definition of *pushout square*.
- Condition (2) implies that $X^n \smallsetminus X^{n-1}$, which is open in X^n, is homeomorphic (with the subspace topology) to $\Lambda_n \times (D^n \smallsetminus S^{n-1})$, or equivalently to $\Lambda_n \times \mathbb{R}^n$. In other words $X^n \smallsetminus X^{n-1}$ is homeomorphic to a disjoint union of copies of $\mathbb{R}^n$. These copies of $\mathbb{R}^n$ are well-defined subspaces of X because they are also the connected components of $X^n \smallsetminus X^{n-1}$. They are called the *$n$-cells* of X. Thus the n-cells of X are homeomorphic to $\mathbb{R}^n$. No specific homeomorphism with $\mathbb{R}^n$ is provided. The vertical arrows in the square of (2) are not *given* as part of the structure of a CW-space, they only *exist*.
- Let S be a subset of X such that the intersection of S with every cell of X is a finite set. Then S is a closed subset of X. Sketch proof: It is enough to show that $S \cap X^n$ is closed in X^n for all n. We proceed by induction on n; so assume for the induction step that $S \cap X^{n-1}$ is closed in X^{n-1}. Now $S \cap X^n$ is the union of $S \cap X^{n-1}$, which is closed in X^{n-1} and therefore closed in X^n, and a subset T of $X^n \smallsetminus X^{n-1}$ which has finite intersection with every n-cell. By condition (2), the set T is closed in X^n.
- Let S be a subset of X such that the intersection of S with every cell of X is a finite set. Then S is discrete with the subspace topology. Proof: Every subset of S is closed in X (by the same reasoning that we applied to S) and therefore closed in S.
- Let C be a compact subspace of X. Then C is contained in a union of finitely many cells of X. Proof: Suppose not. Then there is an infinite subset S of C such that S has at most one point in common with each cell. We know already that S is closed in X and discrete. Therefore S is closed in C and discrete. Therefore S is compact, discrete and infinite, contradiction.

- The closure in X of every cell of X is contained in a finite union of cells. Proof: condition (2) implies that the closure of every n-cell is compact in X^n, being equal to the image of a continuous map from D^n to the Hausdorff space X^n. Therefore it is compact in X and so (by the previous results) it is contained in a finite union of cells.
- Every compact subspace of X (and in particular the closure of any cell in X) is contained in a compact subspace of X which is a finite union of cells. Proof: by the previous it suffices to show that any n-cell E of X is contained in a compact subspace of X which is a finite union of cells. The closure $\bar{E}$ of E in X is compact and therefore contained in a finite union of cells. These cells might be called $E = E_0$, $E_1, E_2, \ldots, E_k$ (where the indexing has nothing to do with their dimension). But we know that $\bar{E} \smallsetminus E$ is contained in X^{n-1} by condition (2). Therefore cells $E_1, E_2, \ldots, E_k$ have dimension $< n$. By inductive assumption (yes, we are doing an induction on n) each E_i where $i = 1, 2, \ldots, k$ is contained in a compact subspace C_k of X which is a finite union of cells of X. Take the union K of $C_1 \cup C_2 \cup \ldots C_k$ and $\bar{E}$, which is the same as the union of $C_1 \cup C_2 \cup \ldots C_k$ and E. Therefore K is compact and it is a finite union of cells of X.

According to Whitehead himself, the letters C and W in CW-space are for *weak topology*, expressed in condition (1), and *closure finiteness*, as in: *the closure of every cell is contained in a finite union of cells*. But perhaps he meant a selection of initials from his full name John Henry Constantine Whitehead. (Against that theory, I believe his preferred first name was Henry, not Constantine.)

In a CW-space X, the subspace X^n is called the n*-skeleton* of X. If $Z \subset X$ is an n-cell, that is to say, a connected component of $X^n \smallsetminus X^{n-1}$, then by condition (2) above we know that there *exists* a continuous map

$$\varphi : D^n \to X$$

which restricts to a homeomorphism from $D^n \smallsetminus S^{n-1}$ to Z. Such a φ is called a *characteristic map* for the cell.

Remark 1.1.2. A commutative square of spaces and maps

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "u"] \\ C \arrow[r, "v"] & D \end{tikzcd}$$

is a *pushout square* if the resulting map

$$\big(B \sqcup C\big)\big/\sim \;\longrightarrow\; D$$

determined by u on B and v on C is a homeomorphism. Here "$\sim$" denotes the equivalence relation on the disjoint union $B \sqcup C$ generated by $f(x) \sim g(x)$ for all $x \in A$. (Intuitively, $f(x) \in B \subset B \sqcup C$ is glued to $g(x) \in C \subset B \sqcup C$.) In such a square, the space D and the maps u and v are in some sense completely determined by A, B, C and f, g, because D is $\big(B \sqcup C\big)\big/\sim$ up to renaming of elements, and u, v are the standard maps from B and C to that. — Note that in this situation a subset E of D is open in D if and only if $u^{-1}(E)$ is open in B and $v^{-1}(E)$ is open in C.
Also note that if $f : A \to B$ happens to be injective, then $v : C \to D$ is injective and $B \smallsetminus f(A)$ is homeomorphic to $D \smallsetminus v(C)$.

Example 1.1.3. Let $(V, \mathcal{S})$ be a vertex scheme. (So V is a set and $\mathcal{S}$ is a collection of nonempty finite subsets of V, and if T, S are nonempty finite subset of V such that $T \subset S$ and $S \in \mathcal{S}$, then $T \in s S$.) Recall that $|V|_{\mathcal{S}}$ is the set of functions $f: V \to [0,1]$ with the property that $V \smallsetminus f^{-1}(0)$ is an element of $\mathcal{S}$ and $\sum_{v \in V} f(v) = 1$. We defined a topology on that (perhaps not the one you think; see lecture notes WS13). Let $X = |V|_{\mathcal{S}}$ with that topology and let X^n consist of all the $f \in X$ such that $V \smallsetminus f^{-1}(0)$ has at most $n+1$ elements. Then X with these subspaces X^n is a CW-space. There is not much to prove here; it is almost true by the definition of $|V|_{\mathcal{S}}$. This CW-space has one n-cell for every element of $\mathcal{S}$ which has cardinality $n+1$ (as a subset of V).

Example 1.1.4. Let Y be a semi-simplicial set. Let $Y^{(n)}$ be the semi-simplicial subset of Y generated by the elements $y \in Y_k$ where $k \le n$. Then the geometric realization $|Y|$ is a CW-space with the subspace $|Y^{(n)}|$ as its n-skeleton. Again there is not much to prove here. This CW-space has one n-cell for every $z \in Y_n$.

Example 1.1.5. The sphere S^k has a structure of CW-space X where X^n is a single point for $n < k$ and $X^n = S^k$ when $n \ge k$. This CW-space has exactly two cells, one of dimension 0 and one of dimension k. (This example is also a special case of example 1.1.4.)

Example 1.1.6. From the sequence of inclusions $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^k$ and the corresponding sequence of inclusions

$$\emptyset = S^{-1} \subset S^0 \subset S^1 \subset S^2 \subset \cdots \subset S^{k-1} \subset S^k$$

we obtain another CW-structure on $X = S^k$ where $X^n = S^n$ if $n \le k$ and $X^n = S^k$ if $n \ge k$. (This example is not a special case of example 1.1.4 if $k > 1$.)

Example 1.1.7. The CW-structure on $X = S^k$ in the previous example is invariant under the antipodal involution on S^k; that is to say, the antipodal map $X \to X$ takes each skeleton X^n to itself. Therefore or (preferably) by inspection, $Y = \mathbb{R}P^k$ has a CW-structure where Y^n is $\mathbb{R}P^n$ for $n \le k$ and $Y^n = \mathbb{R}P^k$ if $n \ge k$.

Example 1.1.8. A more difficult and more interesting example of a CW-space is the Grassmannian $G_{p,q}$ of p-dimensional linear subspaces in $\mathbb{R}^{p+q}$ with the CW-structure due to Schubert. (I believe Schubert found this in the 19th century, long before CW-spaces were invented.) The Grassmannian is probably well known to you from courses on differential topology or differential geometry as a fine example of a smooth manifold. Here we are not so interested in the manifold aspect, but we need to know that $G_{p,q}$ is a topological space. Write $n = p+q$. A p-dimensional linear subspace V of $\mathbb{R}^n$ determines a linear map $\mathbb{R}^n \to \mathbb{R}^n$, orthogonal projection to V. It has the following properties: self-adjoint, idempotent, rank p. In this way, $G_{p,q}$ can be identified with the set of $n \times n$-matrices which are symmetric, idempotent and of rank p. So $G_{p,q}$ is "contained" in the finite dimensional real vector space of real $n \times n$-matrices, which has a standard topology ... and we can give it the subspace topology.
Let $E(k)$ be the linear span of the first k standard basis vectors in $\mathbb{R}^n$. So we have an increasing sequence of real vector spaces

$$0 = E(0) \subset E(1) \subset E(2) \subset \cdots \subset E(n-1) \subset E(n) = \mathbb{R}^n.$$

Now let $V \in G_{p,q}$, that is to say, V is a p-dimensional linear subspace of $\mathbb{R}^n = E(n)$. Let $f_V(k) = \dim(V \cap E(k))$ for $k = 0, 1, 2, \ldots, n$. So V determines a function f_V from $\{0, 1, 2, \ldots, n\}$ to $\{0, 1, \ldots, p\}$. The function is nondecreasing and surjective. Schubert's idea was to say: we put two elements V, W of $G_{p,q}$ in the same equivalence class if

$f_V = f_W$. Let us see whether these equivalence classes are cells and if so, what their dimensions are. So fix a nondecreasing surjective f from $\{0,1,\dots,n\}$ to $\{0,1,\dots,p\}$, and let us be interested in the set of $V \in G_{p,q}$ having $f_V = f$. Let

$$f_! : \{1,\dots,p\} \to \{1,\dots,n\}$$

be the *injective* monotone function such that $f_!(j)$ is the minimal element in $f^{-1}(j)$. Form the set A_f of real $n \times p$-matrices

$$(a_{ij})$$

where $a_{ij} = 0$ if $i > f_!(j)$, $a_{ij} = 1$ if $i = f_!(j)$, and $a_{ij} = 0$ if $i = f_!(k)$ for some $k < j$. The columns are linearly independent. So we can make a map from A_f to $G_{p,q}$ by taking the matrix (a_{ij}) to its column span. Etc. etc. ; this gives a homeomorphism from A_f to the set of $V \in G_{p,q}$ having $f_V = f$. Now clearly A_f is an affine subspace of $\mathbb{R}^{p+q}$ (translate of a linear subspace) and its dimension is

$$\sum_{j=1}^{p}\big(f_!(j) - 1 - (j-1)\big) = \sum_{j=1}^{p}(f_!(j) - j)\ .$$

Therefore we are allowed to say that the set of $V \in G_{p,q}$ having $f_V = f$ is a cell. It will be left as an exercise to show that Schubert's partition of $G_{p,q}$ into cells is in fact a structure of CW-space (where the n-skeleton, obviously, has to be the union of all cells whose dimension is at most n). There are $\binom{n}{p}$ cells in the structure; the maximum of their dimensions is

$$n + (n-1) + \cdots + (n-p+1) - (1 + 2 + \cdots + p) = p(n-p) = pq$$

and there is exactly one cell which has the maximal dimension. It corresponds to the $f : \{0,1,2,\dots,n\} \to \{0,1,2,\dots,p\}$ which has $f(x) = x - (n-p)$ for $x > n-p$ and $f(x) = 0$ otherwise.

1.2. CW-subspaces and CW quotient spaces

Proposition 1.2.1. *Let X be a CW-space and $A \subset X$ a closed subspace such that A is a union of cells of X. Then A becomes a CW-space in its own right if we define $A^n := X^n \cap A$.*

In this situation we call A a *CW-subspace* of X.

Sketch proof. There is not much to prove here. Let $Z \subset X$ be an n-cell which is contained in A. Let $\varphi_Z : D^n \to X$ be a characteristic map for Z, so that φ_Z restricts to a homeomorphism from $D^n \smallsetminus S^{n-1}$ to Z. The image of φ_Z is contained in A because it is the closure $\bar{Z}$ of Z in X, and $\bar{Z} \subset A$ because $Z \subset A$ and A is closed in X. Therefore we can write $\varphi_Z : D^n \to A$ without lying very hard. Now choose characteristic maps for all the n-cells of X, giving a pushout square

$$\begin{tikzcd} \coprod_{\lambda\in\Lambda_n} S^{n-1} \arrow[r, "\text{incl.}"] \arrow[d] & \coprod_{\lambda\in\Lambda_n} D^n \arrow[d] \\ X^{n-1} \arrow[r, "\text{incl.}"] & X^n \end{tikzcd}$$

as in definition 1.1.1. Here Λ_n is in a (chosen) bijection with the set of n-cells of X. Let $\Lambda_n' \subset \Lambda_n$ be the subset corresponding to the n-cells which are contained in A. Then by what we have just seen there is a commutative square

$$\begin{tikzcd} \coprod_{\lambda\in\Lambda'_n} S^{n-1} \arrow[r, "\text{incl.}"] \arrow[d] & \coprod_{\lambda\in\Lambda'_n} D^n \arrow[d] \\ X^{n-1}\cap A \arrow[r, "\text{incl.}"] & X^n \cap A \end{tikzcd}$$

which is obtained from the previous square by appropriate restrictions. It is easy to show that this is again a pushout square. This verifies condition (2) in definition 1.1.1 for the space A. □

PROPOSITION 1.2.2. *Under assumptions as in proposition 1.2.1, the quotient space* X/A *is also a CW-space with the definition*

$$(X/A)^n := X^n/A^n = X^n/(X^n \cap A).$$

Remark. It is wise to define the quotient space X/A as the pushout of $X \leftarrow A \to \star$ where, as usual, $\star$ denotes a singleton space and the left-hand arrow is the inclusion. This removes an ambiguity which would otherwise arise if A is empty. Namely, if A is empty, then X/A is homeomorphic to $X \sqcup \star$. (Consequently it is *not* quite correct to say that X/A is the quotient space of X by the equivalence relation which is generated by $x \sim y$ if $x, y \in A$. That statement is only correct when A is nonempty.) It follows that X/A is always a based space, i.e., it has a distinguished element or singleton subspace which we can again denote by $\star$ without lying too hard.

PROOF OF PROPOSITION 1.2.2. In the notation of the proof of proposition 1.2.1: a choice of characteristic maps for the n-cells of X gives us a pushout square

$$\begin{tikzcd} \coprod_{\lambda\in\Lambda_n} S^{n-1} \arrow[r, "\text{incl.}"] \arrow[d] & \coprod_{\lambda\in\Lambda_n} D^n \arrow[d] \\ X^{n-1} \arrow[r, "\text{incl.}"] & X^n \end{tikzcd}$$

and if $n > 0$ this determines a pushout square

$$\begin{tikzcd} \coprod_{\lambda\in\Lambda_n\smallsetminus\Lambda'_n} S^{n-1} \arrow[r, "\text{incl.}"] \arrow[d] & \coprod_{\lambda\in\Lambda_n\smallsetminus\Lambda'_n} D^n \arrow[d] \\ X^{n-1}/A^{n-1} \arrow[r, "\text{incl.}"] & X^n/A^n \end{tikzcd}$$

Here the vertical maps are obtained by using the chosen characteristic maps for the n-cells of X and composing with the quotient map $X^n \to X^n/A^n$, or $X^{n-1} \to X^{n-1}/A^{n-1}$ where appropriate. The case $n = 0$ is different: we have $(X/A)^0 = X^0/A^0 \cong \Lambda_0/\Lambda'_0$ which is *not* identifiable with $\Lambda_0 \smallsetminus \Lambda'_0$ because it has one extra element. That extra element accounts for the base point of X/A, which is a 0-cell in X/A. □

EXAMPLE 1.2.3. In the notation of example 1.1.7, the quotient space $\mathbb{R}P^k/\mathbb{R}P^n$ where $0 < n < k$ is a CW-space which has one 0-cell (base point), then one cell exactly in each of the dimensions $n+1, n+2, \ldots, k$, and no cells in other dimensions. These based spaces are called *stunted projective spaces.*

CHAPTER 2

Cellular maps and cellular homotopies

2.1. Products of CW-spaces

This is quite an educational topic. Why are we interested in it here? Because we want to say something about homotopies. In connection with that we need to know that for a CW-space Y, the product $Y \times [0,1]$ is also a CW-space in a preferred way.

LEMMA 2.1.1. (Kuratowski) *Let Y be any space and K a compact*[1] *space. Then the projection $p : Y \times K \to Y$ is a closed map, i.e., for any closed subset A of $Y \times K$ the image $p(A)$ is closed in Y.*

PROOF. Choose closed $A \subset Y \times K$. Choose $z \in Y \smallsetminus p(A)$. Then $\{z\} \times K$ has empty intersection with the closed set A in $Y \times K$. So by definition of the topology on $Y \times K$, there exist open sets $U_\lambda \subset Y$ and $V_\lambda \subset K$ (depending on an index $\lambda \in \Lambda$) such that

$$\{z\} \times K \;\subset\; \bigcup_{\lambda \in \Lambda} \big(U_\lambda \times V_\lambda\big) \;\subset\; (Y \times K) \smallsetminus A.$$

By the compactness of K, we can assume that Λ is a finite set. We can also assume $z \in U_\lambda$ for all $\lambda \in \Lambda$. Then $\bigcap_\lambda U_\lambda$ is an open neighborhood of z which has empty intersection with $p(A)$. □

PROPOSITION 2.1.2. (J.H.C. Whitehead) *Let $g : Y \to Z$ be a continuous map of spaces which is a quotient map*[2]*. Let K be a locally compact space. Then the map $f : Y \times K \to Z \times K$ defined by $(y,k) \mapsto (g(y),k)$ is also a quotient map.*

PROOF. (I copied this from an internet page, author Henning Brandsma.) First a simple fact which we will use a few times (nothing to do with topology): if $A \subset K$, $y \in Y$, $B \subset Z \times K$, then

$$(*) \qquad y \times A \;\subset\; f^{-1}(B) \;\text{ if and only if }\; g^{-1}(g(y)) \times A \;\subset\; f^{-1}(B).$$

Now, let $W \subset Z \times K$ be such that $f^{-1}(W)$ is open. We need to show that W is open. So let $(z_0, x_0) \in W$. Choose y_0 such that $g(y_0) = z_0$; can be found since g is onto. Let C be a compact neighbourhood of x_0 such that $\{y_0\} \times C \subset f^{-1}(W)$; can be found since K is locally compact.
From $(*)$ we see that in fact $g^{-1}(z_0) \times C \subset f^{-1}(W)$. Define

$$V = \{z \in Z \mid g^{-1}(z) \times C \;\subset\; f^{-1}(W)\}.$$

By the previous, z_0 belongs to V. Clearly, $V \times C \subset W$ (because g is surjective). If we can show that V is open, then $V \times C$ is a neighbourhood of (z_0, x_0) in W, showing that W is open. To show that V is indeed open, it suffices (as g is a quotient map) to show

[1]Not necessarily Hausdorff.

[2]Means: g is surjective and a subset W of Z is open if and only if $g^{-1}(W)$ is open in Y.

that $g^{-1}(V)$ is open in Y. Note that the projection map $p\colon Y\times C\to Y$ is a closed map, as C is compact (Kuratowski's lemma). Now

$$\begin{aligned} g^{-1}(V) &= \{y\in Y \mid g(y)\in V\} = \{y\in Y \mid g^{-1}(g(y))\times C\subset f^{-1}(W)\} \\ &\overset{(*)}{=} \{y\in Y \mid \{y\}\times C\subset f^{-1}(W)\} = Y\smallsetminus p\big((Y\times C)\smallsetminus f^{-1}(W)\big) \end{aligned}$$

which is open, as p is a closed map and $(Y\times C)\smallsetminus f^{-1}(W)$ is closed in $Y\times C$. So V is indeed open. ☐

Corollary 2.1.3. *If K is a locally compact space and*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is a (commutative) pushout square of spaces and continuous maps, then the resulting commutative square

$$\begin{tikzcd} A\times K \arrow[r] \arrow[d] & B\times K \arrow[d] \\ C\times K \arrow[r] & D\times K \end{tikzcd}$$

is also a pushout square.

Proof. We need to show mainly that the map from $(B\times K)\sqcup(C\times K)$ to $D\times K$ determined by the second square is a quotient map. But we know already that $B\sqcup C\to D$ determined by the first square is a quotient map. ☐

Corollary 2.1.4. *Let X be a CW-space and let Y be a locally compact CW-space. Then the product $X\times Y$, with the product topology, becomes a CW-space if we define*

$$(X\times Y)^n := \bigcup_{p+q=n} X^p\times Y^q\ .$$

Proof. Let Λ be the set of cells of X and Θ the set of cells of Y. Choose characteristic maps

$$\varphi_\lambda\colon D^{n(\lambda)}\to X\ ,\qquad \psi_\theta\colon D^{n(\theta)}\to Y$$

for the cells of X and Y. Then we have (in sloppy notation) maps

$$\varphi_\lambda\times\psi_\theta\colon D^{n(\lambda)}\times D^{n(\theta)}\longrightarrow X\times Y$$

for each pair (λ,θ). We need to show mainly that the resulting map

$$\coprod_{(\lambda,\theta)\in\Lambda\times\Theta} D^{n(\lambda)}\times D^{n(\theta)}\longrightarrow X\times Y$$

is a quotient map. (Everything else that we might want to know follows easily from that. Note in particular that $D^{n(\lambda)}\times D^{n(\theta)}$ is homeomorphic to $D^{n(\lambda)+n(\theta)}$, so we can use the maps $\varphi_\lambda\times\psi_\theta$ as characteristic maps for cells in $X\times Y$.) To show this we write that map as a composition of two:

$$\coprod_{(\lambda,\theta)\in\Lambda\times\Theta} D^{n(\lambda)}\times D^{n(\theta)}\longrightarrow \coprod_{\lambda\in\Lambda} D^{n(\lambda)}\times Y$$

and

$$\coprod_{\lambda\in\Lambda} D^{n(\lambda)}\times Y\longrightarrow X\times Y\ .$$

The first of these maps is a quotient map because for each $\lambda \in \Lambda$ the map (obtained by restriction) from $\coprod_\theta D^{n(\lambda)} \times D^{n(\theta)}$ to $D^{n(\lambda)} \times Y$ is a quotient map (by an easier special case of proposition 2.1.2). The second of these maps is also a quotient map by proposition 2.1.2. □

2.2. The homotopy extension property

LEMMA 2.2.1. *Let* X *be a CW-space and let* A *be a CW-subspace of* X. *Let* Y *be any space,* $f: X \to Y$ *a continuous map and* $(h_t : A \to Y)_{t\in[0,1]}$ *a homotopy such that* $h_0 = f_{|A}$. *Then there exists a homotopy*

$$(\bar{h}_t : X \to Y)_{t\in[0,1]}$$

such that $\bar{h}_t\big|_A = h_t$ *for all* $t \in [0,1]$ *and* $\bar{h}_0 = f$.

Remark. In the language of homotopy theory, this can be stated by saying that the inclusion $A \to X$ has the HEP, homotopy extension property. Equivalently, the inclusion $A \to X$ is a *cofibration.*

PROOF. We construct homotopies

$$(\bar{h}_{t,n} : X^n \to Y)_{t\in[0,1]}$$

by induction on n. These will be compatible, in the sense that $\bar{h}_{t,n-1}$ is the restriction of $\bar{h}_{t,n}$ to $X^{n-1} \times [0,1]$. Then we can define $\bar{h}_t$ so that it agree with $\bar{h}_{t,n}$ on $X^n \times [0,1]$. Because of condition (1) in the definition of CW-space, there is no continuity problem. Therefore, for the induction step, assume that the homotopy

$$(\bar{h}_{t,n-1} : X^{n-1} \to Y)_{t\in[0,1]}$$

has already been constructed, and that it agrees with the prescribed $(h_t)_{t\in[0,1]}$ on $A^{n-1} \times [0,1]$, and also that $\bar{h}_{0,n-1}$ agrees with f on X^{n-1}. We wish to construct

$$(\bar{h}_{t,n} : X^n \to Y)_{t\in[0,1]}$$

which, to be honest, is a map $X^n \times [0,1] \to Y$. This map is already defined for us on $X^{n-1} \times [0,1]$ and on $A^n \times [0,1]$. What this means is that it is not defined on the n-cells of X which are not contained in A. Choose characteristic maps for these to get a pushout square

$$\begin{tikzcd} \coprod_{\Lambda_n \smallsetminus \Lambda'_n} S^{n-1} \arrow[r] \arrow[d] & \coprod_{\Lambda_n \smallsetminus \Lambda'_n} D^n \arrow[d, "\varphi"] \\ X^{n-1} \cup A^n \arrow[r] & X^n \end{tikzcd}$$

where Λ_n is an indexing set for the n-cells of X, and $\Lambda'_n \subset \Lambda_n$ corresponds to the n-cells contained in A. (Reader: explain why this is a pushout square.) By the good properties of pushouts (and here we are using the fact that $X \times [0,1]$ is again a CW-space), it is now enough to define a homotopy

$$(g_t : \coprod D^n \to Y)_{t\in[0,1]}$$

which agrees with $\bar{h}_{t,n-1} \circ \varphi$ on $\coprod S^{n-1}$ and, for $t = 0$, with $f \circ \varphi$ on $\coprod D^n$. The coproducts are indexed by $\Lambda_n \smallsetminus \Lambda'_n$. By the good properties of coproducts, it is then also enough to define for each $\lambda \in \Lambda_n \smallsetminus \Lambda'_n$ a homotopy

$$(g_{t,\lambda} : D^n \to Y)_{t\in[0,1]}$$

which agrees with $\bar{h}_{t,n-1} \circ \varphi$ on that copy of S^{n-1} and, for $t = 0$, with $f \circ \varphi$ on that copy of D^n (where *that copy* refers to the copy corresponding to λ). Of course, the homotopy $(g_{t,\lambda})_{t \in [0,1]}$ is really a map

$$D^n \times [0,1] \to Y$$

to be constructed which is already defined for us on $(D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$. Therefore it suffices to show: *every* continuous map

$$u : (D^n \times \{0\}) \cup (S^{n-1} \times [0,1]) \longrightarrow Y$$

admits an extension to a continuous map $v : D^n \times [0,1] \to Y$. A solution to that is $v = u \circ r$ where

$$r : D^n \times [0,1] \longrightarrow (D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$$

is a map which agrees with the identity on $(D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$. Such a map r can be obtained as follows. View $D^n \times [0,1]$ as a subspace of $\mathbb{R}^n \times \mathbb{R}$ in the most obvious way. Let z be the point $(0,0,0,\dots,0,2)$ in $\mathbb{R}^n \times \mathbb{R}$. Define r in such a way that $r(x)$ is the unique point where the line through x and z intersects $(D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$. □

2.3. Cellular maps

DEFINITION 2.3.1. Let $f : X \to Y$ be a continuous map, where X and Y are CW-spaces. The map f is called *cellular* if $f(X^n) \subset Y^n$ for all $n \geq 0$.

EXAMPLE 2.3.2. View S^1 as the unit circle in $\mathbb{C}$. For $n \in \mathbb{Z}$, the map $f : S^1 \to S^1$ defined by $f(z) = z^n$ is a cellular map if we use the CW-structure on S^1 which has 0-skeleton equal to $\{1\}$ and 1-skeleton equal to all of S^1. If instead we use the CW-structure on S^1 with 0-skeleton S^0 and 1-skeleton equal to all of S^1, then f is also a cellular map.

EXAMPLE 2.3.3. The antipodal map $g : S^n \to S^n$ is not a cellular map if we use a CW-structure on S^n with exactly one 0-cell and exactly one n-cell and no other cells.

2.4. Approximation of maps by cellular maps

LEMMA 2.4.1. *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^{n+k}$ a continuous map such that $f^{-1}(0)$ is compact, where $k > 0$. Then for any $\varepsilon > 0$ there exists a map $g : U \to \mathbb{R}^{n+k}$ such that $\|g - f\| \leq \varepsilon$, the support of $g - f$ is compact and $g^{-1}(0) = \varnothing$.*

PROOF. There are two well-known methods for this. One is to use Sard's theorem. Choose a smooth function $\varphi : U \to [0,1]$ with compact support $\mathrm{supp}(\varphi)$ such that

$$\mathrm{supp}(1 - \varphi) \cap f^{-1}(0) = \varnothing.$$

Without loss of generality, ε is less than the minimum value of $\|f\|$ on the compact set $\mathrm{supp}(\varphi) \cap \mathrm{supp}(1-\varphi)$. It is easy to construct a smooth map g_1 from U to $\mathbb{R}^{n+k}$ such that $\|f(x) - g_1(x)\| < \varepsilon/2$ for all $x \in U$. As a special case of Sard's theorem, the image of g_1 is a set of Lebesgue measure zero in $\mathbb{R}^{n+k}$. Hence there exists $y \in \mathbb{R}^{n+k}$, not in the image of g_1, such that $\|y\| < \varepsilon/2$. Let $g_2 = g_1 - y$, so that 0 is not in the image of g_2. By construction, $\|f(x) - g_2(x)\| < \varepsilon$ for all $x \in U$. Let $g = \varphi \cdot g_2 + (1 - \varphi) \cdot f$. This g has all the properties that we require. (In particular, suppose for a contradiction that $g(x) = 0$ for some $x \in U$. Then clearly $x \in \mathrm{supp}(\varphi) \cap \mathrm{supp}(1 - \varphi)$ and $\varphi(x) g_2(x) = (\varphi(x) - 1) f(x)$, so $\varphi(x)(g_2(x) - f(x)) = -f(x)$, so $\varphi(x) \cdot \varepsilon > \|f(x)\|$, so $\varepsilon > \|f(x)\|$, contradiction.)
The other method would be to use piecewise linear approximation. This is more elementary but also much more tedious. ... *Under construction ... perhaps.* □

COROLLARY 2.4.2. *Let* U *be an open subset of* $\mathbb{R}^n$ *and* $f: U \to \mathbb{R}^{n+k}$ *a continuous map such that* $f^{-1}(0)$ *is compact,* $k > 0$. *Then there exist a map* $g: U \to \mathbb{R}^{n+k}$ *such that* $g^{-1}(0) = \varnothing$ *and a homotopy* $(h_t: U \to \mathbb{R}^{n+k})_{t\in[0,1]}$ *such that* $h_0 = f$, $h_1 = g$ *and* $(h_t)_{t\in[0,1]}$ *is stationary*[3] *outside a compact subset* K *of* U.

PROOF. Take g as in lemma 2.4.1. Put $h_t(x) := (1-t)f(x) + tg(x)$. □

LEMMA 2.4.3. *Let* $f: D^n \to X$ *be a continuous map, where* X *is a CW-space. Suppose that* $f(S^{n-1}) \subset X^{n-1}$. *Then there exists a homotopy*

$$(h_t: D^n \to X)_{t\in[0,1]}$$

which is stationary on S^{n-1} *and such that* $h_0 = f$ *while* $h_1(D^n) \subset X^n$.

PROOF. The image of f is compact, therefore contained in a compact CW-subspace Y of X (which must have finitely many cells only, as it is compact). We choose Y as small as possible. Suppose that the maximal dimension of the cells in Y is $n+k$, where $k > 0$. The $(n+k)$-cells in Y all have nonempty intersection with the image of f, otherwise the choice of Y was not minimal. Choose one of them, say $E \subset Y$, and let $U = f^{-1}(E) \subset D^n \smallsetminus S^{n-1}$, an open set. The restriction of f to U can be viewed as a map from U to $E \cong \mathbb{R}^{n+k}$. This is (after some more reparameterization) the situation of corollary 2.4.2. Therefore we can make a homotopy $(\alpha_t)_{t\in[0,1]}$ from f to a map $f_1: D^n \to X$ as in that corollary. (The homotopy is stationary outside a compact subset K of U, that is to say, it associates a *constant* path $t \mapsto \alpha_t(z)$ in X to every element z of $D^n \smallsetminus K$.) The advantage of f_1 compared with f is that it avoids the point p in $E \subset Y$ which corresponds to the origin of $\mathbb{R}^{n+k}$ in our parametrization of E. But the image of f_1 is still contained in Y. Now it is easy to make a homotopy

$$(\beta_t: Y \smallsetminus \{p\} \to Y)_{t\in[0,1]}$$

where β_0 is the inclusion and β_1 lands in the CW-subspace $Y \smallsetminus E$, and $(\beta_t)_{t\in[0,1]}$ is stationary on $Y \smallsetminus E$. Composing this homotopy with f_1, where we view f_1 as a map from D^n to $Y \smallsetminus \{p\}$ we get a homotopy

$$(\beta_t \circ f_1)_{t\in[0,1]}$$

from f_1 to a map $f_2 = \beta_1 \circ f_1$ which avoids the cell E entirely. The combined homotopy from f to f_2 is stationary on S^{n-1} by construction. We have made progress in the sense that the image of f_2 is contained in $Y \smallsetminus E$, a compact CW-subspace of X with fewer $(n+k)$-dimensional cells than Y. Carry on like this, treating f_2 as we treated f before. □

COROLLARY 2.4.4. *Every map* $f: X \to Y$ *between CW-spaces* X *and* Y *is homotopic to a cellular map.*

PROOF. Let $a(n) = 1 - 2^{-n-1}$ for $n = -1, 0, 1, 2, 3, \ldots$. We write $f = f_{-1}$ and we construct maps $f_n: X \to Y$ such that f_n is cellular on X^n, and for each $n \geq 0$ a homotopy

$$(h_t: X \to Y)_{t\in[a(n-1),a(n)]}$$

which is stationary on X^{n-1} and such that $h_{a(n-1)} = f_{n-1}$ and $h_{a(n)} = f_n$.
Suppose that f_{n-1} and h_t for $0 \leq t \leq a(n-1)$ have already been constructed. By condition (2) in the definition of a CW-space and by lemma 2.4.3, we can define a homotopy

$$(g_t: X^n \to Y)_{t\in[a(n-1),a(n)]}$$

[3] A homotopy $(\gamma_t: A \to B)_{t\in[0,1]}$ is *stationary* on a subspace C of A if the path $t \mapsto \gamma_t(x)$ is constant for every $x \in C$.

which is stationary on X^{n-1} and such that $g_{a(n)}(X^n) \subset Y^n$, and $g_{a(n-1)}$ agrees with f_{n-1} on X^{n-1}. By the homotopy extension property, lemma 2.2.1, that homotopy can be extended to a homotopy $(h_t : X \to Y)_{t\in[a(n-1),a(n)]}$, where $h_{a(n-1)} = f_{n-1}$. This completes the induction step. Now observe that the maps h_t so far constructed define a homotopy

$$(h_t : X \to Y)_{t\in[0,1]}$$

from $f = f_{-1}$ to another map $h_1 = f_\infty$, if we define h_1 so that it agrees with h_t on X^n for all $t \in [a(n), 1[$. The map f_∞ is cellular. □

2.5. Cellular approximation of homotopies

The goal is to prove:

THEOREM 2.5.1. *Let* X *and* Y *be CW-spaces and let* $f, g : X \to Y$ *be cellular maps. Suppose that* f *is homotopic to* g*. Then there exists a* cellular *homotopy from* f *to* g*, that is to say, a cellular map* $H : X \times [0,1] \longrightarrow Y$ *such that* $H(x,0) = f(x)$ *and* $H(x,1) = g(x)$ *for all* $x \in X$.

Here we are using the standard CW-structure on $[0,1]$ with two 0-cells $\{0\}$ and $\{1\}$ and one 1-cell, and we are using the product CW-structure on $X \times [0,1]$. This is the reason why we had to discuss products of CW-spaces in the previous (sub)section.

The proof is a special case of a slight refinement of corollary 2.4.4. The refinement is formulated in the following remark.

REMARK 2.5.2. Let $f : X \to Y$ be a map between CW-spaces and let $A \subset X$ be a CW-subspace such that $f_{|A}$ is already cellular. Then there exists a homotopy h from f to a map $g : X \to Y$ such that g is cellular, and the homotopy is stationary on A. The homotopy can be constructed exactly as in the proof of corollary 2.4.4; in step number n, worry only about the n-cells of X which are not in A.

PROOF OF THEOREM 2.5.1. It is a direct application of remark 2.5.2: but for X, A, f in the remark substitute $X \times [0,1]$, $X \times \{0,1\}$, H as in the statement of the theorem, respectively. □

CHAPTER 3

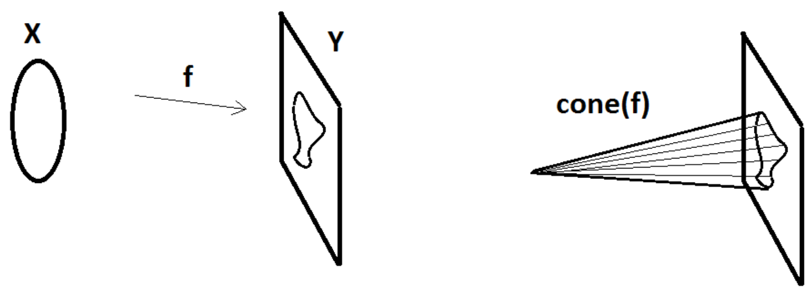
Homology of CW-spaces

3.1. Mapping cones

DEFINITION 3.1.1. Let $f : X \to Y$ be a map of spaces. The *mapping cone* of f is the quotient space

$$\frac{Y \sqcup \big([0,1]\times X\big) \sqcup \{1\}}{\sim}$$

where "$\sim$" is the smallest equivalence relation such that $(0,x) \sim f(x) \in Y$ for all $x \in X$ and $(1,x) \sim 1 \in \{1\}$ for all $x \in X$. Notation: $\mathrm{cone}(f)$.



The mapping cone has a distinguished base point 1; this is sometimes important.[1]

Suppose that X is a closed subset of Y and $f : X \to Y$ is the inclusion map. Then there is a comparison map

$$p : \mathrm{cone}(f) \longrightarrow Y/X$$

where Y/X is understood to be $\{\infty\} \sqcup Y$ modulo the smallest equivalence relation which has $y \sim \infty$ for all $y \in X$. (Note that Y/X also has a distinguished base point ∞ by construction.[2]) The formula for the comparison map is: equivalence class of (t,x) maps

[1] Question for the gentle reader: what does $\mathrm{cone}(f)$ look like when X is empty ?

[2] What does Y/X look like when X is empty ?

to the base point ∞ for all $(t,x) \in [0,1] \times X$; equivalence class of $y \in Y$ maps to equivalence class of y.

PROPOSITION 3.1.2. *If the inclusion* $f\colon X \to Y$ *is a cofibration* (has the homotopy extension property), *then the comparison map* $p\colon \mathrm{cone}(f) \to Y/X$ *is a homotopy equivalence.*

PROOF. Let $j\colon Y \to \mathrm{cone}(f)$ be the obvious inclusion. The composition

$$jf\colon X \to \mathrm{cone}(f)$$

has a nullhomotopy $(h_t\colon X \to \mathrm{cone}(f))_{t\in[0,1]}$ given by

$$h_t(x) = \text{equivalence class of } (t,x) \text{ in } \mathrm{cone}(f),$$

so that $h_0 = jf$ and h_1 is constant (with value 1). Since f has the homotopy extension property, there exists a homotopy $(H_t\colon Y \to \mathrm{cone}(f))_{t\in[0,1]}$ such that $H_0 = j$ and $H_t f = h_t$ for all $t \in [0,1]$. Then H_1 is a map from Y to $\mathrm{cone}(f)$ which maps all of X to the base point 1. So H_1 can be viewed as a map q from Y/X to $\mathrm{cone}(f)$. We will show that $pq \sim \mathrm{id}_{Y/X}$ and $qp \sim \mathrm{id}_{\mathrm{cone}(f)}$. *First claim*: pq is homotopic to $\mathrm{id}_{Y/X}$ by the homotopy $(pH_{1-t})_{t\in[0,1]}$. Strictly speaking pH_{1-t} is a map from Y to Y/X, but it maps all of X to the base point. *Second claim*: qp is homotopic to $\mathrm{id}_{\mathrm{cone}(f)}$ by the homotopy which agrees with $(H_{1-t})_{t\in[0,1]}$ on $Y \subset \mathrm{cone}(f)$ and which agrees with $((s,x) \mapsto (1-t+ts,x))_{t\in[0,1]}$ on points of the form (s,x) in $\mathrm{cone}(f)$, where $x \in X$ and $s \in [0,1]$. □

Let's note that all the maps (and homotopies) in this proof were base-point preserving. So it can be said that $p\colon \mathrm{cone}(f) \to Y/X$ is a pointed homotopy equivalence, in the situation of the proposition.

3.2. Homology of the mapping cone

DEFINITION 3.2.1. The *reduced homology* of a space X with base point $\star$ is

$$\tilde{H}_n(X) := H_n(X)/H_n(\star),$$

more precisely, the cokernel of the inclusion-induced (injective) map from $H_n(\star)$ to $H_n(X)$. Alternatively it can be defined as the kernel of the homomomorphism $H_n(X) \to H_n(\star)$ induced by the unique map $X \to \star$. (These two are isomorphic in a preferred way. The second definition does not require a choice of base point for X, but it should only be used in this form for nonempty X. It was mentioned briefly in the WS2017/18 lecture notes.)

Clearly $H_n(X) = \tilde{H}_n(X)$ for $n \neq 0$, since $H_n(\star)$ is nonzero only for $n = 0$. The tilde notation is therefore mostly welcome when we are tired of making exceptions for $n = 0$.

PROPOSITION 3.2.2. *For a map* $f\colon X \to Y$*, there is a long exact sequence of homology groups*

$$\cdots \longrightarrow H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{j_*} \tilde{H}_n(\mathrm{cone}(f)) \longrightarrow H_{n-1}(X) \xrightarrow{f_*} H_{n-1}(Y) \xrightarrow{j_*} \cdots$$

PROOF. This is essentially the Mayer-Vietoris sequence of the open covering of $\mathrm{cone}(f)$ by open subsets $V = \mathrm{cone}(f) \smallsetminus \star$ and $W = \mathrm{cone}(f) \smallsetminus Y$, where $\star$ is the base point (also known as 1). So let us look at this MV sequence:

$$\cdots \to H_n(V \cap W) \to H_n(V) \oplus H_n(W) \to H_n(\mathrm{cone}(f)) \to H_{n-1}(V \cap W) \to \cdots$$

It should be clear that W is contractible; the picture of $\mathrm{cone}(f)$ above illustrates that well. Also, it is not hard to see that the inclusion $Y \to V$ is a homotopy equivalence; the picture of cone(f) above illustrates that well, too! Last not least, $V \cap W$ is the same as X times open interval, so homotopy equivalent to X. Taking all that into account, we can write the MV sequence in the form

$$\cdots \to H_n(X) \to H_n(Y) \oplus H_n(\star) \to H_n(\text{cone}(f)) \to H_{n-1}(X) \to \cdots$$

Now we observe that exactness is not affected if we put a tilde over each $H_n(\star)$ and over each $H_n(\text{cone}(f))$. Indeed, it means that we are taking out two copies of $\mathbb{Z}$ in adjacent locations of the long exact sequence (only where $n = 0$) and the homomorphism relating them maps one of these copies of $\mathbb{Z}$ isomorphically to the other. Then we have a long exact sequence

$$\cdots \to H_n(X) \to H_n(Y) \oplus \tilde{H}_n(\star) \to \tilde{H}_n(\text{cone}(f)) \to H_{n-1}(X) \to \cdots$$

And now we conclude by observing that $\tilde{H}_n(\star)$ is always zero. So it can be deleted without loss. □

Corollary 3.2.3. *Let X be a closed subspace of Y such that the inclusion $X \to Y$ is a cofibration. Then there is a long exact sequence of homology groups*

$$\cdots \longrightarrow H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{p_*} \tilde{H}_n(Y/X) \longrightarrow H_{n-1}(X) \xrightarrow{f_*} H_{n-1}(Y) \xrightarrow{p_*} \cdots$$

□

Remark. If X and Y are both nonempty, we also get an exact sequence of the form

$$\cdots \longrightarrow \tilde{H}_n(X) \xrightarrow{f_*} \tilde{H}_n(Y) \xrightarrow{p_*} \tilde{H}_n(Y/X) \longrightarrow \tilde{H}_{n-1}(X) \xrightarrow{f_*} \tilde{H}_{n-1}(Y) \xrightarrow{p_*} \cdots$$

Here we define $\tilde{H}_k(X)$ etc. as the kernel of the homomorpism $H_k(X) \to H_k(\star)$ induced by the unique map $X \to \star$.

3.3. The cellular chain complex of a CW-space

Corollary 3.3.1. *Let Y be a CW-space and let $X \subset Y$ be a CW-subspace of Y. Then there is a long exact sequence*

$$\cdots \longrightarrow H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{p_*} \tilde{H}_n(Y/X) \longrightarrow H_{n-1}(X) \xrightarrow{f_*} H_{n-1}(Y) \xrightarrow{p_*} \cdots$$

Proof. The inclusion $X \to Y$ is a cofibration by lemma 2.2.1. □

Let m be a fixed non-negative integer and let Q be a CW-space with a distinguished 0-cell $\star$ (base point). We want to assume that all cells of Q have dimension m, with the possible exception of the distinguished 0-cell. (We allow $m = 0$.)

Lemma 3.3.2. *Then $\tilde{H}_m(Q)$ is a direct sum of infinite cyclic groups, one summand for each m-cell, excluding the base point cell if $m = 0$. Moreover $\tilde{H}_n(Q) = 0$ for $n \neq m$.*

Proof. The case $m = 0$ is easy, so we assume $m > 0$. Let Λ be an indexing set for the m-cells of Z. For each m-cell $E_\lambda \subset Q$ let K_λ be the closure of E_λ. By the axioms for a CW-space, $K_\lambda = E_\lambda \cup \star$. Therefore K_λ is homeomorphic to a sphere S^m and has a distinguished base point. (But we did not *choose* a homeomorphism of K_λ with S^m.) Now let $Y = \coprod_{\lambda \in \Lambda} K_\lambda$ and $X = \coprod_{\lambda \in \Lambda} \star$. Then we can identify Q with Y/X. This leads to a long exact sequence in homology

$$\cdots \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow \tilde{H}_n(Q) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-1}(Y) \longrightarrow \cdots$$

The maps $H_n(X) \to H_n(Y)$ are injective because the inclusion $X \to Y$ admits a left inverse $Y \to X$. Therefore the long exact sequence breaks up into short exact sequences

$$0 \to H_n(X) \to H_n(Y) \to \tilde{H}_n(Q) \to 0.$$

In other words, $H_n(Q)$ is isomorphic to $H_n(Y)$ if $n > 0$, and zero if $n = 0$. Also $H_n(Y) = \bigoplus_{\lambda \in \Lambda} H_n(K_\lambda)$. Because K_λ is homeomorphic to S^m, the group $H_n(K_\lambda)$ is zero if $n > 0$, $n \neq m$ and infinite cyclic if $n = m$. □

Now in order to describe the homology of a CW-space X, we are going to proceed inductively by trying to understand the homology of the skeleton X^n for each n. There is a long exact sequence in homology relating the homology groups of X^{n-1}, X^n and X^n/X^{n-1}. Lemma 3.3.2 tells us what the homology of X^n/X^{n-1} is.

Definition 3.3.3. The *cellular chain complex* $C(X)$ of a CW-space X has $C(X)_m = \tilde{H}_m(X^m/X^{m-1})$ and differential $d: C(X)_m \to C(X)_{m-1}$ equal to the composition

$$\tilde{H}_m(X^m/X^{m-1}) \xrightarrow{3.3.1} H_{m-1}(X^{m-1}) \xrightarrow{\text{projection}_*} \tilde{H}_{m-1}(X^{m-1}/X^{m-2}).$$

For $m = 0$, it is often more illuminating to write $C(X)_0 = H_0(X^0)$. This is justified because the composition $H_0(X^0) \to H_0(X^0/X^{-1}) \to \tilde{H}_0(X^0/X^{-1})$ is an isomorphism. From this point of view, $d: C(X)_1 \to C(X)_0$ is the homomorphism $\tilde{H}_1(X^1/X^0) \to H_0(X^0)$ of 3.3.1.

Do not confuse $C(X)$ with the singular chain complex of X, which was denoted $sC(X)$ in the Topology I lecture notes. The cellular chain complex $C(X)$ depends on the structure of X as a CW-space; the singular chain complex $sC(X)$ does not. Typically $sC(X)$ is gigantic but $C(X)$, if defined, is rather small. For example if X is a compact CW-space, then the groups $C(X)_m$ are *finitely generated* free abelian groups, and only finitely many of them are nonzero.

Remark: We should verify that $dd = 0$. According to the definition $d: C(X)_m \to C(X)_{m-1}$ is a composition of two homomorphisms; let's write it as $p_{m-1}\delta_m$. Therefore $dd = p_{m-2}\delta_{m-1}p_{m-1}\delta_m$. This is zero because $\delta_{m-1}p_{m-1}$ is the composition of two consecutive homomorphisms in the long exact sequence of corollary 3.3.1.

Theorem 3.3.4. *For a CW-space X and integer $m \geq 0$ there is a natural isomorphism*

$$H_m(X) \to H_m(C(X)).$$

Here $H_m(X)$ is the m-th homology group of the space X (which was difficult to define) and $H_m(C(X))$ is the m-th homology group of the chain complex $C(X)$ (which was very easy to define). Therefore, in some sense, the theorem gives a rather good way to calculate the homology of X. Determining the chain groups $C(X)_m$ is typically not hard (you need to know how many m-cells X has), but determining $d: C(X)_m \to C(X)_{m-1}$ can be a little harder.

The word *natural* in theorem 3.3.4 obviously has to be there, but what does it mean? It has meaning only for *cellular* maps $f: X \to Y$ between CW-spaces. Such a cellular map induces base-point preserving maps $X^m/X^{m-1} \to Y^m/Y^{m-1}$ for every $m \geq 0$, therefore homomorphisms $f_*: C(X)_m \to C(Y)_m$ for every $m \geq 0$. These homomorphisms constitute a chain map, i.e., the diagrams

$$\begin{tikzcd} C(X)_m \arrow[r, "d"] \arrow[d, "f_*"] & C(X)_{m-1} \arrow[d, "f_*"] \\ C(Y)_m \arrow[r, "d"] & C(Y)_{m-1} \end{tikzcd}$$

commute. (The reason for that can be traced all the way back to naturality in proposition 3.2.2.)

The proof of theorem 3.3.4 is a combination of several lemmas. The first of these is basic, not specific to CW-spaces.

LEMMA 3.3.5. (Proposition 8.2.1 in the Topology I lecture notes.) *Let X be a space. For any integer k and $z \in H_k(X)$ there exists a compact (not necessarily Hausdorff) subspace $X' \subset X$ such that z is in the image of the inclusion-induced homomorphism $H_k(X') \to H_k(X)$. If X' is any compact subspace of X and $w_1, w_2 \in H_k(X')$ have the same image in $H_k(X)$, then there is another compact subspace X'' of X containing X such that w_1, w_2 already have the same image in $H_k(X'')$.*

We now work with a fixed CW-space X as in theorem 3.3.4.

COROLLARY 3.3.6. *For every $z \in H_k(X)$ there exists $m \geq 0$ such that z is in the image of the homomorphism $H_k(X^m) \to H_k(X)$ induced by the inclusion $X^m \to X$. If two elements of $H_k(X^m)$ have the same image in $H_k(X)$, then there is $n \geq m$ such that they already have the same image in $H_k(X^n)$.*

PROOF. This follows easily from lemma 3.3.5 since every compact subspace of X is contained in some skeleton X^m. □

LEMMA 3.3.7. $H_n(X^m) = 0$ *for* $n > m$.

PROOF. By induction on m. The cases $m = -1$ and/or $m = 0$ are obvious. For the induction step we have the long exact sequence

$$\cdots \longrightarrow H_n(X^{m-1}) \longrightarrow H_n(X^m) \longrightarrow \tilde{H}_n(X^m/X^{m-1}) \longrightarrow H_{n-1}(X^{m-1}) \longrightarrow \cdots$$

which is a special case of corollary 3.3.1. And we have the computation of lemma 3.3.2. □

LEMMA 3.3.8. *The inclusion $X^{m-1} \to X^m$ induces a homomorphism from $H_k(X^{m-1})$ to $H_k(X^m)$ which is an isomorphism if $k < m-1$. There is an exact sequence*

$$0 \longrightarrow H_m(X^m) \xrightarrow{p_m} C(X)_m \xrightarrow{\delta_m} H_{m-1}(X^{m-1}) \longrightarrow H_{m-1}(X^m) \longrightarrow 0 \ .$$

PROOF. Use lemma 3.3.7, and use the same long exact sequence as in the proof of that lemma. □

PROOF OF THEOREM 3.3.4. We use the notation of lemma 3.3.8. By corollary 3.3.6 and lemma 3.3.8 we know that the inclusion $X^{m+1} \to X$ induces an isomorphism

$$H_m(X^{m+1}) \cong H_m(X) \ .$$

Then we compute $H_m(X^{m+1})$ using the exact sequence(s) of lemma 3.3.8:

$$H_m(X^{m+1}) \cong \frac{H_m(X^m)}{\mathrm{im}(\delta_{m+1})} \cong \frac{\mathrm{im}(p_m)}{\mathrm{im}(p_m\delta_{m+1})} = \frac{\ker(\delta_m)}{\mathrm{im}(p_m\delta_{m+1})} = \frac{\ker(p_{m-1}\delta_m)}{\mathrm{im}(p_m\delta_{m+1})} = H_n(C(X)).$$

This looks like the end of the proof, but it is not since we still have to deal with the naturality assertion. So far we have constructed a *specific* isomorphism

$$\alpha_n : H_n(X) \to H_n(C(X)).$$

It is as follows. For $z \in H_n(X)$, there exists $z_1 \in H_n(X^n)$ mapping to z under the inclusion-induced homomorphism $H_n(X^n) \to H_n(X)$. The image of z_1 under the projection-induced homomorphism to $\tilde{H}_n(X^n/X^{n-1}) = C(X)_n$ is a cycle z_2 in $C(X)$. Now $\alpha_n(z) = [z_2]$, homology class of the cycle z_2.

From this description of α_n the naturality is clear. To spell it out once more, let $f : X \to Y$ be a *cellular* map between CW-spaces. Then we obtain induced maps $X^n \to Y^n$ and $X^n/X^{n-1} \to Y^n/Y^{n-1}$. These in turn determine a chain map $C(f) : C(X) \longrightarrow C(Y)$. The following diagram commutes:

$$\begin{tikzcd}
H_n(C(X)) \arrow[r, "\text{induced by } C(f)"] & H_n(C(Y)) \\
H_n(X) \arrow[u, "\alpha_n"] \arrow[r, "\text{the usual } f_*"] & H_n(Y) \arrow[u, "\alpha_n"']
\end{tikzcd}$$

□

3.4. Incidence numbers

The goal is to make the cellular chain complex $C(X)$ more explicit. To that end we choose characteristic maps

$$\varphi_\lambda : D^{n(\lambda)} \longrightarrow X$$

for all the cells of X. (Here λ runs through an indexing set Λ for the cells of X, and $n(\lambda)$ is the dimension of the cell corresponding to λ. We may write Z_λ for the cell. Think of Λ as the disjoint union of subsets Λ_n where Λ_n corresponds to the n-cells for a fixed n.)

More preparations of a bureaucratic nature are required. The group $\tilde{H}_n(D^n/S^{n-1})$ is infinite cyclic (for all $n \geq 0$). We need to choose isomorphisms

$$u_n : \tilde{H}_n(D^n/S^{n-1}) \to \mathbb{Z}$$

for $n \geq 0$, once and for all. (These choices will be discussed later.) Then we obtain an isomorphism

$$\nu_n : \tilde{H}_n(S^n) \longrightarrow \mathbb{Z}$$

by composing u_{n+1} with the inverse of the boundary homomorphism

$$\partial : \tilde{H}_{n+1}(D^{n+1}/S^n) \to \tilde{H}_n(S^n)$$

from the appropriate long exact sequence in reduced homology. (This is the sequence of the remark right after corollary 3.2.3. This instance of ∂ is an isomorphism because $\tilde{H}_k(D^{n+1}) = 0$ for all k.) Finally we choose a map

$$h_n : D^n/S^{n-1} \to S^n$$

which has degree 1 in the following sense: the composition

$$\mathbb{Z} \xrightarrow{u_n^{-1}} \tilde{H}^n(D^n/S^{n-1}) \xrightarrow{(h_n)_*} \tilde{H}_n(S^n) \xrightarrow{\nu_n} \mathbb{Z}$$

is the identity homomorphism of $\mathbb{Z}$. (The choice of h_n will also be discussed later.)

PROPOSITION 3.4.1. *Take* $\lambda_0 \in \Lambda_n$ *and* $\tau_0 \in \Lambda_{n-1}$*, where* $n > 0$*. The entry in row* τ_0 *and column* λ_0 *of the* $\Lambda_{n-1} \times \Lambda_n$ *-matrix corresponding to the homomorphism*

$$\begin{tikzcd} \bigoplus_{\lambda\in\Lambda_n} \mathbb{Z} \arrow[r, dotted] \arrow[d, "\oplus_\lambda(\varphi_\lambda)_*"', "\cong"] & \bigoplus_{\tau\in\Lambda_{n-1}} \mathbb{Z} \arrow[d, "\oplus_\tau(\varphi_\tau)_*"', "\cong"] \\ \tilde{H}_n(X^n/X^{n-1}) \arrow[r, "\partial"] & \tilde{H}_n(X^{n-1}/X^{n-2}) \end{tikzcd}$$

is the degree $[\lambda_0 : \tau_0]$ *of the map*

$$\begin{tikzcd} S^{n-1} \arrow[r, "\text{res.of } \varphi_{\lambda_0}"] & X^{n-1} \arrow[d, "\text{quot.map}"] & & S^{n-1} \\ & X^{n-1}/(X^{n-1} \smallsetminus Z_{\tau_0}) \arrow[rr, "(\varphi_{\tau_0})^{-1}", "\cong"'] & & D^{n-1}/S^{n-2} \arrow[u, "h_{n-1}"'] \end{tikzcd}$$

Remark. The case $n = 1$ must be taken seriously. The degree of a map $f : S^0 \to S^0$ is either 1, -1 or 0. It can be read off by looking at the induced map in reduced homology,

$$f_* : \tilde{H}(S^0) \to \tilde{H}(S^0)$$

where $\tilde{H}_k(Y)$ in general (for nonempty Y) is the kernel of $H_k(Y) \to H_k(\star)$ induced by the unique map $Y \to \star$. This f_* is a homomorphism from an infinite cyclic group to itself, so it is multiplication by an integer (the degree of f). Explicitly, if $f = \text{id}$ then the degree is 1; if f interchanges the two elements of S^0, then the degree is -1; if f is constant, then the degree is 0.

DEFINITION 3.4.2. The degree $[\lambda_0 : \tau_0] \in \mathbb{Z}$ is the *incidence number* (associated with the n-cell corresponding to λ_0 and the $(n-1)$-cell corresponding to τ_0). It depends on a choice of characteristic maps for these cells.

Remark. If we choose another characteristic map for the cell corresponding to λ_0, but leave all selected characteristic maps for the $(n-1)$-cells as they are, then either all incidence numbers $[\lambda_0 : \tau]$ for $\tau \in \Lambda_{n-1}$ remain the same, or they all change sign. If we choose another characteristic map for the cell corresponding to τ_0, but leave all selected characteristic maps for the $(n-1)$-cells as they are, then either all incidence numbers $[\lambda : \tau_0]$ for $\lambda \in \Lambda_n$ remain the same, or they all change sign.

PROOF OF THE PROPOSITION. By naturality we can reduce to a situation where X has exactly three cells: one in dimension n (corresponding to index λ_0), one in dimension $n-1$ (corresponding to τ_0) and one in dimension 0 which we view as the base point. For this reduction we introduce two CW-subspaces

$$B \subset A \subset X.$$

Namely, A is the union of X^{n-1} and the n-cell Z_{λ_0}, while B is X^{n-1} minus the $(n-1)$-cell Z_{τ_0}. The inclusion $A \to X$ is a cellular map which induces an (inclusion) map of the cellular chain complexes, and with that we can see that if the proposition holds for A, then it holds for X. Next we replace A by the CW quotient space A/B. The projection $A \to A/B$ is again a cellular map, and so induces a map of the cellular chain complexes. With that we can see that if the proposition holds for A/B, then it holds for A. Now A/B has exactly three cells.

Therefore we can assume from now on that X has exactly three cells (and more details as specified above). The disk D^n also has a standard CW-structure with three cells: one

0-cell, one $(n-1)$-cell and one n-cell. The characteristic map φ_{λ_0} is nearly cellular: by definition it takes the n-skeleton of D^n to the n-skeleton of X, and it takes the $(n-1)$-skeleton of D^n to the $(n-1)$-skeleton of X. It might fail to take the 0-skeleton of D^n to the 0-skeleton of X. But we can find a homotopy $(f_t)_{t\in[0,1]}$ from $\varphi_{\lambda_0} = f_0$ to a cellular map $f_1 : D^n \to X$ such that each f_t takes the $(n-1)$-skeleton of D^n to the $(n-1)$-skeleton of X. (For the construction of such a homotopy, proceed as follows. First make a homotopy from the map

$$S^{n-1} \to X^{n-1}$$

obtained by restricting φ_{λ_0} to a *cellular* map $S^{n-1} \to X^{n-1}$. Then use the homotopy extension property to extend this to a homotopy $(f_t)_{t\in[0,1]}$ where $f_0 = \varphi_{\lambda_0}$.)
The map f_1 determines a chain map from the cellular chain complex of D^n to the cellular chain complex of X. The following diagram depicts the interesting part of this chain map (i.e. it depicts what is happening in degrees n and $n-1$):

$$\begin{tikzcd} \tilde{H}_{n-1}(S^{n-1}) \arrow[d] & \tilde{H}_n(D^n/S^{n-1}) \arrow[l] \arrow[d, "\cong"] \\ \tilde{H}_{n-1}(X^{n-1}) & \tilde{H}_n(X^n/X^{n-1}) \arrow[l] \end{tikzcd}$$

The vertical arrows are the maps induced by f_1. All the groups in the square are identified, for one reason or another, with $\mathbb{Z}$. Using these identifications, we find that the right-hand vertical arrow is multiplication by 1 (because f_1 is homotopic to φ_{λ_0} by a special homotopy). The top horizontal arrow is also multiplication by 1. The left-hand vertical arrow is multiplication by the degree $[\lambda_0 : \tau_0]$. Therefore the lower horizontal arrow is also multiplication by the degree $[\lambda_0 : \tau_0]$. □

Now we turn to the choices of $u_n : \tilde{H}_n(D^n/S^{n-1}) \to \mathbb{Z}$ and $h_n : D^n/S^{n-1} \to S^n$. It may look as if we need to choose the u_n for all n and the h_n will then be more or less determined; but we can also proceed the other way round. I propose to define h_n by

$$D^n \ni v = (v_1, \dots, v_n) \quad \mapsto \quad (1 - 2\|v\|, tv_1, tv_2, \dots, tv_n) \in S^n \subset \mathbb{R}^{n+1}$$

where t is the non-negative solution of $(1-2\|v\|)^2 + t^2 = 1$. This takes $0 \in D^n$ to $(1, 0, \dots, 0)$ and takes all points in S^{n-1} to $(-1, 0, \dots, 0)$. Now we can define u_n by recursion. The choice of of u_0 is easy, since there is a canonical isomorphism

$$\tilde{H}_0(D^0/S^{-1}) \cong H_0(D^0)$$

and $H_0(D^0)$ is identified with the free abelian group generated by (the set of path components of) D^0, and so with $\mathbb{Z}$. If we have selected u_n for some n, then we can pin down v_n since the composition

$$\tilde{H}^n(D^n/S^{n-1}) \xrightarrow{(h_n)_*} \tilde{H}_n(S^n) \xrightarrow{v_n} \mathbb{Z}$$

must agree with u_n. And v_n determines u_{n+1} since u_{n+1} must be equal to the composition

$$\tilde{H}_{n+1}(D^{n+1}/S^n) \xrightarrow{\partial} \tilde{H}_n(S^n) \xrightarrow{v_n} \mathbb{Z}.$$

Remark 3.4.3. This procedure does not answer all questions since we may think of other ways to choose u_n. I believe the following is everybody's favorite.

- Let Δ^n be the geometric n-simplex, with boundary $\partial\Delta^n$. The identity map $\iota\colon\Delta^n \to \Delta^n$ is an element of $sC(\Delta^n)/sC(\partial\Delta^n)$ and in fact an n-cycle in that chain complex. Its homology class is a generator of the infinite cyclic group
$$H_n(\Delta^n, \partial\Delta^n).$$
(Longish proof by induction on n.) Therefore we have a preferred isomorphism from $H_n(\Delta^n, \partial\Delta^n)$ to $\mathbb{Z}$.
- There is a map σ from the geometric n-simplex Δ^n to $\mathbb{R}^n$ given by
$$\Delta^n \ni (t_0, t_1, \dots, t_n) \mapsto (t_1 - t_0, t_2 - t_0, \dots, t_n - t_0) \in D^n.$$
This induces an isomorphism from $H_n(\Delta^n, \partial\Delta^n)$ to $H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\})$.
- We have more obvious isomorphisms
$$\tilde{H}_n(D^n/S^{n-1}) \cong H_n(D^n, S^{n-1}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\}),$$
the second one induced by inclusion.

Putting these preferred isomorphisms together, we have a preferred isomorphism
$$\tilde{H}_n(D^n/S^{n-1}) \to \mathbb{Z}.$$
Now the question emerges: is this isomorphism equal to the u_n specified above? I believe the answer is yes, and again there ought to be a tedious proof by induction, but I am not inclined to write out the details.

3.5. Products and cellular chain complexes

Definition 3.5.1. The tensor product $C \otimes D$ of two chain complexes C and D is defined as follows:
$$(C \otimes D)_r = \bigoplus_{p+q=r} C_p \otimes D_q$$
and the differential $(C \otimes D)_r \to (C \otimes D)_{r-1}$ is determined by
$$d(x \otimes y) := \big(d(x) \otimes y\big) + (-1)^p\big(x \otimes d(y)\big)$$
for $x \in C_p$ and $y \in D_q$, assuming $p+q = r$. (A generic d has been used for the differentials in C, D and $C \otimes D$.)

Remark 3.5.2. With notation as above we have
$$\begin{aligned} & d\big(d(x \otimes y)\big) = d\big(d(x) \otimes y\big) + (-1)^p\big(x \otimes d(y)\big) \\ = \; & d(d(x)) \otimes y + (-1)^{p-1} d(x) \otimes d(y) + (-1)^p(d(x) \otimes d(y)) + x \otimes d(d(y)) = 0. \end{aligned}$$
Obviously the sign $(-1)^p$ is important to ensure that $dd = 0$ holds in $C \otimes D$. There is a rule of thumb for this: if, in a product-like expression you move a term of degree u past a term of degree v, then you should probably introduce a sign $(-1)^{uv}$. For example $d(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes d(y)$ because it feels like moving the d, which has degree -1, past the x which was assumed to have degree p. Another application of this useful rule: $C \otimes D$ is isomorphic to $D \otimes C$ by the isomorphism taking $x \otimes y$ to $(-1)^{pq} y \otimes x$, where $x \in C_p$ and $y \in D_q$.

Theorem 3.5.3. *Let X and Y be compact CW-spaces. Use the product CW-structure on $X \times Y$. Then for the cellular chain complexes we have $C(X \times Y) \cong C(X) \otimes C(Y)$.*

In an earlier edition of these lecture notes, there was a bare-hands proof. But it did not stand the test of time. We will prove this later using product machinery. See section 6.1.

CHAPTER 4

Rudimentary homological algebra

4.1. Tensor product and Hom of chain complexes

DEFINITION 4.1.1. For chain complexes C and D of abelian groups, graded over $\mathbb{Z}$, a chain complex $\hom(C,D)$ is defined as follows:

$$\hom(C,D)_r := \prod_{n\in\mathbb{Z}} \hom(C_n, D_{n+r})$$

with differential given by $d((f_n : C_n \to D_{n+r})_{n\in\mathbb{Z}}) := (df_n - (-1)^r f_{n-1} d)_{n\in\mathbb{Z}}$.

EXAMPLE 4.1.2. A 0-dimensional cycle in $\hom(C,D)$ is a collection of homomorphisms $f_n : C_n \to D_n$, where $n \in \mathbb{Z}$, which satisfy $df_n - f_{n-1}d = 0$ for all n. This is exactly the same thing as a chain map f from C to D. Two chain maps $f, g : C \to D$, viewed as 0-dimensional cycles in $\hom(C,D)$, are homologous (belong to the same homology class) in $\hom(C,D)$ if and only if there exists a family of homomorphisms

$$h = (h_n : C_n \to D_{n+1})_{n\in\mathbb{Z}}$$

such that $dh_n + h_{n-1}d = g_n - f_n$ for all n. This is precisely a chain homotopy from f to g. Therefore $H_0(\hom(C,D))$ is isomorphic to $[C,D]$, the abelian group of chain homotopy classes of chain maps from C to D. (The other homology groups of $\hom(C,D)$ can be interpreted in a similar way. Details omitted.)

PROPOSITION 4.1.3. *For chain complexes* A, B, C *there is an isomorphism*

$$\hom(A, \hom(B,C)) \cong \hom(A \otimes B, C).$$

PROOF. Exercise. □

4.2. Homology and cohomology with coefficients

Let X be a topological space. We write $sC(X)$ for the singular chain complex as usual. Let A be an abelian group. We write

$$H_n(X;A) := H_n(sC(X) \otimes A)$$

and call this the singular homology of X with coefficients in A. We write

$$H^n(X;A) = H_{-n}(\hom(sC(X), A))$$

and call this the singular cohomology of X with coefficients in A. Note that $\hom(sC(X), A)$ can be understood as a special case of definition 4.1.1, if we promote A to a chain complex $(A,0)$ which has $(A,0)_r = 0$ for $r \neq 0$ and $(A,0)_0 = A$. In any case $(\hom(sC(X),A))_r$ is $\hom(sC(X)_{-r}, A)$, if we want to think of $\hom(sC(X),A)$ as a chain complex.[1]
Singular homology with coefficients in A is a covariant functor (from spaces to graded abelian groups); singular cohomology with coefficients in A is a contravariant functor.

[1]As opposed to *cochain complex*; in a cochain complex the differential d raises degree by 1.

They satisfy the standard "axioms" except for some obvious adjustments. For example, there is the homotopy invariance property: homotopic maps $f, g : X \to Y$ induce identical homomorphisms $H_n(X;A) \to H_n(Y;A)$ and $H^n(Y;A) \to H^n(X;A)$. If X comes with open subsets U, V such that $X = \mathrm{int}(U) \cup \mathrm{int}(V)$, then there is a long exact Mayer-Vietoris sequence involving the singular homology groups with coefficients in A of X, U, V and $U \cap V$. There is also a long exact Mayer-Vietoris sequence in singular cohomology with coefficients in A; this has the form

$$\cdots \to H^n(X;A) \to H^n(U;A) \oplus H^n(V;A) \to H^n(U \cap V;A) \xrightarrow{\partial} H^{n+1}(X;A) \to \cdots$$

The most important adjustment concerns the homology groups of a point; they are given by $H_n(*;A) \cong A$ if $n = 0$ and $H_n(*;A) = 0$ otherwise, and similarly $H^n(*;A) = A$ if $n = 0$ and $H^n(*;A) = 0$ otherwise. Equivalently, $H_n(*;A) \cong H_n(*) \otimes A$ and $H^n(*;A) \cong \mathrm{hom}(H_n(*), A)$ for all n. For the singular homology and cohomology of spheres with coefficients, we have similarly

$$H_n(S^k;A) \cong H_n(S^k) \otimes A, \qquad H^n(S^k;A) \cong \mathrm{hom}(H_n(S^k, A)$$

for all n.
And finally we can define the singular homology and cohomology of pairs with coefficients in A. Let (X, Y) be a pair of spaces (so that Y is a subspace of X). Then

$$H_n(X,Y;A) := H_n((sC(X)/sC(Y))\otimes A) \qquad \text{and } H^n(X,Y;A) := H_{-n}(\mathrm{hom}(sC(X)/sC(Y)), A).$$

The long exact sequence in singular homology for pairs has an analogue for singular homology and cohomology with coefficients in A. For example, in the cohomology case this takes the form

$$\cdots \to H^n(X,Y;A) \to H^n(X;A) \to H^n(Y;A) \xrightarrow{\partial} H^{n+1}(X,Y;A) \to \cdots$$

4.3. Tor and Ext

Let A and B be abelian groups.

DEFINITION 4.3.1. Choose a free abelian group F_0 and a surjection $p : F_0 \to A$; this defines a short exact sequence

$$A \xleftarrow{p} F_0 \xleftarrow{q} F_1$$

where $F_1 = \ker(p)$ and q is the inclusion. (It is important that F_1 is again a free abelian group; subgroups of free abelian groups are free abelian.) Then form

$$F_0 \otimes B \xleftarrow{q_*} F_1 \otimes B .$$

Then take homology, which in this case amounts to taking the cokernel and kernel of q_*. We write

$$\mathrm{Tor}_0(A, B) := \mathrm{coker}(q_* : F_1 \otimes B \to F_0 \otimes B),$$
$$\mathrm{Tor}_1(A, B) := \ker(q_* : F_1 \otimes B \to F_0 \otimes B) .$$

Instead of using $\otimes B$, we can also use $\mathrm{hom}(-, B)$; this gives

$$\mathrm{hom}(F_0, B) \xrightarrow{q^*} \mathrm{hom}(F_1, B)$$

and we write

$$\mathrm{Ext}^0(A, B) := \ker(q^* : \mathrm{hom}(F_0, B) \to \mathrm{hom}(F_1, B),$$
$$\mathrm{Ext}^1(A, B) := \mathrm{coker}(q^* : \mathrm{hom}(F_0, B) \to \mathrm{hom}(F_1, B)).$$

This definition is obviously somewhat incomplete; we need to show that $\mathrm{Tor}_*(A,B)$ and $\mathrm{Ext}^*(A,B)$ do not depend on the choice of a surjection $p{:}F_0 \to A$ (where F_0 is free abelian). In order to fill this in we take the following view. We regard q as part of a chain complex $\Phi(A)$ by adding zeros where necessary:

$$\cdots \leftarrow 0 \leftarrow F_0 \xleftarrow{q} F_1 \leftarrow 0 \leftarrow \cdots$$

The homology groups of this chain complex are all zero, except $H_0(\Phi(A))$ which is identified with A. (It is customary to say that $\Phi(A)$ is a *free resolution* of A. The map $p{:}F_0 \to A$ is called the *augmentation.*)

Instead of just trying to show that $\mathrm{Tor}_*(A,B)$ and $\mathrm{Ext}^*(A,B)$ are well defined, we aim to show that they are functors of the variable A. This suggests that we need to show the following.

LEMMA 4.3.2. *Let $f{:}A \to A'$ be a homomorphism of abelian groups. There is a chain map $g = \Phi(f){:}\Phi(A) \to \Phi(A')$ making the following diagram commutative:*

$$\begin{tikzcd}
A \arrow[d, "f"] & F_0 \arrow[l, "p"'] \arrow[d, dotted, "g_0"] & F_1 \arrow[l, "q"'] \arrow[d, dotted, "g_1"] \\
A' & F'_0 \arrow[l, "p'"'] & F'_1 \arrow[l, "q'"']
\end{tikzcd}$$

(where the rows describe the resolutions $\Phi(A)$ and $\Phi(A')$, respectively). Such a chain map is unique up to chain homotopy.

PROOF. Existence: it is enough to construct g_0 so that $p'g_0 = fp$. (Then g_1 must be defined as the restriction of g_1 and it will automatically take F_1 to $F'_1 \subset F'_0$.) But g_0 is easy to construct since we can specify it on the elements of a chosen basis for the free abelian group F_0. (For every basis element z we choose $z' \in F'_0$ such that $p'(z') = f(p(z))$. Then we can map z to z'.) Uniqueness up to homotopy: suppose that g and $g^\sharp$ are two solutions. Then $g^\sharp_0 - g_0$ can be viewed as a homomorphism h_0 from F_0 to the kernel of $p'{:}F'_0 \to A'$, which is the same as a map h_0 from F_0 to F'_1. That homomorphism $h_0{:}F_0 \to F'_1$ all by itself constitutes a chain homotopy from g to $g^\sharp$. □

This completes definition 4.3.1 because we can now reason as follows. For an abelian group A, the choice of free resolution $\Phi(A)$ is unique up to unique chain homotopy equivalence (take $A = A'$ and $f = \mathrm{id}$ in lemma 4.3.2). Therefore $\Phi(A) \otimes B$ and $\hom(\Phi(A), B)$ are also well defined up to preferred chain homotopy equivalence, and consequently $\mathrm{Tor}_*(A,B) = H_*(\Phi(A) \otimes B)$ and $\mathrm{Ext}^*(A,B) = H_{-*}(\hom(\Phi(A), B)$ are well defined. Moreover they are functors of the variable A, for fixed B, since a homomorphism $f{:}A \to A'$ determines a chain map $\Phi(A) \to \Phi(A')$, well defined up to homotopy. (It is clear that they are also functors of the variable B.)

EXAMPLE 4.3.3. Take $A = B = \mathbb{Z}/2$. For the free resolution $\Phi(A)$ we can take

$$\mathbb{Z} = F_0 \xleftarrow{\cdot 2} F_1 = \mathbb{Z}.$$

Then $\Phi(A) \otimes B$ takes the form $\mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2$, so that $\mathrm{Tor}_0(A,B) = \mathbb{Z}/2$ and $\mathrm{Tor}_1(A,B) = \mathbb{Z}/2$. By a similar calculation, $\mathrm{Ext}^0(A,B) = \mathbb{Z}/2$ and $\mathrm{Ext}^1(A,B) = \mathbb{Z}/2$.

REMARK 4.3.4. It is easy to verify that $\mathrm{Tor}_0(A,B)$ is naturally isomorphic to $A \otimes B$ and $\mathrm{Ext}^0(A,B)$ is naturally isomorphic to $\hom(A,B)$. Therefore $\mathrm{Tor}_1(A,B)$ and $\mathrm{Ext}^1(A,B)$ are the interesting new objects for us.

Proposition 4.3.5. *A short exact sequence of abelian groups*

$$0 \to A' \to A \xrightarrow{f} A'' \to 0$$

and a choice of abelian group B *naturally determine exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}_1(A',B) \arrow[r] & \mathrm{Tor}_1(A,B) \arrow[r] & \mathrm{Tor}_1(A'',B) \arrow[d] \\ 0 & \mathrm{Tor}_0(A'',B) \arrow[l] & \mathrm{Tor}_0(A,B) \arrow[l] & \mathrm{Tor}_0(A',B) \arrow[l] \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^0(A'',B) \arrow[r] & \mathrm{Ext}^0(A,B) \arrow[r] & \mathrm{Ext}^0(A',B) \arrow[d] \\ 0 & \mathrm{Ext}^1(A',B) \arrow[l] & \mathrm{Ext}^1(A,B) \arrow[l] & \mathrm{Ext}^1(A'',B) \arrow[l] \end{tikzcd}$$

Proof. Without loss of generality, $A' \to A$ is an inclusion, so that $A' = \ker(f)$. The idea is that we can choose free resolutions $\Phi(A')$, $\Phi(A)$ and $\Phi(A'')$ in such a way that they are arranged in a short exact sequence of chain complexes

$$0 \to \Phi(A') \to \Phi(A) \to \Phi(A'') \to 0.$$

In each degree r (for us, only $r = 0$ and $r = 1$ are of interest) these sequences are split exact. (The homomorphism $\Phi(A)_j \to \Phi(A'')_j$ admits a right inverse because $\Phi(A'')_j$ is a free abelian group.) Therefore the sequences

$$0 \to \Phi(A') \otimes B \to \Phi(A) \otimes B \to \Phi(A'') \otimes B \to 0$$

$$0 \to \hom(\Phi(A''), B) \to \hom(\Phi(A), B) \to \hom(\Phi(A'), B) \to 0$$

are also split exact in each degree, and consequently they are still short exact sequences of chain complexes. Then we obtain the long exact sequences of Tor and Ext groups as the long exact homology group sequences of these short exact sequences of chain complexes. — It remains to show that $\Phi(A')$, $\Phi(A)$ and $\Phi(A'')$ can be arranged in a short exact sequence. We begin by setting up $\Phi(A)$ and $\Phi(A'')$ and the broken arrows to make the following commutative:

$$\begin{tikzcd} A \arrow[d, "f"] & F_0 \arrow[l, "p"'] \arrow[d, dashed, "g_0"] & F_1 \arrow[l] \arrow[d, dashed, "g_1"] \\ A'' & F_0'' \arrow[l, "p''"] & F_1'' \arrow[l] \end{tikzcd}$$

In addition we take care to ensure not only that g_0 is surjective, but that the resulting homomorphism from F_0 to the pullback

$$\{(a, b) \in A \times F_0'' \mid f(a) = p''(b)\}$$

is surjective. Now, if we take kernels of the vertical homomorphisms, we obtain a free resolution $\Phi(A')$:

$$\begin{tikzcd}
A' \arrow[d] & F'_0 = \ker(g_0) \arrow[l, "p'"'] \arrow[d, "\text{incl.}"] & F'_1 = \ker(g_1) \arrow[l] \arrow[d, "\text{incl.}"] \\
A \arrow[d, "f"] & F_0 \arrow[l] \arrow[d, "g_0"] & F_1 \arrow[l] \arrow[d, "g_1"] \\
A'' & F''_0 \arrow[l] & F''_1 \arrow[l]
\end{tikzcd}$$

(The important thing is that p' is surjective.) This seems to complete the proof.
But a few comments on uniqueness are in order. To make these exact sequences of Tor and Ext groups from the short exact sequence $A' \to A \to A''$, or simply from the surjection $f : A \to A''$, we had to make the following choices: choice of a surjection $F''_0 \to A''$ where F''_0 is free abelian, and choice of a surjection $F_0 \to \{(a, b) \in A \times F''_0 \mid f(a) = p''(b)\}$ where F_0 is free abelian. We can make these choices in several ways; suppose that we have made two choices

$$\begin{tikzcd}
A \arrow[d, "f"] & F_{0,0} \arrow[l] \arrow[d] \\
A'' & F''_{0,0} \arrow[l]
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
A \arrow[d, "f"] & F_{0,1} \arrow[l] \arrow[d] \\
A'' & F''_{0,1} \arrow[l]
\end{tikzcd}$$

Then it is easy to choose homomorphisms $F''_{0,0} \to F''_{0,1}$ and $F_{0,0} \to F_{0,1}$ so that the resulting diagram in the shape of a prism (with 6 vertices $F''_{0,0}, F''_{0,1}, F_{0,0}, F_{0,1}, A, A''$) is commutative. This leads us to a commutative diagram relating the two candidates of long exact sequence, for example in the Tor case:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Tor}_1(A,B) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{Tor}_1(A'',B) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathrm{Tor}_0(A',B) \arrow[r] \arrow[d, "\mathrm{id}"] & \cdots \\
\cdots \arrow[r] & \mathrm{Tor}_1(A,B) \arrow[r] & \mathrm{Tor}_1(A'',B) \arrow[r] & \mathrm{Tor}_0(A',B) \arrow[r] & \cdots
\end{tikzcd}$$

Therefore the "new" arrow from $\mathrm{Tor}_1(A'', B)$ to $\mathrm{Tor}_0(A', B)$ is well defined. □

PROPOSITION 4.3.6. *There is a natural isomorphism* $\mathrm{Tor}_*(A, B) \cong \mathrm{Tor}_*(B, A)$*, for abelian groups* A *and* B*.*

PROOF. We gave an asymmetrical definition of $\mathrm{Tor}_*(A, B)$, as follows: $\mathrm{Tor}_*(A, B)$ is the homology of $\Phi(A) \otimes B$, where $\Phi(A)$ is a free resolution of A. (More precisely $\Phi(A)$ is a chain complex of free abelian groups which is allowed to be nontrivial only in degrees 0 and 1. Its homology $H_0(\Phi(A))$ comes with a chosen isomorphism to A; the other homology groups are zero.) But there is a more symmetrical definition as follows: $\mathrm{Tor}^*(A, B)$ is the homology of the chain complex $\Phi(A) \otimes \Phi(B)$. If this is correct, then obviously $\mathrm{Tor}_*(A, B) \cong \mathrm{Tor}_*(B, A)$.
To prove that it is correct, we place $\Phi(B)$ in a short exact sequence of chain complexes

$$(*) \qquad \Psi(B) \to \Phi(B) \to (B, 0)$$

Here $(B,0)$ is just B, viewed as a chain complex in such a way that $(B,0)_0 = B$ and $(B,0)_r = 0$ for $r \neq 0$. The map $\Phi(B) \to (B,0)$ is the augmentation, now viewed as a chain map. This induces an isomorphism in homology (by construction). Last not least $\Psi(B)$ is defined as the kernel of $\Phi(B) \to (B,0)$. It is easy to see that it is a contractible chain complex (admits a chain homotopy equivalence to 0). In fact $\Psi(B)_j$ is zero for $j \neq 0,1$ and the differential $\Psi(B)_1 \to \Psi(B)_0$ is an isomorphism. The short exact sequence $(*)$ determines another short exact sequence of chain complexes

$$\Phi(A) \otimes \Psi(B) \to \Phi(A) \otimes \Phi(B) \to \Phi(A) \otimes B.$$

This has a long exact sequence of homology groups. But since $\Psi(B)$ is a contractible chain complex, $\Phi(A)\otimes\Psi(B)$ is also a contractible chain complex. Therefore the homology groups of $\Phi(A) \otimes \Psi(B)$ are zero. Therefore the long exact sequence of homology groups gives us isomorphisms

$$H_n(\Phi(A) \otimes \Phi(B)) \to H_n(\Phi(A) \otimes B)$$

for all $n \in \mathbb{Z}$. □

4.4. The universal coefficient theorem

The following is the covariant version of the universal coefficient theorem.

THEOREM 4.4.1. *Let C be a chain complex of free abelian groups and let A be an abelian group. For every $n \in \mathbb{Z}$ there is a natural short exact sequence*

$$0 \to H_n(C) \otimes A \longrightarrow H_n(C \otimes A) \longrightarrow \mathrm{Tor}_1(H_{n-1}(C), A) \to 0.$$

It admits a non-natural splitting.

PROOF. I use the following without proof (but see remark 4.4.2 below): a chain map between two chain complexes of free abelian groups is a chain homotopy equivalence if and only if it induces an isomorphism in homology H_n for all n.
For $j \in \mathbb{Z}$, we can easily construct a chain complex $E(j)$ of free abelian groups with the following properties:

$E(j)_r = 0$ for $r \notin \{j, j+1\}$;
$H_j(E(j)) \cong H_j(C)$ and $H_r(E(j)) = 0$ for $r \neq j$.

(Essentially $E(j)$ is a free resolution of the abelian group $H_j(C)$, but shifted in such a way that the interesting homology is in degree j.) Furthermore it is easy to construct chain maps

$$f^{(j)} : E(j) \to C$$

inducing an isomorphism $H_j(E(j)) \to H_j(C)$. *For example*, we can define

$$E(j)_j = \ker(d : C_j \to C_{j+1}), \quad E(j)_{j+1} = \mathrm{im}(d : C_{j+1} \to C_j).$$

Then we have an inclusion $E(j)_{j+1} \to E(j)_j$, and this is our choice of differential d. To define $f^{(j)}$ we take the inclusion

$$\ker(d : C_j \to C_{j+1}) \longrightarrow C_j$$

in degree j. In degree $j+1$ we choose a homomorphism

$$\mathrm{im}(d : C_{j+1} \to C_j) \longrightarrow C_{j+1}$$

which is right inverse to d as a surjective map from C_{j+1} to $\mathrm{im}(d : C_{j+1} \to C_j)$. — Now write E for the direct sum of the $E(j)$. Then the direct sum of the chain maps $f^{(j)}$ is a chain map $E \to C$ which induces an isomorphism in homology H_n for all $n \in \mathbb{Z}$. Therefore it is a chain homotopy equivalence. It follows that the induced chain map $E \otimes A \to C \otimes A$ is also a chain homotopy equivalence. We deduce isomorphisms

$$H_n(C \otimes A) \cong H_n(E \otimes A) \cong \bigoplus_j H_n(E(j) \otimes A) \cong H_n(E(n) \otimes A) \oplus H_n(E(n-1) \otimes A).$$

By construction of the $E(j)$, this last expression can also be written in the form

$$\mathrm{Tor}_0(H_n(C), A) \oplus \mathrm{Tor}_1(H_{n-1}(C), A) = H_n(C) \otimes A \ \oplus \ \mathrm{Tor}_1(H_{n-1}(C), A).$$

Therefore it looks as if we have proved the theorem. But we have not yet addressed the naturality question.

So let us look at this carefully. Fix j in $\mathbb{Z}$. The chain complex $E(j) = E^C(j)$ is fairly well defined as a shifted free resolution of the abelian group $H_j(C)$. (It is well defined up to unique chain homotopy equivalence. Moreover, if we have a chain map $C \to D$ of chain complexes of free abelian groups, then we get a homomorphism $H_j(C) \to H_j(D)$ and we can construct a compatible chain map $E^C(j) \to E^D(j)$, unique up to chain homotopy.) But the chain maps $f^{(j)}: E(j) \to C$ are not unique up to chain homotopy, even if we say that the induced homomorphism from $H_j(E(j))$ to $H_j(C)$ is an isomorphism and more precisely, that it agrees with the augmentation. Suppose that

$$f^{(j)}, g^{(j)} : E(j) \to C$$

are two competing choices. Then it is easy to see (see remark 4.4.3 below) that they induce the same homomorphism

$$H_j(E(j) \otimes A) \to H_j(C \otimes A)$$

and homomorphisms $H_{j+1}(E(j) \otimes A) \to H_{j+1}(C \otimes A)$ which are the same *modulo* the image of $H_{j+1}(C) \otimes A$ in $H_{j+1}(C \otimes A)$. The conclusion is that the isomorphism

$$H_n(C \otimes A) \longrightarrow H_n(C) \otimes A \ \oplus \ \mathrm{Tor}_1(H_{n-1}(C), A)$$

which we found above is somewhat dependent on choices, but parts of it are independent; the resulting injection $H_n(C) \otimes A \to H_n(C \otimes A)$ is well defined and the resulting projection $H_n(C \otimes A) \to \mathrm{Tor}_1(H_{n-1}(C), A)$ is well defined. □

REMARK 4.4.2. If the chain maps $f^{(j)}: E(j) \to C$ in the above proof are constructed exactly as suggested (in *For example ...*), then the resulting map from $E = \bigoplus_j E(j)$ to C is an isomorphism of chain complexes. This proves that C is isomorphic to a direct sum of "elementary" chain complexes (chain complexes of free abelian groups whose chain groups are nonzero only in two adjacent degrees j and $j+1$).

REMARK 4.4.3. Let $u = g^{(j)} - f^{(j)}$. By assumption, u induces the zero homomorphism from $H_j(E(j))$ to $H_j(C)$. Therefore u takes all of $E(j)_j$ to the subgroup of boundaries in C_j. It follows, by the freeness of $E(j)_j$, that there exists a homomorphism $h: E(j)_j \to C_{j+1}$ such that

$$dh = u_j \text{ on } E(j)_j\,.$$

We can use h as a chain homotopy to modify u. More precisely, we replace u_j by $u_j - dh = 0$ and u_{j+1} by $u_{j+1} - hd$. Therefore we can assume without loss of generality from now on that the chain map u satisfies $u_j = 0$. In that case, u_{j+1} is a homomorphism from $E(j)_{j+1}$ to the subgroup of cycles in C_{j+1}.

This has the following consequence(s). It is clear that u induces the zero homomorphism $H_j(E(j) \otimes A) \to H_j(C \otimes A)$. Since u_{j+1} takes $E(j)_{j+1}$ to the subgroup of cycles of C_{j+1}, it is clear that it takes $H_{j+1}(E(j) \otimes A)$ to the image of $H_{j+1}(C) \otimes A$ in $H_{j+1}(C \otimes A)$.

We now come to the contravariant edition of the universal coefficient theorem.

THEOREM 4.4.4. *Let* C *be a chain complex of free abelian groups and let* A *be an abelian group. For every* $n \in \mathbb{Z}$ *there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1(H_{n-1}(C), A) \longrightarrow H_{-n}(\mathrm{hom}(C, A)) \longrightarrow \mathrm{hom}(H_n(C), A) \to 0.$$

It admits a non-natural splitting.

The proof is very similar to that of the covariant edition. □

REMARK 4.4.5. Both versions of the universal coefficient theorem have a down-to-earth formulation which you may find easier to remember. Covariant case: there is an easy-to-describe natural homomorphism from $H_n(C) \otimes A$ to $H_n(C \otimes A)$. It is injective and its cokernel is naturally isomorphic to $\mathrm{Tor}^1(H_{n-1}, A)$. Contravariant case: there is an easy-to-describe natural homomorphism from $H_{-n}(\mathrm{hom}(C, A))$ to $\mathrm{hom}(H_n(C), A)$. It is surjective and its kernel is naturally isomorphic to $\mathrm{Ext}^1(H_{n-1}(C), A)$.

4.5. Ext^* is a "ring" and Tor_* is a "module"

Notation: For an abelian group A and an integer r we sometimes write (A, r) to mean the chain complex which has A in degree r and 0 in all other degrees. For a chain complex C, we may write $\Sigma C := (\mathbb{Z}, 1) \otimes C$. Then we have

$$H_{-r}(\mathrm{hom}(C, D)) \cong [C, \Sigma^r D].$$

PROPOSITION 4.5.1. *Let* A *and* B *be abelian groups; choose free resolutions* $\Phi(A)$ *and* $\Phi(B)$. *Then there are natural isomorphisms*

$$\mathrm{Ext}^0(A, B) \cong H_0(\mathrm{hom}(\Phi(A), \Phi(B)) = [\Phi(A), \Phi(B)] ;$$

$$\mathrm{Ext}^1(A, B) \cong H_{-1}(\mathrm{hom}(\Phi(A), \Phi(B)) \cong [\Phi(A), \Sigma\Phi(B)].$$

For all $r \neq 0, 1$ *we have* $[\Phi(A), \Sigma^r \Phi(B)] = 0$.

PROOF. We are going to show that the augmentation of $\Phi(B)$, which we view as a chain map from $\Phi(B)$ to $(B, 0)$, induces a chain map $\mathrm{hom}(\Phi(A), \Phi(B)) \to \mathrm{hom}(\Phi(A), B)$ which in turn induces an isomorphism in homology. As in the proof of proposition 4.3.6, let $\Psi(B)$ be the kernel of $\Phi(B) \to (B, 0)$. There is a short exact sequence of chain complexes

$$\mathrm{hom}(\Phi(A), \Psi(B)) \to \mathrm{hom}(\Phi(A), \Phi(B)) \to \mathrm{hom}(\Phi(A), B).$$

Since $\Psi(B)$ is (clearly) contractible, the resulting long exact sequence of homology groups tells us that $\mathrm{hom}(\Phi(A), \Phi(B)) \to \mathrm{hom}(\Phi(A), B)$ induces isomorphisms in homology. □

COROLLARY 4.5.2. *The additive category of abelian groups can be enhanced/enlarged to a category enriched over graded abelian groups such that* $\mathrm{mor}(A, B) := \mathrm{Ext}^*(A, B)$. □

Without going into the detailed meaning of *enriched*: this means that the "new" $\mathrm{mor}(A, B)$ is viewed as a graded abelian group consisting of the usual $\mathrm{hom}(A, B)$ in degree 0 and $\mathrm{Ext}^1(A, B)$ in degree 1. Composition is a graded biadditive map of the form

$$\mathrm{Ext}^*(B, C) \times \mathrm{Ext}^*(A, B) \longrightarrow \mathrm{Ext}(A, C).$$

COROLLARY 4.5.3. *There are bilinear composition maps*

$$\mathrm{Ext}^q(B, C) \times \mathrm{Tor}_p(A, B) \longrightarrow \mathrm{Tor}_{p-q}(A, C).$$

These are associative w.r.t. the composition product on Ext^*.

PROOF. If we think of $\mathrm{Tor}_*(A,B)$ as $H_*(A\otimes\Phi(B))$, and if we think of $\mathrm{Ext}^q(B,C)$ as $[\Phi(B),\Sigma^q\Phi(C)]$, then we can define these composition maps as follows: for a chain map $f\colon\Phi(B)\to\Sigma^q\Phi(C)$ and a p-dimensional cycle $z\in A\otimes\Phi(B)$ we take the pair $([f],[z])$ to the homology class of

$$f_*(z)\in A\otimes\Sigma^q\Phi(B),$$

so that we land in $H_p(A\otimes\Sigma^q\Phi(B))\cong H_{p-q}(A\otimes\Phi(B))$. □

4.6. Homology of a tensor product

This is about the homology of a tensor product of chain complexes. The main result is usually called the Künneth theorem. But let us begin with a review of the universal cofficient theorem.

Let us (temporarily) write $\mathcal{C}$ for the category of chain complexes of free abelian groups, graded over $\mathbb{Z}$. The morphisms are chain maps as usual. We know already that every C in $\mathcal{C}$ is isomorphic to a direct sum of *elementary* chain complexes $C(j)$. More precisely $C(j)$ is a chain complex of free abelian groups which has $C(j)_r=0$ for $r\notin\{j,j+1\}$ and the differential $C_{j+1}\to C_j$ is injective. It follows that $H_r(C(j))=0$ for $r\neq j$ and it follows that the inclusion $C(j)\to C$ induces an isomorphism $H_j(C(j))\to H_j(C)$.

Such a splitting is not unique, let alone natural, and this is important. But in the following we tend to assume that objects C of $\mathcal{C}$ come with a choice of a splitting $C=\bigoplus_j C(j)$.

LEMMA 4.6.1. *For $C=\bigoplus_j C(j)$ and $D=\bigoplus_j D(j)$ in $\mathcal{C}$ there is a canonical isomorphism of abelian groups*

$$[C,D]\cong\prod_{j\in\mathbb{Z}}\Big(\mathrm{Ext}^0(H_j(C),H_j(D))\oplus\mathrm{Ext}^1(H_j(C),H_{j+1}(D))\Big).$$

We may write this in the form $[f]\mapsto([f]^j_0,[f]^j_1)_{j\in\mathbb{Z}}$.

PROOF. We begin by noting $[C,D]\cong\prod_j[C(j),D]$. By inspection,

$$[C(j),D]\cong[C(j),D(j-1)]\oplus[C(j),D(j)]\oplus[C(j),D(j+1)].$$

By inspection, $[C(j),D(j-1)]=0$. By proposition 4.5.1,

$$[C(j),D(j)]\cong\hom(H_j(C),H_j(D))=\mathrm{Ext}^0(H_j(C),H_j(D))$$

and $[C(j),D(j+1)]=\mathrm{Ext}^1(H_j(C),H_{j+1}(D))$. □

Now let's go for a new formulation of the universal coefficient theorem, covariant case.

PROPOSITION 4.6.2. *For $C=\bigoplus_j C(j)$ in $\mathcal{C}$ and an abelian group A there is a canonical isomorphism*

$$H_j(C\otimes A)\cong\big(H_j(C)\otimes A\big)\ \oplus\ \mathrm{Tor}_1(H_{j-1}(C),A)=\mathrm{Tor}_0(H_j(C),A)\oplus\mathrm{Tor}_1(H_{j-1}(C),A).$$

For a morphism $f\colon C\to D$ in $\mathcal{C}$, where $D=\bigoplus_j D(j)$, the following is commutative,

$$\begin{array}{ccc}
H_j(C\otimes A) & \xrightarrow[\text{canon.}]{\cong} & \mathrm{Tor}_0(H_j(C),A)\ \oplus\ \mathrm{Tor}_1(H_{j-1}(C),A)\\
\Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle \begin{pmatrix}[f]^j_0 & [f]^{j-1}_1\\ 0 & [f]^{j-1}_0\end{pmatrix}}\\
H_j(D\otimes A) & \xrightarrow[\text{canon.}]{\cong} & \mathrm{Tor}_0(H_j(D),A)\ \oplus\ \mathrm{Tor}_1(H_{j-1}(D),A)
\end{array}$$

in the notation of lemma 4.6.1 and corollary 4.5.3.

PROOF. It is enough to try this out if $C = C(j)$ for some j and $D = D(j+i)$ for the same j and some $i \in \{0,1\}$. Then $[f]$ is fully described by $[f]_i^j$ for this j and i. Then we are back in the situation of corollary 4.5.3. □

Notice how this explains the presence or absence of naturality in the standard formulation of the universal coefficient theorem. If we arrange the splitting

$$H_j(C \otimes A) \cong \mathrm{Tor}_0(H_j(C), A) \oplus \mathrm{Tor}_1(H_{j-1}(C), A)$$

in the shape of a short exact sequence

$$0 \longrightarrow \mathrm{Tor}_0(H_j(C), A) \longrightarrow H_j(C \otimes A) \longrightarrow \mathrm{Tor}_1(H_{j-1}(C), A) \longrightarrow 0$$

(and ask whether this is natural), then we are suppressing the entry in row 1, column 2 of the 2×2-matrix in the diagram of proposition 4.6.2. Then we are only left with entries which can be described in terms of $f_* : H_*(C) \to H_*(D)$. Therefore our answer is: yes, this short exact sequence is natural.

Now it is easy to formulate a generalization of proposition 4.6.2. This is the Künneth theorem.

THEOREM 4.6.3. *For* $C = \bigoplus_j C(j)$ *in* $\mathcal{C}$ *and* $E = \bigoplus E(j)$ *in* $\mathcal{C}$ *there is a canonical isomorphism*

$$H_j(C \otimes E) \cong \bigoplus_{\substack{p,q,i \\ p+q+i=j}} \mathrm{Tor}_i(H_p(C), H_q(E))$$

where $p, q \in \mathbb{Z}$ *and* $i \in \{0,1\}$. *For a morphism* $f : C \to D$ *in* $\mathcal{C}$, *where* $D = \bigoplus_j D(j)$, *the following is commutative,*

$$\begin{tikzcd}
H_j(C \otimes E) \arrow[r, "\cong", "\text{canon.}"'] \arrow[d, "f_*"'] & \bigoplus\limits_{\substack{p,q,i \\ p+q+i=j}} \mathrm{Tor}_i(H_p(C), H_q(E)) \arrow[d, "\begin{pmatrix} [f]_0^p & [f]_1^{p-1} \\ 0 & [f]_0^{p-1} \end{pmatrix}_{p \in \mathbb{Z}}"] \\
H_j(D \otimes E) \arrow[r, "\cong", "\text{canon.}"'] & \bigoplus\limits_{\substack{p,q,i \\ p+q+i=j}} \mathrm{Tor}_i(H_p(D), H_q(E))
\end{tikzcd}$$

in the notation of lemma 4.6.1 and corollary 4.5.3.

PROOF. It is easy to reduce this to the situation where $C = C(0)$ and $E = E(0)$. Then we can write $C = \Phi(A)$ and $E = \Phi(B)$, where $A = H_0(C)$ and $B = H_0(E)$. Then we are back to the situation of definition 4.3.1 and proposition 4.3.6. □

Once again this formulation of the Künneth theorem leads to a cautious naturality statement. If we arrange the splitting

$$H_j(C \otimes E) \cong \bigoplus_{\substack{p,q,i \\ p+q+i=j}} \mathrm{Tor}_i(H_p(C), H_q(E))$$

in the shape of a short exact sequence

$$\bigoplus_{\substack{p,q\\p+q=j}} \mathrm{Tor}_0(H_p(C),H_q(E)) \longrightarrow H_j(C\otimes E) \longrightarrow \bigoplus_{\substack{p,q\\p+q+1=j}} \mathrm{Tor}_1(H_p(C),H_q(E))$$

(and ask whether this is natural), then we are suppressing the entries in row 1, column 2 of the 2×2-matrices in the diagram of theorem 4.6.3. Then we are only left with entries which can be described in terms of $f_*\colon H_*(C)\to H_*(D)$. Therefore our answer is: yes, this short exact sequence is natural. The argument covers naturality in the first variable, here C, but by a similar argument we can also establish naturality in the second variable, here E, since $\otimes$ has certain symmetry properties.

The Künneth theorem has an easier variant for chain complexes defined over fields. By a chain complex defined over a field K, we mean a chain complex C where each chain group C_r comes with the additional structure of a K-vector space, and the differentials $C_r\to C_{r-1}$ are K-linear.

Proposition 4.6.4. *For chain complexes C,D over a field K, there is a natural isomorphism of vector spaces*

$$H_j(C\otimes_K D) \xleftarrow{\ \cong\ } \bigoplus_{\substack{p,q\\p+q=j}} H_p(C)\otimes_K H_q(D)$$

given by the formula

$$H_{p+q}(C\otimes_K D \ni [x\otimes y] \quad \leftarrow\!\shortmid \quad [x]\otimes[y]\in H_p(C)\otimes_K H_q(D).$$

Proof. Let $Z_pC\subset C_p$ be the subgroup of p-cycles. The map from $Z_pC\times Z_qD$ to $Z_{p+q}(C\otimes_K D)$ given by $(x,y)\mapsto x\otimes y$ is K-bilinear. If one of x,y is a boundary, then $\mu(x,y)$ is also a boundary (by the definition of the differential in $C\otimes_K D$). Therefore we obtain an induced K-bilinear map $H_p(C)\times H_q(D)\to H_{p+q}(C\otimes_K D)$. Assembling these maps for all p,q satisfying $p+q=j$, we obtain a K-linear map

$$\bigoplus_{\substack{p,q\\p+q=j}} H_p(C)\otimes_K H_q(D) \longrightarrow H_j(C\otimes_K D).$$

If the differentials in C and D are all zero, then there is no need to make a distinction between C and $H_*(C)$, or between D and $H_*(D)$, and it is clear that this last map is an isomorphism. The general case follows from this special case because every chain complex over K is chain homotopy equivalent (as such) to a chain complex with differential zero. (Exercise. Begin with the observation that every short exact sequence $0\to A\to B\to C\to 0$ splits, i.e., the K-linear map $B\to C$ admits a right inverse as a K-linear map.) □

4.7. Various products in homology and cohomology

We return to chain complexes $C,D,E,\dots$ of abelian groups.
For chain complexes C and D, there is a homomorphism

$$H_p(C)\otimes H_q(D) \longrightarrow H_{p+q}(C\otimes D)$$

defined by $[x]\otimes[y]\mapsto[x\otimes y]$ (for a p-cycle x in C and a q-cycle y in D). This is called the *exterior product in homology.* Here we are not assuming that all chain groups are *free* abelian. To make this look more general we can select abelian groups A and B and substitute $C \otimes A$ for C, and $D \otimes B$ for D. Then we have an exterior homology product

$$H_p(C \otimes A) \otimes H_q(D \otimes B) \longrightarrow H_{p+q}(C \otimes D \otimes (A \otimes B)).$$

(But it is not really more general.)

For chain complexes C, D, E and F there is a homomorphism

$$H_{-p}(\mathrm{hom}(C,E)) \otimes H_{-q}(\mathrm{hom}(D,F)) \longrightarrow H_{-(p+q)}(\mathrm{hom}(C \otimes D, E \otimes F)).$$

It is defined by

$$[f] \otimes [g] \mapsto (-1)^{q|x|}\big[(x \otimes y \mapsto f(x) \otimes g(y)\big]$$

where $f : C \to \Sigma^p D$ and $g : D \to \Sigma^q F$ are chain maps and $x \in C$, $y \in D$ are chains in arbitrary degrees. If we adopt cochain complex notation, writing H^p instead of H_{-p} etc., then we can write this in the form

$$H^p(\mathrm{hom}(C,E)) \otimes H^q(\mathrm{hom}(D,F)) \longrightarrow H^{p+q}(\mathrm{hom}(C \otimes D, A \otimes B)).$$

Then we can also specialize some more by assuming that E and F are concentrated in degree zero (i.e., they are just abelian groups). This specialization would be called the *exterior product in cohomology.*

[I am a little reluctant to use these terms here, *exterior product in homology/cohomology*, because they are commonly used in the context of singular chain complexes, say $C = sC(X)$ and $D = sC(Y)$ for spaces X and Y. In that situation we have, in addition to the exterior products

$$H_p(C) \otimes H_q(D) \longrightarrow H_{p+q}(C \otimes D),$$

a preferred chain homotopy equivalence from $C \otimes D = sC(X) \otimes sC(Y)$ to $sC(X \times Y)$, which will be the topic of subsequent chapters. Then we can think of the exterior homology product as a homomorphism from $H_p(X) \otimes H_q(Y)$ to $H_{p+q}(X \times Y)$, or more generally, from $H_p(X;A) \otimes H_q(Y;B)$ to $H_{p+q}(X \times Y; A \otimes B)$. Similarly the exterior product in cohomology turns into a homomorphism from $H^p(X) \otimes H^q(Y)$ to $H^{p+q}(X \times Y)$, or more generally, from $H^p(X;A) \otimes H^q(Y;B)$ to $H^{p+q}(X \times Y; A \otimes B)$.]

4.8. The algebraic mapping cone and the meaning of Ext^1

In topology, we have the important constructions of *mapping cylinder* and *mapping cone* of a map $f : X \to Y$. The mapping cylinder $Z(f)$ of f is the quotient of the disjoint union

$$[0,1] \times X \;\sqcup\; Y$$

by the relations $[0,1] \times X \ni (1,x) \sim f(x) \in Y$. (We are gluing points $(1,x)$ in $[0,1] \times X$ to $f(x) \in Y$.) Useful properties: the map f has a factorization of the form

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & Z(f) \arrow[ur, "q"'] & \end{tikzcd}$$

where j is a cofibration (has the homotopy extension property, HEP) and q is a homotopy equivalence. The maps j and q are given by $j(x) = (0,x)$ and $q(t,x) = f(x)$ for $(t,x) \in [0,1] \times X$, and $q(y) = y$ for $y \in Y$.

The mapping cone of f, denoted $\mathrm{cone}(f)$, is the quotient $Z(f)/j(X)$, or more precisely, the pushout of

$$Z(f) \xleftarrow{j} X \to \{0\}$$

(In relation to definition 3.1.1 I seem to have switched the endpoints 0 and 1 of $[0,1]$ ... but I am in better agreement with the Dold book as a result. Must fix something in definition 3.1.1.) If f is a cellular map between CW-spaces, then $Z(f)$ is also a CW-space (in such a way that the quotient map $[0,1] \times X \sqcup Y \to Z(f)$ is cellular, and the inclusion $Y \to Z(f)$ is the inclusion of a CW-subspace, and the cells of $[0,1] \times X$ not in $\{1\} \times X$ map homeomorphically to cells of $Z(f)$). Then $\mathrm{cone}(f)$ is also a CW-space, being the quotient of $Z(f)$ by the CW-subspace $j(X)$.

If we write $C(f)\colon C(X) \to C(Y)$ for the induced map of cellular chain complexes, then we see that $C(Z(f))$ can be described as follows: it is the quotient of

$$C([0,1]) \otimes C(X) \ \oplus \ C(Y)$$

by relations $z_1 \otimes x \sim C(f)(x) \in C(Y)$, where $z_1 \in C([0,1])$ denotes the 0-cycle corresponding to the 0-cell $\{1\} \subset [0,1]$. Therefore we are led to the following definitions.

Definition 4.8.1. Let $f\colon C \to D$ be a chain map and let $B = C([0,1])$ be the cellular chain complex of the unit interval:

$$B_0 = \mathbb{Z} \oplus \mathbb{Z}, \ B_1 = \mathbb{Z}, \ d\colon B_1 \to B_0; \ t \mapsto (-t, t)$$

and $B_j = 0$ for all $j \notin \{0,1\}$. We write $z_0, z_1 \in B_0$ and $w \in B_1$ for the preferred generators. The *mapping cylinder* $Z(f)$ of f is the quotient

$$B \otimes C \ \oplus \ D$$

modulo the relations $z_1 \otimes x \sim f(x) \in D$, where $x \in C$. The chain map f has a factorization

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "j"'] & & D \\ & Z(f) \arrow[ur, "q"'] & \end{tikzcd}$$

where $j(x) = z_0 \otimes x$ for $x \in C$ and $q(z_0 \otimes x) = f(x) = q(z_1 \otimes x,\ q(w \otimes x) = 0,\ q(y) = y$ for $y \in D$. (In this factorization, j is an injection and q is a chain homotopy equivalence.) The mapping cone of f is the quotient of $Z(f)$ by the subcomplex $j(C)$. Therefore we have the following explicit description of the mapping cone, $\mathrm{cone}(f)$:

$$(\mathrm{cone}(f))_r = D_r \oplus C_{r-1}$$
$$d(y, x) = (d(y) + f(x), -d(x)).$$

Proposition 4.8.2. *For a chain map $f\colon C \to D$, there is a long exact sequence of homology groups*

$$\cdots \to H_k(C) \xrightarrow{f_*} H_k(D) \to H_k(\mathrm{cone}(f)) \to H_{k-1}(C) \to \cdots$$

Proof. This is essentially the long exact sequence of homology groups associated with the short exact sequence of chain complexes

$$j(C) \hookrightarrow Z(f) \longrightarrow \mathrm{cone}(f).$$

We only need to observe that $q\colon Z(f) \to D$ is a chain homotopy equivalence and so induces an isomorphism in homology groups. (To show that q is a homotopy equivalence: we have the inclusion $e: D \to Z(f)$ and clearly $qe = \mathrm{id}_D$. Therefore we need a chain homotopy from id to $eq: Z(f) \to Z(f)$. A solution is h defined by $h(z_0 \otimes x) = w \otimes x$, $h(w \otimes x) = 0$ for $x \in C$, and $h(y) = 0$ for $y \in D$.) □

It is rather obvious that the definition of the algebraic mapping cone is natural. If we have a commutative diagram of chain maps

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d] & B \arrow[d] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

then there is an induced chain map $\mathrm{cone}(e) \to \mathrm{cone}(f)$. The following is an important special case: if $e: A \to B$ is the inclusion of a subcomplex, then we can make a commutative diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & B/A \end{tikzcd}$$

This leads to a chain map $\mathrm{cone}(e) \to \mathrm{cone}(0 \to B/A)$, but of course $\mathrm{cone}(0 \to B/A)$ is just B/A. In short we get a canonical projection from $\mathrm{cone}(e)$ to B/A (in the case where $e: A \to B$ is an inclusion).

COROLLARY 4.8.3. *If* $e: A \to B$ *is the inclusion of a chain subcomplex, then the projection from* $\mathrm{cone}(e)$ *to* B/A *induces an isomorphism in homology.*

PROOF. Follows from the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "="] & Z(e) \arrow[r] \arrow[d, "q"', "\simeq"] & \mathrm{cone}(e) \arrow[d] \\ A \arrow[r] & B \arrow[r] & B/A \end{tikzcd}$$

with short exact rows. (Form the long exact homology group sequence for each row and apply the five lemma.) □

Now we turn to the meaning of Ext^1. It is supposed to have something to do with group extensions (in the setting of abelian groups). So let us clarify what we mean by that. Given abelian groups A and B, an *extension of* B *with kernel* A shall mean: a short exact sequence of abelian groups

$$A \xrightarrow{j} E \xrightarrow{p} B.$$

Two such extensions (of B with kernel A), say $A \xrightarrow{j} E \xrightarrow{p} B$ and $A \xrightarrow{k} F \xrightarrow{q} B$, will be regarded as *equivalent* if there exists an isomorphism $u: E \to F$ such that $uj = k$ and $qu = p$. (It is indeed an equivalence relation although we cannot honestly claim that it is defined on a *set*.)

THEOREM 4.8.4. *The equivalence classes of such extensions are in canonical bijection with* $\mathrm{Ext}^1(B, A)$.

PROOF. This proof is merely a sketch. An element x of $\mathrm{Ext}^1(B, A)$ can be represented by a chain map

$$f^x : \Phi(B) \longrightarrow \Sigma^1 \Phi(A)$$

where $\Phi(B)$ and $\Phi(A)$ are free resolutions of B and A, respectively. (That was one of the definitions; $\mathrm{Ext}^1(B, A) = \Phi(B), \Sigma^1\Phi(A)]$.) We form the algebraic mapping cone cone(f^x). By proposition 4.8.2, there is a long exact sequence of homology groups

$$\cdots \longrightarrow H_1(\Sigma^1\Phi(A)) \longrightarrow H_1(\mathrm{cone}(f^x)) \longrightarrow H_0(B) \xrightarrow{(f^x)_*} H_0(\Sigma^1\Phi(A)) \longrightarrow \cdots$$

which immediately simplifies to a short exact sequence $A \to H_1(\mathrm{cone}(f^x)) \to B$. This is the extension associated to the element x. □

The theorem has a curious consequence. By construction $\mathrm{Ext}^1(B, A)$ is an abelian group. Therefore we obtain a structure of abelian group on the set of equivalence classes of extensions of B with kernel A. Does this have a direct definition? Here is a sketch of such a direct definition. Start with extensions (short exact sequences)

$$A \xrightarrow{j} E \xrightarrow{p} B \qquad \text{and} \qquad A \xrightarrow{k} F \xrightarrow{q} B.$$

Form the quotient

$$D := \frac{\{\,(x, y) \in E \oplus F \mid p(x) = q(y)\,\}}{\{\,(j(a), -k(a)) \mid a \in A\,\}}$$

This has an obvious homomorphism to B given by $(x, y) \mapsto p(x) = q(y)$, and receives an obvious homomorphism from A given by $a \mapsto$ (class of $(j(a), 0)$) alias (class of $(0, k(a))$). Together, these homomorphisms form a new exact sequence $A \to D \to B$. The equivalence class of this is our candidate for the "sum" of the equivalence classes of $A \to E \to B$ and $A \to F \to B$.

CHAPTER 5

The Eilenberg-Zilber theorem

5.1. The method of acyclic models

The lecture notes for my Topology I course, WS 2017, contain a section with the same title, *the method of acyclic models*. The main result there was as follows:

- *Let* $\alpha: sC(X) \to sC(X)$ *be a natural chain map. If* $\alpha: sC(*)_0 \to sC(*)_0$ *is the zero homomorphism, then* α *admits a natural chain homotopy to zero.*

(The emphasis is on *natural*,. i.e., we are assuming that $\alpha: sC(X) \to sC(X)$ is defined for every space X, and behaves naturally with respect to continuous maps $f: X \to Y$.)
The one and only application of this in the Topology I lecture notes was to $\alpha = \beta - \mathrm{id}$, where $\mathrm{id}: sC(X) \to sC(X)$ is the identity map and $\beta: sC(X) \to sC(X)$ is/was a natural chain map called *barycentric subdivision*. The conclusion was: β is naturally chain homotopic to the identity map.

Here we wish to prove more results of that type using the same technology. A major goal will be to show that the functors $(X, Y) \mapsto sC(X) \otimes sC(Y)$ and $(X, Y) \mapsto sC(X \times Y)$ are related by a natural chain homotopy equivalence (where X and Y are arbitrary topological spaces).

Definition 5.1.1. Let $\mathcal{K}$ be a category and let F be a functor from $\mathcal{K}$ to the category of abelian groups. A *basis* for F consists of a set S and an assignment which for every $s \in S$ selects

- an object x_s
- and an element $b_s \in F(x_s)$.

These are subject to the following condition: for every y in $\mathcal{K}$, the map

$$\coprod_{s \in S} \mathrm{mor}(x_s, y) \longrightarrow F(y)$$

defined by $\mathrm{mor}(x_s, y) \ni f \mapsto f(b_s)$ is injective and its image freely generates the abelian group $F(y)$ (in particular $F(y)$ is a free abelian group).
In such a case we say that F is free, and more precisely, that it is free with basis

$$(b_s \in F(x_s))_{s \in S}.$$

Equivalent formulation. The functor F is *free* if it is isomorphic to a direct sum

$$\bigoplus_{s \in S} \Psi \circ \mathrm{mor}_{\mathcal{K}}(x_s, -).$$

Here Ψ is the well known functor from the category of sets to the category of abelian groups which takes a set V to the free abelian group generated by V. And $\mathrm{mor}_{\mathcal{K}}(x_s, -)$ is the representable (covariant) functor from $\mathcal{K}$ to the category of sets determined by an object x_s of $\mathcal{K}$. The functor $\mathrm{mor}_{\mathcal{K}}(x_s, -)$ takes an object y of $\mathcal{K}$ to the set of morphisms $\mathrm{mor}_{\mathcal{K}}(x_s, y)$.

Example 5.1.2. In this example, $\mathcal{K}$ is the category of topological spaces. Fix an integer $n \geq 0$. The functor taking a topological space X to the n-th chain group $(sC(X))_n$ of the singular chain complex of X is free, and it has a basis consisting of *one* element. Indeed, $(sC(X))_n$ is the free abelian group generated by $\mathrm{mor}_{\mathcal{K}}(\Delta^n, X)$. So we can take $S = \{s\}$ and $x_s = \Delta^n$ and $b_s = \mathrm{id} \in (sC(\Delta^n))_n$.

Proposition 5.1.3. *Suppose that F and G are functors from $\mathcal{K}$ to the category of abelian groups. If F is free with basis $(b_s \in F(x_s))_{s\in S}$, then for any selection $(c_s \in G(x_s))_{s\in S}$ there exists a unique natural transformation $\nu : F \Rightarrow G$ such that $\nu(b_s) = c_s \in G(x_s)$.*

Proof. By the *equivalent formulation*, we may assume that F is a direct sum of functors of the form $\Psi \circ \mathrm{mor}_{\mathcal{K}}(x_s, -)$. It is easy to reduce to the case where there is only one summand. Then we can write x for x_s; consequently $F = \Psi \circ \mathrm{mor}_{\mathcal{K}}(x, -)$. (So F is the functor taking $y \in \mathcal{K}$ to the free abelian group generated by the set $\mathrm{mor}(x, y)$.) The functor Ψ has a right adjoint L, known as the forgetful functor from the category of abelian groups to the category of sets. Therefore natural transformations

$$\Psi \circ \mathrm{mor}_{\mathcal{K}}(x, -) \longrightarrow G$$

correspond bijectively to natural transformations

$$\mathrm{mor}_{\mathcal{K}}(x, -) \longrightarrow L \circ G$$

(between functors from $\mathcal{K}$ to the category of sets). Now we are in a position to apply the Yoneda lemma: natural transformations from $\mathrm{mor}_{\mathcal{K}}(x, y)$ to any other functor E from $\mathcal{K}$ to the category of sets are fully determined by their value on $\mathrm{id}_x \in \mathrm{mor}_{\mathcal{K}}(x, x)$, which can be any prescribed element of $E(x)$. Our case is $E = L \circ G$ and so we are selecting an element c in the underlying set of $G(x)$. □

Remark. In general, if F and G are functors from $\mathcal{K}$ to the category of abelian groups, we cannot take it for granted that there is a *set* of natural transformations from F to G. But in the situation of the proposition, we can, and the set of natural transformations from F to G is in canonical bijection with $\prod_{s\in S} G(x_s)$. It is in fact an abelian group, and this makes sense.

Corollary 5.1.4. *Let F, G_1 and G_2 be functors from $\mathcal{K}$ to the category of abelian groups. Suppose that F is free with basis $(b_s \in F(x_s))_{s\in S}$. Let $p : G_1 \to G_2$ be a natural transformation such that $p : G_1(x_s) \to G_2(x_s)$ is surjective for every x_s, where $s \in S$. Then every natural transformation $e : F \to G_2$ admits a factorization $e = p\bar{e}$, where $\bar{e}$ is a natural transformation from F to G_1.* □

$$\begin{tikzcd} & G_1 \arrow[d, "p"] \\ F \arrow[ur, dotted, "\bar{e}"] \arrow[r, "e"'] & G_2 \end{tikzcd}$$

Theorem 5.1.5. (Acyclic model theorem) *Let F and G be functors from $\mathcal{K}$ to the category of chain complexes; write F_j and G_j for the degree j parts. Suppose that $F_j = 0$ for $j < 0$, and F_j for $j \geq 0$ is free with basis*

$$(b_s \in F_j(x_s))_{s\in S_j}$$

where the sets S_j are pairwise disjoint. Suppose that $H_k(G(x_s)) = 0$ for $k > 0$ and $s \in S_j$, where $j \geq 0$. Then every natural transformation $H_0 \circ F \to H_0 \circ G$ of functors from $\mathcal{K}$ to abelian groups is induced by a natural transformation $F \to G$, unique up to natural chain homotopy.

PROOF. We begin with a natural transformation $u\colon H_0\circ F\to H_0\circ G$ between functors from $\mathcal{K}$ to the category of abelian groups, and we are hoping to construct a natural transformation $\nu\colon F\to G$. The idea is to construct $\nu_n\colon F_n\to G_n$ (natural transformation between functors from $\mathcal{K}$ to the category of abelian groups) by induction on n. Write informally Z for subgroup(s) of cycles (in chain complexes).
The construction of ν_0 can be understood from the (commutative) diagram

$$\begin{tikzcd} F_0 = ZF_0 \arrow[r, dotted] \arrow[d] & ZG_0 \arrow[d, two heads] \\ H_0F \arrow[r, "u"] & H_0G \end{tikzcd}$$

Since the right-hand vertical arrow is (always) surjective, the dotted arrow can be found by corollary 5.1.4. For the induction step, let us assume that $\nu_k\colon F_k\to G_k$ has already been constructed for $k=0,1,2,\dots,n$ and that $d\nu_k=\nu_{k-1}d$ holds where applicable. Then in particular $d\nu_n=\nu_{n-1}d$, so that ν_n takes ZF_n to ZG_n. The construction of ν_{n+1} can then be understood from the commutative diagram

$$\begin{tikzcd} F_{n+1} \arrow[r, dotted, "\nu_{n+1}"] \arrow[d] & G_{n+1} \arrow[d, "d"] \\ ZF_n \arrow[r, "\text{restr. of } \nu_n"] & ZG_n \end{tikzcd}$$

Here the right-hand vertical arrow is surjective if we evaluate it on any of the special objects x_s (where $s\in S_n$), because we assumed $H_n(G(x_s))=0$. Therefore the dotted arrow can be found by corollary 5.1.4. This completes the induction step and the construction of ν.
Now we have to show that if $\nu\colon F\to G$ is a natural chain map inducing the zero transformation from H_0F to H_0G, then ν is naturally nullhomotopic. In other words we have to construct natural transformations

$$h_n\colon F_n\to G_{n+1}$$

so that $dh_n+h_{n-1}d=\nu_n$ for all $n\ge 0$ (where h_{-1} is necessarily zero). We will construct them by induction on n. Write informally B for subgroups of boundaries (in chain complexes). — The construction of h_0 can be understood from the (commutative) diagram

$$\begin{tikzcd} F_0 \arrow[r, dotted, "h_0"] \arrow[d, "="] & G_1 \arrow[d, two heads] \\ F_0 \arrow[r, "\nu_0"] & BG_0 \end{tikzcd}$$

For the induction step, let us assume that $h_k\colon F_k\to G_{k+1}$ has already been constructed for $k=0,1,2,\dots,n-1$ and that $dh_k+h_{k-1}d=\nu_k$ holds where applicable. Then in particular $dh_{n-1}+h_{n-2}d=\nu_{n-1}$ and so $dh_{n-1}d=\nu_{n-1}d=d\nu_n$, and so we can write $\nu_n-h_{n-1}d\colon F_n\to ZG_n$. The construction of h_n can be understood from the (commutative) diagram

$$\begin{tikzcd} F_n \arrow[r, dotted, "h_n"] \arrow[d, "="] & G_{n+1} \arrow[d, "d"] \\ F_n \arrow[r, "\nu_n-h_{n-1}d"] & ZG_n \end{tikzcd}$$

Here the right-hand vertical arrow is surjective if we evaluate it on any of the special objects x_s (where $s \in S_n$), because we assumed $H_n(G(x_s)) = 0$. Therefore the dotted arrow can be found by corollary 5.1.4. This completes the induction step and so it completes the construction of h. □

REMARK 5.1.6. As Dold observes in his book, the conditions on G can be weakened. We did not have to know that $H_n(G(x_s)) = 0$ whenever $n > 0$ and $s \in S_j$ for some $j \geq 0$. We only used $H_j(G(x_s)) = 0$ and $H_{j+1}(G(x_s)) = 0$ when $s \in S_j$, where $j \geq 1$, and also $H_1(G(x_s)) = 0$ when $s \in S_0$.
But if we insist on the stronger condition, $H_n(G(x_s)) = 0$ whenever $n > 0$ and $s \in S_j$ for some $j \geq 0$, the we can make a stronger statement. We can speak of a chain complex $\hom(F, G)$ of natural transformations. We proved that an obvious homomorphism

$$H_0(\hom(F, G)) \to \hom(H_0 F, H_0 G)$$

is an isomorphism. But the same method also proves that $H_j(\hom(F, G)) = 0$ for all $j > 0$. This has many useful consequences, and therefore I rather like the stronger condition. (Although ... it is still excessive. It would be enough to know $H_k(G(x_s)) = 0$ when $s \in S_j$ and $k \geq j$, where $j \geq 1$, and also $H_k(G(x_s)) = 0$ when $s \in S_0$ and $k > 0$.)

5.2. Products of spaces and tensor products of chain complexes

Recall that we/I write $sC(X)$ for the singular chain complex of a space X and $\mathcal{T}op$ for the category of topological spaces.

THEOREM 5.2.1. (Eilenberg-Zilber theorem.) *The functors* $(X, Y) \mapsto sC(X) \otimes sC(Y)$ *and* $(X, Y) \mapsto sC(X \times Y)$ *from* $\mathcal{T}op \times \mathcal{T}op$ *to the category of chain complexes are related by a natural homotopy equivalence. More precisely, there exists a natural chain map*

$$U : sC(X) \otimes sC(Y) \longrightarrow sC(X \times Y)$$

such that U_0 *from* $sC(*)_0 \otimes sC(*)_0 = \mathbb{Z} \otimes \mathbb{Z}$ *to* $sC(* \times *)_0 = \mathbb{Z}$ *is the standard isomorphism determined by the multiplication on* $\mathbb{Z}$. *Any natural chain map satisfying this condition is a natural homotopy equivalence; i.e., there exists a natural chain map*

$$V : sC(X \times Y) \longrightarrow sC(X) \otimes sC(Y)$$

and natural chain homotopies from VU *to* id *and from* UV *to* id.

PROOF. Write $\mathcal{K} = \mathcal{T}op \times \mathcal{T}op$. Let F be the functor $(X, Y) \mapsto sC(X) \otimes sC(Y)$ and let G be the functor $(X, Y) \mapsto sC(X \times Y)$. These are both functors from $\mathcal{K}$ to the category of chain complexes. We proceed to verify that both F_n and G_n are free (in the sense of definition 5.1.1) for all n. (The cases where $n < 0$ are trivial.) Indeed, F_n for $n \geq 0$ is free with basis

$$(\iota_p \otimes \iota_{n-p} \in F_n(\Delta^p, \Delta^{n-p}))_{p=0,1,\ldots,n}$$

where ι_p is the identity map of Δ^p, viewed as an element of $sC(\Delta^p)_p$. And G_n is free with basis

$$(\iota_n, \iota_n) \in G_n(\Delta^n, \Delta^n).$$

Furthermore $H_k(G(\Delta^p, \Delta^{n-p})) = 0$ for $k > 0$ and $p, n - p \geq 0$ and $H_k(F(\Delta^n, \Delta^n)) = 0$ for $k > 0$ and $n \geq 0$ since $sC(\Delta^n) \otimes sC(\Delta^n)$ and $sC(\Delta^p \times \Delta^{n-p})$ are both chain homotopy equivalent to the chain complex which we often call $(\mathbb{Z}, 0)$. Therefore the conditions of the acyclic model theorem 5.1.5 are satisfied (in both directions), and we learn from the acyclic model theorem that the classification up to natural chain homotopy of natural chain maps from F to G, from G to F, for F to F and from G to G coincides with the

classification of natural transformations from H_0F to H_0G, from H_0G to H_0F, from H_0F to H_0F and from H_0G to H_0G, respectively. So we can finish this proof comfortably by taking a look a the possibilities for such natural transformations $H_0F \to H_0G$, etc.
Every element of $H_0F(X,Y)$ is a finite linear combination (with integer coefficients) of elements in the image of a homomorphism $H_0F(*,*) \to H_0F(X,Y)$ induced by a morphism $(*,*) \to (X,Y)$ in $\mathcal{K}$. It follows that a natural transformation from H_0F to H_0G is determined by how it specializes to a homomorphism from $H_0F(*,*) \cong \mathbb{Z}\times\mathbb{Z}$ to $H_0G(*,*) \cong \mathbb{Z}$. Conversely, every homomorphism from $H_0F(*,*) \cong \mathbb{Z}\times\mathbb{Z}$ to $H_0G(*,*) \cong \mathbb{Z}$ is an integer multiple of a certain preferred *isomorphism* α which corresponds to the multiplication on the ring $\mathbb{Z}$. It is rather clear that α extends to a natural isomorphism $H_0F \to H_0G$, and therefore integer multiples of α also extend to natural homomorphisms $H_0F \to H_0G$. (Here it may be useful to remember the description of H_0 of a space as the free abelian group generated by the set of path components of that space.) Therefore we can say that natural transformations from H_0F to H_0G are classified by their *degree*, which is an integer. The analysis of natural transformations from H_0G to H_0F, from H_0F to H_0F and from H_0G to H_0G is similar. In all cases they are classified by their degree, an integer. If that integer is invertible, i.e. if it is ± 1, then the natural transformation in question is invertible. □

Let us make some observations about commutativity and associativity properties of the natural chain maps in the Eilenberg-Zilber theorem. These are important, but I prefer not to formalize them very much.
By *commutativity*, I mean the following. Let u be a natural chain map as in the theorem. Then we can make a new natural transformation, call it u' for now, by composing the arrows in the following diagram (for every object (X,Y) in $\mathcal{T}op \times \mathcal{T}op$):

$$\begin{tikzcd} sC(X)\otimes sC(Y) \arrow[rr, "a\otimes b\mapsto \pm b\otimes a", "\cong"'] & & sC(Y)\otimes sC(X) \arrow[d, "u_{(Y,X)}"] \\ sC(X\times Y) & & sC(Y\times X) \arrow[ll, "\text{induced by } (y,x)\mapsto(x,y)"', "\cong"] \end{tikzcd}$$

(Here $\pm b\otimes a$ is short for $(-1)^{|a|\cdot|b|} b\otimes a$.) We ask whether u' is naturally chain homotopic to u. The answer is yes. It is an easy answer because H_0u' and H_0u agree.
By associativity, I mean the following. Let u be a natural chain map as in the theorem. Then we have *two* easy ways to make a natural transformation

$$sC(X)\otimes sC(Y)\otimes sC(Z) \longrightarrow sC(X\times Y\times Z).$$

One of these is

$$(sC(X)\otimes sC(Y))\otimes sC(Z) \xrightarrow{u_{(X,Y)}\otimes \mathrm{id}} sC(X\times Y)\otimes sC(Z) \xrightarrow{u_{(X\times Y,Z)}} sC(X\times Y\times Z)$$

and the other is

$$sC(X)\otimes (sC(Y)\otimes sC(Z)) \xrightarrow{\mathrm{id}\otimes u_{(Y,Z)}} sC(X)\otimes sC(Y\times Z) \xrightarrow{u_{(X,Y\times Z)}} sC(X\times Y\times Z).$$

We ask whether these two are naturally chain homotopic (as natural transformations between functors from $\mathcal{T}op\times\mathcal{T}op\times\mathcal{T}op$ to chain complexes). The answer is again yes. This does not follow directly from theorem 5.2.1, though. Instead, theorem 5.1.5 can be applied in a situation where $\mathcal{K} = \mathcal{T}op\times\mathcal{T}op\times\mathcal{T}op$. Then it is necessary to verify that certain functors from $\mathcal{T}op \times \mathcal{T}op \times \mathcal{T}op$ to the category of chain complexes are degreewise free in the sense of 5.1.1.

Similar things can be said about V, homotopy inverse of U in theorem 5.2.1. Here another important observation can be made. Using V we can make a natural transformation (call it K temporarily) between functors on $\mathcal{T}op$ as follows:

$$sC(X) \xrightarrow{\text{induced by } x \mapsto (x,x)} sC(X \times X) \xrightarrow{V_{(X,X)}} sC(X) \otimes sC(X)$$

Again this has certain commutativity and associativity properties up to natural chain homotopy. For example, commutativity up to natural chain homotopy means that K is naturally chain homotopic to $\tau \circ K$, where τ from $sC(X) \otimes sC(X)$ to $sC(X) \otimes sC(X)$ is given by $a \otimes b \mapsto (-1)^{|a|\cdot|b|} b \otimes a$. Associativity up to natural chain homotopy means that the compositions

$$sC(X) \xrightarrow{K} sC(X) \otimes sC(X) \xrightarrow{K \otimes \mathrm{id}} (sC(X) \otimes sC(X)) \otimes sC(X)$$

and

$$sC(X) \xrightarrow{K} sC(X) \otimes sC(X) \xrightarrow{\mathrm{id} \otimes K} sC(X) \otimes (sC(X) \otimes sC(X))$$

are naturally chain homotopic.

This raises an interesting question. Since K is only unique up to natural chain homotopy, we seem to have a choice. Can we make that choice in such a way that K is strictly commutative and/or strictly associative? It turns out, and we will probably see some explicit formulae later, that we *can* make a choice of K which is strictly associative. Surprisingly, it is not possible to make a choice of K which is strictly commutative (even if we are willing to sacrifice associativity). Steenrod exploited the noncommutative features of K to construct some cohomology *operations*, more precisely, natural transformations $H^n(-;\mathbb{F}_p) \to H^{n+k}(-;\mathbb{F}_p)$. (Going into that would take us too far, but remember that Steenrod operations are very important.)

Remark 5.2.2. We have learned that the singular chain complex $sC(X)$ of a space X has a preferred "diagonal" chain map $K: sC(X) \to sC(X) \otimes sC(X)$, well defined at least up to chain homotopy, and with various good properties such as commutativity and associativity up to chain homotopy. At the risk of stating the obvious, I just want to point out that "random" chain complexes C do not come with a preferred diagonal chain map $C \to C \otimes C$. This has something to do with the fact that the tensor product of chain complexes is not a product in the sense of category theory. (It does not have the universal property that a categorical product should have; there are no obvious "projections" from $C \otimes D$ to C and/or D, and so on.)

5.3. A zoo of products

Scalar product. This is an easy product which has the form of a biadditive map

$$H^n(X;A) \times H_n(X) \longrightarrow A\ ;\ (f,x) \mapsto \langle f,x \rangle$$

where A can be any abelian group. Idea: take a cocycle f of degree $-n$ in $\mathrm{hom}(sC(X),A)$; take cycle c of degree n in $sC(X)$; then $f(c) \in A$ depends only on $[f] \in H^n(X;A)$ and $[c] \in H_n(X)$. — We have already seen this in connection with the universal coefficient theorem, UCT. We can write it in adjoint form

$$H^n(X;A) \longrightarrow \mathrm{hom}(H_n(X), A)$$

and the UCT tells us that this is onto with kernel isomorphic to $\mathrm{Ext}^1(H_{n-1}(X), A)$.

Exterior product in homology. This is a more serious product which in the simplest case has the form of a bi-additive map

$$H_k(X) \times H_\ell(Y) \longrightarrow H_{k+\ell}(X \times Y).$$

The standard notation for this is $(u, v) \mapsto u \times v$.
It is obtained as follows. Take a k-cycle $a \in sC(X)$ and an ℓ-cycle $b \in sC(Y)$. Then $a \otimes b$ is a $(k+\ell)$-cycle in $sC(X) \otimes sC(Y)$. To this we apply an Eilenberg-Zilber chain map

$$U : sC(X) \otimes sC(Y) \longrightarrow sC(X \times Y)$$

as in theorem 5.2.1. Then $U(a \otimes b)$ is a $(k+\ell)$-cycle in $sC(X \times Y)$. The homology class $[U(a \otimes b)]$ depends only on the homology classes $[a]$ and $[b]$.
We can write the exterior product in homology in the form of a homomorphism

$$H_k(X) \otimes H_\ell(Y) \longrightarrow H_{k+\ell}(X \times Y).$$

This makes the connection with the Künneth theorem 4.6.3 (and following half-page) and therefore we can state at no cost to us:

Corollary 5.3.1. *The exterior product in homology*

$$\bigoplus_{k=0,1,\dots,m} H_k(X) \otimes H_{m-k}(Y) \longrightarrow H_m(X \times Y)$$

is injective and its cokernel is isomorphic to $\bigoplus_{k=0,\dots,m-1} \mathrm{Tor}_1(H_k(X), H_{m-k-1}(Y))$. □

A straightforward generalization of the exterior homology product is the exterior homology product with coefficients. For that we choose abelian groups A and B and we get a biadditive map

$$H_k(X; A) \times H_\ell(Y; B) \longrightarrow H_{k+\ell}(X \times Y; A \otimes B).$$

There is a variant of corollary 5.3.1 for homology with coefficients which I am not going to write out.

A more surprising generalization arises if we turn to the homology of pairs. So let (X, P) be a pair of spaces (and now this means that P is a subspace of X) and let (Y, Q) be a pair of spaces. The *categorical* product of (X, P) and (Y, Q), in the category of such pairs, would simply be the pair $(X \times Y, P \times Q)$. But this is not exactly what we need here. Instead we need $(X \times Y,\ X \times Q\ \cup P \times Y)$. The exterior product in homology for pairs has the form of a biadditive map

$$H_k(X, P) \times H_\ell(Y, Q) \longrightarrow H_{k+\ell}(X \times Y,\ X \times Q\ \cup\ P \times Y).$$

It is obtained as follows. Take a k-cycle $a \in sC(X)/sC(P)$ and an ℓ-cycle $b \in sC(Y)/sC(Q)$. Then $a \otimes b$ is a $(k+\ell)$-cycle in

$$(sC(X)/sC(P)) \otimes (sC(Y)/sC(Q)) = \frac{sC(X) \otimes sC(Y)}{sC(X) \otimes sC(Q)\ +\ sC(P) \otimes sC(Y)}.$$

(Here the $+$ symbol indicates an internal sum of chain subcomplexes of $sC(X) \otimes sC(X)$, typically not a direct sum since the two may have a nontrivial intersection.) We apply a natural Eilenberg-Zilber map U to this and obtain a $(k+\ell)$-cycle

$$U(a \otimes b) \in \frac{sC(X \times Y)}{sC(X \times Q) + sC(P \times Y)}.$$

Since $sC(X\times Q)+sC(P\times Y)$ is certainly *contained* in $sC(X\times Q\ \cup\ P\times Y)$, we may pass from there to

$$\frac{sC(X\times Y)}{sC(X\times Q\ \cup\ P\times Y)}.$$

So we have a $(k+\ell)$-cycle in $sC(X\times Y)/sC(X\times Q\ \cup\ P\times Y)$. Its homology class depends only on the homology classes $[a]$ and $[b]$.

Something awfully disruptive happens when we try to formulate a variant of corollary 5.3.1 for pairs of spaces. To start with the good news, the EZ-map gives us a chain map from

$$sC(X)\otimes sC(Q)\ +\ sC(P)\otimes sC(Y),$$

chain subcomplex of $sC(X)\otimes sC(Y)$, to

$$sC(X\times Q)+sC(P\times Y),$$

chain subcomplex of $sC(X\times Y)$. This map of chain subcomplexes induces an isomorphism in homology. The reason is that we understand in both cases (source and target) how the internal sum differs from the direct sum. Namely, we have one pushout square of inclusions

$$\begin{tikzcd}
sC(P)\otimes sC(Q) \arrow[r] \arrow[d] & sC(X)\otimes sC(Q) \arrow[d] \\
sC(P)\otimes sC(Y) \arrow[r] & sC(X)\otimes sC(Q)\ +\ sC(P)\otimes sC(Y)
\end{tikzcd}$$

(giving rise to a long exact Mayer-Vietoris sequence) and another pushout square of inclusions

$$\begin{tikzcd}
sC(P\times Q) \arrow[r] \arrow[d] & sC(X\times Q) \arrow[d] \\
sC(P\times Y) \arrow[r] & sC(X\times Q)+sC(P\times Y)
\end{tikzcd}$$

(also giving rise to a long exact Mayer-Vietoris sequence). The natural EZ-map specializes to a map from one pushout square to the other. Since it induces isomorphisms in homology in the upper left-hand, upper right-hand and lower left-hand terms, it will induce isomorphisms in homology for the lower right-hand terms, too. So far, so good. It follows (use the five lemma) that the composition of chain maps

$$\begin{tikzcd}
(sC(X)/sC(P))\otimes(sC(Y)/sC(Q)) \arrow[d, "="] \\
\dfrac{sC(X)\otimes sC(Y)}{sC(X)\otimes sC(Q)\ +\ sC(P)\otimes sC(Y)} \arrow[d, "u"] \\
\dfrac{sC(X\times Y)}{sC(X\times Q)+sC(P\times Y)}
\end{tikzcd}$$

induces an isomorphism in homology, too. — Therefore, if we *could* say that the inclusion

$$sC(X\times Q)+sC(P\times Y)\longrightarrow sC(X\times Q\ \cup P\times Y)$$

induces an isomorphism in homology, then we could conclude that

$$\begin{tikzcd} (sC(X)/sC(P)) \otimes (sC(Y)/sC(Q)) \arrow[d, "="] & \\ \dfrac{sC(X) \otimes sC(Y)}{sC(X) \otimes sC(Q) \;+\; sC(P) \otimes sC(Y)} \arrow[d, "u"] & \\ \dfrac{sC(X \times Y)}{sC(X \times Q) + sC(P \times Y)} \arrow[r] & \dfrac{sC(X \times Y)}{sC(X \times Q \;\cup\; P \times Y)} \end{tikzcd}$$

induces an isomorphism in homology. That would give us the connection with the Künneth theorem. So let us cut our losses and make a conditional statement.

COROLLARY 5.3.2. *Suppose that the inclusion of* $sC(X{\times}Q){+}sC(P{\times}Y)$ *in* $sC(X{\times}Q \cup P{\times}Y)$ *induces an isomorphism in homology. Then the exterior product in homology*

$$\bigoplus_{k=0,1,\dots,m} H_k(X,P) \otimes H_{m-k}(Y,Q) \longrightarrow H_m(X \times Y,\ X \times Q \;\cup\; P \times Y)$$

is injective and its cokernel is isomorphic to $\displaystyle\bigoplus_{k=0,\dots,m-1} \mathrm{Tor}_1(H_k(X,P), H_{m-k-1}(Y,Q))$. □

The condition here, that the inclusion of $sC(X \times Q) + sC(P \times Y)$ in $sC(X \times Q \;\cup P \times Y)$ should induce an isomorphism in homology, is a typical *excision* condition. It is satisfied for example if P is open in X and Q is open in Y (because then $X \times Q$ and $P \times Y$ make up an open cover of $X \times Q \;\cup\; P \times Y$).

Of course we can combine the two generalizations; then we have pairs of spaces *and* coefficients, and we get an exterior homology product in the form of a biadditive map

$$H_k(X,P;A) \times H_\ell(Y,Q;B) \longrightarrow H_{k+\ell}(X \times Y,\ X \times Q \;\cup\; P \times Y;\ A \otimes B).$$

To mention the most basic algebraic properties of the exterior homology product, it is *associative* and it is *graded commutative*, and it has a *unit*. The meaning of *associative* is easy to unravel, and the reason for associativity of the exterior homology product is the associativity property (up to natural homotopy) of the Eilenberg-Zilber maps. The meaning of *commutativity* is roughly as follows. In the simplest case (no strange coefficients, no "pairs"), if we have $u \in H_k(X)$ and $v \in H_\ell(Y)$, then we can form $u \times v \in H_{k+\ell}(X \times Y)$ and $v \times u \in H_{k+\ell}(Y \times X)$. The homeomorphism from $X \times Y$ to $Y \times X$ given by $(x,y) \mapsto (y,x)$ induces an isomorphism from $H_{k+\ell}(X{\times}Y)$ to $H_{k+\ell}(Y{\times}X)$ which takes $u{\times}v$ to $(-1)^{|u|\cdot|v|}v{\times}u$. The meaning of *unit* is again easier to unravel; the unit is the standard generator (call it e for now) of the infinite cyclic group $H_0(\star)$. If we make no careful distinction between a space Y and the products $\star \times Y$ and $Y \times \star$, then we may write $e \times u = u = u \times e$ for $u \in H_k(X)$. These statements generalize easily to situations with coefficients, or pairs, or both. — This is not the complete list of good properties of the exterior homology product that one should know, but we can leave the remaining ones for later.

Exterior product in cohomology. This is a product which in the simplest case has the form of a bi-additive map

$$H^k(X) \times H^\ell(Y) \longrightarrow H^{k+\ell}(X \times Y).$$

The standard notation for this is again $(u,v) \mapsto u \times v$.

It is obtained as follows. Take $f \in \hom(sC(X), \mathbb{Z})$ representing an element of $H^k(X)$ and take $g \in \hom(sC(Y), \mathbb{Z})$ representing an element of $H^\ell(X)$. Then we have $f \otimes g$ in $\hom(sC(X) \otimes sC(Y), \mathbb{Z})$, a cocycle of degree $k+\ell$, or a cycle of degree $-(k+\ell)$ according to taste.[1] More precisely, $(f \otimes g)(x \otimes y)$ is defined to be $(-1)^{k\ell} f(x) \cdot g(y)$ if x is in degree k and y is in degree ℓ; otherwise it is defined to be 0. We choose a natural Eilenberg-Zilber chain map

$$V : sC(X \times Y) \longrightarrow sC(X) \otimes sC(Y)$$

as in theorem 5.2.1. Then $(f \otimes g) \circ V$ is a cocycle of degree $k + \ell$ in $sC(X \times Y)$. Its cohomology class lives in $H^{k+\ell}(X \times Y)$ and depends only on the cohomology classes $[f]$ and $[g]$.

The exterior product in cohomology satisfies graded commutativity, associativity and has a unit element (much as in the case of the exterior product in homology).

Let us see what we can do for pairs. Let P be a subspace of X and let Q be a subspace of Y. Unfortunately the Eilenberg-Zilber map $V : sC(X \times Y) \longrightarrow sC(X) \otimes sC(Y)$ does not restrict to a chain map from $sC(X \times Q \ \cup \ P \times Y)$ to $sC(X) \otimes sC(Y)$. To get around this we impose the condition of corollary 5.3.2. Then we can write

$$\begin{tikzcd}
\dfrac{sC(X \times Y)}{sC(X \times Q \ \cup \ P \times Y)} & (sC(X)/sC(P)) \otimes (sC(Y)/sC(Q)) \\
\dfrac{sC(X \times Y)}{sC(X \times Q) + sC(P \times Y)} \arrow[u, "\simeq"'] \arrow[r, "\simeq", "V"'] & \dfrac{sC(X) \otimes sC(Y)}{sC(X) \otimes sC(Q) + sC(P) \otimes sC(Y)} \arrow[u, "\cong"]
\end{tikzcd}$$

COROLLARY 5.3.3. *Under the condition of corollary 5.3.2, there is a natural isomorphism of the relative cohomology group*

$$H^n(X \times Y, X \times Q \ \cup \ P \times Y) = H_{-n}\left(\hom\left(\frac{sC(X \times Y)}{sC(X \times Q \ \cup \ P \times Y)}, \mathbb{Z}\right)\right)$$

with $H_{-n}(\hom((sC(X)/sC(P)) \otimes (sC(Y)/sC(Q)), \mathbb{Z}))$*, for all* n*. This leads to an exterior cohomology product* $H^k(X, P) \times H^\ell(Y, Q) \longrightarrow H^{k+\ell}(X \times Y, X \times Q \cup P \times Y)$*, and more generally with coefficients,* $H^k(X, P;\ A) \times H^\ell(Y, Q;\ B) \longrightarrow H^{k+\ell}(X \times Y, X \times Q \ \cup \ P \times Y;\ A \otimes B)$.

What about Künneth theorems in cohomology? The formulation and the success depend a little on what we want. It is easy to give a formula for the cohomology of a product $X \times Y$ in terms of the *homology* groups of the factors X.

[1]We had this in section 4.7, but I made a sign mistake there, which is now corrected. Failure to apply the sign rule.

PROPOSITION 5.3.4. *There is a natural diagram of the following shape,*

$$\begin{tikzcd}
\mathrm{Ext}^1\Big(\bigoplus_{k+\ell=m-2}\mathrm{Tor}_1(H_k(X),H_\ell(Y)),\ \mathbb{Z}\Big) \arrow[d] & & \\
\mathrm{Ext}^1(H_{m-1}(X\times Y),\mathbb{Z}) \arrow[r] \arrow[d] & H^m(X\times Y) \arrow[r] & \hom\Big(\bigoplus_{k+\ell=m}H_k(X)\otimes H_\ell(Y),\ \mathbb{Z}\Big) \\
\mathrm{Ext}^1\Big(\bigoplus_{k+\ell=m-1}H_k(X)\otimes H_\ell(Y),\ \mathbb{Z}\Big) & &
\end{tikzcd}$$

in which the row and the column are short exact, and un-naturally split.

PROOF. The row in the diagram is essentially what the universal coeffcient theorem tells us about $H_m(X\times Y)$. Strictly speaking we should then see $\hom(H_m(X\times Y),\mathbb{Z})$ in the right-hand side. But now $H_m(X\times Y)$ sits in a short exact sequence

$$\bigoplus_{k+\ell=m}H_k(X)\otimes H_\ell(Y)\longrightarrow H_m(X\times Y)\longrightarrow\bigoplus_{k+\ell=m-1}\mathrm{Tor}_1(H_k(X),H_\ell(Y))$$

(which is un-naturally split). We know from exercises that the groups $\mathrm{Tor}_1(H_k(X),H_\ell(Y))$ are torsion groups. Therefore they have no nontrivial homomorphisms to $\mathbb{Z}$, and we may write

$$\hom(H_m(X\times Y),\mathbb{Z})\cong\hom\Big(\bigoplus_{k+\ell=m}H_k(X)\otimes H_\ell(Y),\ \mathbb{Z}\Big)$$

(the isomorphism is given by restriction, from left to right). Next, to obtain the column of the diagram, we apply the Künneth theorem to $H_{m-1}(X\times Y)$. The short exact sequence

$$\oplus_{k+\ell=m-1}H_k(X)\otimes H_\ell(Y)\longrightarrow H_{m-1}(X\times Y)\longrightarrow\oplus_{k+\ell=m-2}\mathrm{Tor}_1(H_k(X),H_\ell(Y))$$

is un-naturally split and so determines a short exact sequence of groups $\mathrm{Ext}^1(-,\mathbb{Z})$ which is also un-naturally split. □

By contrast, if we want a Künneth formula expressing the cohomology of $X\times Y$ in terms of the cohomology of X and Y, then we have to impose some conditions.

PROPOSITION 5.3.5. *Suppose that either $H_j(X)$ is finitely generated for all j, or $H_j(Y)$ is finitely generated for all j. Then the map*

$$\bigoplus_{\substack{k,\ell\\ k+\ell=m}}H^k(X)\otimes H^\ell(Y)\longrightarrow H^m(X\times Y)$$

given by the exterior cohomology product is injective and its cokernel is isomorphic to

$$\bigoplus_{\substack{k,\ell\\ k+\ell=m+1}}\mathrm{Tor}_1(H^k(X),H^\ell(Y)).$$

PROOF. We assume that $H_j(X)$ is finitely generated for all j. Then it is easy to construct a chain complex C of free abelian groups such that each C_j is finitely generated, and a chain map $g\colon C\to sC(X)$ which induces an isomorphism in homology. We know that g must be a chain homotopy equivalence. It follows that

$$g^*\colon\hom(sC(X),\mathbb{Z})\longrightarrow\hom(C,\mathbb{Z})$$

is a chain homotopy equivalence. Here $\hom(C,\mathbb{Z})$ is still a chain complex of finitely generated free abelian groups, whereas $\hom(sC(X),\mathbb{Z})$ will in most cases not have that property. Now we can set up a diagram

$$\begin{tikzcd} \hom(sC(X\otimes Y),\mathbb{Z}) \arrow[r, "U^*", "\simeq"'] & \hom(sC(X)\otimes sC(Y),\mathbb{Z}) \arrow[d, "(g\otimes \mathrm{id})^*", "\simeq"'] \\ \hom(C,\mathbb{Z})\otimes\hom(sC(Y),\mathbb{Z}) & \hom(C\otimes sC(Y),\mathbb{Z}) \arrow[l, "\cong"'] \end{tikzcd}$$

where $U\colon \mathcal{C}(X)\otimes sC(Y) \longrightarrow sC(X\times Y)$ is an Eilenberg-Zilber map. Therefore the chain complex $\hom(sC(X\otimes Y),\mathbb{Z})$ is chain homotopy equivalent to $\hom(C,\mathbb{Z})\otimes\hom(sC(Y),\mathbb{Z})$, and now we can hope to understand the homology of the latter using the Künneth theorem 4.6.3 in the usual homological form.

But now we notice that our formulation of the Künneth theorem is not strong enough for the present purpose. There we assumed that both tensor factors are chain complexes of free abelian groups. Fortunately it is easy to generalize to the setting where only one of the tensor factors is a chain complex of free abelian groups. Namely, we use the following observation.

> Let D be a chain complex of free abelian groups. Let E and E' be any chain complexes and let $f\colon E\to E'$ be a chain map which induces isomorphisms in H_n for all n. Then $\mathrm{id}\otimes f\colon D\otimes E\to D\otimes E'$ induces isomorphisms in H_n for all n.

(Proof of this: first reduce to the case where D is elementary, i.e., it is concentrated in two adjacent degrees r and $r+1$, and $d\colon D_{r+1}\to D_r$ is injective. Then, by putting this elementary D in the middle of a short exact sequence, reduce further to the case where D is concentrated is a single dimension.) Using the observation, we can get around the difficulty that $\hom(sC(Y),\mathbb{Z})$ is unlikely to be a chain complex of free abelian groups by choosing a chain complex E of free abelian groups and a chain map $f\colon E\to\hom(sC(Y),\mathbb{Z})$ which induces an isomorphism in homology. Then we can say that $\hom(sC(X\times Y),\mathbb{Z})$ has the same homology as $\hom(C,\mathbb{Z})\otimes\hom(sC(Y),\mathbb{Z})$, and consequently the same homology as $\hom(C,\mathbb{Z})\otimes E$, and now we can apply theorem 4.6.3. □

Interior product in cohomology, alias cup product. This is a product which in the simplest case has the form of a bi-additive map

$$H^k(X)\times H^\ell(X)\longrightarrow H^{k+\ell}(X).$$

The standard notation for this is $(u,v)\mapsto u\smile v$. Definition:

$$u\smile v := \mathrm{dia}^*(u\times v)$$

where $u\times v\in H^{k+\ell}(X\times X)$ is the exterior product and $\mathrm{dia}\colon X\to X\times X$ is the diagonal map, $\mathrm{dia}(x):=(x,x)$.

The cup product satisfies graded commutativity, associativity and has a unit element. It makes $H^*(X)$ into a graded ring (associative and graded commutative) with unit. The unit of the ring (neutral element for the multiplication) lives in $H^0(X)$.

Remark 5.3.6. Suppose that P and Q are subsets of X such that the inclusion of the subcomplex $sC(P\times X)+sC(X\times Q)$ in $sC(P\times X\ \cup\ X\times Q)$ induces an isomorphism in homology. (For example, this is the case if P and Q are open in X.) Then we have a cup product of the form

$$H^k(X,P)\times H^\ell(X,Q)\longrightarrow H^{k+\ell}(X,P\cup Q)\,.$$

This is simply the composition of the exterior product

$$H^k(X,P)\times H^\ell(X,Q)\longrightarrow H^{k+\ell}(X\times X,\ P\times X\ \cup\ X\times Q)$$

(as in 5.3.3) with $\mathrm{dia}^*\colon H^{k+\ell}(X\times X,\ P\times X\ \cup\ X\times Q)\longrightarrow H^{k+\ell}(X,P\cup Q)$.

Products and (co)boundary operators.

PROPOSITION 5.3.7. *For spaces* X,Y *and subspaces* $P\subset X$, $Q\subset Y$ *and integers* k,ℓ,m *such that* $m=k+\ell$, *the following diagrams are commutative:*

$$\begin{tikzcd}
H_k(X,P)\otimes H_\ell(Y) \arrow[r, "\times"] \arrow[d, "\partial\otimes\mathrm{id}"'] & H_m(X\times Y,P\times Y) \arrow[d, "\partial"] \\
H_{k-1}(P)\otimes H_\ell(Y) \arrow[r, "\times"] & H_{m-1}(P\times Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
H^k(X,P)\otimes H^\ell(Y) \arrow[r, "\times"] & H_m(X\times Y,P\times Y) \\
H^{k-1}(P)\otimes H^\ell(Y) \arrow[r, "\times"] \arrow[u, "\partial\otimes\mathrm{id}"] & H^{m-1}(P\times Y) \arrow[u, "\partial"']
\end{tikzcd}$$

$$\begin{tikzcd}
H_k(X)\otimes H_\ell(Y,Q) \arrow[r, "\times"] \arrow[d, "(-1)^k\mathrm{id}\otimes\partial"'] & H_m(X\times Y,X\times Q) \arrow[d, "\partial"] \\
H_k(X)\otimes H_{\ell-1}(Q) \arrow[r, "\times"] & H_{m-1}(X\times Q)
\end{tikzcd}
\qquad
\begin{tikzcd}
H^k(X)\otimes H^\ell(Y,Q) \arrow[r, "\times"] & H_m(X\times Y,X\times Q) \\
H^k(X)\otimes H^{\ell-1}(Q) \arrow[r, "\times"] \arrow[u, "(-1)^k\mathrm{id}\otimes\partial"] & H^{m+1}(X\times X) \arrow[u, "\partial"']
\end{tikzcd}$$

$$\begin{tikzcd}
H_k(X,P)\otimes H_\ell(Y,Q) \arrow[r, "\times"] \arrow[dd, "(\partial\otimes\mathrm{id},(-1)^k\mathrm{id}\otimes\partial)"'] & H_m(X\times Y,\ P\times Y\ \cup\ X\times Q) \arrow[d, "\partial"] \\
& H_{m-1}(P\times Y\ \cup\ X\times Q,\ P\times Q) \\
{[H_{k-1}(P)\otimes H_\ell(Y,Q)]\oplus[H_k(X,P)\otimes H_{\ell-1}(Q)]} \arrow[r, "\times"] & H_{m-1}(P\times Y,P\times Q)\oplus H_{m-1}(X\times Q,P\times Q) \arrow[u, "(j_1)_*\oplus(j_2)_*"']
\end{tikzcd}$$

where $j_1\colon X\times Q\longrightarrow X\times Q\ \cup\ P\times Y$ *and* $j_2\colon P\times Y\longrightarrow X\times Q\ \cup\ P\times Y$ *are the inclusions. If the condition of corollary 5.3.2 is satisfied, then we also have commutativity in the diagram*

$$\begin{tikzcd}
H^k(X,P)\otimes H^\ell(Y,Q) \arrow[r, "\times"] & H^m(X\times Y,\ P\times Y\ \cup\ X\times Q) \\
& H^{m-1}(P\times Y\ \cup\ X\times Q,\ P\times Q) \arrow[u, "\partial"'] \arrow[d, "(j_1^*,j_2^*)"] \\
{[H^{k-1}(P)\otimes H^\ell(Y,Q)]\oplus[H^k(X,P)\otimes H^{\ell-1}(Q)]} \arrow[r, "\times"] \arrow[uu, "(\partial\otimes\mathrm{id})\oplus((-1)^k\mathrm{id}\otimes\partial)"] & H^{m-1}(P\times Y,P\times Q)\oplus H^{m-1}(X\times Q,P\times Q)
\end{tikzcd}$$

PROOF. There is not much to say here. For example, to establish commutativity of the first diagram, we observe that the following diagram of chain complexes is commutative:

$$\begin{tikzcd}
sC(X)\otimes sC(Y) \arrow[r, "u"] & sC(X\times Y) \\
sC(P)\otimes sC(Y) \arrow[r, "u"] \arrow[u, "\subset"] & sC(P\times Y) \arrow[u, "\subset"]
\end{tikzcd}$$

This is due to the naturality of the Eilenberg-Zilber map U. Each column in this diagram of chain complexes can be expanded to a short exact sequence of chain complexes. These

short exact sequences of chain complexes determine long exact sequences of homology groups. The horizontal arrows (in the above diagram of chain complexes) determine a map between the long exact sequences which can be described as a commutative diagram in the shape of a ladder. One of the squares in that ladder is the diagram that we are investigating.
The proof of commutativity in the third diagram is quite similar, although one might expect that the sign $(-1)^k$ attached to the left-hand vertical arrow requires additional thought. It does not require much additional thought. It just reflects the definition of the differential in $sC(X) \otimes sC(Y)$, which is $d(u \otimes v) = du \otimes v + (-1)^{|u|} u \otimes dv$.
In the last two diagrams (which have five terms each), a look at the right-hand columns suggests that we should work in a setting relative to $P \times Q$. Consequently a good starting point (for a proof of commutativity) is the commutative diagram of chain complexes

$$\begin{tikzcd}
\dfrac{sC(X) \otimes sC(Y)}{sC(P) \otimes sC(Q)} \arrow[rr, "u"] & & \dfrac{sC(X \times Y)}{sC(P \times Q)} \\
sC(P) \otimes (sC(Y)/sC(Q)) \ \oplus \ (sC(X)/sC(P)) \otimes sC(Q) \arrow[u, "\subset"] \arrow[rr, "u"] & & \dfrac{sC(P \times Y \ \cup \ X \times Q)}{sC(P \times Q)} \arrow[u, "\subset"']
\end{tikzcd}$$

where u is again a (natural) Eilenberg-Zilber map. □

The list of products and their important properties is not yet complete, but we pause for some applications and examples.

CHAPTER 6

First applications of products

6.1. Cellular chain complexes revisited

Let X and Y be CW-spaces, and for simplicity assume that both X and Y are compact. Then $X \times Y$ is a CW-space again in such a way that $(X \times Y)^n$ is the union of the subspaces $X^p \times Y^q$ where $p + q \leq n$. We want to describe an isomorphism

$$C(X) \otimes C(Y) \longrightarrow C(X \times Y).$$

In other words we want to prove theorem 3.5.3 at last. — Choose integers $k, \ell, m \geq 0$ such that $m = k + \ell$. A natural homomorphism

$$C(X)_k \otimes C(Y)_\ell \longrightarrow C(X \times Y)_m$$

can be defined as the composition

$$\begin{tikzcd} H_k(X^k, X^{k-1}) \otimes H_\ell(Y^\ell, Y^{\ell-1}) \arrow[d, "\cong"', "\times"] & \\ H_m(X^k \times Y^\ell,\ X^k \times Y^{\ell-1} \ \cup\ X^{k-1} \times Y^\ell) \arrow[r] & H_m((X \times Y)^m, (X \times Y)^{m-1}) \end{tikzcd}$$

The vertical arrow is an isomorphism by corollary 5.3.2 and the horizontal arrow (induced by an inclusion) is injective. More precisely the horizontal arrow is the inclusion of the (sum of the) infinite cyclic summands corresponding to the m-cells of $X \times Y$ which are products $V \times W$ where V is a k-cell of X and W is an ℓ-cell of Y. Therefore this recipe gives us a natural isomorphism from

$$\bigoplus_{\substack{k,\ell \\ k+\ell=m}} C(X)_k \otimes C(Y)_\ell \ = \ (C(X) \otimes C(Y))_m$$

to $C(X \times Y)_m$. Letting m vary, we have an isomorphism of graded abelian groups, call it α, and we want to show that it respects the differentials. But it turns out that this is a special case of diagram number 5 in proposition 5.3.7. We have to make the following substitutions:

$$X \rightsquigarrow X^k,\ P \rightsquigarrow X^{k-1},\ Y \rightsquigarrow Y^\ell,\ Q \rightsquigarrow Y^{\ell-1}.$$

This means that in the middle of the right-hand column we will see

$$H_m(X^{k-1} \times Y^\ell \ \cup\ X^k \times Y^{\ell-1},\ X^{k-1} \times Y^{\ell-1})$$

which should be understood as a direct summand of $C(X \times Y)_m$. Then one of the compositions taking us from top of the left-hand column to middle of the right-hand column is the differential in $C(X) \times C(Y)$ followed by our isomorphism α, and the other is α followed by the differential in $C(X \times Y)$. This completes the verification.

There is another thing that should be said about cellular chain complexes. This is somewhat indirectly related to products. Again let X be a CW-space, with cellular chain complex $C(X)$. We have established isomorphisms $H_n(X) \cong H_n(C(X))$ for all n, where $H_n(X)$ is short for $H_n(sC(X))$. In other words we have established isomorphisms $H_n(sC(X)) \cong H_n(C(X))$ for all n. We showed (with some difficulty) that these isomorphisms are natural. But we have not produced a natural chain homotopy equivalence from $sC(X)$ to $C(X)$. In fact this is difficult in one step (I suspect, impossible). But we can proceed in two steps:

$$C(X) \xleftarrow{\simeq} sC'(X) \xrightarrow{\simeq} sC(X).$$

For this, let $sC'(X)$ be the chain subcomplex of $sC(X)$ such that

$$sC'(X)_n := sC(X^n)_n \cap d^{-1}(sC(X^{n-1})_{n-1}).$$

In words: in $sC'(X)_n$ we allow only chains w in $sC(X)_n$ which are $\mathbb{Z}$-linear combinations of singular n-simplices in X^n and we impose the condition that $d(w)$ be a linear combination of singular $(n-1)$-simplices in X^{n-1}. Now the chain map $sC'(X) \to C(X)$ is clear, since every n-chain in $sC'(X)$ represents a class in $H_n(X^n, X^{n-1})$. The other one, $sC'(X) \to sC(X)$, is the inclusion.

Proposition 6.1.1. *These chain maps* $sC'(X) \to C(X)$ *and* $sC'(X) \to sC(X)$ *are chain homotopy equivalences.*

(The proof is left as an exercise.)

Now we can show that our isomorphism $\alpha\colon C(X) \otimes C(Y) \longrightarrow C(X \times Y)$ (for CW-spaces X and Y subject to mild conditions, such as compactness) is compatible with Eilenberg-Zilber! I bet you did not think of that. Here is the proof in a commutative diagram:

$$\begin{tikzcd}
C(X\times Y) & sC'(X\times Y) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & sC(X\times Y) \\
C(X)\otimes C(Y) \arrow[u, "\alpha", "\cong"'] & sC'(X)\otimes sC'(Y) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[u, "u"'] & sC(X)\otimes sC(Y) \arrow[u, "\simeq", "u"']
\end{tikzcd}$$

This has the consequence that we can calculate external products in cohomology and homology of CW-spaces using cellular chain complexes and the isomorphism α. (I bet you took this for granted.)

6.2. H-spaces and Hopf algebras

Definition 6.2.1. An H-space (named after Heinz Hopf) is a path-connected space X with base point $*$ together with a map $\mu\colon X \times X \longrightarrow X$ such that $\mu(*, y) = y = \mu(y, *)$ for all $y \in X$.

Example 6.2.2. A path-connected topological group is an H-space. The sphere S^7 admits a structure of H-space (because it can be viewed as a subspace of the Cayley Octonion algebra, closed under multiplication).

Definition 6.2.3. Let S be a commutative ring. A Hopf algebra over S is a graded S-algebra with unit together with

- a homomorphism $c\colon T \to S$ of graded algebras (where S is viewed as something concentrated in degree 0), called the *counit*;

- an algebra homomorphism $\gamma\colon T \to T \otimes_S T$, called the *comultiplication*, such that the homomorphism $(c\otimes \mathrm{id})\circ\gamma$ from T to $S\otimes_S T \cong T$ is the identity, and $(\mathrm{id}\otimes c)\circ\gamma$ from T to $T\otimes_S S \cong T$ is the identity.

If T is graded commutative as an S-algebra, we speak of a *commutative* Hopf algebra.

EXAMPLE 6.2.4. Let S be a commutative ring as before. Then $H^*(X;S)$ is an S-algebra which as such is graded commutative. If $H_*(X;S)$ is degreewise finitely generated and free, then $sC(X)\otimes S$ is chain homotopy equivalent to a chain complex over R with zero differential, and it follows (using the Eilenberg-Zilber theorem) that

$$H^*(X\times X;S) \cong H^*(X;S)\otimes_S H^*(X;S).$$

If in addition to all that X is an H-space with base point $*$ and multiplication $\mu\colon X\times X \to X$, then $H^*(X;S)$ is a Hopf algebra. The counit $H^*(X;S) \to S$ is the ring homomorphism $H^*(X;S) \to H^*(\star;S)$ determined by the inclusion of the base point. The comultiplication is $\mu^*\colon H^*(X;S) \to H^*(X\times X;S)$.

This leads to many examples of path-connected based spaces X which do *not* admit a structure of H-space. If $H^*(X)$ is degreewise finitely generated and free, but does not admit a structure of Hopf ring, then X does not admit a structure of H-space.

EXAMPLE 6.2.5. An even-dimensional sphere $X = S^{2n}$ does not admit a structure of H-space. To see this we choose $S=\mathbb{Z}$ as the ground ring. In this case $H^*(X)$ is degreewise finitely generated and free. There is only one possibility for a counit $H^*(X) \to \mathbb{Z}$. If $\gamma\colon H^*(X)\to H^*(X)\times H^*(X)$ is a ring homomorphism which satisfies the Hopf ring conditions, then we must have $\gamma(z) = 1\otimes z + z\otimes 1$ for the generator $z\in H^{2n}(X)\cong\mathbb{Z}$. But since $z^2=0$ in $H^*(X)$, we obtain

$$0=\gamma(z^2)=\gamma(z)\gamma(z)=2(z\otimes z)\neq 0\in H^*(X)\otimes H^*(X),$$

a contradiction.

EXAMPLE 6.2.6. Let $X=\mathbb{R}P^n$. If $n+1$ is not a power of 2, then this X does not admit a structure of H-space. To prove this we take $S=\mathbb{F}_2$ as the ground ring. Then $H^*(X;S)$ is isomorphic to a graded polynomial ring $T := S[x]/(x^{n+1})$, where x is an element degree 1. (This will be shown in the next section.) Suppose that the latter admits a Hopf algebra structure:

$$\gamma\colon T \longrightarrow T\otimes_S T.$$

The Hopf algebra conditions imply that $\gamma(x) = (1\otimes x)+(x\otimes 1)$. Since $x^{n+1}=0$ in T, we must have $((1\otimes x)+(x\otimes 1))^{n+1}=0$ in $T\otimes_S T$. This is certainly true if $n+1$ is a power of 2, because then $(-)^{n+1}$ is an S-linear map and so

$$((1\otimes x)+(x\otimes 1))^{n+1} = (1\otimes x)^{n+1}+(x\otimes 1)^{n+1} = 1\otimes x^{n+1}+x^{n+1}\otimes 1 = 0.$$

But it fails in all other cases.

6.3. Cohomology of projective spaces

THEOREM 6.3.1. *The cohomology ring* $H^*(\mathbb{C}P^n;\mathbb{Z})$ *is isomorphic to a truncated polynomial graded ring* $\mathbb{Z}[y]/(y^{n+1})$, *where* y *has degree* 2.

I only give a sketchy proof. (Some of this is/was a homework problem). — Take $n\geq 1$. There is a map

$$f\colon \mathbb{C}P^n\times\mathbb{C}P^1 \longrightarrow \mathbb{C}P^{n+1}$$

defined by

$$\begin{array}{c}\big((a_0 : a_1 : \cdots : a_{n-1} : a_n), (b_0 : b_1)\big) \\ \Big\downarrow \\ (a_0 b_0 : a_1 b_0 + a_0 b_1 : a_2 b_0 + a_1 b_1 : \cdots : a_n b_0 + a_{n-1} b_1 : a_n b_1)\end{array}$$

The map is cellular if we use the standard (Schubert) cell structures on $\mathbb{C}P^n$, $\mathbb{C}P^1$ and $\mathbb{C}P^{n+1}$. By induction we can assume that $H^*(\mathbb{C}P^n)$ is isomorphic to $\mathbb{Z}[w]/(w^{n+1})$ as a ring, where $|w| = 2$. We know also that $H^*(\mathbb{C}P^1) \cong \mathbb{Z}[x]/(x^2)$, where $|x| = 2$. Therefore

$$H^*(\mathbb{C}P^n \times \mathbb{C}P^1) \cong \mathbb{Z}[w, x]/(w^{n+1}, x^2)$$

by the cohomology Künneth theorem. Let $y \in H^2(\mathbb{C}P^{n+1})$ be the element which maps to $w \in H^2(\mathbb{C}P^n)$ under the ring homomorphism induced by the inclusion of $\mathbb{C}P^n$ in $\mathbb{C}P^{n+1}$. Looking at the map of cellular chain complexes determined by f, we get

$$f^*(y) = w + x \in H^2(\mathbb{C}P^n \times \mathbb{C}P^1).$$

Therefore

$$f^*(y^{n+1}) = (w + x)^{n+1} = \sum_{j=0}^{n+1} \binom{n+1}{j} w^j x^{n+1-j} = (n+1) w^n x.$$

If we can show that f^* from $H^{2n+2}(\mathbb{C}P^{n+1}) \cong \mathbb{Z}$ to $H^{2n+2}(\mathbb{C}P^n \times \mathbb{C}P^1) \cong \mathbb{Z}$ is multiplication by $\pm(n+1)$, then we may conclude that y^{n+1} is an additive generator of the infinite cyclic group $H^{2n+2}(\mathbb{C}P^{n+1})$, which is exactly what we want to know. This means that we have to determine the degree of the map

$$D^{2n+2}/S^{2n+1} \cong \frac{\mathbb{C}P^n \times \mathbb{C}P^1}{(\mathbb{C}P^n \vee \mathbb{C}P^1)} \longrightarrow \frac{\mathbb{C}P^{n+1}}{\mathbb{C}P^n} \cong D^{2n+2}/S^{2n+1}$$

induced by f. (The denominators describe the $(2n+1)$-skeletons of the numerators in both cases.) For that we have a standard method, counting the elements in the preimage of a regular value with their multiplicities (which are ± 1). This uses smoothness properties of f. More precisely, if we choose characteristic maps for the $2n+2$-dimensional cells of $\mathbb{C}P^n \times \mathbb{C}P^1$ and $\mathbb{C}P^{n+1}$ wisely, then the above map from D^{2n+2}/S^{2n+1} to itself is smooth on $D^{2n+2} \smallsetminus S^{2n+1}$. The count of elements in the preimage of a regular value gives us exactly $(n+1)$ points, each with multiplicity 1. See remark 6.3.2 below. □

Remark 6.3.2. (This summarizes remarks made by students.) The proof above is closely related to complex polynomials and the fundamental theorem of algebra. To see this, think of a point $(a_0 : a_1 : \cdots : a_{n-1} : a_n)$ in $\mathbb{C}P^n$ as a polynomial $\sum_{i=0}^n a_i t^i$ in t (well defined only up to multiplication by a nonzero scalar). Now it emerges that the map f describes the multiplication of degree $\le n$ polynomials with degree ≤ 1 polynomials. It follows immediately that f is surjective. This also explains why the preimage count in the proof above gives us $n+1$ points. The reason is that a "generic" polynomial of degree $(n+1)$ with complex coefficients should have $n+1$ *distinct* linear factors. Consequently, if we look for a regular value of f in $\mathbb{C}P^{n+1} \smallsetminus \mathbb{C}P^n$, we should try a polynomial of degree $n+1$ with $n+1$ *distinct* roots. For example, $-1 + t^{n+1}$ appears to be a good choice (corresponding to $(-1 : 0 : 0 : \cdots : 0 : 1) \in \mathbb{C}P^{n+1}$).

This method does not work equally well for *real* projective spaces. For example the map $\mathbb{R}P^n \times \mathbb{R}P^1 \to \mathbb{R}P^{n+1}$ analogous to the above f in the complex setting is *not* surjective for $n \ge 1$. Then again, we can make a map $g : \mathbb{R}P^n \times \mathbb{R}P^2 \longrightarrow \mathbb{R}P^{n+2}$ analogous to the above

f. This would describe the multiplication of real degree $\le n$ polynomials by real degree ≤ 2 polynomials. It is surjective because every real polynomial of degree $\le n+2$ can be written as a product of a polynomial of degree $\le 2n$ and one of degree ≤ 2. Whether this map g can be used to show anything useful about the cohomology of $\mathbb{R}P^n$, especially about $H^*(\mathbb{R}P^n;\mathbb{F}_2)$, I don't know.

COROLLARY 6.3.3. *The ring* $H^*(\mathbb{C}P^\infty;\mathbb{Z})$ *is isomorphic to a graded polynomial ring* $\mathbb{Z}[y]$*, where* $|y| = 2$.

This follows from theorem 6.3.1 since the inclusion $\mathbb{C}P^n \to \mathbb{C}P^\infty$ induces isomorphisms $H^k(\mathbb{C}P^\infty;\mathbb{Z}) \to H^k(\mathbb{C}P^n)$ for $k \le 2n$. In fact it induces isomorphisms

$$C(\mathbb{C}P^n)_k \to C(\mathbb{C}P^\infty)_k$$

for $k \le 2n+1$, where $C(\ldots)$ denotes the cellular chain complexes. (Therefore it induces an isomorphism in homology of degree $\le 2n$, and then an isomorphism in cohomology of degree $\le 2n$ by the universal coefficient theorem.)

THEOREM 6.3.4. *The cohomology ring* $H^*(\mathbb{R}P^n;\mathbb{F}_2)$ *is isomorphic to a truncated graded polynomial ring* $\mathbb{F}_2[x]/(x^{n+1})$*, where* x *has degree* 1*. The cohomology ring* $H^*(\mathbb{R}P^\infty;\mathbb{F}_2)$ *is isomorphic to a graded polynomial ring* $\mathbb{F}_2[x]$*, where* x *has degree* 1.

As indicated in remark 6.3.2, there is no proof of this which resembles the proof of theorem 6.3.1. Instead we will rely much more on CW structures and indeed we will reason using cellular approximations to the diagonal, $\mathbb{R}P^n \to \mathbb{R}P^n \times \mathbb{R}P^n$ or $\mathbb{R}P^\infty \to \mathbb{R}P^\infty \times \mathbb{R}P^\infty$. Fortunately we can get away without making a specific *choice* of such a cellular approximation.

PROOF. We start by stating what we know already. Using the Schubert CW structure on $\mathbb{R}P^n$ (with exactly one cell in dimension k for $k = 0,1,\ldots,n$) we know what the cellular chain complex of $\mathbb{R}P^n$ looks like and in particular we know that all differentials are divisible by 2. It follows immediately that $H^k(\mathbb{R}P^n;\mathbb{F}_2)$ is isomorphic to $\mathbb{F}_2$ for $k \le n$ and zero otherwise. It follows that the inclusion $\mathbb{R}P^n \to \mathbb{R}P^\infty$ induces an isomorphism in $H^k(-;\mathbb{F}_2)$ for $k \le n$. Therefore, in proving theorem 6.3.4, it is enough to handle the case of $\mathbb{R}P^\infty$. In the following we view $\mathbb{R}P^\infty$ as the quotient space obtained from S^∞ by identifying antipodal points. In more detail, $S^\infty = \bigcup_{j\ge 0} S^j$ is also a CW-space. The j-skeleton of S^∞ is exactly the standard $S^j \subset S^\infty$. The j-cells are therefore the connected components of $S^j \smallsetminus S^{j-1}$, of which there are two. We have exactly two j-cells in S^∞ for every $j \ge 0$. The antipodal involution $S^\infty \to S^\infty$ is a cellular homeomorphism which interchanges the two j-cells, for every $j \ge 0$. Therefore the cellular chain complex $C(S^\infty)$ has the form

$$\cdots \longleftarrow 0 \longleftarrow \mathbb{Z}\oplus\mathbb{Z} \longleftarrow \mathbb{Z}\oplus\mathbb{Z} \longleftarrow \mathbb{Z}\oplus\mathbb{Z} \longleftarrow \mathbb{Z}\oplus\mathbb{Z} \longleftarrow \cdots$$

and it comes with an action of the group $\mathbb{Z}/2$ (respecting differentials) which interchanges the two $\mathbb{Z}$ summands in each degree. We can understand this better by introducing $R = \mathbb{Z}[\mathbb{Z}/2]$, the group ring of the group $\mathbb{Z}/2$. For this purpose write $\mathbb{Z}/2$ multiplicatively, i.e, call the elements 1 and T, multiplication given by $1^2 = T^2 = 1$ and $1T = T1 = T$. Then $R = \mathbb{Z}[\mathbb{Z}/2]$ consists of formal linear combinations $a1 + bT$, where $a, b \in \mathbb{Z}$, and the multiplication is given by $(a1+bT)(a'1+b'T) = (aa'+bb')1 + (ab'+ba')T$. Now we see that each chain group $C(S^\infty)_k$ is a free R-module of rank 1, so we can write this in the form of a chain complex of R-modules

$$\cdots \longleftarrow 0 \longleftarrow R \longleftarrow R \longleftarrow R \longleftarrow R \longleftarrow \cdots$$

What else do we need to know? We can write out the differentials (which amounts to determining incidence numbers):

$$\cdots \longleftarrow 0 \longleftarrow R \xleftarrow{1\mapsto 1-T} R \xleftarrow{1\mapsto 1+T} R \xleftarrow{1\mapsto 1-T} R \xleftarrow{1\mapsto 1+T} \cdots$$

(According to remark 6.3.5 below, it is not absolutely necessary to determine the incidence numbers.) The important thing is that $H_0(C(S^\infty)) \cong \mathbb{Z}$ and all other homology groups are zero. Here $H_0(C(S^\infty))$ obviously still comes with a structure of R-module, but it is the so-called *trivial* R-module structure., i.e., scalar multiplication with T is the identity in this module. We can say briefly that $C(S^\infty)$ is a *free resolution* over R of the trivial R-module $\mathbb{Z}$.

Now we want to think about diagonals, both in topology and in (homological) algebra. The diagonal map

$$S^\infty \to S^\infty \times S^\infty$$

is not cellular, but we can choose a cellular approximation. More precisely, we can choose a homotopy $(h_t : S^\infty \to S^\infty \times S^\infty)_{t\in[0,1]}$ such that h_0 is the diagonal and h_1 is a cellular map. But more precisely still, we want to ensure that each map h_t in the homotopy respects the antipodal involutions, so that if $h_t(x) = (y, z)$ then $h_t(-x) = (-y, -z)$. (Such a homotopy is easy to construct by induction over skeleta, using the HEP, homotopy extension property, for the induction step.) In particular h_1 is then a cellular map respecting the (antipodal) involutions, and the map of cellular chain complexes induced by h_1 is a chain map

$$\delta : C(S^\infty) \longrightarrow C(S^\infty) \otimes_{\mathbb{Z}} C(S^\infty)$$

which we can view as a map of R-module chain complexes. (Here we should make two little remarks. Number one, we have used the principle that cellular chain complexes of products are identified with tensor products of cellular chain complexes. Number two, we have used a canonical "diagonal" ring homomorphism $R \to R \otimes_{\mathbb{Z}} R$ given by $1 \mapsto 1 \otimes 1$ and $T \mapsto T \otimes T$. In fact, every group ring $\mathbb{Z}[G]$ is a Hopf algebra over $\mathbb{Z}$, with a "diagonal" homomorphism $\mathbb{Z}[G] \to \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ given by $a \mapsto a \otimes a$ for $a \in G$. We have used this ring homomorphism $R \to R \otimes R$ in order to view $C(S^\infty) \otimes C(S^\infty)$, which is really a chain complex of $R \otimes R$-modules, as a chain complex of R-modules.)

Now let's look at the chain map δ from the viewpoint of homological algebra. We have already observed that the source $C(S^\infty)$ is a free resolution over R of the trivial R-module $\mathbb{Z}$. In fact the target $C(S^\infty) \otimes C(S^\infty)$, viewed as an R-module chain complex, is *also* a free resolution over R of the trivial R-module $\mathbb{Z}$. Indeed its homology is isomorphic to $\mathbb{Z} \otimes \mathbb{Z}$ in degree 0 and equal to zero in all other dimensions (for which one could use the Künneth theorem). And the chain modules of $C(S^\infty) \otimes C(S^\infty)$ in each degree are direct sums of modules of the form $R \otimes R$ (with the "diagonal" structure of R-module); these summands happen to be free R-modules again (but of rank 2).

A chain map between two free (or projective) resolutions of left modules M_0 and M_1 (over some ring) is determined up to chain homotopy (over the same ring) by the induced map in H_0 , which is a module homomorphism $M_0 \to M_1$. (This is a very basic fact from homological algebra of which we have seen special cases in earlier chapters.) In our situation, M_0 and M_1 are both $\mathbb{Z}$, the trivial R-module, and the module map induced by δ is the identity. Summarizing: δ is an R-module chain map between two free resolutions of the trivial R-module $\mathbb{Z}$, and the homomorphism in H_0 induced by δ is the identity $\mathbb{Z} \to \mathbb{Z}$.

The most useful consequence of this partial description of δ for us is as follows. We may not be able to say exactly what δ is and does, but we know that it is homotopic as an

R-module chain map to *any other* R-module chain map $C(S^\infty) \to C(S^\infty) \otimes C(S^\infty)$ which induces the identity homomorphism $\mathbb{Z} \to \mathbb{Z}$ in H_0.
It is not very hard to construct an example of an R-module chain map

$$\delta' : C(S^\infty) \to C(S^\infty) \otimes C(S^\infty)$$

which induces the identity $\mathbb{Z} \to \mathbb{Z}$ on H_0. But now we really have to do it. For this purpose write D for $C(S^\infty)$. We need to define R-module homomorphisms

$$\delta'_k : D_k \longrightarrow \bigoplus_{j=0}^{k} D_j \otimes D_{k-j}\,.$$

One solution appears to be

$$\delta'_0(1) = 1_0 \otimes 1_0$$

$$\delta'_1(1) = (1_0 \otimes 1_1) + (1_1 \otimes T_0)$$

$$\delta'_2(1) = (1_0 \otimes 1_2) + (-1_1 \otimes T_1) + (1_2 \otimes 1_0)$$

$$\delta'_3(1) = (1_0 \otimes 1_3) + (1_1 \otimes T_2) + (1_2 \otimes 1_1) + (1_3 \otimes T_0)$$

$$\delta'_4(1) = (1_0 \otimes 1_4) + (-1_1 \otimes T_3) + (1_2 \otimes 1_2) + (-1_3 \otimes T_1) + (1_4 \otimes 1_0)$$

$$\delta'_k(1) = \sum_{j=0}^{k} (-1)^{kj-j}\ 1_j \otimes (T^j)_{k-j}$$

where I have written x_s for $x \in D_s \cong R$, especially $x = 1$ or $x = T$.
Now we can finish as follows. Let $E = C(\mathbb{R}P^\infty) \otimes_{\mathbb{Z}} \mathbb{F}_2$. This is the cellular chain complex of $\mathbb{R}P^\infty$ with $\mathbb{F}_2$ coefficients, and we know that its differentials are all zero. We can write $E = D \otimes_R \mathbb{F}_2$ where $D = C(S^\infty)$ as before. (Since R is commutative, we need not distinguish between left and right R-modules.) We obtain a chain map $E \longrightarrow E \otimes E$ in the form

$$D \otimes_R \mathbb{F}_2 \xrightarrow{\delta \otimes \mathbb{F}_2} (D \otimes D) \otimes_R \mathbb{F}_2 \longrightarrow (D \otimes D) \otimes_{R \otimes R} \mathbb{F}_2$$

This is also the chain map determined by a certain *cellular* map $\mathbb{R}P^\infty \to \mathbb{R}P^\infty \times \mathbb{R}P^\infty$ which is homotopic to the diagonal map. (It is the cellular map which we get from $h_1 : S^\infty \to S^\infty \times S^\infty$ by passing to suitable quotient spaces.) Clearly this chain map is chain homotopic to (therefore equal to) the composition

$$D \otimes_R \mathbb{F}_2 \xrightarrow{\delta' \otimes \mathbb{F}_2} (D \otimes D) \otimes_R \mathbb{F}_2 \longrightarrow (D \otimes D) \otimes_{R \otimes R} \mathbb{F}_2$$

From the definition of δ' this last composition is given in degree k by

$$E_k \longrightarrow \bigoplus_{j=0}^{k} E_j \otimes E_{k-j}\ ;\ 1 \ \mapsto\ \sum_{j=0}^{k} (1_j \otimes 1_{k-j})\,.$$

Since we need not distinguish between $H^k(\mathbb{R}P^\infty; \mathbb{F}_2)$ and $\hom(E_k, \mathbb{F}_2)$, this amounts to saying that the cup product multiplication in $H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$ restricts to isomorphisms

$$H^j(\mathbb{R}P^\infty; \mathbb{F}_2) \otimes H^{k-j}(\mathbb{R}P^\infty; \mathbb{F}^2) \longrightarrow H^k(\mathbb{R}P^\infty; \mathbb{F}_2)$$

for every $k \geq 0$ and $j \in \{0, 1, \ldots, k\}$. Clearly then $H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$ is a graded polynomial ring in one generator x, where $|x| = 1$. $\square$

REMARK 6.3.5. We could have been more radical in the proof of theorem 6.3.4, as follows. The commutative ring $R = \mathbb{Z}[\mathbb{Z}/2]$ is a Hopf algebra over $\mathbb{Z}$, as explained above. In particular it has a preferred " diagonal" ring homomorphism $R \to R \otimes_{\mathbb{Z}} R$. Let D be a free resolution of $\mathbb{Z}$, the "trivial" R-module (as explained above). The diagonal homomorphism $R \to R \otimes R$ allows us to think of $D \otimes_{\mathbb{Z}} D$ as a chain complex over R, which turns out to be another free R-resolution of $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$, the "trivial" R-module. Therefore we obtain a chain map $\delta : D \to D \otimes D$ of R-module chain complexes, unique up to chain homotopy, which induces the identity from $H_0(D) \cong \mathbb{Z}$ to $H_0(D \otimes D) \cong \mathbb{Z}$. Now define $E := D \otimes_R \mathbb{F}_2$ and use δ to make a "diagonal" chain map $\gamma : E \to E \times E$, as explained above. We have given arguments to show that E is a correct model (i.e., chain homotopy equivalent to) the cellular chain complex of $\mathbb{R}P^\infty$ with $\mathbb{F}_2$ coefficients. We have also given arguments to show that γ is a correct model (up to chain homotopy) for the chain map determined by a cellular approximation of the diagonal $\mathbb{R}P^\infty \to \mathbb{R}P^\infty \times \mathbb{R}P^\infty$. Therefore we can use E and γ to determine the cup product in $H^*(\mathbb{R}P^\infty; \mathbb{F}_2)$. (And this is what we have done.)

6.4. The Borsuk-Ulam theorem

The Borsuk-Ulam theorem is the following statement. (Wikipedia says that it was first stated by Lyusternik-Schnirelmann in 1930, but Borsuk proved it first in 1933 after Ulam had told him about it.)

THEOREM 6.4.1. *Let $f : S^n \to \mathbb{R}^n$ be a continuous map. Then there exists $z \in S^n$ such that $f(z) = f(-z)$.*

PROOF. Suppose that $f(z) \neq f(-z)$ for all $z \in S^n$. Define a continuous map

$$g : S^n \to S^{n-1}$$

by $g(z) = (f(z) - f(-z))/\|f(z) - f(-z)\|$. This has the property that $g(-z) = -g(z)$ for all $z \in S^n$. In other words it respects the antipodal involution(s). It is obvious that this is impossible for $n = 1$, so we may assume $n \geq 2$ from now on.

As in the proof of theorem 6.3.4, we can equip S^n and S^{n-1} with standard CW structures which make the antipodal maps into cellular homeomorphisms. (In the case of S^n this has two j-cells for every $j \in \{0, 1, \ldots, n\}$ and in the case of S^{n-1} it has two j-cells for every $j \in \{0, 1, \ldots, n-1\}$.) We can find a homotopy

$$(h_t : S^n \to S^{n-1})_{t \in [0,1]}$$

where each h_t respects the antipodal involutions, and h_1 is cellular. Summing up, h_1 is a cellular map $S^n \to S^{n-1}$ which respects the antipodal involutions. Now we will show that this cannot be.

Looking at the cellular chain complexes we get an induced R-module chain map (notation as in the proof of theorem 6.3.4) from D' , the cellular chain complex of S^n (which has $D'_0 \cong D'_1 \cong \cdots \cong D'_n$ equal or isomorphic to R, all other chain modules zero) to D'' , the cellular chain complex of S^{n-1} (which has $D''_0 \cong D''_1 \cong \cdots \cong D''_{n-1}$ equal or isomorphic to R, all other chain modules zero). The following commutative diagram describes this chain map (to some extent).

$$\begin{tikzcd} R \arrow[d] & R \arrow[l, "1 \mapsto 1-T"'] \arrow[d] & \cdots \arrow[l, "1 \mapsto 1+T"'] \\ R & R \arrow[l, "1 \mapsto 1-T"'] & \cdots \arrow[l, "1 \mapsto 1+T"'] \end{tikzcd}$$

We describe the R-module homomorphism from $R = D'_k$ to $R = D''_k$ by $1 \mapsto a_k 1 + b_k T$. To begin with a_k and b_k seem to be unknown integers, but we know something. We know that the chain map induces an isomorphism in H_0. Therefore $a_0 + b_0$ must be an odd integer (in fact it must be 1, but this is more than we need). We see $(1 \pm T)(a_k 1 + b_k T) = (1 \pm T)(a_{k-1} 1 + b_{k-1} T)$ and we deduce that $a_k - a_{k-1} \equiv b_k - b_{k-1}$ modulo 2, so that by induction, $a_k + b_k$ is always an odd integer. But if we use this for $k = n-1$ we see that the composition of differential $D'_{n-1} \leftarrow D'_n$ with the map $D'_{n-1} \to D''_{n-1}$ is nonzero. So our chain map is not a chain map. Contradiction. □

Corollary 6.4.2. (Ham sandwich theorem.) *Let $K_1, K_2, \dots, K_{n+1}$ be compact subsets of $\mathbb{R}^{n+1}$. Then there exists an affine n-dimensonal hyperplane in $\mathbb{R}^{n+1}$ which divides each K_i into portions of equal Lebesgue measure.*

Proof. For $z \in S^n$ determine $c_z \in \mathbb{R}$ in such a way that the hyperplane

$$L_z := \{w \in \mathbb{R}^{n+1} \mid \langle w, z\rangle = c_z\}$$

divides the compact set K_{n+1} into portions of equal Lebesgue measure. Define $f(z) \in \mathbb{R}^n$ in such a way that the i-th coordinate of $f(z)$ is the Lebesgue measure of

$$\{w \in K_i \mid \langle w, z\rangle \geq c_z\}$$

for $i = 1, 2, \dots, n$. By the Borsuk-Ulam theorem there exists $z \in S^n$ such that $f(z) = f(-z)$. The hyperplane L_z associated with this z solves the problem. □

Remark 6.4.3. There is another way to organize the proof of the Borsuk-Ulam theorem which avoids cellular approximation but uses some knowledge of fundamental groups and the relation to covering spaces instead. As before we start with a hypothetical $f : S^n \to \mathbb{R}^n$ which has $f(x) \neq f(-x)$ for all $x \in S^n$, and we define $g : S^n \to S^{n-1}$ by the same formula as before, and we note that $g(z) = -g(-z)$ for all $z \in S^n$. (Therefore we may assume $n \geq 2$.) Let $q : \mathbb{R}P^n \to \mathbb{R}P^{n-1}$ be the map obtained from g by passing to quotients. We observe that q^* of the covering space $S^{n-1} \to \mathbb{R}P^{n-1}$ is identified with the covering space $S^n \to \mathbb{R}P^n$; in particular it is not a trivial covering space. This means that the homomorphism of fundamental groups (once we choose base points) determined by q is nontrivial. Since the fundamental group of $\mathbb{R}P^n$ is $\cong \mathbb{Z}/2$, we can deduce that $n \geq 3$ and that the homomorphism of fundamental groups induced by q is an isomorphism. It follows that the map induced by q on $H_1(-;\mathbb{Z})$ is an isomorphism (because $H_1(-;\mathbb{Z})$ is naturally isomorphic to the abelianized fundamental group). We can use the universal coefficient theorem to deduce that the map induced by q on $H^1(-;\mathbb{F}_2)$ is an isomorphism. Therefore

$$q^* : H^*(\mathbb{R}P^{n-1}; \mathbb{F}^2) \longrightarrow H^*(\mathbb{R}P^n; \mathbb{F}_2)$$

is a homomorphism of graded rings which is an isomorphism in degree 1. By theorem 6.3.4, this is impossible.

Since our proof of theorem 6.3.4 relies on cellular approximation, one could say that this new argument for theorem 6.4.1 does not shed more light on the matter. But there are other proofs of theorem 6.3.4 available.

CHAPTER 7

The cap product and Poincaré duality

7.1. The cap product

The cap product in the simplest form is a natural bi-additive map

$$\mathsf{H}^k(\mathsf{X}) \times \mathsf{H}_\ell(\mathsf{X}) \longrightarrow \mathsf{H}_{\ell-k}(\mathsf{X}).$$

The preferred notation for that is $(a, b) \mapsto a \frown b$.
The idea is very simple. We take a class in $\mathsf{H}^k(\mathsf{X})$ and we think of it as a homotopy class of chain maps $a : s\mathsf{C}(\mathsf{X}) \to (\mathbb{Z}, k)$. Here, as before, $(\mathbb{Z}, k)$ is the chain complex which has $\mathbb{Z}$ in degree k and 0 in all other dimensions. We take a class in $\mathsf{H}_\ell(\mathsf{X})$ and think of it as a homotopy class of chain maps $b : (\mathbb{Z}, \ell) \to s\mathsf{C}(\mathsf{X})$. Then the following composition describes $b \cap a$:

$$\begin{tikzcd}
(\mathbb{Z},\ell) \arrow[r, "b"] & s\mathsf{C}(\mathsf{X}) \arrow[r, "\text{dia}"] & s\mathsf{C}(\mathsf{X}\times\mathsf{X}) \arrow[d, "\text{EZ}", "\simeq"'] & \\
& & s\mathsf{C}(\mathsf{X})\otimes s\mathsf{C}(\mathsf{X}) \arrow[r, "a\otimes\text{id}"] & (\mathbb{Z},k)\otimes s\mathsf{C}(\mathsf{X})
\end{tikzcd}$$

Therefore $a \frown b$ is represented by an ℓ-cycle in $(\mathbb{Z}, k) \otimes s\mathsf{C}(\mathsf{X})$, which we can also view as an $(\ell - k)$-cycle in $s\mathsf{C}(\mathsf{X})$.
More formally, we can describe this at the level of chain maps. I use $s\mathsf{C}$ as an abbreviation for $s\mathsf{C}(\mathsf{X})$.

$$\begin{tikzcd}
\hom(s\mathsf{C},\mathbb{Z})\otimes s\mathsf{C} \arrow[r, "\text{id}\otimes(\text{EZ}\circ\text{dia})"] & \hom(s\mathsf{C},\mathbb{Z})\otimes(s\mathsf{C}\otimes s\mathsf{C}) \arrow[d, equal] & s\mathsf{C} \\
& (\hom(s\mathsf{C},\mathbb{Z})\otimes s\mathsf{C})\otimes s\mathsf{C} \arrow[r, "\text{evaluation}\otimes\text{id}"] & \mathbb{Z}\otimes s\mathsf{C} \arrow[u, "\cong"']
\end{tikzcd}$$

There is also a relative version, which will be important for us. Let Y be a subspace of X. Let $a \in \mathsf{H}^k(\mathsf{X}, \mathsf{Y})$ and $b \in \mathsf{H}_\ell(\mathsf{X}, \mathsf{Y})$. Then we can define $a \frown b \in \mathsf{H}_{\ell-k}(\mathsf{X})$ by composing

$$\begin{tikzcd}
(\mathbb{Z},\ell) \arrow[r, "b"] & \dfrac{s\mathsf{C}(\mathsf{X})}{s\mathsf{C}(\mathsf{Y})} \arrow[r, "\text{dia}"] & \dfrac{s\mathsf{C}(\mathsf{X}\times\mathsf{X})}{s\mathsf{C}(\mathsf{Y}\times\mathsf{X})} \arrow[d, "\text{EZ}", "\simeq"'] & (\mathbb{Z},k)\otimes s\mathsf{C}(\mathsf{X}) \\
& & \dfrac{s\mathsf{C}(\mathsf{X})\otimes s\mathsf{C}(\mathsf{X})}{s\mathsf{C}(\mathsf{Y})\otimes s\mathsf{C}(\mathsf{X})} \arrow[r, "\cong"] & \dfrac{s\mathsf{C}(\mathsf{X})}{s\mathsf{C}(\mathsf{Y})}\otimes s\mathsf{C}(\mathsf{X}) \arrow[u, "a\otimes\text{id}"']
\end{tikzcd}$$

There are other more general relative variants of the cap product, but this is the one which we will need most.

PROPOSITION 7.1.1. *The cap product is associative: for example*

$$(a \cup b) \cap c = a \cap (b \cap c)$$

if $a \in H^k(X)$, $b \in H^\ell(X)$ *and* $c \in H^m(X)$, *so that the two sides of the equation describe elements in* $H_{m-k-\ell}(X)$. *In the absolute case, it has* $1 \in H^0(X)$ *as a unit. Therefore the cap product makes* $H_*(X)$ *into a graded module over the graded ring* $H^*(X)$.

PROOF. *under construction* □

PROPOSITION 7.1.2. *In the case where* $|a| = |b|$, *the cap product* $a \frown b$ *agrees with the scalar product. More precisely let* $f : X \to \star$ *be the only possible map to a point, let* $a \in H^k(X)$ *and* $b \in H_k(X)$. *Then* $f_*(a \frown b) \in H_0(\star) \cong \mathbb{Z}$ *agrees with* $\langle a, b \rangle$.

PROOF. In the commutative diagram

$$\begin{tikzcd}
(\mathbb{Z},k) \arrow[r, "b"] & sC(X) \arrow[r, "\mathrm{dia}"] & sC(X \times X) \arrow[d, "\mathrm{EZ}", "\simeq"'] & \\
& & sC(X) \otimes sC(X) \arrow[r, "a\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes f_*"] & (\mathbb{Z},k) \otimes sC(X) \arrow[d, "\mathrm{id}\otimes f_*"] \\
& & sC(X) \otimes sC(\star) \arrow[r, "a\otimes\mathrm{id}"] & (\mathbb{Z},k) \otimes sC(\star)
\end{tikzcd}$$

the upper itinerary from $(\mathbb{Z}, k)$ to $(\mathbb{Z}, k) \otimes sC(\star)$ gives us $f_*(a \frown b)$ and the lower itinerary gives us $\langle a, b \rangle$. □

7.2. Orientations and fundamental classes

Let M be an n-dimensional topological manifold. To be very precise, we mean by that a topological space which is Hausdorff, locally homeomorphic to $\mathbb{R}^n$, and admits a countable base for its topology. (We are not assuming that M comes with a differentiable structure or that it is triangulated.)
In discussing what an *orientation* for M might be, we take a strictly homological viewpoint. This begins with the following observation.

PROPOSITION 7.2.1. *For every* $p \in M$, *the homology group* $H_n(M, M \smallsetminus p)$ *is an infinite cyclic group, while* $H_k(M, M \smallsetminus p) = 0$ *for* $k \neq n$.

PROOF. Choose an open neighborhood U of p such that there exists a homeomorphism $U \to \mathbb{R}^n$ (taking p to the origin). By excision, $H_*(M, M \smallsetminus p) \cong H_*(U, U \smallsetminus p)$ and this is isomorphic to $H_*(\mathbb{R}^n, \mathbb{R}^n \smallsetminus 0)$ which we can calculate and have calculated in a number of ways. □

DEFINITION 7.2.2. An orientation of M at p is a choice of generator of the infinite cyclic group $H_n(M, M \smallsetminus p)$.

Speaking intuitively, an *orientation of* M is a compatible choice of orientations of M at p for all $p \in M$. The difficulty here is to make sense of the word *compatible*. Let

$$E := \{ (p, z) \mid p \in M,\ z \in H_n(M, M \smallsetminus p) \}.$$

There is a projection map $g : E \to M$ given by $(p, z) \mapsto p$. Each fiber $g^{-1}(p)$ is of course in canonical bijection with the abelian group $H_n(M, M \smallsetminus p)$. Now we would like to introduce a topology on E. This will make $g : E \to M$ into a fiber bundle (indeed a fiber bundle with discrete fibers, also know as covering space).

DEFINITION 7.2.3. A subset U of E is *open* if for every $(p,z) \in U$ there exists an open neighborhood W of p in M and an element $y \in H_n(M, M \smallsetminus W)$ such that all

$$(q, y_q),\ \text{where } q \in W \text{ and } y_q \text{ is the image of } y \text{ in } H_n(M, M \smallsetminus q),$$

belong to U, and (p,z) is one of them, so that $y_p = z$.

It is easy to verify that this notion of open subset makes up a topology on E.

LEMMA 7.2.4. *Let V be an open subset of M and let $z \in H_n(M, M \smallsetminus V)$. The map taking $q \in V$ to $(q, z_q) \in E$ is continuous.* [Here z_q is the image of z in $H_n(M, M \smallsetminus q)$.]

PROOF. Call that map s. Let $U \subset E$ be open in M and let $q \in V$ be such that (q, z_q) belongs to U. Since U is open, we can find an open neighborhood W of $q \in M$ and an element $y \in H_n(M, M \smallsetminus W)$ such that for every $t \in W$, the pair (t, y_t) is an element of U, and moreover $y_q = z_q$. Making W smaller, we can assume that it is contained in V and that the inclusion-induced $H_n(M, M \smallsetminus W) \to H_n(M, M \smallsetminus q)$ is an isomorphism. Then it follows that $y \in H_n(M, M \smallsetminus W)$ is the image of $z \in H_n(M, M \smallsetminus V)$ under the inclusion-induced homomorphism. This implies that $s(t) = (t, z_t) = (t, y_t) \in U$ for all $t \in W$. Therefore W is an open neighborhood of q contained in $s^{-1}(U)$. □

COROLLARY 7.2.5. *With the above choice of topology on E, the projection $g : E \to M$ is a fiber bundle with discrete fibers.*

PROOF. Choose $p \in M$ and an open neighborhood V of $p \in M$ and a homeomorphism $\kappa : V \to \mathbb{R}^n$. Let $V_1 \subset V$ be the preimage under κ of the open unit ball in $\mathbb{R}^n$. Then $H_n(M, M \smallsetminus V_1) \cong \mathbb{Z}$ by excision. We obtain a map h from $V_1 \times H_n(M, M \smallsetminus V_1)$ to E by taking (q,z) to (q, z_q) where $z_q \in H_n(M, M \smallsetminus q)$ denotes the image of z under the inclusion-induced homomorphism $H_n(M, M \smallsetminus V_1) \to H_n(M, M \smallsetminus q)$. Now we have to verify that this map h is a homeomorphism from $V_1 \times H_n(M, M \smallsetminus V_1)$ to $g^{-1}(V_1)$. Here $H_n(M, M \smallsetminus V_1) \cong \mathbb{Z}$ comes with the discrete topology by definition.
It is clear from the definition of the topology on E that h is an open map, i.e., that it takes open subsets of $V_1 \times H_n(M, M \smallsetminus V_1)$ to open subsets of E, hence to open subsets of $g^{-1}(V_1) \subset E$. By lemma 7.2.4, it is also continuous. Finally, it is a bijection from $V_1 \times H_n(M, M \smallsetminus V_1)$ to $g^{-1}(V_1)$. □

DEFINITION 7.2.6. An *orientation of* M is a continuous map $s : M \to E$ such that $gs = \mathrm{id}_M$ and such that, for every $p \in M$, the element $s(p) = (p,z) \in E$ has the property that $z \in H_n(M, M \smallsetminus p)$ is a generator (of the infinite cyclic group $H_n(M, M \smallsetminus p)$).

REMARK 7.2.7. A manifold M is *orientable* if it admits an orientation, and *oriented* if it is equipped with a choice of orientation. Let $E^\times \subset E$ consist of all pairs $(p,z) \in E$ where z is a generator of $H_n(M, M \smallsetminus p)$. The restriction of g to $E^\times$ is a two-sheeted covering space, $E^\times \to M$. Clearly M is orientable if and only if $E^\times \to M$ is a trivial covering space, i.e., isomorphic as a bundle to the projection $M \times \{\pm 1\} \longrightarrow M$.
If M is connected, it admits either exactly two (distinct) orientations, or none. For example, in the cases where M is the (open) Möbius strip or the projective plane $\mathbb{R}P^2$, it admits none.

So much for orientations. Now we obtain somewhat mechanically a definition for the notion of *fundamental class* which underlines the close relationship with orientations.

DEFINITION 7.2.8. An element $z \in H_n(M)$ is a *fundamental class* for M if, for every $p \in M$, the image z_p of z in $H_n(M, M \smallsetminus p)$ is a generator of the infinite cyclic group $H_n(M, M \smallsetminus p)$.

Therefore, if $z \in H_n(M)$ is a fundamental class for M, then the map $s: M \to E$ defined by $s(p) = (p, z_p)$ is an *orientation* for M, and in particular M is orientable. Consequently, if M is not orientable, then it cannot have a fundamental class. However, the converse does not hold; it can happen that M is orientable although it does not have a fundamental class. (The manifold $M = \mathbb{R}^1$ is the easiest example; clearly it is orientable and clearly it does not have a fundamental class since $H_1(\mathbb{R}^1) = 0$.)

This easy example suggests that the existence of a fundamental class for M has something to do with compactness. In order to turn this idea into an investigation we choose a subset K of M. Let $\Gamma(g, K)$ be the abelian group of continuous sections of $g: E \to M$ over K; that is, continuous maps $s: K \to E$ such that $gs \equiv \mathrm{id}$ on K. Addition is pointwise; more precisely, if $s_1, s_2 \in \Gamma(g, K)$ and $s_1(p) = (p, y)$, $s_2(p) = (p, z)$, then $(s_1 + s_2)(p) := (p, y + z)$. There is a homomorphism

$$\lambda: H_n(M, M \smallsetminus K) \longrightarrow \Gamma(g, K)$$

defined by $z \mapsto (p \mapsto (p, z_p))$, where as usual z_p denotes the image of z in $H(M, M \smallsetminus p)$. (There is an inclusion-induced homomorphism $H_n(M, M \smallsetminus K) \to H_n(M, M \smallsetminus p)$ since we are assuming $p \in K$.)

Theorem 7.2.9. *For compact* $K \subset M$*, this homomorphism* $\lambda: H_n(M, M \smallsetminus K) \to \Gamma(g, K)$ *is an isomorphism. Besides,* $H_j(M, M \smallsetminus K)$ *is zero if* $j > n$*.*

Proof. This is an induction story. Let us write $\lambda_{M,K}$ instead of just λ if there is a need to be specific. There are two types of induction steps.

(i) If P and Q are compact subsets of M such that $\lambda_{M,P}$, $\lambda_{M,Q}$ and $\lambda_{M,P\cap Q}$ are isomorphisms, and if the groups $H_j(M, M \smallsetminus K)$ are zero for $j > n$ and $K = P, Q, P \cap Q$, then $\lambda_{M,P\cup Q}$ is an isomorphism, too, and $H_j(M, M \smallsetminus (P \cup Q)) = 0$ for $j > n$.

(ii) If K is the intersection of a descending sequence $K_0 \supset K_1 \supset K_2 \supset K_3 \supset \dots$ of compact subsets of M, and if λ_{M,K_i} is an isomorphism for $i = 0, 1, 2, \dots$, then $\lambda_{M,K}$ is an isomorphism, too.

We prove (i) first. To save space I write P' etc. for $M \smallsetminus P$. Remember $P' \cup Q' = (P \cap Q)'$ and all that. There is an exact sequence

$$\cdots \longrightarrow H_{n+1}(M, P' \cup Q') \longrightarrow H_n(M, P' \cap Q') \longrightarrow \begin{matrix} H_n(M, P') \\ \oplus H_n(M, Q') \end{matrix} \longrightarrow H_n(M, P' \cup Q')$$

(which is part of a Mayer-Vietoris sequence) and there is an exact sequence of abelian groups

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \Gamma(g, P \cup Q) \longrightarrow \Gamma(g, P) \oplus \Gamma(g, Q) \longrightarrow \Gamma(g, P \cap Q)$$

The homomorphisms $\lambda_{M,K}$ for $K = P \cup Q$, $K = P$, $K = Q$ and $K = P \cap Q$ give us a map (a commutative diagram in the shape of a ladder) from the first of these exact sequences to the second. Since we are assuming that $\lambda_{M,K}$ is an isomorphism for $K = P$, $K = Q$ and $K = P \cap Q$, and that the groups $H_j(M, M \smallsetminus K)$ are zero for $K = P$, $K = Q$ and $K = P \cap Q$, the five lemma implies that $\lambda_{M,K}$ is an isomorphism for $K = P \cup Q$ and that the groups $H_j(M, M \smallsetminus K)$ are zero for $j > n$ and $K = P \cup Q$.

Now we prove (ii). This is more challenging. By a relative version of Proposition 8.2.1 of the Topo I lecture notes, we may write

$$(*) \qquad H_n(M, M \smallsetminus \bigcap_i K_i) \xleftarrow{\ \cong\ } \operatorname*{colim}_{i \to \infty} H_n(M, M \smallsetminus K_i)$$

Let me first explain the meaning of the right hand side. (If you are familiar with limits and colimits in category theory, you will not need this explanation.) We can view the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ as a category with objects $0, 1, 2, 3, \dots$, in such a way that there is exactly one morphism from m to n if $m \le n$, otherwise none. Then we can view the rule $i \mapsto H_n(M, M \smallsetminus K_i)$ as a covariant functor F from the category $\mathbb{N}$ to the category of abelian groups. Such a functor F (in fact any functor from a small category to the category of abelian groups) has a *colimit*, denoted colim F or less formally $\mathrm{colim}_i F(i)$, which is again an abelian group. I resist the temptation to define this in general. In the case where the small category is $\mathbb{N}$, the meaning is as follows. An element of $\mathrm{colim}_i F(i)$ can be *represented* by a pair (i, y) where $y \in F(i)$. Two such pairs (i, y) and (j, z) are *equivalent*, in other words represent the same element of $\mathrm{colim}_i F(i)$, if and only if there exists $k \ge i, j$ such that the images of y and z in $F(k)$ agree.

What does statement $(*)$ have to do with a relative version of Proposition 8.2.1 of the Topo I lecture notes? The point is that every element $z \in H_n(M, M \smallsetminus \bigcap_i K_i)$ comes from $H_n(M, L)$ for some compact $L \subset M \smallsetminus \bigcap_i K_i$. We have

$$L \supset K_j \supset \bigcap_i K_i$$

for some j large enough. (This is true by the compactness of L, since the sets $M \smallsetminus K_j$ for all j cover L.) Therefore $M \smallsetminus L \subset M \smallsetminus K_j \subset M \smallsetminus \bigcap_i K_i$ and we deduce that $z \in H_n(M, M \smallsetminus \bigcap_i K_i)$ comes from $H_n(M, M \smallsetminus K_j)$ for some j large enough. This means that the homomorphism $(*)$ is surjective. A similar argument proves injectivity. — We can also write

$$(**) \qquad \Gamma(g, \bigcap_i K_i) \xleftarrow{\ \cong\ } \operatorname*{colim}_{i \to \infty} \Gamma(g, K_i)\,.$$

By writing this, we are saying two things. Firstly, every $s \in \Gamma(g, \bigcap_i K_i)$ is the restriction of some $t \in \Gamma(g, K_j)$, for some large enough j. (Indeed lemma 7.2.10 below says that we can extend s to some t defined on an open neighborhood U of $\bigcap_i K_i$ in M, where $gt \equiv \mathrm{id}$ on U. Then we need to verify that U contains one of the K_j, but this is just one of these standard exercises in compactness.) Secondly, if $s_1 \in \Gamma(g, K_a)$ and $s_2 \in \Gamma(g, K_b)$ agree on $\bigcap_i K_i$, then they already agree on K_c for some $c \ge a, b$. (Without loss of generality, $b \ge a$. The set of $x \in K_b$ which have $s_1(x) = s_2(x)$ is an open neighborhood of $\bigcap_i K_i$ in K_b. Therefore it contains K_c for some $c \ge b$ large enough.) — Finally the maps

$$\lambda_{M, K_i} : H_n(M, M \smallsetminus K_i) \longrightarrow \Gamma(g, K_i)$$

taken together can be viewed as a natural transformation between two covariant functors on the category $\mathbb{N}$. Since that natural transformation is a natural isomorphism by our assumption, the induced map of colimits is also an isomorphism. This completes the proof of (ii).

Now we come to the proof of the theorem proper. The compact subset K of M is contained in a finite union $V_1 \cup V_2 \cup \cdots \cup V_r$ where each $V_i \subset M$ is open and homeomorphic to $\mathbb{R}^n$. Using (i) we can reduce easily to the situation $r = 1$. This means $K \subset V \subset M$ where V is homeomorphic to $\mathbb{R}^n$. Since $H_n(M, M \smallsetminus K)$ and $\Gamma(g, K)$ remain unchanged if we replace M by the smaller V (without making changes to K), we may assume $M = V$, in other words $M = \mathbb{R}^n$. Using (ii) we can further reduce to the situation where K is a union of finitely many standard boxes

$$\prod_{j=1}^{n} [a_j, b_j] \subset \mathbb{R}^n$$

(where $a_j \le b_j \in \mathbb{R}$). Using (i) again we can further reduce to the situation where the number of boxes is exactly one. (If $K \subset \mathbb{R}^n$ is the union of ℓ such standard boxes, then we can write $K = K_1 \cup K_2$ where K_1 is one of the boxes and K_2 is the union of the remaining $\ell - 1$ boxes. Then $K_1 \cap K_2$ is again a union of at most $\ell - 1$ standard boxes, due to the fact that an intersection of standard boxes is again a standard box, or empty.) In the case where $M = \mathbb{R}^n$ and K is a single standard box, the theorem is trivially true. □

LEMMA 7.2.10. *Let $f: A \to B$ be a covering space, also known as fiber bundle with discrete fibers, where B is paracompact*[1]*. Let C be a closed subset of B and let $s: C \to A$ be a continuous map such that $fs \equiv \mathrm{id}$ on C. Then there are an open neighborhood U of C in B and a continuous map $t: U \to A$ such that $ft \equiv \mathrm{id}$ on U and $t \equiv s$ on C.*

PROOF. The covering space property of f implies that for each $x \in C$ there exist an open neighborhood V_x of x in B and a continuous map $t_x: V_x \to A$ such that $ft_x \equiv \mathrm{id}$ on V_x and $t_x \equiv s$ on $V_x \cap C$. Choose V_x and t_x for each x. Paracompactness of B implies that there is a family of open subsets $(W_i)_{i \in \Lambda}$ of B such that $C \subset \bigcup_i W_i$ and every $x \in C$ admits a neighborhood which has nonempty intersection with only finitely many W_i , and moreover, each W_i is contained in V_x for some x. Choose $x(i) \in C$ for each $i \in \Lambda$ such that $W_i \subset V_{x(i)}$. The set

$$\{\ y \in B \mid y \in \textstyle\bigcup_i W_i \text{ and } t_{x(i)}(y) = t_{x(j)}(y) \text{ whenever } y \in W_i \cap W_j\ \}$$

is an open subset U of B. (This uses the covering space property of f again.) Clearly U contains C. Define $t: U \to A$ unambiguously by $t(y) := t_{x(i)}(y)$ where $W_i \ni y$. □

7.3. Cohomology with compact supports

DEFINITION 7.3.1. Let X be a locally compact Hausdorff space. The k-th cohomology group of X with *compact supports* is defined as

$$H^k_c(X) := \operatorname*{colim}_{K \subset X} H^k(X, X \smallsetminus K)$$

where K runs over the compact subsets of X. In other words, elements of $H^k_c(X)$ can be represented by pairs (K, y) where $K \subset X$ is compact and $y \in H^k(X, X \smallsetminus K)$. Two such pairs (K, y) and (L, z) represent the same element of $H^k_c(X)$ if there is a compact subset J of X containing K and L such that y and z have the same image in $H^k(X, X \smallsetminus J)$.

EXAMPLE 7.3.2. In many cases of interest it is possible to exhaust X by an increasing sequence of compact subsets $K_0 \subset K_1 \subset K_2 \subset \dots$ such that $K_i \subset \mathrm{int}(K_{i+1})$ for all i and $\bigcup K_i = X$. Then any other compact subset L of X is contained in K_j for some (perhaps large) j. This implies that elements of $H^k_c(X)$ can always be represented by pairs (K_i, y) where $y \in H^k(X, X \smallsetminus K_i)$, and two such pairs (K_i, y), (K_j, z) represent the same element of $H^k_c(X)$ if and only if they map to the same element of $H^k(X, X \smallsetminus K_\ell)$ for some $\ell \ge i, j$. To illustrate this, we can take $X = \mathbb{R}^n$ and we can take K_i to be the closed disk of radius i in $\mathbb{R}^n$. Then $H^k(\mathbb{R}^n, \mathbb{R}^n \smallsetminus K_i)$ is zero for $k \ne n$, and for $k = n$ it is identified with $\mathbb{Z}$. For $j > i$, the inclusion-induced homomorphism

$$H^k(\mathbb{R}^n, \mathbb{R}^n \smallsetminus K_i) \longrightarrow H^k(\mathbb{R}^n, \mathbb{R}^n \smallsetminus K_j)$$

is an isomorphism. Consequently $H^k_c(\mathbb{R}^n)$ is isomorphic to $H^k(\mathbb{R}^n, \mathbb{R}^n \smallsetminus K_i)$ for every i. In this way we find that $H^k_c(\mathbb{R}^n)$ is zero for $k \ne n$, and isomorphic to $\mathbb{Z}$ for $k = n$.

[1] For me this includes "Hausdorff".

DEFINITION 7.3.3. Let X be a locally compact Hausdorff space. The k-th cohomology group of X with compact supports and coefficients in an abelian group A is defined as

$$H_c^k(X;A) := \operatorname*{colim}_{K\subset X} H^k(X, X\smallsetminus K; A)$$

where K runs over the compact subsets of X.

Suppose that $g\colon Y\to Z$ is a *proper* map between locally compact Hausdorff spaces. Let A be an abelian group. Unsurprisingly, g induces homomorphisms

$$g^*\colon H_c^k(Z;A)\to H_c^k(Y;A)$$

in cohomology with compact supports. Indeed, for every compact $K\subset Z$ the preimage $g^{-1}(K)$ is a compact subset of Y and we have

$$H^k(Z, Z\smallsetminus K; A) \longrightarrow H^k(Y, Y\smallsetminus g^{-1}(K); A)$$

induced contravariantly by g. Letting K run over all compact subsets and passing to (co)limits, we obtain $g^*\colon H_c^k(Z;A)\to H_c^k(Y;A)$.
Now let X be a locally compact Hausdorff space, Y an open subspace of X. Surprisingly, the inclusion $Y\to X$ induces *covariantly* a homomorphism in cohomology with compact supports,

$$H_c^k(Y;A)\longrightarrow H_c^k(X;A)$$

for each $k\in\mathbb{Z}$. Reason: every compact subset K of Y is also a compact subset of X. The inclusion of pairs $(Y, Y\smallsetminus K)\to(X, X\smallsetminus K)$ induces an excision *isomorphism*

$$u_K\colon H^k(X, X\smallsetminus K; A)\longrightarrow H^k(Y, Y\smallsetminus K; A)$$

so that the inverses of the u_K determine a homomorphism $H_c^k(Y;A)\to H_c^k(X;A)$. — Slightly more generally, if X and Y are locally compact spaces, and if $f\colon Y\to X$ is a continuous open injection, then there is an induced wrong-way map $f_!\colon H_c^k(Y;A)\to H_c^k(X;A)$. Briefly, cohomology with compact supports is a *covariant* functor for open embeddings of locally compact Hausdorff spaces.
There is some compatibility between these two kinds of functoriality. Let $g\colon Y\to Z$ be a proper map of locally compact Hausdorff spaces and let W be an open subset of Z. Then $V=g^{-1}(W)$ is an open subset of Y. Let $f\colon V\to W$ be the restriction of g. Write $a\colon V\to Y$ and $b\colon W\to Z$ for the inclusions. We obtain a commutative square

$$\begin{tikzcd}
H_c^k(W;A) \arrow[r, "f^*"] \arrow[d, "a_!"] & H_c^k(V;A) \arrow[d, "b_!"] \\
H_c^k(Z;A) \arrow[r, "g^*"] & H_c^k(Y;A)
\end{tikzcd}$$

REMARK 7.3.4. The two-way functoriality of cohomology with compact supports can also be formulated as follows. The category $\mathcal{C}$ of locally compact Hausdorff spaces and proper maps can be viewed as a "subcategory" of the category $\mathcal{D}$ of compact Hausdorff spaces with base point (where the morphisms are base-point preserving maps). This can be done by replacing each locally compact space X by its one-point compactification, $X\cup\infty$, where ∞ is viewed as the base point. Write $F\colon\mathcal{C}\to\mathcal{D}$ for this functor. Every object of $\mathcal{D}$ is isomorphic to an object of the form $F(X)$ where X is in $\mathcal{C}$. (Just delete the base point.) But F is not a full embedding; if X and Y are locally compact spaces, there can be based maps $X\cup\infty\to Y\cup\infty$ which are not induced by a proper map $X\to Y$. For example, if Y is

an open subset of X, then there is a map $X \cup \infty \to Y \cup \infty$ which is the identity on Y and takes all points of $X \cup \infty$ which are not in Y to the base point ∞ of $Y \cup \infty$.
We started by observing that compactly supported cohomology is a contravariant functor on $\mathcal{C}$, but then we discovered that it is actually a contravariant functor on the "bigger" category $\mathcal{D}$.

PROPOSITION 7.3.5. *Let X be a locally compact Hausdorff space and V, W open subsets of X such that $V \cup W = X$. Let A be an abelian group. There is a long exact Mayer-Vietoris sequence*

$$\cdots \longrightarrow H^k_c(V \cap W; A) \longrightarrow H^k_c(V; A) \oplus H^k_c(W; A) \longrightarrow H^k_c(X; A) \longrightarrow H^{k+1}_c(V \cap W; A) \longrightarrow \cdots$$

PROOF. Choose compact subsets $K \subset V$ and $L \subset V$. Then we have a long exact MV sequence relating the cohomology groups (with coefficients in A) of the pairs $(X, X \smallsetminus K)$, $(X, X \smallsetminus L)$, $(X, X \smallsetminus (K \cup L))$ and $(X, X \smallsetminus (K \cap L))$. This behaves covariantly in the variables K and L. We pass to the (co)limit over all compact K in V and all compact L in W. The result is the long exact sequence of the proposition. Remember that $H^k(X, X \smallsetminus K)$ for example can be identified with $H^k(W, W \smallsetminus K)$, by excision. See also remark 7.3.6 just below. □

REMARK 7.3.6. Let B be a partially ordered set (poset for short) which is also *directed*, i.e., for every finite subset S of B there exists $x \in B$ such that $x \geq y$ for all $y \in S$. This condition ensures in particular that B is nonempty. We may view B as a category whose objects are the elements of B and where the morphism sets $\mathrm{mor}(x, y)$ have exactly one element if $x \leq y$, otherwise no element at all. Let F be a (covariant) functor from B to the category of abelian groups. The colimit of F, denoted colim F or more informally $\mathrm{colim}_{x \in B} F(x)$, is an abelian group which we can define (or have already defined) as follows. Elements of colim F can be represented by pairs (x, u) where $x \in B$ and $u \in F(x)$. Two such, say (x, u) and (y, v), are equivalent if there exists $z \in B$ such that $z \geq x$, $z \geq y$ and the images of u and v in $F(z)$ agree. The abelian group of equivalence classes is colim F. Suppose that A is another *directed* poset and that $Q : A \to B$ is an order-preserving map, i.e., whenever $x \leq y$ in A then $Q(x) \leq Q(y)$ in B. We wish to compare colim F with colim $F \circ Q$. There is a (well defined) homomorphism

$$\mathrm{colim}\ F \circ Q \longrightarrow \mathrm{colim}\ F$$

which takes an element represented by (x, u), where $x \in A$ and $u \in F(Q(x))$, to the element of colim F represented by $(Q(x), u)$. The following condition on Q ensures that this homomorphism is an isomorphism. *For every $y \in B$ there is $x \in A$ such that $Q(x) \geq y$.* (The proof is straightforward.)

7.4. Poincaré duality

Let M be an oriented n-dimensional manifold. We aim for a Poincaré duality statement along the following lines:

> *For all $j \in \mathbb{Z}$, the cohomology group with compact supports $H^j_c(M)$ is isomorphic to the homology group $H_{n-j}(M)$.*

Note that M is not required to be compact. (And at this stage, it is not allowed to have a boundary.) Having a statement which does not ban noncompact manifolds will allow us to develop a proof by a form of induction, starting with the case $M = \mathbb{R}^n$. Nevertheless, the proposed statement could be criticized as being too vague because it does not specify an isomorphism. It would be better to say: *we have such-and-such a homomorphism from* *the compactly supported cohomology of* M *to the homology to* M*, easy to understand, and we have clever arguments to show that it is always an isomorphism.*

So let us look for an easy-to-understand homomorphism from $H^j_c(M)$ to $H_{n-j}(M)$. According to definition 7.2.6, the orientation of M *is* a section $s: M \to E$ of $g: E \to M$, where g and E are as in corollary 7.2.5. For every compact $K \subset M$, we can restrict s to K and we obtain $s|_K \in \Gamma(g, K)$. By theorem 7.2.9 this corresponds to an element $z_K \in H_n(M, M \smallsetminus K)$. These elements z_K will be very valuable for us. By construction, they satisfy compatibility: whenever K, L are compact subsets of M and $K \subset L$, then the inclusion-induced homomorphism $H_n(M, M \smallsetminus L) \to H_n(M, M \smallsetminus K)$ takes z_L to z_K.
For compact $K \subset M$ and $j \in \mathbb{Z}$ we obtain a homomorphism

$$\frown z_K : H^j(M, M \smallsetminus K) \longrightarrow H_{n-j}(M)$$

by cap product on the right with z_K ; more precisely, $y \in H^j(M, M \smallsetminus K)$ is taken to $y \frown z_K \in H_{n-j}(M)$. Now suppose again that K and L are compact subsets of M, where $K \subset L$. It follows from general properties of the cap product that the diagram

$$\begin{tikzcd} H^j(M, M \smallsetminus K) \arrow[r, "\frown z_K"] \arrow[d, "\text{inclusion-induced}"'] & H_{n-j}(M) \arrow[d, equal] \\ H^j(M, M \smallsetminus L) \arrow[r, "\frown z_L"] & H_{n-j}(M) \end{tikzcd}$$

is commutative. (The abstract formula that we can use here is $f^*(a) \frown b = a \frown f_*(b)$. For f we can take the inclusion of pairs $(M, M \smallsetminus L) \to (M, M \smallsetminus K)$ and for b we take z_L , so that $f_*(b) = z_K$.) Therefore we can pass to the (co)limit over all compact $K \subset M$ and we obtain a homomorphism

$$\operatorname*{colim}_{\text{cpct } K \subset M} H^j(M, M \smallsetminus K) = H^j_c(M) \xrightarrow{\quad \wp \quad} H_{n-j}(M).$$

THEOREM 7.4.1. (Poincaré duality.) *The homomorphism* $\wp$ *is an isomorphism.*

PROOF. We use an induction mechanism similar to, but easier than, the one used in the proof of theorem 7.2.9. First it is worth saying that $\wp$ is *natural* with respect to inclusions of open subsets. So if N is an open subset of M, then we have a commutative diagram

$$\begin{tikzcd} H^j_c(N) \arrow[r] \arrow[d, "\wp"] & H^j_c(M) \arrow[d, "\wp"] \\ H_{n-j}(N) \arrow[r] & H_{n-j}(M) \end{tikzcd}$$

The upper horizontal arrow is the wrong-way map of 7.3.4. The lower horizontal arrow is simply induced by the inclusion $N \to M$. — There are two kinds of induction steps:

(i) If M is the union of an ascending sequence of open subsets $M_0 \subset M_1 \subset M_2 \subset \dots$, and if $\wp$ is an isomorphism for $M_0, M_1, M_2, \dots$ and a specific j, then it is also an isomorphism for M and that specific j.
(ii) If $M = V \cup W$ and $\wp$ is an isomorphism for V, W and $V \cap W$ (for all $j \in \mathbb{Z}$), then $\wp$ is an isomorphism for $M = V \cup W$ and all $j \in \mathbb{Z}$.

We prove (i). Every compact subset K of M is contained in M_r for some $r \geq 0$. It follows that we can write

$$H_c^j(M) = \operatorname*{colim}_{r\to\infty} H_c^j(M_r)$$

and also

$$H_{n-j}(M) = \operatorname*{colim}_{r\to\infty} H_{n-j}(M_r).$$

So $\wp$ for M (and a specific degree j) is just the homomorphism of colimits obtained from the maps $\wp$ for each M_r (and the specific j), which we can view together as a natural transformation between functors from the category $\mathbb{N}$ to the category of abelian groups. But that natural transformation is a natural isomorphism by assumption. So the induced map of colimits is also an isomorphism.

We prove (ii). This is mainly a consequence of proposition 7.3.5. More precisely, we can arrange the compactly supported cohomology groups of $V, W, V\cap W$ and $V\cup W$ in a long exact sequence as stated there. We can also arrange the homology groups of $V, W, V\cap W$ and $V\cup W$ in a more ordinary long exact Mayer-Vietoris sequence. The maps $\wp$ amount to a map (a commutative diagram in the shape of a ladder) between the two Mayer-Vietoris sequences (which are the rows of the diagram). Out of every three adjacent vertical arrows in the diagram, two are isomorphisms by our assumption. Therefore the remaining ones are isomorphisms, by the Five lemma.

Now we prove the theorem proper. Using (i), we can reduce to the situation where M has a *finite* open cover with subsets $U_1, U_2, \dots, U_r$ such that each U_i is homeomorphic to an open subset of $\mathbb{R}^n$. Using (ii), we can then reduce to the situation where r is 1, in other words, M is an open subset of $\mathbb{R}^n$. Using (i) again, we can then reduce to the situation where $M \subset \mathbb{R}^n$ is a *finite* union of standard open boxes

$$\prod_{i=1}^{n}(a_i, b_i)$$

(where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$.) Using (ii) again, we can reduce further to the situation where M is just one such open box. In that case M is homeomorphic to $\mathbb{R}^n$, so we have reduced to the situation $M = \mathbb{R}^n$.

That case is worth looking at in detail! We know already that $H_c^j(\mathbb{R}^n)$ is zero for $j \neq n$, an isomorphic to $\mathbb{Z}$ for $j = n$. Of course it is also true that $H_{n-j}(\mathbb{R}^n)$ is zero for $j \neq n$ and isomorphic to $\mathbb{Z}$ for $j = n$. But we still need to show that

$$\wp : H_c^n(\mathbb{R}^n) \to H_0(\mathbb{R}^n)$$

is an isomorphism. Unraveling the definitions, and using example 7.3.2 if necessary, we find that this amounts to showing that *the cap product with* $z_K \in H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus K)$ (where $K \subset \mathbb{R}^n$ is the closed unit disk of some radius ≥ 0) *is an isomorphism*

$$H^n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus K) \longrightarrow H_0(\mathbb{R}^n).$$

We can choose the radius as we like and it is convenient to choose it to be zero, so that we can write $\mathbb{R}^n \smallsetminus 0$ instead of $\mathbb{R}^n \smallsetminus K$ if we wish. Remember that z_K is determined by the orientation of $\mathbb{R}^n$; we are assuming that an orientation has been selected. To show this thing, we start by observing that z_K is a generator of the infinite cyclic group $H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus 0)$. This is true by construction. We are looking at cap products of n-dimensional (relative) cohomology classes with a fixed n-dimensional (relative) homology class, landing in H_0 of a path-connected space, here $\mathbb{R}^n$. This is a particularly simple case of cap product; in fact it is the scalar product. Therefore we just have to convince ourselves that the scalar product

$$H^n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus 0) \times H_n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus 0) \longrightarrow \mathbb{Z} = H_0(\mathbb{R}^n)$$

is a nondegenerate bilinear map. This can be viewed as a special case of the universal coefficient theorem for cohomology, theorem 4.4.4. (Take $C = sC(\mathbb{R}^n)/sC(\mathbb{R}^n \smallsetminus 0)$ there, and $A = \mathbb{Z}$.) □

7.5. Poincaré duality for manifolds with boundary

Now we discuss the case of a possibly noncompact manifold *with boundary.* The goal is to get an isomorphism

$$\wp : H^j_c(M) \longrightarrow H_{n-j}(M, \partial M)$$

under the assumption that M is oriented. This generalizes theorem 7.4.1, that being the case where ∂M is empty. The proof is similar (and most of it will therefore be omitted) but some of the necessary definitions are slightly more intricate and will therefore be given.

For a start we need a more general form of cap product. Suppose that X is a space and Y, Z are two subspaces of X. For want of a better idea I shall write $H_k(X, Y + Z)$ for the k-th homology group of the quotient

$$\frac{sC(X)}{sC(Y) + sC(Z)}.$$

There is an obvious comparison homomorphism from $H_k(X, Y + Z)$ to $H_k(X, Y \cup Z)$ and in many cases it is an isomorphism (for example when Y and Z are open in $Y \cup Z$). But we cannot always impose that condition. The more general cap product has the form

$$H^k(X, Y) \times H_\ell(X, Y + Z) \longrightarrow H_{\ell-k}(X, Z).$$

Namely, let $a \in H^k(X, Y)$ and $b \in H_\ell(X, Y + Z)$. Then we can define $a \frown b \in H_{\ell-k}(X, Z)$ by composing

$$\begin{tikzcd}
(\mathbb{Z}, \ell) \arrow[r, "b"] & \dfrac{sC(X)}{sC(Y) + sC(Z)} \arrow[r, "\text{dia}"] & \dfrac{sC(X \times X)}{sC(Y \times X) + sC(X \times Z)} \arrow[d, "\simeq"', "\text{EZ}"] & (\mathbb{Z}, k) \otimes \dfrac{sC(X)}{sC(Z)} \\
& & \dfrac{sC(X) \otimes sC(X)}{sC(Y) \otimes sC(X) + sC(X) \otimes sC(Z)} \arrow[r, "\cong"] & \dfrac{sC(X)}{sC(Y)} \otimes \dfrac{sC(X)}{sC(Z)} \arrow[u, "a \otimes \text{id}"']
\end{tikzcd}$$

Next thing to do: we need to reconsider the construction of the covering space $E \to M$ just before definition 7.2.3. Let E be the set of pairs (p, z) where $p \in M$ and $z \in H_n(M, (M \smallsetminus p) + \partial M)$. Note that, if $p \in M \smallsetminus \partial M$, then $H_n(M, M \smallsetminus p)$ is strictly the same as $H_n(M, (M \smallsetminus p) + \partial M)$, and it is an infinite cyclic group. But if $p \in \partial M$, then $H_n(M, M \smallsetminus p)$ is zero whereas $H_n(M, (M \smallsetminus p) + \partial M)$ is still infinite cyclic.

Definition 7.5.1. A subset U of E is *open* if for every $(p, z) \in U$ there exists an open neighborhood W of p in M and an element $y \in H_n(M, (M \smallsetminus W) + \partial M)$ such that all

$$(q, y_q), \text{ where } q \in W \text{ and } y_q \text{ is the image of } y \text{ in } H_n(M, (M \smallsetminus q) + \partial M),$$

belong to U, and (p, z) is one of them, so that $y_p = z$.

It is easy to verify that this notion of open subset makes up a topology on E. The following definitions and statements (up to and including theorem 7.5.7) are analogous to definitions and statements given in section 7.2.

LEMMA 7.5.2. *Let* V *be an open subset of* M *and let* $z \in H_n(M, (M \smallsetminus V) + \partial M)$. *The map taking* $q \in V$ *to* $(q, z_q) \in E$ *is continuous.* □

COROLLARY 7.5.3. *With the above choice of topology on* E*, the projection* $g : E \to M$ *is a fiber bundle with discrete fibers.* □

DEFINITION 7.5.4. An *orientation of* M is a continuous map $s : M \to E$ such that $gs = \mathrm{id}_M$ and such that, for every $p \in M$, the element $s(p) = (p, z) \in E$ has the property that $z \in H_n(M, (M \smallsetminus p) + \partial M)$ is a generator.

REMARK 7.5.5. A manifold with boundary M is *orientable* if it admits an orientation, and *oriented* if it is equipped with a choice of orientation. Let $E^\times \subset E$ consist of all pairs $(p, z) \in E$ where z is a generator of $H_n(M, (M \smallsetminus p) + \partial M)$. The restriction of g to $E^\times$ is a two-sheeted covering space, $E^\times \to M$. Clearly M is orientable if and only if $E^\times \to M$ is a trivial covering space, i.e., isomorphic as a bundle to the projection $M \times \{\pm 1\} \longrightarrow M$.

DEFINITION 7.5.6. An element $z \in H_n(M, \partial M)$ is a *fundamental class* for M if, for every $p \in M$, the image z_p of z in $H_n(M, (M \smallsetminus p) + \partial M)$ is a generator of that infinite cyclic group.

For a subset K of M let $\Gamma(g, K)$ be the abelian group of continuous sections of $g : E \to M$ over K; that is, continuous maps $s : K \to E$ such that $gs \equiv \mathrm{id}$ on K. Addition is pointwise. There is a homomorphism

$$\lambda : H_n(M, (M \smallsetminus K) + \partial M) \longrightarrow \Gamma(g, K)$$

defined by $z \mapsto (p \mapsto (p, z_p))$.

THEOREM 7.5.7. *For compact* $K \subset M$*, this homomorphism* λ *from* $H_n(M, (M \smallsetminus K) + \partial M)$ *to* $\Gamma(g, K)$ *is an isomorphism. Besides,* $H_j(M, M \smallsetminus K)$ *is zero if* $j > n$. □

Now we are ready for the construction of a homomorphism $H^j_c(M) \to H_{n-j}(M, \partial M)$ which has a chance to be an isomorphism. This is under the assumption that M is oriented. Because M is oriented, we have a preferred section s of $E \to M$ which we can restrict to any compact subset K of M. By theorem 7.5.7, this determines (for every compact $K \subset M$) an element $z_K \in H_n(M, (M \smallsetminus K) + \partial M)$. If K and L are compact subsets of M such that $K \subset L$, then the inclusion-induced homomorphism

$$H_n(M, (M \smallsetminus L) + \partial M) \longrightarrow H_n(M, (M \smallsetminus K) + \partial M)$$

takes z_L to z_K. Cap product with z_K (on the right) is a homomorphism from $H^j(M, M \smallsetminus K)$ to $H_{n-j}(M, \partial M)$. We pass to the (co)-limit over all compact K in M and obtain a well defined homomorphism

$$\wp : H^j_c(M) \longrightarrow H_{n-j}(M, \partial M).$$

THEOREM 7.5.8. *The homomorphism* $\wp$ *is an isomorphism.* □

7.6. Poincaré duality for compact manifolds with boundary

If M is compact and oriented, there are some simplifications. There is no need to distinguish between H^j_c and H^j. In the case where ∂M is empty, the family of elements

$Z_K \in H_n(M, M \smallsetminus K)$ which we needed to define $\wp$ is now mostly superfluous. We can just go for the maximal K, which is M itself, and we obtain

$$z \in H_n(M)$$

which is the fundamental class determined by the orientation. The Poincaré duality homomorphism $\wp$ is then just $\frown z$, cap product with the fundamental class.

THEOREM 7.6.1. *For a compact oriented manifold without boundary, with fundamental class $z \in H_n(M)$, the cap product $\frown z\colon H^j(M) \to H_{n-j}(M)$ is an isomorphism for all $j \in \mathbb{Z}$.*

In the case where ∂M is nonempty, we still have the following simplifications. There is no need to distinguish between H^j_c and H^j. The family of elements $Z_K \in H_n(M, (M \smallsetminus K) + \partial M)$ which we needed to define $\wp$ is now mostly superfluous. We can just go for the maximal K, which is M itself, and we obtain

$$z \in H_n(M, \partial M)$$

which is the fundamental class determined by the orientation. The Poincaré duality homomorphism $\wp$ is then just $\frown z$, cap product with the fundamental class.

THEOREM 7.6.2. *For a compact oriented manifold without boundary, with fundamental class $z \in H_n(M)$, the cap product $\frown z\colon H^j(M) \to H_{n-j}(M, \partial M)$ is an isomorphism for all $j \in \mathbb{Z}$.*

(Obviously this is a generalization of theorem 7.6.1.) — There is yet another type of Poincaré duality for oriented compact manifolds with boundary.

THEOREM 7.6.3. *For a compact oriented manifold without boundary, with fundamental class $z \in H_n(M)$, the cap product $\frown z\colon H^j(M, \partial M) \to H_{n-j}(M)$ is an isomorphism for all $j \in \mathbb{Z}$.*

PROOF. We need the following commutative diagram:

$$\begin{tikzcd}
H^j(M, \partial M) \arrow[r, "\frown z"] & H_{n-j}(M) \\
H^j_c(M \smallsetminus \partial M) \arrow[u] \arrow[r, "\wp"] & H_{n-j}(M \smallsetminus \partial M) \arrow[u]
\end{tikzcd}$$

where $z \in H_n(M, \partial M)$ is the fundamental class. The lower horizontal arrow comes from theorem 7.4.1. The right-hand vertical arrow is induced by the inclusion of $M \smallsetminus \partial M$ in M. The left-hand vertical arrow is induced by the maps $(v_*)(u^*)^{-1}$ in

$$H^j(M \smallsetminus \partial M, (M \smallsetminus \partial M) \smallsetminus K) \xleftarrow[\cong]{u^*} H^j(M, M \smallsetminus K) \xrightarrow{v^*} H^j(M, \partial M)$$

where K is a compact subset $M \smallsetminus \partial M$ and u, v are obvious inclusions:

$$u\colon (M \smallsetminus \partial M, (M \smallsetminus \partial M) \smallsetminus K) \longrightarrow (M, M \smallsetminus K), \quad v\colon (M, \partial M) \longrightarrow (M, M \smallsetminus K).$$

Commutativity of the diagram is a consequence of the fact that, under the inclusion-induced homomorphism

$$H_n(M, \partial M) \to H_n(M, M \smallsetminus K)$$

the fundamental class z maps to the element which we have called z_K. (This follows from the characterizations of z and z_K in terms of associated sections of $g\colon E \to M$.) Here again K is a compact subset of $M \smallsetminus \partial M$.

Clearly we need to show that the vertical arrows in this diagram are isomorphisms. This is not completely trivial. I will use the existence of a *collar* for M, remark 7.6.4 below, without proof. It is an easy consequence of this that the inclusion $M \smallsetminus \partial M \to M$ is a homotopy equivalence. Therefore the right-hand vertical arrow in our diagram is an isomorphism. For the left-hand vertical arrow, we can reason as follows. Every compact subset K of $M \smallsetminus \partial M$ is contained in a compact subset of the form

$$M \smallsetminus f\big(\partial M \times [0, 1/r)\big) = (M \smallsetminus \partial M) \smallsetminus f\big(\partial M \times (0, 1/r)\big)$$

where $r = 1, 2, 3, \ldots$ (and f is the homeomorphism of remark 7.6.4). Therefore

$$\operatorname*{colim}_{K} \; H^j(M, M \smallsetminus K) \xleftarrow{\;\cong\;} \operatorname*{colim}_{r \geq 1} \; H^j(M,\; M \smallsetminus f(\partial M \times [0, 1/r)))$$

is an isomorphism. The composition of that with colim ν_* from $\mathrm{colim}_K H^j(M, M \smallsetminus K)$ to $H^j(M, \partial M)$ is also an isomorphism. Therefore colim ν_* is an isomorphism, and that is what we need. □

REMARK 7.6.4. A compact manifold M with boundary admits a *collar*. This consists of an open neighborhood U of ∂M in M and a homeomorphism $f\colon \partial M \times [0, 1) \to U$ such that $f(x, 0) = x$ for all $x \in \partial M$. The existence of collars (even without the compactness assumption on M) was proved by Morton Brown (1962).

Finally one wants to understand to understand the precise relationship between Poincaré duality for M and Poincaré duality for ∂M. Here it should not be necessary to assume that M is compact, but we will assume it for simplicity. We write $\lambda_M : H_n(M, \partial M) \to \Gamma(g_M, M)$ and $\lambda_{\partial M} : H_{n-1}(\partial M) \to \Gamma(g_{\partial M}, \partial M)$ for the isomorphisms of theorems 7.5.7 and 7.2.9, special instances where K is the maximal choice.

LEMMA 7.6.5. *For a compact n-dimensional manifold M with boundary, there is an isomorphism $\alpha\colon E_M|_{\partial M} \cong E_{\partial M}$ of bundles over ∂M, respecting fiberwise abelian group structures and making the following diagram commutative*

$$\begin{tikzcd}
H_n(M, \partial M) \arrow[r, "\lambda_M"] \arrow[d, "\partial"] & \Gamma(g_M, M) \arrow[d, "\alpha_* \circ \text{restriction}"] \\
H_{n-1}(\partial M) \arrow[r, "\lambda_{\partial M}"] & \Gamma(g_{\partial M}, \partial M)
\end{tikzcd}$$

PROOF. The fiber of $g_M : E_M \to M$ over $p \in \partial M$ is identified with

$$H_n(M, (M \smallsetminus p) + \partial M) = H_n\Big(\frac{sC(M)}{sC(M \smallsetminus p) + sC(\partial M)}\Big).$$

(More precisely it consists of all pairs (p,z) where $z \in H_n(M,(M \smallsetminus p)+\partial M)$.) There are homomorphisms

$$\begin{tikzcd}
H_n\Big(\dfrac{sC(M)}{sC(M \smallsetminus p)+sC(\partial M)}\Big) \arrow[r, "\partial"] & H_{n-1}\big(sC(M \smallsetminus p)+sC(\partial M)\big) \arrow[d] \\
H_{n-1}\Big(\dfrac{sC(\partial M)}{sC(\partial M)\cap sC(M \smallsetminus p)}\Big) \arrow[d, equal] & H_{n-1}\Big(\dfrac{sC(M \smallsetminus p)+sC(\partial M)}{sC(M \smallsetminus p)}\Big) \arrow[l, "\cong"'] \\
H_{n-1}\Big(\dfrac{sC(\partial M)}{sC(\partial M \smallsetminus p)}\Big) &
\end{tikzcd}$$

The composition is an isomorphism, and it is what we take for α. (To show that it is an isomorphism, use an excision argument to replace M by a standard neighborhood of p throughout. This can be chosen so that it is homeomorphic to $\mathbb{R}^{n-1}\times[0,\infty)$. Then ∂M gets replaced by $\mathbb{R}^{n-1}\times\{0\}$. Details omitted.) □

(Intuitively, the lemma is just trying to say that an orientation for M determines an orientation for ∂M, and a fundamental class from $(M,\partial M)$ determines a fundamental class for ∂M by application of the boundary operator ∂, and these two procedures are *compatible*.)

Corollary 7.6.6. *The image of a fundamental class $z_M \in H_n(M,\partial M)$ under the boundary operator $\partial\colon H_n(M,\partial M)\to H_{n-1}(\partial M)$ is a fundamental class for ∂M.* □

Corollary 7.6.7. *For an oriented compact n-dimensional manifold with boundary there is a commutative diagram relating the long exact cohomology sequence of the pair $(M,\partial M)$ to the long exact homology sequence:*

$$\begin{tikzcd}
\cdots \arrow[r] & H^j(M) \arrow[r] \arrow[d, "\frown (-1)^j z_M"] & H^j(\partial M) \arrow[r, "\partial"] \arrow[d, "\frown z_{\partial M}"] & H^{j+1}(M,\partial M) \arrow[r] \arrow[d, "\frown (-1)^{j+1} z_M"] & H^{j+1}(M) \arrow[r] \arrow[d, "\frown (-1)^{j+1} z_M"] & \cdots \\
\cdots \arrow[r] & H_{n-j}(M,\partial M) \arrow[r, "\partial"] & H_{n-j-1}(\partial M) \arrow[r] & H_{n-1-j}(M) \arrow[r] & H_{n-1-j}(M,\partial M) \arrow[r] & \cdots
\end{tikzcd}$$

Here $z_M \in H_n(M,\partial M)$ and $z_{\partial M}$ are the fundamental classes. All vertical arrows are isomorphisms.

Proof. We know already that the vertical arrows (given by cap product with the appropriate fundamental classes, up to sign) are isomorphisms. Commutativity of the little squares which do not involve a vertical arrow of type $\frown z_{\partial M}$ is clear. Commutativity of the little squares with a right-hand vertical arrow of type $\frown z_{\partial M}$ follows from the general law

$$\partial(a \frown b) = (-1)^{|a|} f^*(a) \frown \partial b$$

which is valid for a pair of spaces (X,Y) and $a \in H^j(X)$, $b \in H_k(X,Y)$, on the understanding that f denotes the inclusion $Y \to X$. Then we get $a \frown b \in H_{k-j}(X,Y)$ and $f^*(a) \frown \partial b \in H_{k-j-1}(Y)$. Commutativity of the little squares with a left-hand vertical arrow of type $\frown z_{\partial M}$ follows from the law

$$\partial a \frown b = (-1)^{|a|+1} f_*(a \frown \partial b)$$

which is valid for $a \in H^j(Y)$, $b \in H_k(X,Y)$, so that $\partial a \in H^{j+1}(X,Y)$ and $\partial b \in H_{k-1}(Y)$. Sketch proofs for these equations: just below. ☐

Equation $\partial(a \frown b) = (-1)^{|a|} f^*(a) \frown \partial b$.

$$
\begin{tikzcd}
(\mathbb{Z},k) \arrow[r, "b"] & \dfrac{sC(X)}{sC(Y)} \arrow[r, "\mathrm{dia}"] & \dfrac{sC(X\times X)}{sC(X\times Y)} \arrow[d, "\simeq"', "\mathrm{EZ}"] & (\mathbb{Z},j)\otimes \dfrac{sC(X)}{sC(Y)} \\
 & & \dfrac{sC(X)\otimes sC(X)}{sC(X)\otimes sC(Y)} \arrow[r, "\cong"] & sC(X)\otimes \dfrac{sC(X)}{sC(Y)} \arrow[u, "a\otimes \mathrm{id}"']
\end{tikzcd}
$$

To determine $\partial(a \frown b)$ we apply d to a chain in $sC(X)$ which lifts a $(k-j)$-cycle in $sC(X)/sC(Y)$, which in turn corresponds to the k-cycle in the top right-hand expression

$$(\mathbb{Z},j)\otimes \frac{sC(X)}{sC(Y)}$$

given by the diagram. To determine $f^*(a) \frown \partial b$ we first apply d to a k-chain in $sC(X)$ which lifts the k-cycle in $sC(X)/sC(Y)$ given by or representing b, and proceed through the diagram. Since all maps in the diagram are chain maps (respect d), this amounts to applying d to a k-chain in the top right-hand term $(\mathbb{Z},j)\otimes sC(X)$ which lifts a k-cycle in $(\mathbb{Z},j)\otimes(sC(X)/sC(Y))$. We get the same result up to multiplication by $(-1)^j$.

Equation $\partial a \frown b = (-1)^{|a|+1} f_*(a \frown \partial b)$. Dold writes δ for the (co)boundary operator in cohomology and ∂ for the boundary operator in homology. It is not absolutely necessary but it helps to avoid confusion. If we adopt this, then the equation turns into

$$\delta a \frown b = (-1)^{|a|+1} f_*(a \frown \partial b).$$

In this form it offends less than before against the sign rule. I use the Σ notation from section 4.5. In that connection, I want to think of ∂ as a chain map from B/A to ΣA, assuming that we have a short exact sequence of chain complexes $A \hookrightarrow B \to B/A$, but this is only defined if A/B is chain complex of *free* abelian groups and then only well defined as a homotopy class of chain maps.

$$
\begin{tikzcd}
(\mathbb{Z},k) \arrow[r, "b"] & \dfrac{sC(X)}{sC(Y)} \arrow[r, "\mathrm{dia}"] \arrow[dd, "\partial"] & \dfrac{sC(X\times X)}{sC(Y\times X)} \arrow[d, "\simeq"', "\mathrm{EZ}"] & (\mathbb{Z},j+1)\otimes sC(X) \\
 & & \dfrac{sC(X)\otimes sC(X)}{sC(Y)\otimes sC(X)} \arrow[r, "\partial"] & \Sigma(sC(Y)\otimes sC(X)) \arrow[u, "a\otimes \mathrm{id}"] \\
 & \Sigma sC(Y) \arrow[r] & \Sigma sC(Y\times Y) \arrow[r, "\mathrm{EZ}"] & \Sigma(sC(Y)\otimes sC(Y)) \arrow[u, "\mathrm{incl.}"]
\end{tikzcd}
$$

Taking the lower itinerary in this homotopy commutative diagram (from top left to top right), we get $f_*(a \frown \partial b)$. Taking the upper itinerary we seem to get $\delta a \frown b$. But this is only true up to sign. Without going too much into details, the composition

$$sC(X)/sC(Y) \xrightarrow{\partial} \Sigma sC(Y) \xrightarrow{a} (\mathbb{Z},j+1)$$

is equal (better, homotopic) to $(-1)^{|a|+1}\delta a$. This may come as a surprise, and I must admit it did for me. But it is a consequence of the fact that we defined δ in terms of the short exact sequence of chain complexes

$$\hom(sC(X)/sC(Y),\mathbb{Z}) \longrightarrow \hom(sC(X),\mathbb{Z}) \longrightarrow \hom(sC(Y),\mathbb{Z})$$

(not directly in terms of ∂) and we used some interesting signs in defining the hom chain complexes. Of course I also learned this in the Dold book. In fact Dold preaches this sermon already where he writes about the most elementary product, the scalar product. Namely,

$$\langle \delta u, v\rangle = (-1)^{|u|+1}\langle u, \partial v\rangle.$$

This is applicable if we have any pair of spaces (X,Y) and $u \in H^k(Y)$, $v \in H_{k+1}(X,Y)$. We conclude that I have been very unsystematic. □

7.7. The homology of a compact manifold is finitely generated

Proposition 7.7.1. *Let M be a compact n-dimensional manifold, with or without boundary. Then $H_k(M)$ is a finitely generated abelian group for all $k \in \mathbb{Z}$.*

It is strange that we did not have to prove this *before* establishing Poincaré duality. Here is a sketch proof. Suppose that $\partial M = \emptyset$ for simplicity. It is enough to show the following.

- If $U_1, U_2, \dots, U_r$ are open subsets of M, each of them homeomorphic to an open subset of $\mathbb{R}^n$, and if $V_1, V_2, \dots, V_r$ are open subsets of M such that the closure of V_i is contained in U_i for all $i \in \{1,2,\dots,r\}$, then the image of the inclusion-induced homomorphism

  $$H_k(V_1 \cup V_2 \cup \dots \cup V_r) \longrightarrow H_k(U_1 \cup U_2 \cup \dots \cup U_r)$$

  is finitely generated (for all $k \in \mathbb{Z}$).

We prove this by induction on r. The induction beginning is the case $r = 1$. Here we can pretend that U_1 is an open subset of $\mathbb{R}^n$ and that V_1 is an open subset of U_1 with compact closure in U_1. Then we can find a finite union K_i of standard compact cubes in $\mathbb{R}^n$ contained in U_1 and containing V_i. So the inclusion $V_u \to U_i$ can be written as a composition $V_i \to K_i \to U_i$. The homology groups of K_i are finitely generated. Therefore the image of $H_k(V_1)$ in $H_k(U_1)$ is finitely generated. — The induction step uses a Mayer-Vietoris argument. It is straightforward and left to the reader. □

7.8. The signature

Let M be a compact *oriented* topological manifold (without boundary) of dimension $4k$. Then we have a symmetric bilinear form Q on $H^{2k}(M,\mathbb{R})$ given by

$$(x,y) \mapsto \langle x \smile y, z_M\rangle$$

for $x,y \in H^{2k}(M;\mathbb{R})$, where z_M is the fundamental class. We have

$$\langle x \smile y, z_M\rangle = (x \smile y) \frown z_M = x \frown (y \frown z_M) = \langle x, y \frown z_M\rangle.$$

Since $y \mapsto y \frown z_M$ is an isomorphism from $H^{2k}(M;\mathbb{R})$ to $H_{2k}(M;\mathbb{R})$, and the scalar product specialized to a nondegenerate pairing between the (finite dimensional) real vector spaces $H_{2k}(M;\mathbb{R})$ and $H^{2k}(M;\mathbb{R})$, the symmetric bilinear form Q is nondegenerate. By standard theorems of linear algebra we can write $H^{2k}(M;\mathbb{R})$ as a direct sum of linear subspaces V and W such that Q is positive definite on V and negative definite on W and $Q(v,w) = 0$ for $v \in V$, $w \in W$. The dimensions of V and W are determined by Q

alone and their difference, $\dim(V) - \dim(W)$, is the *signature* of Q. We also call it the *signature of* M.
Note that the signature of M changes sign if we replace the chosen orientation by its opposite (better, *negative*).

The signature of M is clearly a homeomorphism invariant. But it is much more invariant than that, as was pointed out by Thom.

THEOREM 7.8.1. *Suppose that* M *is the boundary of a compact oriented manifold* N *of dimension* $4k+1$. *Then the signature of* M *is* 0.

PROOF. I follow Dold once again. Let $f : M \to N$ be the inclusion. Let A be the image of the restriction homomorphism

$$H^{2k}(N) \longrightarrow H^{2k}(M)$$

(cohomology with real coefficients throughout). We have $x \in A$ if and only if $\delta x = 0 \in H^{2k+1}(N, M)$. By corollary 7.6.7 this holds if and only if $f_*(x \cap z_M) = 0$ in $H_{2k}(N)$, if and only if $f_*(x \frown z_M)$ has zero scalar product with all $y \in H^{2k}(N)$, if and only if $x \frown z_M$ has zero scalar product with all $f^*(y)$, if and only if $x \smile f^*(y)$ has zero scalar product with z_M for all $y \in H^{2k}(N)$, if and only $Q(x, v) = 0$ for all $v \in A$. So the linear subspace A of $H^{2k}(M; \mathbb{R})$ is its own annihilator (for the symmetric form Q). We also say that A is a Lagrangian subspace for Q. In such a case the signature of Q must be zero. □

For example if $M = \mathbb{C}P^{2k}$ for some $k \in \{1, 2, \dots\}$, then the signature of M is 1 or -1 depending on the orientation. Therefore these M are not of the form ∂N for any compact oriented N.