05.02.2018

# Lecture Notes
# Topology 1, WS 2017/18 (Weiss)

CHAPTER 1

# Homotopy

## 1.1. The homotopy relation

Let $X$ and $Y$ be topological spaces. (If you are not sufficiently familiar with topological spaces, you should assume that $X$ and $Y$ are metric spaces.) Let $f$ and $g$ be continuous maps from $X$ to $Y$. Let $[0,1]$ be the unit interval with the standard topology, a subspace of $\mathbb{R}$.

Definition 1.1.1. A *homotopy* from $f$ to $g$ is a continuous map

$$h : X \times [0,1] \to Y$$

such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x \in X$. If such a homotopy exists, we say that $f$ and $g$ are *homotopic*, and write $f \simeq g$. We also sometimes write $h : f \simeq g$ to indicate that $h$ is a homotopy from the map $f$ to the map $g$.

Remark 1.1.2. If you made the assumption that $X$ and $Y$ are metric spaces, then you should use the product metric on $X \times [0,1]$ and $Y \times [0,1]$, so that for example

$$d((x_1,t_1),(x_2,t_2)) := \max\{d(x_1,x_2), |t_1 - t_2|\ \}$$

for $x_1, x_2 \in X$ and $t_1, t_2 \in [0,1]$. If you were happy with the assumption that $X$ and $Y$ are "just" topological spaces, then you need to know the definition of *product of two topological spaces* in order to make sense of $X \times [0,1]$ and $Y \times [0,1]$.

Remark 1.1.3. A homotopy $h : X \times [0,1] \to Y$ from $f : X \to Y$ to $g : X \to Y$ can be seen as a "family" of continuous maps

$$h_t : X \to Y\ ;\ h_t(x) = h(x,t)$$

such that $h_0 = f$ and $h_1 = g$. The important thing is that $h_t$ depends continuously on $t \in [0,1]$.

Example 1.1.4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the identity map. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be the map such that $g(x) = 0 \in \mathbb{R}^n$ for all $x \in \mathbb{R}^n$. Then $f$ and $g$ are homotopic. The map $h : \mathbb{R}^n \times [0,1]$ defined by $h(x,t) = tx$ is a homotopy from $f$ to $g$.

Example 1.1.5. Let $f : S^1 \to S^1$ be the identity map, so that $f(z) = z$. Let $g : S^1 \to S^1$ be the antipodal map, $g(z) = -z$. Then $f$ and $g$ are homotopic. Using complex number notation, we can define a homotopy by $h(z,t) = e^{\pi i t} z$.

Example 1.1.6. Let $f : S^2 \to S^2$ be the identity map, so that $f(z) = z$. Let $g : S^2 \to S^2$ be the antipodal map, $g(z) = -z$. Then $f$ and $g$ are *not* homotopic. We will prove this later in the course.

Example 1.1.7. Let $f : S^1 \to S^1$ be the identity map, so that $f(z) = z$. Let $g : S^1 \to S^1$ be the constant map with value $1$. Then $f$ and $g$ are *not* homotopic. We will prove this quite soon.

PROPOSITION 1.1.8. *"Homotopic" is an equivalence relation on the set of continuous maps from* $X$ *to* $Y$.

PROOF. Reflexive: For every continuous map $f : X \to Y$ define the *constant homotopy* $h : X \times [0,1] \to Y$ by $h(x,t) = f(x)$.
Symmetric: Given a homotopy $h : X \times [0,1] \to Y$ from a map $f : X \to Y$ to a map $g : X \to Y$, define the *reverse homotopy* $\bar{h} : X \times [0,1] \to Y$ by $\bar{h}(x,t) = h(x, 1-t)$. Then $\bar{h}$ is a homotopy from $g$ to $f$.
Transitive: Given continuous maps $e, f, g : X \to Y$, a homotopy $h$ from $e$ to $f$ and a homotopy $k$ from $f$ to $g$, define the *concatenation homotopy* $k * h$ as follows:

$$(x,t) \mapsto \begin{cases} h(x,2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ k(x,2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \ . \end{cases}$$

Then $k * h$ is a homotopy from $e$ to $g$. □

DEFINITION 1.1.9. The equivalence classes of the above relation "homotopic" are called *homotopy classes*. The homotopy class of a map $f : X \to Y$ is often denoted by $[f]$. The set of homotopy classes of maps from $X$ to $Y$ is often denoted by $[X, Y]$.

PROPOSITION 1.1.10. *Let* $X$, $Y$ *and* $Z$ *be topological spaces. Let* $f : X \to Y$ *and* $g : X \to Y$ *and* $u : Y \to Z$ *and* $v : Y \to Z$ *be continuous maps. If* $f$ *is homotopic to* $g$ *and* $u$ *is homotopic to* $v$, *then* $u \circ f : X \to Z$ *is homotopic to* $v \circ g : X \to Z$.

PROOF. Let $h : X \times [0,1] \to Y$ be a homotopy from $f$ to $g$ and let $w : Y \times [0,1] \to Z$ be a homotopy from $u$ to $v$. Then $u \circ h$ is a homotopy from $u \circ f$ to $u \circ g$ and the map $X \times [0,1] \to Z$ given by $(x,t) \mapsto w(g(x),t)$ is a homotopy from $u \circ g$ to $v \circ g$. Because the homotopy relation is transitive, it follows that $u \circ f \simeq v \circ g$. □

DEFINITION 1.1.11. Let $X$ and $Y$ be topological spaces. A (continuous) map $f : X \to Y$ is a *homotopy equivalence* if there exists a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.
We say that $X$ is *homotopy equivalent* to $Y$ if there exists a map $f : X \to Y$ which is a homotopy equivalence.

DEFINITION 1.1.12. If a topological space $X$ is homotopy equivalent to a point, then we say that $X$ is *contractible*. This amounts to saying that the identity map $X \to X$ is homotopic to a constant map from $X$ to $X$.

EXAMPLE 1.1.13. $\mathbb{R}^m$ is contractible, for any $m \geq 0$.

EXAMPLE 1.1.14. $\mathbb{R}^m \smallsetminus \{0\}$ is homotopy equivalent to $S^{m-1}$.

EXAMPLE 1.1.15. The general linear group of $\mathbb{R}^m$ is homotopy equivalent to the orthogonal group $O(m)$. The Gram-Schmidt orthonormalisation process leads to an easy proof of that.

## 1.2. Homotopy classes of maps from the circle to itself

Let $p \colon \mathbb{R} \to S^1$ be the (continuous) map given in complex notation by $p(t) = \exp(2\pi i t)$ and in real notation by $p(t) = (\cos(2\pi t), \sin(2\pi t))$. In the first formula we think of $S^1$ as a subset of $\mathbb{C}$ and in the second formula we think of $S^1$ as a subset of $\mathbb{R}^2$.
Note that $p$ is surjective and $p(t+1) = p(t)$ for all $t \in \mathbb{R}$. We are going to use $p$ to understand the homotopy classification of continuous maps from $S^1$ to $S^1$. The main lemma is as follows.

LEMMA 1.2.1. *Let $\gamma\colon [0,1] \to S^1$ be continuous, and $a \in \mathbb{R}$ such that $p(a) = \gamma(0)$. Then there exists a unique continuous map $\tilde{\gamma}\colon [0,1] \to \mathbb{R}$ such that $\gamma = p \circ \tilde{\gamma}$ and $\tilde{\gamma}(0) = a$.*

PROOF. The map $\gamma$ is *uniformly continuous* since $[0,1]$ is compact. It follows that there exists a positive integer $n$ such that $d(\gamma(x),\gamma(y)) < 1/100$ whenever $|x-y| \le 1/n$. Here $d$ denotes the standard (euclidean) metric on $S^1$ as a subset of $\mathbb{R}^2$. We choose such an $n$ and write

$$[0,1] = \bigcup_{k=1}^{n} [t_{k-1}, t_k]$$

where $t_k = k/n$. We try to define $\tilde{\gamma}$ on $[0,t_k]$ by induction on $k$. For the induction beginning we need to define $\tilde{\gamma}$ on $[0,t_1]$ where $t_1 = 1/n$. Let $U \subset S^1$ be the open ball of radius $1/100$ with center $\gamma(0)$. (Note that *open ball* is a metric space concept.) Then $\gamma([0,t_1]) \subset U$. Therefore, in defining $\tilde{\gamma}$ on $[0,t_1]$, we need to ensure that $\tilde{\gamma}([0,t_1])$ is contained in $p^{-1}(U)$. Now $p^{-1}(U) \subset \mathbb{R}$ is a disjoint union of open intervals which are mapped homeomorphically to $U$ under $p$. One of these, call it $V_a$ , contains $a$, since $p(a) = \gamma(0) \in U$. The others are translates of the form $\ell + V_a$ where $\ell \in \mathbb{Z}$. Since $[0,t_1]$ is connected, its image under $\tilde{\gamma}$ will also be connected, whatever $\tilde{\gamma}$ is, and so it must be contained entirely in exactly one of the intervals $\ell + V_a$. Since we want $\tilde{\gamma}(0) = a$, we must have $\ell = 0$, that is, image of $\tilde{\gamma}$ contained in $V_a$. Since the map $p$ restricts to a homeomorphism from $V_a$ to $U$, we must have $\tilde{\gamma} = q\gamma$ where $q$ is the inverse of the homeomorphism from $V_a$ to $U$. This formula determines the map $\tilde{\gamma}$ on $[0,t_1]$.
The induction steps are like the induction beginning. In the next step we define $\tilde{\gamma}$ on $[t_1,t_2]$, using a "new" $a$ which is $\tilde{\gamma}(t_1)$ and a "new" $U$ which is the open ball of radius $1/100$ with center $\gamma(t_1)$. □

Now let $g\colon S^1 \to S^1$ be any continuous map. We want to associate with it an integer, the degree of $g$. Choose $a \in \mathbb{R}$ such that $p(a) = g(1)$. Let $\gamma = g \circ p$ on $[0,1]$; this is a map from $[0,1]$ to $S^1$. Construct $\tilde{\gamma}$ as in the lemma. We have $p\tilde{\gamma}(1) = \gamma(1) = \gamma(0) = p\tilde{\gamma}(0)$, which implies $\tilde{\gamma}(1) = \tilde{\gamma}(0) + \ell$ for some $\ell \in \mathbb{Z}$.

DEFINITION 1.2.2. This $\ell$ is the degree of $g$, denoted $\deg(g)$.

It looks as if this might depend on our choice of $a$ with $p(a) = g(1)$. But if we make another choice then we only replace $a$ by $m + a$ for some $m \in \mathbb{Z}$, and we only replace $\tilde{\gamma}$ by $m + \tilde{\gamma}$. Therefore our calculation of $\deg(g)$ leads to the same result.

*Remark.* Suppose that $g\colon S^1 \to S^1$ is a continuous map which is not surjective. Then $\deg(g) = 0$. *Reason*: choose $z \in S^1$ which is not in the image of $g$. Then $\mathbb{R} \smallsetminus p^{-1}(z)$ is the disjoint union of many open intervals of length exactly 1. The image of $[0,1]$ under $\tilde{\gamma}$ is connected and has empty intersection with $p^{-1}(z)$; therefore it is contained in one of these intervals. It follows that $|\tilde{\gamma}(1) - \tilde{\gamma}(0)| < 1$, that is to say, $|\deg(g)| < 1$.

*Remark.* Suppose that $f, g\colon S^1 \to S^1$ are continuous maps. Let $w\colon S^1 \to S^1$ be defined by $w(z) = f(z) \cdot g(z)$ (using the multiplication in $S^1 \subset \mathbb{C}$). Then $\deg(w) = \deg(f) + \deg(g)$. The verification is mechanical. Define $\varphi, \gamma, \omega\colon [0,1] \to S^1$ by $\varphi(t) = f(p(t))$, $\gamma(t) = g(p(t))$ and $\omega(t) = w(p(t))$. Construct $\tilde{\varphi}\colon [0,1] \to \mathbb{R}$ and $\tilde{\gamma}\colon [0,1] \to \mathbb{R}$ as in lemma 1.2.1. Put $\tilde{\omega} := \tilde{\varphi} + \tilde{\gamma}$. Then $p \circ \tilde{\omega} = \omega$, so

$$\deg(w) = \tilde{\omega}(1) - \tilde{\omega}(0) = \cdots = \deg(f) + \deg(g).$$

LEMMA 1.2.3. *If $f, g\colon S^1 \to S^1$ are continuous maps which are homotopic, $f \sim g$, then they have the same degree.*

PROOF. Let $h\colon S^1 \times [0,1] \to S^1$ be a homotopy from $f$ to $g$. As usual let $h_t\colon S^1 \to S^1$ be the map defined by $h_t(z) = h(z,t)$, for fixed $t \in [0,1]$. By uniform continuity of $h$, we can find $\delta > 0$ such that $d(h_t(z), h_s(z)) < 1/1000$ for all $z \in S^1$ and all $s,t \in [0,1]$ which satisfy $|s-t| < \delta$. Therefore $h_s(z) = g(z)\cdot h_t(z)$ for such $t$ and $s$, where $g\colon S^1 \to S^1$ is a map (depending on $s$ and $t$) which satisfies $d(g(z),1) < 1/1000$ for all $z \in S^1$. Then $g$ is not surjective and so $\deg(g) = 0$ by the remark above, and so $\deg(h_s) = \deg(g) + \deg(h_t) = \deg(h_t)$.
We have now shown that the the map $[0,1] \to \mathbb{Z}$ given by $t \mapsto \deg(h_t)$ is locally constant (equivalently, *continuous* as a map of metric spaces) and so it is constant (since $[0,1]$ is connected). In particular $\deg(f) = \deg(h_0) = \deg(h_1) = \deg(g)$. $\square$

LEMMA 1.2.4. *If $f, g\colon S^1 \to S^1$ are continuous maps which have the same degree, then they are homotopic.*

PROOF. Certainly $f$ is homotopic to a map which takes $1$ to $1$ and $g$ is homotopic to a map which takes $1$ to $1$ (using complex notation, $1 \in S^1 \subset \mathbb{C}$). Therefore we can assume without loss of generality that $f(1) = 1$ and $g(1) = 1$.
Let $\varphi\colon [0,1] \to S^1$ and $\gamma\colon [0,1] \to S^1$ be defined by $\varphi(t) = f(p(t))$ and $\gamma(t) = g(p(t))$. Construct $\tilde{\varphi}$ and $\tilde{\gamma}$ as in the lemma, using $a = 0$ in both cases, so that $\tilde{\varphi}(0) = 0 = \tilde{\gamma}(0)$. Then

$$\tilde{\varphi}(1) = \deg(f) = \deg(g) = \tilde{\gamma}(1).$$

Note that $f$ can be recovered from $\tilde{\varphi}$ as follows. For $z \in S^1$ choose $t \in [0,1]$ such that $p(t) = z$. Then $f(z) = f(p(t)) = \varphi(t) = p\tilde{\varphi}(t)$. If $z = 1 \in S^1$, we can choose $t = 0$ or $t = 1$, but this ambiguity does not matter since $p\tilde{\varphi}(1) = p\tilde{\varphi}(0)$. Similarly, $g$ can be recovered from $\tilde{\gamma}$. Therefore we can show that $f$ is homotopic to $g$ by showing that $\tilde{\varphi}$ is homotopic to $\tilde{\gamma}$ *with endpoints fixed*. In other words we need a continuous

$$H\colon [0,1] \times [0,1] \to \mathbb{R}$$

where $H(s,0) = \tilde{\varphi}(s)$, $H(s,1) = \tilde{\gamma}(s)$ and $H(0,t) = 0$ for all $t \in [0,1]$ and $H(1,t) = \tilde{\varphi}(1) = \tilde{\gamma}(1)$ for all $t \in [0,1]$. This is easy to do: let $H(s,t) = (1-t)\tilde{\varphi}(s) + t\tilde{\gamma}(s)$. $\square$

Summarizing, we have shown that the degree function gives us a well defined map from $[S^1, S^1]$ to $\mathbb{Z}$, and moreover, that this map is injective. It is not hard to show that this map is also surjective! Namely, for arbitrary $\ell \in \mathbb{Z}$ the map $f\colon S^1 \to S^1$ given by $f(z) = z^\ell$ (complex notation) has $\deg(f) = \ell$. (Verify this.)

COROLLARY 1.2.5. *The degree function is a bijection from $[S^1, S^1]$ to $\mathbb{Z}$.* $\square$

CHAPTER 2

# Fiber bundles and fibrations

## 2.1. Fiber bundles and bundle charts

DEFINITION 2.1.1. Let $p\colon E \to B$ be a continuous map between topological spaces and let $x \in B$. The subspace $p^{-1}(\{x\})$ is sometimes called the *fiber* of $p$ over $x$.

DEFINITION 2.1.2. Let $p\colon E \to B$ be a continuous map between topological spaces. We say that $p$ is a *fiber bundle* if for every $x \in B$ there exist an open neighborhood $U$ of $x$ in $B$, a topological space $F$ and a homeomorphism $h\colon p^{-1}(U) \to U \times F$ such that $h$ followed by projection to $U$ agrees with $p$.

Note that $h$ restricts to a homeomorphism from the fiber of $f$ over $x$ to $\{x\} \times F$. Therefore $F$ must be homeomorphic to the fiber of $p$ over $x$.

*Terminology.* Often $E$ is called the *total space* of the fiber bundle and $B$ is called the *base space*. A homeomorphism $h\colon p^{-1}(U) \to U \times F$ as in the definition is called a *bundle chart*. A fiber bundle $p\colon E \to B$ whose fibers are discrete spaces (intuitively, just sets) is also called a *covering space*. (A *discrete space* is a topological space $(X, \mathcal{O})$ in which $\mathcal{O}$ is the entire power set of $X$.)
Here is an easy way to make a fiber bundle with base space $B$. Choose a topological space $F$, put $E = B \times F$ and let $p\colon E \to B$ be the projection to the first factor. Such a fiber bundle is considered unexciting and is therefore called *trivial*. Slightly more generally, a fiber bundle $p\colon E \to B$ is *trivial* if there exist a topological space $F$ and a homeomorphism $h\colon E \to B \times F$ such that $h$ followed by the projection $B \times F \to B$ agrees with $p$. Equivalently, the bundle is trivial if it admits a bundle chart $h\colon p^{-1}(U) \to U \times F$ where $U$ is all of $B$.
Two fiber bundles $p_0\colon E_0 \to B$ and $p_1\colon E_1 \to B$ with the same base space $B$ are considered *isomorphic* if there exists a homeomorphism $g\colon E_0 \to E_1$ such that $p_1 \circ g = p_0$. In that case $g$ is an *isomorphism* of fiber bundles.

According to the definition above a fiber bundle is a *map*, but the expression is often used informally for a space rather than a map (the total space of the fiber bundle).

PROPOSITION 2.1.3. *Let $p\colon E \to B$ be a fiber bundle where $B$ is a connected space. Let $x_0, y_0 \in B$. Then the fibers of $p$ over $x_0$ and $y_0$, respectively, are homeomorphic.*

PROOF. For every $x \in B$ choose an open neighborhood $U_x$ of $x$, a space $F_x$ and a bundle chart $h^x\colon p^{-1}(U_x) \to U_x \times F_x$. The open sets $U_x$ for all $x \in B$ form an open cover of $B$. We make an equivalence relation $R$ on the set $B$ in the following manner: $xRy$ means that there exist elements

$$z_0, z_1, \dots, z_k \in B$$

such that $z_0 = x$, $z_k = y$ and $U_{z_{j-1}} \cap U_{z_j} \neq \emptyset$ for $j = 1, \dots, k$. Clearly $xRy$ implies that $F_x$ is homeomorphic to $F_y$. Therefore it suffices to show that $R$ has only one equivalence class. Each equivalence class is open, for if $x \in B$ belongs to such an equivalence class, then $U_x$ is contained in the equivalence class. Each equivalence class is closed, since its complement is open, being the union of the other equivalence classes. Since $B$ is connected, this means that there can only be one equivalence class. $\square$

EXAMPLE 2.1.4. One example of a fiber bundle is $p\colon \mathbb{R} \to S^1$, where $p(t) = \exp(2\pi it)$. We saw this in section 1. To show that it is a fiber bundle, select some $z \in S^1$ and some $t \in \mathbb{R}$ such that $p(t) = z$. Let $V = ]t-\delta, t+\delta[$ where $\delta$ is a positive real number, not greater than $1/2$. Then $p$ restricts to a homeomorphism from $V \subset \mathbb{R}$ to an open neighborhood $U = p(V)$ of $z$ in $S^1$; let $q\colon U \to V$ be the inverse homeomorphism. Now $p^{-1}(U)$ is the disjoint union of the translates $\ell + V$, where $\ell \in \mathbb{Z}$. This amounts to saying that

$$g\colon U \times \mathbb{Z} \to p^{-1}(U)$$

given by $(y, m) \mapsto m + q(y)$ is a homeomorphism. The inverse $h$ of $g$ is then a bundle chart. Moreover $\mathbb{Z}$ plays the role of a discrete space. Therefore this fiber bundle is a covering space. It is not a trivial fiber bundle because the total space, $\mathbb{R}$, is not homeomorphic to $S^1 \times \mathbb{Z}$.

EXAMPLE 2.1.5. The Möbius strip leads to another popular example of a fiber bundle. Let $E \subset S^1 \times \mathbb{C}$ consist of all pairs $(z, w)$ where $w^2 = c^2 z$ for some $c \in \mathbb{R}$. This is a (non-compact) implementation of the Möbius strip. There is a projection

$$q\colon E \to S^1$$

given by $q(z, w) = z$. Let us look at the fibers of $q$. For fixed $z \in S^1$, the fiber of $q$ over $z$ is identified with the space of all $w \in \mathbb{C}$ such that $w^2 = c^2 z$ for some real $c$. This is equivalent to $w = c\sqrt{z}$ where $\sqrt{z}$ is one of the two roots of $z$ in $\mathbb{C}$. In other words, $w$ belongs to the one-dimensional linear *real* subspace of $\mathbb{C}$ spanned by the two square roots of $z$. In particular, each fiber of $q$ is homeomorphic to $\mathbb{R}$. The fact that all fibers are homeomorphic to each other should be taken as an indication (though not a proof) that $q$ is a fiber bundle. The full proof is left as an exercise, along with another exercise which is slightly harder: show that this fiber bundle is not trivial.

In preparation for the next example I would like to recall the concept of *one-point compactification*. Let $X = (X, \mathcal{O})$ be a locally compact topological space. (That is to say, $X$ is a Hausdorff space in which every element $x \in X$ has a compact neighborhood.) Let $X^c = (X^c, \mathcal{U})$ be the topological space defined as follows. As a set, $X^c$ is the disjoint union of $X$ and a singleton (set with one element, which in this case we call $\infty$). The topology $\mathcal{U}$ on $X^c$ is defined as follows. A subset $V$ of $X^c$ belongs to $\mathcal{U}$ if and only if

- either $\infty \notin V$ and $V \in \mathcal{O}$;
- or $\infty \in V$ and $X^c \smallsetminus V$ is a *compact* subset of $X$.

Then $X^c$ is compact Hausdorff and the inclusion $u\colon X \to X^c$ determines a homeomorphism of $X$ with $u(X) = X^c \smallsetminus \{\infty\}$. The space $X^c$ is called the *one-point compactification* of $X$. The notation $X^c$ is not standard; instead people often write $X \cup \infty$ and the like. The one-point compactification can be characterized by various good properties; see books on point set topology. For use later on let's note the following, which is clear from the definition of the topology on $X^c$. Let $Y = (Y, \mathcal{W})$ be any topological space. A map $g\colon Y \to X^c$ is continuous if and only if the following hold:

- $g^{-1}(X)$ is open in $Y$
- the map from $g^{-1}(X)$ to $X$ obtained by restricting $g$ is continuous

- for every compact subset $K$ of $X$, the preimage $g^{-1}(K)$ is a closed subset of $Y$ (that is, its complement is an element of $\mathcal{W}$).

EXAMPLE 2.1.6. A famous example of a fiber bundle which is also a crucial example in homotopy theory is the Hopf map from $S^3$ to $S^2$, so named after its inventor Heinz Hopf. (Date of invention: around 1930.) Let's begin with the observation that $S^2$ is homeomorphic to the one-point compactification $\mathbb{C} \cup \infty$ of $\mathbb{C}$. (The standard homeomorphism from $S^2$ to $\mathbb{C} \cup \infty$ is called *stereographic projection.*) We use this and therefore describe the Hopf map as a map

$$p\colon S^3 \to \mathbb{C} \cup \infty.$$

Also we like to think of $S^3$ as the unit sphere in $\mathbb{C}^2$. So elements of $S^3$ are pairs $(z, w)$ where $z, w \in \mathbb{C}$ and $|z|^2 + |w|^2 = 1$. To such a pair we associate

$$p(z, w) = z/w$$

using complex division. This is the Hopf map. Note that in cases where $w = 0$, we must have $z \neq 0$ as $|z|^2 = |z|^2 + |w|^2 = 1$; therefore $z/w$ can be understood and must be understood as $\infty \in \mathbb{C} \cup \infty$ in such cases. In the remaining cases, $z/w \in \mathbb{C}$.
Again, let us look at the fibers of $p$ before we try anything more ambitious. Let $s \in \mathbb{C} \cup \infty$. If $s = \infty$, the preimage of $s$ under $p$ consists of all $(z, w) \in S^3$ where $w = 0$. This is a circle. If $s \neq \infty$, the preimage of $s$ under $p$ consists of all $(z, w) \in S^3$ where $w \neq 0$ and $z/w = s$. This is the intersection of $S^3 \subset \mathbb{C}^2$ with the one-dimensional complex linear subspace $\{(z, w) \mid z = sw\} \subset \mathbb{C}^2$. It is also a circle! Therefore all the fibers of $p$ are homeomorphic to the same thing, $S^1$. We take this as an indication (though not a proof) that $p$ is a fiber bundle.
Now we *show* that $p$ is a fiber bundle. First let $U = \mathbb{C}$, which we view as an open subset of $\mathbb{C} \cup \infty$. Then

$$p^{-1}(U) = \{(z, w) \in S^3 \subset \mathbb{C}^2 \mid w \neq 0\}.$$

A homeomorphism $h$ from there to $U \times S^1 = \mathbb{C} \times S^1$ is given by

$$(z, w) \mapsto (z/w, w/|w|).$$

This has the properties that we require from a bundle chart: the first coordinate of $h(z, w)$ is $z/w = p(z, w)$. (The formula $g(y, z) = (yz, z)/\|(yz, z)\|$ defines a homeomorphism $g$ inverse to $h$.) Next we try $V = (\mathbb{C} \cup \infty) \smallsetminus \{0\}$, again an open subset of $\mathbb{C} \cup \infty$. We have the following commutative diagram

$$\begin{tikzcd} S^3 \arrow[r, "\alpha"] \arrow[d, "p"] & S^3 \arrow[d, "p"] \\ \mathbb{C} \cup \infty \arrow[r, "\zeta"] & \mathbb{C} \cup \infty \end{tikzcd}$$

where $\alpha(z, w) = (w, z)$ and $\zeta(s) = s^{-1}$. (This amounts to saying that $p \circ \alpha = \zeta \circ p$.) Therefore the composition

$$p^{-1}(V) \xrightarrow{\alpha} p^{-1}(U) \xrightarrow{h} U \times S^1 \xrightarrow{(s, w) \mapsto (s^{-1}, w)} V \times S^1$$

has the properties required of a bundle chart. Since $U \cup V$ is all of $\mathbb{C} \cup \infty$, we have produced enough charts to know that $p$ is a fiber bundle. $\square$

## 2.2. Restricting fiber bundles

Let $p\colon E \to B$ be a fiber bundle. Let $A$ be a subset of $B$. Put $E_{|A} = p^{-1}(A)$. This is a subset of $E$. We want to regard $A$ as a subspace of $B$ (with the subspace topology) and $E_{|A}$ as a subspace of $E$.

PROPOSITION 2.2.1. *The map* $p_A\colon E_{|A} \to A$ *obtained by restricting* $p$ *is also a fiber bundle.*

PROOF. Let $x \in A$. Choose a bundle chart $h\colon p^{-1}(U) \to U \times F$ for $p$ such that $x \in U$. Let $V = U \cap A$, an open neighborhood of $x$ in $A$. By restricting $h$ we obtain a bundle chart $h_A\colon p^{-1}(V) \to V \times F$ for $p_A$. □

*Remark.* In this proof it is important to remember that a bundle chart as above is not just *any* homeomorphism $h\colon p^{-1}(U) \to U \times F$. There is a condition: for every $y \in p^{-1}(U)$ the $U$-coordinate of $h(y) \in U \times F$ must be equal to $p(y)$. The following informal point of view is recommended: A bundle chart $h\colon p^{-1}(U) \to U \times F$ for $p$ is just a way to specify, simultaneously and continuously, homeomorphisms $h_x$ from the fibers of $p$ over elements $x \in U$ to $F$. Explicitly, $h$ determines the $h_x$ and the $h_x$ determine $h$ by means of the equation

$$h(y) = (x, h_x(y)) \in U \times F$$

when $y \in p^{-1}(x)$, that is, $x = p(y)$.

Let $p\colon E \to B$ be any fiber bundle. Then $B$ can be covered by open subsets $U_i$ such that $E_{|U_i}$ is a trivial fiber bundle. This is true by definition: choose the $U_i$ together with bundle charts $h_i\colon p^{-1}(U_i) \to U_i \times F_i$. Rename $p^{-1}(U_i) = E_{|U_i}$ if you must. Then each $h_i$ is a bundle isomorphism of $p_{|U_i}\colon E_{|U_i} \to U_i$ with a trivial fiber bundle $U_i \times F_i \to U_i$. There are cases where we can say more. One such case merits a detailed discussion because it takes us back to the concept of homotopy.

LEMMA 2.2.2. *Let* $B$ *be any space and let* $q\colon E \to B \times [0,1]$ *be a fiber bundle. Then* $B$ *admits a covering by open subsets* $U_i$ *such that*

$$q_{|U_i \times [0,1]}\colon E_{|U_i \times [0,1]} \longrightarrow U_i \times [0,1]$$

*is a trivial fiber bundle (for all* $i$*).*

PROOF. We fix $x_0 \in B$ for this proof. We try to construct an open neighborhood $U$ of $x_0$ in $B$ such that $q_{|U \times [0,1]}\colon E_{|U \times [0,1]} \longrightarrow U \times [0,1]$ is a trivial fiber bundle. This is enough.

We introduce a relation $R$ on $[0,1]$ as follows. Two elements $s, t \in [0,1]$ satisfy $sRt$ if and only if there exists an open neighborhood $U$ of $\{x_0\} \times [s,t]$ in $B \times [0,1]$ such that $E|_U \to U$ (restriction of $q$) is a trivial fiber bundle. (Here we have assumed $s \le t$; if not, write $[t,s]$ instead of $[s,t]$.)

The main point of the proof is to show that the relation $R$ is *transitive*. To show this, let us suppose that we have $r, s, t \in [0,1]$ where $r < s < t$, and $rRs$ holds as well as $sRt$. Choose $U$, open neighborhood of $\{x_0\} \times [r,s]$, and $V$, open neighborhood of $\{x_0\} \times [s,t]$, such that $E|_U \to U$ and $E|_V \to V$ (the restrictions of $q$) are both trivial fiber bundles. Let

$$g\colon E|_U \longrightarrow U \times F_U$$

be a trivialization of the fiber bundle $E|_U \to U$ and let

$$h\colon E|_V \longrightarrow V \times F_V$$

be a trivialization of the fiber bundle $E|_V \to V$. Without loss of generality, $U = U' \times U''$ where $U'$ is an open neighborhood of $x_0$ in $B$ and $U''$ is a connected open subset of $[0,1]$ containing $[r,s]$. Without loss of generality, $V = V' \times V''$ where $V'$ is an open neighborhood of $x_0$ in $B$ and $V''$ is a connected open subset of $[0,1]$ containing $[s,t]$. Without loss of generality, $U' = V'$. Without loss of generality, $F_U = F_V$ since both are homeomorphic to $p^{-1}(x_0, s)$; we write $F$ for both. Now $U'' \cup V''$ is a connected open subset of $[0,1]$ containing $[r,t]$ and $W := U' \times (U'' \cup V'')$ is an open neighborhood of $\{x\} \times [r,t]$ in $B \times [0,1]$. We make a trivialization

$$k\colon E|_W \longrightarrow W \times F$$

as follows. For $(x,t) \in U' \times U''$ with $t \le r$ we take $k_{(x,t)} = g_{(x,t)}$, where $g_{(x,t)}$ denotes the $F$-coordinate of $g$ restricted to $p^{-1}(x,t)$. (This is notation as in the remark following proposition 2.2.1.) For $(x,t) \in U' \times V''$ with $t \ge r$ we take

$$k_{(x,t)} = g_{(x,r)} \circ h^{-1}_{(x,r)} \circ h_{(x,t)}\,.$$

Therefore $R$ is transitive. It is clearly also symmetric and reflexive; so it is an equivalence relation. The equivalence classes are clearly open! Then there is only one equivalence class, since $[0,1]$ is connected. □

## 2.3. Pullbacks of fiber bundles

Let $p\colon E \to B$ be a fiber bundle. Let $g\colon X \to B$ be any continuous map of topological spaces.

Definition 2.3.1. The pullback of $p\colon E \to B$ along $g$ is the space

$$g^*E := \{\,(x,y) \in X \times E \mid g(x) = p(y)\}.$$

It is regarded as a subspace of $X \times E$ with the subspace topology.

Lemma 2.3.2. *The projection* $g^*E \to X$ *given by* $(x,y) \mapsto x$ *is a fiber bundle.*

Proof. First of all it is helpful to write down the obvious maps that we have in a commutative diagram:

$$\begin{tikzcd} g^*E \arrow[r, "r"] \arrow[d, "q"] & E \arrow[d, "p"] \\ X \arrow[r, "g"] & B \end{tikzcd}$$

Here $q$ and $r$ are the projections given by $(x,y) \mapsto x$ and $(x,y) \mapsto y$. *Commutative* means that the two compositions taking us from $g^*E$ to $B$ agree. Suppose that we have an open set $V \subset B$ and a bundle chart

$$h\colon p^{-1}(V) \xrightarrow{\ \cong\ } V \times F.$$

Now $U := g^{-1}(V)$ is open in $X$. Also $q^{-1}(U)$ is an open subset of $g^*E$ and we describe elements of that as pairs $(x,y)$ where $x \in U$ and $y \in E$, with $g(x) = p(y)$. We make a homeomorphism

$$q^{-1}(U) \to U \times F$$

by the formula $(x,y) \mapsto (x, h_{g(x)}(y)) = (x, h_{p(y)}(y))$. It is a homeomorphism because the inverse is given by

$$(x,z) \mapsto (x, (h_{g(x)})^{-1}(z))$$

for $x \in U$ and $z \in F$, so that $(g(x), z) \in V \times F$. Its is also clearly a bundle chart. In this way, every bundle chart

$$h\colon p^{-1}(V) \xrightarrow{\cong} V \times F$$

for $p\colon E \to B$ determines a bundle chart

$$q^{-1}(U) \xrightarrow{\cong} U \times F$$

with the same $F$, where $U$ is the preimage of $V$ under $g$. Since $p\colon E \to B$ is a fiber bundle, we have many such bundle charts $p^{-1}(V_j) \to V_j \times F_j$ such that the union of the $V_j$ is all of $B$. Then the union of the corresponding $U_j$ is all of $X$, and we have bundle charts $q^{-1}(U_j) \to U_j \times F_j$. This proves that $q$ is a fiber bundle. $\square$

This proof was too long and above all too formal. Reasoning in a less formal way, one should start by noticing that the fiber of $q$ over $z \in X$ is essentially the same (and certainly homeomorphic) to the fiber of $p$ over $g(z) \in B$. Namely,

$$q^{-1}(z) = \{(x, y) \in X \times E \mid g(x) = p(y),\ x = z\} \ = \ \{z\} \times p^{-1}(\{g(z)\})\ .$$

Now recall once again that a bundle chart $h\colon p^{-1}(U) \to U \times F$ for $p$ is just a way to specify, simultaneously and continuously, homeomorphisms $h_x$ from the fibers of $p$ over elements $x \in U$ to $F$. If we have such a bundle chart for $p$, then for any $z \in g^{-1}(U)$ we get a homeomorphism from the fiber of $q$ over $z$, which “is” the fiber of $p$ over $g(z)$, to $F$. And so, by letting $z$ run through $g^{-1}(U)$, we get a bundle chart for $q$.

Example 2.3.3. Restriction of fiber bundles is a special case of pullback, up to isomorphism of fiber bundles. More precisely, suppose that $p\colon E \to B$ is a fiber bundle and let $A \subset B$ be a subspace, with inclusion $g\colon A \to B$. Then there is an isomorphism of fiber bundles from $p_A\colon E_{|A} \to A$ to the pullback $g^*E \to A$. This takes $y \in E_{|A}$ to the pair $(p(y), y) \in g^*E \subset A \times E$.

## 2.4. Homotopy invariance of pullbacks of fiber bundles

Theorem 2.4.1. *Let $p\colon E \to B$ be a fiber bundle. Let $f, g\colon X \to B$ be continuous maps, where $X$ is a compact Hausdorff space. If $f$ is homotopic to $g$, then the fiber bundles $f^*E \to X$ and $g^*E \to X$ are isomorphic.*

Remark 2.4.2. The compactness assumption on $X$ is unnecessarily strong; *paracompact* is enough. But paracompactness is also a more difficult concept than compactness. Therefore we shall prove the theorem as stated, and leave a discussion of improvements for later.

Remark 2.4.3. Let $X$ be a compact Hausdorff space and let $U_0, U_1, \dots, U_n$ be open subsets of $X$ such that the union of the $U_i$ is all of $X$. *Then there exist continuous functions*

$$\varphi_0, \varphi_1, \dots, \varphi_n\colon X \to [0, 1]$$

*such that* $\sum_{j=0}^{n} \varphi_j \equiv 1$ *and such that* supp$(\varphi_j)$*, the support of* $\varphi_j$ *, is contained in* $U_j$ *for* $j = 0, 1, \dots, n$. Here supp$(\varphi_j)$ is the closure in $X$ of the open set

$$\{x \in X \mid \varphi_j(x) > 0\}.$$

A collection of functions $\varphi_0, \varphi_1, \dots, \varphi_n$ with the stated properties is called a *partition of unity subordinate to the open cover of* $X$ *given by* $U_0, \dots, U_n$. For readers who are not aware of this existence statement, here is a reduction (by induction) to something which they might be aware of.

First of all, if $X$ is a compact Hausdorff space, then it is a *normal* space. This means,

in addition to the Hausdorff property, that any two disjoint closed subsets of $X$ admit disjoint open neighborhoods. Next, for any normal space $X$ we have the *Tietze-Urysohn extension lemma*. This says that if $A_0$ and $A_1$ are disjoint closed subsets of $X$, then there is a continuous function $\psi\colon X \to [0,1]$ such that $\psi(x) = 1$ for all $x \in A_1$ and $\psi(x) = 0$ for all $x \in A_0$. Now suppose that a normal space $X$ is the union of two open subsets $U_0$ and $U_1$. Because $X$ is normal, we can find an open subset $V_0 \subset U_0$ such that the closure of $V_0$ in $X$ is contained in $U_0$ and the union of $V_0$ and $U_1$ is still $X$. Repeating this, we can also find an open subset $V_1 \subset U_1$ such that the closure of $V_1$ in $X$ is contained in $U_1$ and the union of $V_1$ and $V_0$ is still $X$. Let $A_0 = X \smallsetminus V_0$ and $A_1 = X \smallsetminus V_1$. Then $A_0$ and $A_1$ are disjoint closed subsets of $X$, and so by Tietze-Urysohn there is a continuous function $\psi\colon X \to [0,1]$ such that $\psi(x) = 1$ for all $x \in A_1$ and $\psi(x) = 0$ for all $x \in A_0$. This means that $\mathrm{supp}(\psi)$ is contained in the closure of $X \smallsetminus A_0 = V_0$, which is contained in $U_0$. We take $\varphi_1 = \psi$ and $\varphi_0 = 1 - \psi$. Since $1 - \psi$ is zero on $A_1$, its support is contained in the closure of $V_1$, which is contained in $U_1$. This establishes the induction beginning (case $n = 1$).
For the induction step, suppose that we have an open cover of $X$ given by $U_0, \dots, U_n$ where $n \geq 2$. By inductive assumption we can find a partition of unity subordinate to the cover $U_0 \cup U_1, U_2, \dots, U_n$ and by the induction beginning, another partition of unity subordinate to $U_0, U_1 \cup U_2 \cup \cdots U_n$. Call the functions in the first partition of unity $\varphi_{01}, \varphi_2, \dots, \varphi_n$ and those in the second $\psi_0, \psi_1$, we see that the functions $\psi_0\varphi_{01}, \psi_1\varphi_{01}, \varphi_2, \dots, \varphi_n$ form a partition of unity subordinate to the cover by $U_0, \dots, U_n$. $\square$

Proof of theorem 2.4.1. Let $h\colon X \times [0,1] \to B$ be a homotopy from $f$ to $g$, so that $h_0 = f$ and $h_1 = g$. Then $h^*E \to X \times [0,1]$ is a fiber bundle. We give this a new name, say $q\colon L \to X \times [0,1]$. Let $\iota_0$ and $\iota_1$ be the maps from $X$ to $X \times [0,1]$ given by $\iota_0(x) = (x,0)$ and $\iota_1(x) = (x,1)$. It is not hard to verify that the fiber bundle $f^*E \to X$ is isomorphic to $\iota_0^*L \to X$ and $g^*E \to X$ is isomorphic to $\iota_1^*L \to X$. Therefore all we need to prove is the following.
*Let* $q\colon L \to X \times [0,1]$ *be a fiber bundle, where* $X$ *is compact Hausdorff. Then the fiber bundles* $\iota_0^*L \to X$ *and* $\iota_1^*L \to X$ *obtained from* $q$ *by pullback along* $\iota_0$ *and* $\iota_1$ *are isomorphic.* To make this even more explicit: given the fiber bundle $q\colon L \to X \times [0,1]$, we need to produce a homeomorphism from $L_{|X\times\{0\}}$ to $L_{|X\times\{1\}}$ which fits into a commutative diagram

$$\begin{tikzcd}
L_{|X\times\{0\}} \arrow[r, "\text{our homeom.}"] \arrow[d, "\text{res. of } q"'] & L_{|X\times\{1\}} \arrow[d, "\text{res. of } q"] \\
X \times \{0\} \arrow[r, "{(x,0)\mapsto(x,1)}"] & X \times \{1\}
\end{tikzcd}$$

Here $L_{|K}$ means $q^{-1}(K)$, for any $K \subset X \times [0,1]$.
By a lemma proved last week (lecture notes week 2), we can find a covering of $X$ by open subsets $U_i$ such that that $q_{U_i \times [0,1]}\colon L_{|U_i \times [0,1]} \to U_i \times [0,1]$ is a trivial bundle, for each $i$. Since $X$ is compact, finitely many of these $U_i$ suffice, and we can assume that their names are $U_1, \dots, U_n$. Let $\varphi_1, \dots, \varphi_n$ be continuous functions from $X$ to $[0,1]$ making up a partition of unity subordinate to the open covering of $X$ by $U_1, \dots, U_n$. For $j = 0, 1, 2, \dots, n$ let $\nu_j = \sum_{k=1}^{j} \varphi_k$ and let $\Gamma_j \subset X \times [0,1]$ be the graph of $\nu_j$. Note that $\Gamma_0$ is $X \times \{0\}$ and $\Gamma_n$ is $X \times \{1\}$. It suffices therefore to produce a homeomorphism

$e_j \colon L_{|\Gamma_{j-1}} \to L_{|\Gamma_j}$ which fits into a commutative diagram

$$\begin{tikzcd} L_{|\Gamma_{j-1}} \arrow[r, "e_j"] \arrow[d, "\text{res. of } q"'] & L_{|\Gamma_j} \arrow[d, "\text{res. of } q"] \\ \Gamma_{j-1} \arrow[r, "(x,\nu_{j-1}(x)) \mapsto (x,\nu_j(x))"] & \Gamma_j \end{tikzcd}$$

(for $j = 1, 2, \dots, n$). Since $q_{U_j \times [0,1]} \colon L_{|U_j \times [0,1]} \to U_j \times [0,1]$ is a trivial fiber bundle, we have a single bundle chart for it, a homeomorphism

$$g \colon L_{|U_j \times [0,1]} \longrightarrow (U_i \times [0,1]) \times F$$

with the additional good property that we require of bundle charts. Fix $j$ now and write $L = L' \cup L''$ where $L'$ consists of the $y \in L$ for which $q(y) = (x,t)$ with $x \notin \mathrm{supp}(\varphi_j)$, and $L''$ consists of the $y \in L$ for which $q(y) = (x,t)$ with $x \in U_j$. Both $L'$ and $L''$ are open subsets of $L$. Now we make our homeomorphism $e = e_j$ as follows. By inspection, $L_{|\Gamma_{j-1}} \cap L' = L_{|\Gamma_j} \cap L'$, and we take $e$ to be the identity on $L_{|\Gamma_{j-1}} \cap L'$. By restricting the bundle chart $g$, we have a homeomorphism $L_{|\Gamma_{j-1}} \cap L'' \to U_j \times F$; more precisely, a homeomorphism from $L_{|\Gamma_{j-1}} \cap L''$ to $(\Gamma_{j-1} \cap U_j \times [0,1]) \times F$. By the same reasoning, we have a homeomorphism $L_{|\Gamma_j} \cap L'' \to U_j \times F$; more precisely, a homeomorphism from $L_{|\Gamma_j} \cap L''$ to $(\Gamma_j \cap U_j \times [0,1]) \times F$. Therefore we have a preferred homeomorphism from $L_{|\Gamma_{j-1}} \cap L''$ to $L_{|\Gamma_j} \cap L''$, and we use that as the definition of $e$ on $L_{|\Gamma_{j-1}} \cap L''$. By inspection, the two definitions of $e$ which we have on the overlap $L_{|\Gamma_{j-1}} \cap L' \cap L''$ agree, so $e$ is well defined. ☐

COROLLARY 2.4.4. *Let $p \colon E \to B$ be a fiber bundle where $B$ is compact Hausdorff and contractible. Then $p$ is a trivial fiber bundle.*

PROOF. By the contractibility assumption, the identity map $f \colon B \to B$ is homotopic to a constant map $g \colon B \to B$. By the theorem, the fiber bundles $f^*E \to B$ and $g^*E \to B$ are isomorphic. But clearly $f^*E \to B$ is isomorphic to the original fiber bundle $p \colon E \to B$. And clearly $g^*E \to B$ is a trivial fiber bundle. ☐

COROLLARY 2.4.5. *Let $q \colon E \to B \times [0,1]$ be a fiber bundle, where $B$ is compact Hausdorff. Suppose that the restricted bundle*

$$q_{B \times \{0\}} \colon E_{|B \times \{0\}} \to B \times \{0\}$$

*admits a section, i.e., there exists a continuous map $s \colon B \times \{0\} \to E_{|B \times \{0\}}$ such that $q \circ s$ is the identity on $B \times \{0\}$. Then $q \colon E \to B \times [0,1]$ admits a section $\bar{s} \colon B \times [0,1] \to E$ which agrees with $s$ on $B \times \{0\}$.*

PROOF. Let $f, g \colon B \times [0,1] \to B \times [0,1]$ be defined by $f(x,t) = (x,t)$ and $g(x,t) = (x,0)$. These maps are clearly homotopic. Therefore the fiber bundles $f^*E \to B \times [0,1]$ and $g^*E \to B \times [0,1]$ are isomorphic fiber bundles. Now $f^*E \to B \times [0,1]$ is clearly isomorphic to the original fiber bundle

$$q \colon E \to B \times \{0,1\}$$

and $g^*E \to B \times [0,1]$ is clearly isomorphic to the fiber bundle

$$E_{|B \times \{0\}} \times [0,1] \to B \times [0,1]$$

given by $(y,t) \mapsto (q(y),t)$ for $y \in E_{|B \times \{0\}}$, that is, $y \in E$ with $q(y) = (x,0)$ for some $x \in B$. Therefore we may say that there is a homeomorphism $h \colon E_{|B \times \{0\}} \times [0,1] \to E$

which is over $B \times [0,1]$, in other words, which satisfies

$$(q \circ h)(y,t) = (q(y),t)$$

for all $y \in E_{|B\times\{0\}}$ and $t \in [0,1]$. Without loss of generality, $h$ satisfies the additional condition $h(y,0) = y$ for all $y \in E_{|B\times\{0\}}$. (In any case we have a homeomorphism $u\colon E_{|B\times\{0\}} \to E_{|B\times\{0\}}$ defined by $u(y) = h(y,0)$. If it is not the identity, use the homeomorphism $(y,t) \mapsto h(u^{-1}(y),t)$ instead of $(y,t) \mapsto h(y,t)$.) Now define $\bar{s}$ by $\bar{s}(x,t) = h(s(x),t)$ for $x \in B$ and $t \in [0,1]$. □

## 2.5. The homotopy lifting property

Definition 2.5.1. A continuous map $p\colon E \to B$ between topological spaces is said to have the *homotopy lifting property* (HLP) if the following holds. Given any space $X$ and continuous maps $f\colon X \to E$ and $h\colon X\times[0,1] \to B$ such that $h(x,0) = p(f(x))$ for all $x \in X$, there exists a continuous map $H\colon X\times[0,1] \to E$ such that $p\circ H = h$ and $H(x,0) = f(x)$ for all $x \in X$. A map with the HLP can be called a *fibration* (sometimes *Hurewicz fibration*).

It is customary to summarize the HLP in a commutative diagram with a dotted arrow:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x\mapsto(x,0)"'] & E \arrow[d, "p"] \\
X\times[0,1] \arrow[r, "h"] \arrow[ur, dotted, "H"] & B
\end{tikzcd}$$

Indeed, the HLP for the map $p$ means that once we have the data in the outer commutative square, then the dotted arrow labeled $H$ can be found, making both triangles commutative. More associated customs: we think of $h$ as a homotopy between maps $h_0$ and $h_1$ from $X$ to $B$, and we think of $f\colon X \to E$ as a *lift* of the map $h_0$ , which is just a way of saying that $p \circ f = h_0$.

More generally, or less generally depending on point of view, we say that $p\colon E \to B$ satisfies the HLP for a class of spaces $\Omega$ if the dotted arrow in the above diagram can always be supplied when the space $X$ belongs to that class $\Omega$.

Proposition 2.5.2. *Let $p\colon E \to B$ be a fiber bundle. Then $p$ has the HLP for compact Hausdorff spaces.*

Proof. Suppose that we have the data $X$, $f$ and $h$ as in the above diagram, but we are still trying to construct or find the diagonal arrow $H$. We are assuming that $X$ is compact Hausdorff. The pullback of $p$ along $h$ is a fiber bundle $h^*E \to X \times [0,1]$. The restricted fiber bundle

$$(h^*E)_{|X\times\{0\}} \to X \times \{0\}$$

has a continuous section $s$ given essentially by $f$, and if we say it very carefully, by the formula

$$(x,0) \mapsto ((x,0), f(x)) \in h^*E \subset (X \times [0,1]\,) \times E\ .$$

The section $s$ extends to a continuous section $\bar{s}$ of $h^*E \to X \times [0,1]$ by corollary 2.4.5. Now we can define $H := r \circ \bar{s}$, where $r$ is the standard projection from $h^*E$ to $E$. □

EXAMPLE 2.5.3. Let $p\colon S^3 \to S^2$ be the Hopf fiber bundle. Assume if possible that $p$ is nullhomotopic; we shall try to deduce something absurd from that. So let

$$h\colon S^3 \times [0,1] \to S^2$$

be a nullhomotopy for $p$. Then $h_0 = p$ and $h_1$ is a constant map. Applying the HLP in the situation

$$\begin{tikzcd} S^3 \arrow[r, "\mathrm{id}"] \arrow[d, "x \mapsto (x,0)"'] & S^3 \arrow[d, "p"] \\ S^3 \times [0,1] \arrow[r, "h"] \arrow[ur, dotted, "H"] & S^2 \end{tikzcd}$$

we deduce the existence of $H\colon S^3 \times [0,1] \to S^3$, a homotopy from the identity map $H_0 = \mathrm{id}\colon S^3 \to S^3$ to a map $H_1\colon S^3 \to S^3$ with the property that $p \circ H_1$ is constant. Since $p$ itself is certainly not constant, this means that $H_1$ is not surjective. If $H_1$ is not surjective, it is nullhomotopic. (A non-surjective map from any space to a sphere is nullhomotopic; that's an exercise.) Consequently $\mathrm{id}\colon S^3 \to S^3$ is also nullhomotopic, being homotopic to $H_1$. This means that $S^3$ is contractible.
Is that absurd enough? We shall prove later in the course that $S^3$ is not contractible. Until then, what we have just shown can safely be stated like this: *if $S^3$ is not contractible, then the Hopf map $p\colon S^3 \to S^2$ is not nullhomotopic.* (I found this argument in Dugundji's book on topology. Hopf used rather different ideas to show that $p$ is not nullhomotopic.)

Let $p\colon E \to B$ be a fibration (for a class of spaces $\mathcal{Q}$) and let $f\colon X \to B$ be any continous map between topological spaces. We define the pullback $f^*E$ by the usual formula,

$$f^*E = \{(x,y) \in X \times E \mid f(x) = p(y)\ \}.$$

LEMMA 2.5.4. *The projection $f^*E \to X$ is also a fibration for the class of spaces $\mathcal{Q}$.*

The proof is an exercise. □

In example 2.5.3, the HLP was used for something resembling a computation with homotopy classes of maps. Let us try to formalize this, as an attempt to get hold of some algebra in homotopy theory. So let $p\colon E \to B$ be a continuous map which has the HLP for a class of topological spaces $\mathcal{Q}$. Let $f\colon X \to B$ be any continuous map of topological spaces. Now we have a commutative square

$$\begin{tikzcd} f^*E \arrow[r, "q_2"'] \arrow[d, "q_1"] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

where $q_1$ and $q_2$ are the projections. Take any space $W$ in the class $\mathcal{Q}$. There is then a commutative diagram of sets and maps

$$\begin{tikzcd} {[W, f^*E]} \arrow[r] \arrow[d] & {[W, E]} \arrow[d] \\ {[W, X]} \arrow[r] & {[W, B]} \end{tikzcd}$$

PROPOSITION 2.5.5. *The above diagram of sets of homotopy classes is "half exact" in the following sense: given* $\mathfrak{a} \in [W, X]$ *and* $\mathfrak{b} \in [W, E]$ *with the same image in* $[W, B]$*, there exists* $\mathfrak{c} \in [W, f^*E]$ *which is taken to* $\mathfrak{a}$ *and* $\mathfrak{b}$ *by the appropriate maps in the diagram.*

PROOF. Represent $\mathfrak{a}$ by a map $\alpha\colon W \to X$, and $\mathfrak{b}$ by some map $\beta\colon W \to E$. By assumption, $f \circ \alpha$ is homotopic to $p \circ \beta$. Let $h = (h_t)_{t\in[0,1]}$ be a homotopy, so that $h_0 = p \circ \beta$ and $h_1 = f \circ \alpha$, and $h_t\colon W \to B$ for $t \in [0, 1]$. By the HLP for $p$, there exists a homotopy $H\colon W \times [0, 1] \to E$ such that $p \circ H = h$ and $H_0 = \beta$. Then $H_1$ is homotopic to $H_0 = \beta$, and $p \circ H_1 = f \circ \alpha$. Therefore the formula $w \mapsto (\alpha(w), H_1(w))$ defines a map $W \to f^*E$. The homotopy class $\mathfrak{c}$ of that is the solution to our problem. $\square$

Looking back, we can say that example 2.5.3 is an application of proposition 2.5.5 with $p\colon E \to B$ equal to the Hopf fibration and $f$ equal to the inclusion of a point (and $\mathcal{Q}$ equal to the class of compact Hausdorff spaces, say). We made some unusual choices: $W = E$ and $\mathfrak{b} = [\mathrm{id}] \in [W, E]$.

$$\begin{tikzcd} {[S^3, S^1]} \arrow[r] \arrow[d] & {[S^3, S^3]} \arrow[d, "{[p]\circ}"] \\ {[S^3, *]} \arrow[r] & {[S^3, S^2]} \end{tikzcd}$$

In the lower right-hand term $[S^3, S^2]$, we have the homotopy class of $p$. The assumption that $p$ is homotopic to a constant map implies that this is the image of an element $\mathfrak{a}$ in the lower left-hand term $[S^3, *]$. It also the image of $\mathfrak{b} = [\mathrm{id}]$ in the upper right-hand term $[S^3, S^3]$. Therefore (by proposition 2.5.5) we should be able to find $\mathfrak{c}$ in the upper left-hand term $[S^3, S^1]$ which maps to $\mathfrak{b}$. This implies that $\mathrm{id}\colon S^3 \to S^3$ is homotopic to a non-surjective map, and therefore that $S^3$ is contractible. (All under the assumption that the Hopf map $p\colon S^3 \to S^2$ is homotopic to a constant map.)

## 2.6. Remarks on paracompactness and fiber bundles

Quoting from many books on point set topology: a topological space $X = (X, \mathcal{O})$ is *paracompact* if it is Hausdorff and every open cover $(U_i)_{i\in\Lambda}$ of $X$ admits a locally finite refinement $(V_j)_{j\in\Psi}$.

There is a fair amount of open cover terminology in that definition. In this formulation, we take the view that an open cover of $X$ is a *family*, i.e., a map from a set to $\mathcal{O}$ (with a special property). This is slightly different from the equally reasonable view that an open cover of $X$ is a subset of $\mathcal{O}$ (with a special property), and it justifies the use of round brackets as in $(U_i)_{i\in\Lambda}$, as opposed to curly brackets. Here the map in question is from $\Lambda$ to $\mathcal{O}$. There is an understanding that $(V_j)_{j\in\Psi}$ is also an open cover of $X$, but $\Psi$ need not coincide with $\Lambda$. *Refinement* means that for every $j \in \Psi$ there exists $i \in \Lambda$ such that $V_j \subset U_i$. *Locally finite* means that every $x \in X$ admits an open neighborhood $W$ in $X$ such that the set $\{j \in \Psi \mid W \cap V_j \neq \emptyset\}$ is a finite subset of $\Psi$.

It is wonderfully easy to get confused about the meaning of paracompactness. There is a strong similarity with the concept of compactness, and it is obvious that *compact* (together with Hausdorff) implies paracompact, but it is worth emphasizing the differences. Namely, where compactness has something to do with open covers and *sub*-covers, the definition of paracompactness uses the notion of *refinement* of one open cover by another open cover. We require that every $V_j$ is *contained* in some $U_i$ ; we do not require that every $V_j$ is *equal* to some $U_i$. And *locally finite* does not just mean that for every $x \in X$ the set $\{j \in \Psi \mid x \in V_j\}$ is a finite subset of $\Psi$. It means more.

For some people, the Hausdorff condition is not part of *paracompact*, but for me, it is.

An important theorem: every metrizable space is paracompact. This is due to A.H. Stone who, as a Wikipedia page reminds me, is not identical with Marshall Stone of the Stone-Weierstrass theorem and the Stone-Čech compactification. The proof is not very complicated, but you should look it up in a book on point-set topology which is not too ancient, because it was complicated in the A.H. Stone version.

Another theorem which is very important for us: in a paracompact space $X$, every open cover $(U_i)_{i\in\Lambda}$ admits a subordinate partition of unity. In other words there exist continuous functions $\varphi_i\colon X\to[0,1]$, for $i\in\Lambda$, such that

- every $x\in X$ admits an open neighborhood $W$ in $X$ for which the set
$$\{\,i\in\Lambda\mid W\cap\operatorname{supp}(\varphi_i)\neq\emptyset\}$$
is finite;
- $\sum_{i\in\Lambda}\varphi_i\equiv 1$;
- $\operatorname{supp}(\varphi_i)\subset U_i$ .

The second condition is meaningful if we assume that the first condition holds. (Then, for every $x\in X$, there are only finitely many nonzero summands in $\sum_{i\in\Lambda}\varphi_i(x)$. The first condition also ensures that for any subset $\Xi\subset\Lambda$, the sum $\sum_{i\in\Xi}\varphi_i$ is a continuous function on $X$.)
The proof of this theorem (existence of subordinate partition of unity for any open cover of a paracompact space) is again not very difficult, and boils down mostly to showing that paracompact spaces are *normal*. Namely, in a normal space, locally finite open covers admit subordinate partitions of unity, and this is easy.

Many of the results about fiber bundles in this chapter rely on partitions of unity, and to ensure their existence, we typically assumed compactness here and there. But now it emerges that paracompactness is enough.
Specifically, in theorem 2.4.1 it is enough to assume that $X$ is paracompact. In corollary 2.4.4 it is enough to assume that $B$ is paracompact (and contractible). In corollary 2.4.5 it is enough to assume that $B$ is paracompact. In proposition 2.5.2 we have the stronger conclusion that $p$ has the HLP for paracompact spaces.

Proof of variant of thm. 2.4.1. Here we assume only that $X$ is paracompact (previously we assumed that it was compact). By analogy with the case of compact $X$, we can easily reduce to the following statement. *Let $q\colon L\to X\times[0,1]$ be a fiber bundle, where $X$ is paracompact. Then the fiber bundles $\iota_0^*L\to X$ and $\iota_1^*L\to X$ obtained from $q$ by pullback along $\iota_0$ and $\iota_1$ are isomorphic.* And to make this more explicit: given the fiber bundle $q\colon L\to X\times[0,1]$, we need to produce a homeomorphism $h$ from $L_{|X\times\{0\}}$ to $L_{|X\times\{1\}}$ which fits into a commutative diagram

$$\begin{tikzcd}
L_{|X\times\{0\}} \arrow[r, "h"] \arrow[d, "\text{res. of } q"'] & L_{|X\times\{1\}} \arrow[d, "\text{res. of } q"] \\
X\times\{0\} \arrow[r, "(x,0)\mapsto(x,1)"] & X\times\{1\}
\end{tikzcd}$$

By a lemma proved in lecture notes week 2, we can find an open cover $(U_i)_{i\in\Lambda}$ of $X$ such that that $q_{U_i\times[0,1]}\colon L_{|U_i\times[0,1]} \to U_i \times [0,1]$ is a trivial bundle, for each $i \in \Lambda$. Let $(\varphi_i)_{i\in\Lambda}$ be a partition of unity subordinate to $(U_i)_{i\in\Lambda}$. So $\varphi_i\colon X \to [0,1]$ is a continuous function with $\mathrm{supp}(\varphi_i) \subset U_i$, and $\sum_i \varphi_i \equiv 1$. Every $x \in X$ admits a neighborhood $W$ in $X$ such that the set

$$\{i \in \Lambda \mid \mathrm{supp}(\varphi_i) \cap W \neq \emptyset\}$$

is finite.

Now choose a total ordering on the set $\Lambda$. (A total ordering on $\Lambda$ is a relation $\leq$ on $\Lambda$ which is transitive and reflexive, and has the additional property that for any distinct $i, j \in \Lambda$, precisely one of $i \leq j$ or $j \leq i$ holds. We need to assume something here to get such an ordering: for example the Axiom of Choice in set theory is equivalent to the Well-Ordering Principle, which states that every set can be well-ordered. A well-ordering is also a total ordering.) Given $x \in X$, choose an open neighborhood $W$ of $x$ such that the set of $i \in \Lambda$ having $\mathrm{supp}(\varphi_i) \cap W \neq \emptyset$ is finite; say it has $n$ elements. We list these elements in their order (provided by the total ordering on $\Lambda$ which we selected):

$$i_1 \leq i_2 \leq i_3 \leq \cdots i_n \ .$$

The functions $\varphi_{i_1}, \varphi_{i_2}, \ldots, \varphi_{i_n}$ (restricted to $W$) make up a partition of unity on $W$ which is subordinate to the covering by open subsets $W \cap U_{i_1}, W \cap U_{i_2}, \ldots W \cap U_{i_n}$. Now we can proceed exactly as in the proof of theorem 2.4.1 to produce (in $n$ steps) a homeomorphism $h_W$ which makes the following diagram commute:

$$\begin{tikzcd}
L_{|W\times\{0\}} \arrow[r, "h_W"] \arrow[d, "\text{res. of } q"'] & L_{|W\times\{1\}} \arrow[d, "\text{res. of } q"] \\
W \times \{0\} \arrow[r, "(x,0)\mapsto(x,1)"] & W \times \{1\}
\end{tikzcd}$$

Finally we can regard $W$ or $x$ as variables. If we choose, for every $x \in X$, an open neighborhood $W_x$ with properties like $W$ above, then the $W_x$ for all $x \in X$ constitute an open cover of $X$. For each $W_x$ we get a homeomorphism $h_{W_x}$ as above. These homeomorphisms agree with each other wherever this is meaningful, and so define together a homeomorphism $h\colon L_{|X\times\{0\}} \to L_{|X\times\{1\}}$ with the property that we require. $\square$

CHAPTER 3

# Categories, functors and natural transformations

The concept of a *category* and the related notions *functor* and *natural transformation* emerged in the middle of the 20th century (Eilenberg-MacLane, 1945) and were immediately used to re-organize algebraic topology (Eilenberg-Steenrod, 1952). Later these notions became very important in many other branches of mathematics, especially algebraic geometry. Category theory has many definitions of great depth, I think, but in the foundations very few theorems and fewer proofs of any depth. Among those who love difficult proofs, it has a reputation of shallowness, boring-ness; for many of the theorizers who appreciate good definitions, it is an ever-ongoing revelation. Young mathematicians tend to like it better than old mathematicians ... probably because it helps them to see some order in a multitude of mathematical facts.

## 3.1. Categories

DEFINITION 3.1.1. A *category* $\mathcal{C}$ consists of a class $\mathrm{Ob}(\mathcal{C})$ whose elements are called the *objects of* $\mathcal{C}$ and the following additional data.

- For any two objects $c$ and $d$ of $\mathcal{C}$, a set $\mathrm{mor}_{\mathcal{C}}(c, d)$ whose elements are called the *morphisms* from $c$ to $d$.
- For any object $c$ in $\mathcal{C}$, a distinguished element $\mathrm{id}_c \in \mathrm{mor}_{\mathcal{C}}(c, c)$, called the *identity morphism* of $c$.
- For any three objects $b, c, d$ of $\mathcal{C}$, a map from $\mathrm{mor}_{\mathcal{C}}(c, d) \times \mathrm{mor}_{\mathcal{C}}(b, c)$ to $\mathrm{mor}_{\mathcal{C}}(b, d)$ called *composition* and denoted by $(f, g) \mapsto f \circ g$.

These data are subject to certain conditions, namely:

- Composition of morphisms is associative.
- The identity morphisms act as two-sided neutral elements for the composition.

The associativity condition, written out in detail, means that

$$(f \circ g) \circ h = f \circ (g \circ h)$$

whenever $a, b, c, d$ are objects of $\mathcal{C}$ and $f \in \mathrm{mor}_{\mathcal{C}}(c, d)$, $g \in \mathrm{mor}_{\mathcal{C}}(b, c)$, $h \in \mathrm{mor}_{\mathcal{C}}(a, b)$. The condition on identity morphisms means that $f \circ \mathrm{id}_c = f = \mathrm{id}_d \circ f$ whenever $c$ and $d$ are objects in $\mathcal{C}$ and $f \in \mathrm{mor}_{\mathcal{C}}(c, d)$. Saying that $\mathrm{Ob}(\mathcal{C})$ is a *class*, rather than a *set*, is a subterfuge to avoid problems which are likely to arise if, for example, we talk about *the set of all sets* (Russell's paradox). If the object class is a set, which sometimes happens, we speak of a *small category*.
*Notation*: we shall often write $\mathrm{mor}(c, d)$ instead of $\mathrm{mor}_{\mathcal{C}}(c, d)$ if it is obvious that the category in question is $\mathcal{C}$. Morphisms are often denoted by arrows, as in $f\colon c \to d$ when $f \in \mathrm{mor}(c, d)$. It is customary to say in such a case that $c$ is the *source* or *domain* of $f$, and $d$ is the *target* or *codomain* of $f$.

A morphism $f\colon c \to d$ in a category $\mathcal{C}$ is said to be an *isomorphism* if there exists a morphism $g\colon d \to c$ in $\mathcal{C}$ such that $g \circ f = \mathrm{id}_c \in \mathrm{mor}_{\mathcal{C}}(c, c)$ and $f \circ g = \mathrm{id}_d \in \mathrm{mor}_{\mathcal{C}}(d, d)$.

EXAMPLE 3.1.2. The prototype is $\mathcal{S}$ets, the category of sets. The objects of that are the sets. For two sets $S$ and $T$, the set of morphisms $\mathrm{mor}(S, T)$ is the set of all maps from $S$ to $T$. Composition is composition of maps as we know it and the identity morphisms are the identity maps as we know them.
Another very important example for us is $\mathcal{T}$op, the category of topological spaces. The objects are the topological spaces. For topological spaces $X = (X, \mathcal{O}_X)$ and $Y = (Y, \mathcal{O}_Y)$, the set of morphisms $\mathrm{mor}(X, Y)$ is the set of continuous maps from $X$ to $Y$. Composition is composition of continuous maps as we know it and the identity morphisms are the identity maps as we know them.
Another very important example for us is $\mathcal{H}$o$\mathcal{T}$op, the homotopy category of topological spaces. The objects are the topological spaces, as in $\mathcal{T}$op. But the set of morphisms from $X = (X, \mathcal{O}_X)$ to $Y = (Y, \mathcal{O}_Y)$ is $[X, Y]$, the set of *homotopy classes* of continuous maps from $X$ to $Y$. Composition $\circ$ is defined by the formula

$$[f] \circ [g] = [f \circ g]$$

for $[f] \in [Y, Z]$ and $[g] \in [X, Y]$. Here $f\colon Y \to Z$ and $g\colon X \to Y$ are continuous maps representing certain elements of $[Y, Z]$ and $[X, Y]$, and $f \circ g\colon X \to Z$ is their composition. There is an issue of well-defined-ness here, but fortunately we settled this long ago in chapter 1. As a result, associativity of composition is not in doubt because it is a consequence of associativity of composition in $\mathcal{T}$op. The identity morphisms in $\mathcal{H}$o$\mathcal{T}$op are the homotopy classes of the identity maps.
Another popular example is $\mathcal{G}$roups, the category of groups. The objects are the groups. For groups $G$ and $H$, the set of morphisms $\mathrm{mor}(G, H)$ is the set of group homomorphisms from $G$ to $H$. Composition of morphisms is composition of group homomorphisms.
The definition of a category as above permits some examples which are rather strange. One type of strange example is as follows. Let $(P, \leq)$ be a partially ordered set, alias poset. That is to say, $P$ is a set and $\leq$ is a relation on $P$ which is transitive ($x \leq y$ and $y \leq z$ forces $x \leq z$), reflexive ($x \leq x$ holds for all $x$) and antisymmetric (in the sense that $x \leq y$ and $y \leq x$ together implies $x = y$). We turn this setup into a small category (nameless) such that the object set is $P$. We decree that, for $x, y \in P$, the set $\mathrm{mor}(x, y)$ shall be empty if $x$ is not $\leq y$, and shall contain exactly one element, denoted $*$, if $x \leq y$. Composition

$$\circ : \mathrm{mor}(y, z) \times \mathrm{mor}(x, y) \longrightarrow \mathrm{mor}(x, z)$$

is defined as follows. If $y$ is not $\leq z$, then $\mathrm{mor}(y, z)$ is empty and so $\mathrm{mor}(y, z) \times \mathrm{mor}(x, y)$ is empty, too. There is exactly one map from the empty set to $\mathrm{mor}(x, z)$ and we take that. If $x$ is not $\leq y$, then $\mathrm{mor}(y, z) \times \mathrm{mor}(x, y)$ is empty, and we have only one choice for our composition map, and we take that. The remaining case is the one where $x \leq y$ and $y \leq z$. Then $x \leq z$ by transitivity. Therefore $\mathrm{mor}(y, z) \times \mathrm{mor}(x, y)$ has exactly one element, but more importantly, $\mathrm{mor}(x, z)$ has also exactly one element. Therefore there is exactly one map from $\mathrm{mor}(y, z) \times \mathrm{mor}(x, y)$ to $\mathrm{mor}(x, z)$ and we take that.
Another type of strange example (less important for us but still instructive) can be constructed by starting with a specific group $G$, with multiplication map $\mu\colon G \times G \to G$. From that we construct a small category (nameless) whose object set has exactly one element, denoted $*$. We let $\mathrm{mor}(*, *) = G$. The composition map

$$\mathrm{mor}(*, *) \times \mathrm{mor}(*, *) \to \mathrm{mor}(*, *)$$

now has to be a map from $G \times G$ to $G$, and for that we choose $\mu$, the multiplication of $G$. Since $\mu$ has an associativity property, composition of morphisms is associative. For the identity morphism $\mathrm{id}_* \in \mathrm{mor}(*, *)$ we take the neutral element of $G$.
There are also some easy ways to make new categories out of old ones. One important example: let $\mathcal{C}$ be any category. We make a new category $\mathcal{C}^{\mathrm{op}}$, the *opposite* category of $\mathcal{C}$. It has the same objects as $\mathcal{C}$, but we let

$$\mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(c, d) := \mathrm{mor}_{\mathcal{C}}(d, c)$$

when $c$ and $d$ are objects of $\mathcal{C}$, or equivalently, objects of $\mathcal{C}^{\mathrm{op}}$. The identity morphism of an object $c$ in $\mathcal{C}^{\mathrm{op}}$ is the identity morphism of $c$ in $\mathcal{C}$. Composition

$$\mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(c, d) \times \mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(b, c) \longrightarrow \mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(b, d)$$

is defined by noting $\mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(c, d) \times \mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(b, c) = \mathrm{mor}_{\mathcal{C}}(d, c) \times \mathrm{mor}_{\mathcal{C}}(c, b)$ and going from there to $\mathrm{mor}_{\mathcal{C}}(c, b) \times \mathrm{mor}_{\mathcal{C}}(d, c)$ by an obvious bijection, and from there to $\mathrm{mor}_{\mathcal{C}}(d, b) = \mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(b, d)$ using composition of morphisms in the category $\mathcal{C}$.

## 3.2. Functors

It turns out that there is something like a *category of all categories*. Let us not try to make that very precise because there are some small difficulties and complications in that. In any case there is a concept of morphism between categories, and the name of that is *functor*.

DEFINITION 3.2.1. A *functor* from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a rule $F$ which to every object $c$ of $\mathcal{C}$ assigns an object $F(c)$ of $\mathcal{D}$, and to every morphism $g\colon b \to c$ in $\mathcal{C}$ a morphism $F(g)\colon F(b) \to F(c)$ in $\mathcal{D}$, subject to the following conditions.

- For any object $c$ in $\mathcal{C}$ with identity morphism $\mathrm{id}_c$ , we have $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.
- Whenever $a, b, c$ are objects in $\mathcal{C}$ and $h \in \mathrm{mor}_{\mathcal{C}}(a, b)$, $g \in \mathrm{mor}_{\mathcal{C}}(b, c)$, we have $F(g \circ h) = F(g) \circ F(h)$ in $\mathrm{mor}_{\mathcal{D}}(F(a), F(c))$.

EXAMPLE 3.2.2. A functor $F$ from the category $\mathcal{T}op$ to the category $\mathcal{S}ets$ can be defined as follows. For a topological space $X$ let $F(X)$ be the set of path components of $X$. A continuous map $g\colon X \to Y$ determines a map $F(g)\colon F(X) \to F(Y)$ like this: $F(g)$ applied to a path component $C$ of $X$ is the unique path component of $Y$ which contains $g(C)$.
Fix a positive integer $n$. Let $\mathcal{R}ings$ be the category of rings and ring homomorphisms. (For me, a ring does not have to be commutative, but it should have distinguished elements $0$ and $1$ and in this example I require $0 \neq 1$.) A functor $F$ from $\mathcal{R}ings$ to $\mathcal{G}roups$ can be defined by $F(R) = \mathrm{GL}_n(R)$, where $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices with entries in $R$. A ring homomorphism $g\colon R_1 \to R_2$ determines a group homomorphism $F(g)$ from $F(R_1)$ to $F(R_2)$. Namely, in an invertible $n \times n$-matrix with entries in $R_1$ , apply $g$ to each entry to obtain an invertible $n \times n$-matrix with entries in $R_2$.
Let $G$ be a group which comes with an action on a set $S$. In example 3.1.2 we constructed from $G$ a category with one object $*$ and $\mathrm{mor}(*, *) = G$. A functor $F$ from that category to $\mathcal{S}ets$ can now be defined by $F(*) = S$, and $F(g) =$ translation by $g$, for $g \in \mathrm{mor}(*, *) = G$. More precisely, to $g \in G = \mathrm{mor}(*, *)$ we associate the map $F(g)$ from $S = F(*)$ to $S = F(*)$ given by $x \mapsto g \cdot x$ (which has a meaning because we are assuming an action of $G$ on $S$).
Let $\mathcal{C}$ be any category and let $x$ be any object of $\mathcal{C}$. A functor $F_x$ from $\mathcal{C}$ to $\mathcal{S}ets$ can be defined as follows. Let $F_x(c) = \mathrm{mor}_{\mathcal{C}}(x, c)$. For a morphism $g\colon c \to d$ in $\mathcal{C}$ define $F_x(g)\colon F_x(c) \to F_x(d)$ by $F_x(g)(h) = g \circ h$. In more detail, we are assuming $h \in F_x(c) = \mathrm{mor}_{\mathcal{C}}(x, c)$ and $g \in \mathrm{mor}_{\mathcal{C}}(c, d)$, so that $g \circ h \in \mathrm{mor}_{\mathcal{C}}(x, d) = F_x(d)$.

The functors of definition 3.2.1 are also called *covariant functors* for more precision. There is a related concept of *contravariant functor*. A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is simply a (covariant) functor from $\mathcal{C}^{op}$ to $\mathcal{D}$ (see example 3.1.2). If we write this out, it looks like this. A contravariant functor $F$ from $\mathcal{C}$ to $\mathcal{D}$ is a rule which to every object $c$ of $\mathcal{C}$ assigns an object $F(c)$ of $\mathcal{D}$, and to every morphism $g\colon c \to d$ in $\mathcal{C}$ a morphism $F(g)\colon F(d) \to F(c)$; note that the source of $F(g)$ is $F(d)$, and the target is $F(c)$. And so on.

EXAMPLE 3.2.3. Let $\mathcal{C}$ be any category and let $x$ be any object of $\mathcal{C}$. A contravariant functor $F^x$ from $\mathcal{C}$ to $\mathcal{S}$ets can be defined as follows. Let $F^x(c) = \mathrm{mor}_{\mathcal{C}}(c, x)$. For a morphism $g\colon c \to d$ in $\mathcal{C}$ define

$$F^x(g)\colon F^x(d) \to F^x(c)$$

by $F^x(g)(h) = h \circ g$. In more detail, we are assuming $h \in F^x(d) = \mathrm{mor}_{\mathcal{C}}(d, x)$ and $g \in \mathrm{mor}_{\mathcal{C}}(c, d)$, so that $h \circ g \in \mathrm{mor}_{\mathcal{C}}(c, x) = F^x(c)$.
There is a contravariant functor $P$ from $\mathcal{S}$ets to $\mathcal{S}$ets given by $P(S) =$ power set of $S$, for a set $S$. In more detail, a morphism $g\colon S \to T$ in $\mathcal{S}$ets determines a map $P(g)\colon P(T) \to P(S)$ by "preimage". That is, $P(g)$ applied to a subset $U$ of $T$ is $g^{-1}(U)$, a subset of $S$. (You may have noticed that this example of a contravariant functor is not very different from a special case of the preceding one; we will return to this point later.)
Next, let $\mathcal{M}$an be the category of smooth manifolds. The objects are the smooth manifolds (of any dimension). The morphisms from a smooth manifold $M$ to a smooth manifold $N$ are the smooth maps from $M$ to $N$. For any fixed integer $k \geq 0$ the rule which assigns to a smooth manifold $M$ the real vector space $\Omega^k(M)$ of smooth differential $k$-forms is a contravariant functor from $\mathcal{M}$an to the category $\mathcal{V}$ect of real vector spaces (with linear maps as morphisms). Namely, a smooth map $f\colon M \to N$ determines a linear map $f^*\colon \Omega^k(N) \to \Omega^k(M)$. (You must have seen the details if you know anything about differential forms.)

## 3.3. Natural transformations

The story does not end there. The functors from a category $\mathcal{C}$ to a category $\mathcal{D}$ also form something like a category. There is a concept of morphism between functors from $\mathcal{C}$ to $\mathcal{D}$, and the name of that is *natural transformation.*

DEFINITION 3.3.1. Let $F$ and $G$ be functors, both from a category $\mathcal{C}$ to a category $\mathcal{D}$. A *natural transformation* from $F$ to $G$ is a rule $\nu$ which for every object $c$ in $\mathcal{C}$ selects a morphism $\nu_c\colon F(c) \to G(c)$ in $\mathcal{D}$, subject to the following condition. Whenever $u\colon c \to d$ is a morphism in $\mathcal{C}$, the square of morphisms

$$\begin{tikzcd} F(c) \arrow[r, "\nu_c"] \arrow[d, "F(u)"] & G(c) \arrow[d, "G(u)"] \\ F(d) \arrow[r, "\nu_d"] & G(d) \end{tikzcd}$$

in $\mathcal{D}$ commutes; that is, the equation $G(u) \circ \nu_c = \nu_d \circ F(u)$ holds in $\mathrm{mor}_{\mathcal{D}}(F(c), G(d))$.

EXAMPLE 3.3.2. MacLane (in his book *Categories for the working mathematician*) gives the following pretty example. For a fixed integer $n \geq 1$ the rule which to a ring $R$ assigns the group $\mathrm{GL}_n(R)$ can be viewed as a functor $\mathrm{GL}_n$ from the category of rings to the category of groups, as was shown earlier. There we allowed non-commutative rings, but here we need commutative rings, so we shall view $\mathrm{GL}_n$ as a functor from the category $c\mathcal{R}$ings of commutative rings to $\mathcal{G}$roups. Note that $\mathrm{GL}_1(R)$ is essentially the group of units of the ring $R$. The group homomorphisms

$$\det\colon \mathrm{GL}_n(R) \to \mathrm{GL}_1(R)$$

(one for every commutative ring $R$) make up a natural transformation from the functor $\mathrm{GL}_n\colon c\mathcal{R}\mathrm{ings} \to \mathcal{G}\mathrm{roups}$ to the functor $\mathrm{GL}_1\colon c\mathcal{R}\mathrm{ings} \to \mathcal{G}\mathrm{roups}$.
Returning to smooth manifolds and differential forms: we saw that for any fixed $k \geq 0$ the assignment $M \mapsto \Omega^k(M)$ can be viewed as a contravariant functor from $\mathcal{M}$an to $\mathcal{V}$ect. The exterior derivative maps

$$d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

(one for each object $M$ of $\mathcal{M}$an) make up a natural transformation from the contravariant functor $\Omega^k$ to the contravariant functor $\Omega^{k+1}$.

*Notation*: let $F$ and $G$ be functors from $\mathcal{C}$ to $\mathcal{D}$. Sometimes we describe a natural transformation $\nu$ from $F$ to $G$ by a strong arrow, as in $\nu\colon F \Rightarrow G$.

*Remark*: one reason for being a little cautious in saying *category of categories* etc. is that the functors from one big category (such as $\mathcal{T}$op for example) to another big category (such as $\mathcal{S}$ets for example) do not obviously form a set. Of course, some people would not exercise that kind of caution and would instead say that the definition of category as given in 3.1.1 is not bold enough. In any case, it must be permitted to say *the category of small categories*.

CHAPTER 4

# Combinatorial description of some spaces

## 4.1. Vertex schemes and simplicial complexes

DEFINITION 4.1.1. A *vertex scheme* consists of a set $V$ and a subset $\mathcal{S}$ of the power set $\mathcal{P}(V)$, subject to the following conditions: every $T \in \mathcal{S}$ is finite and nonempty, every subset of $V$ which has exactly one element belongs to $\mathcal{S}$, and if $T'$ is a nonempty subset of some $T \in \mathcal{S}$, then $T' \in \mathcal{S}$.
The elements of $V$ are called *vertices* (singular: *vertex*) of the vertex scheme. The elements of $\mathcal{S}$ are called *distinguished subsets* of $V$.

EXAMPLE 4.1.2. The following are examples of vertex schemes:

(i) Let $V = \{1, 2, 3, \dots, 10\}$. Define $\mathcal{S} \subset \mathcal{P}(V)$ so that the elements of $\mathcal{S}$ are the following subsets of $V$: all the singletons, that is to say $\{1\}, \{2\}, \dots, \{10\}$, and $\{1, 2\}$, $\{2, 3\}$, $\dots$, $\{9, 10\}$ as well as $\{10, 1\}$.
(ii) Let $V = \{1, 2, 3, 4\}$ and define $\mathcal{S} \subset \mathcal{P}(V)$ so that the elements of $\mathcal{S}$ are exactly the subsets of $V$ which are nonempty and not equal to $V$.
(iii) Let $V$ be any set and define $\mathcal{S}$ so that the elements of $\mathcal{S}$ are exactly the nonempty finite subsets of $V$.
(iv) Take a regular icosahedron. Let $V$ be the set of its vertices (which has 12 elements). Define $\mathcal{S} \subset \mathcal{P}(V)$ in such a way that the elements of $\mathcal{S}$ are all singletons, all doubletons which are connected by an edge, and all tripletons which make up a triangular face of the icosahedron. (There are twenty such tripletons, which is supposed to explain the name *icosahedron*.)

The *simplicial complex* determined by a vertex scheme $(V, \mathcal{S})$ is a topological space $X = |V|_{\mathcal{S}}$. We describe it first as a set. An element of $X$ is a function $f \colon V \to [0, 1]$ such that

$$\sum_{v \in V} f(v) = 1$$

and the set $\{v \in V \mid f(v) > 0\}$ is an element of $\mathcal{S}$.
It should be clear that $X$ is the union of certain subsets $\Delta(T)$, where $T \in \mathcal{S}$. Namely, $\Delta(T)$ consists of all the functions $f \colon V \to [0, 1]$ for which $\sum_{v \in V} f(v) = 1$ and $f(v) = 0$ if $v \notin T$. The subsets $\Delta(T)$ of $X$ are not always disjoint. Instead we have $\Delta(T) \cap \Delta(T') = \Delta(T \cap T')$ if $T \cap T'$ is nonempty; also, if $T \subset T'$ then $\Delta(T) \subset \Delta(T')$.
The subsets $\Delta(T)$ of $X$, for $T \in \mathcal{S}$, come equipped with a preferred topology. Namely, $\Delta(T)$ is (identified with) a subset of a finite dimensional real vector space, the vector space of all functions from $T$ to $\mathbb{R}$, and as such gets a subspace topology. (For example, $\Delta(T)$ is a single point if $T$ has one element; it is homeomorphic to an edge or closed interval if $T$ has two elements; it looks like a compact triangle if $T$ has three elements; etc. We say that $\Delta(T)$ is a *simplex* of dimension $m$ if $T$ has cardinality $m + 1$.) These topologies are compatible in the following sense: if $T \subset T'$, then the inclusion $\Delta(T) \to \Delta(T')$ makes a

homeomorphism of $\Delta(T)$ with a subspace of $\Delta(T')$.
We decree that a subset $W$ of $X$ shall be *open* if and only if $W \cap \Delta(T)$ is open in $\Delta(T)$, for every $T$ in $\mathcal{S}$. Equivalently, and perhaps more usefully: a map $g$ from $X$ to another topological space $Y$ is continuous if and only if the restriction of $g$ to $\Delta(T)$ is a continuous from $\Delta(T)$ to $Y$, for every $T \in \mathcal{S}$.

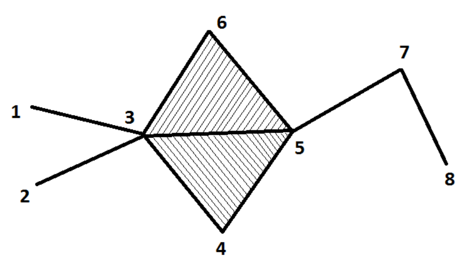
Every $v \in V$ determines a map $\beta_v \colon |V|_{\mathcal{S}} \to [0,1]$ by $f \mapsto f(v)$. This is continuous (almost by definition). It is called the *barycentric coordinate* associated with $v \in V$.

EXAMPLE 4.1.3. The simplicial complex associated to the vertex scheme (i) in example 4.1.2 is homeomorphic to $S^1$. In (ii) and (iv) of example 4.1.2, the associated simplicial complex is homeomorphic to $S^2$.

EXAMPLE 4.1.4. The simplicial complex associated to the vertex scheme $(V, \mathcal{S})$ where $V = \{1,2,3,4,5,6,7,8\}$ and

$$\mathcal{S} = \left\{ \begin{array}{l} \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{1,3\},\{2,3\},\{3,4\}, \\ \{3,5\},\{3,6\},\{4,5\},\{5,6\},\{5,7\},\{7,8\},\{3,4,5\},\{3,5,6\} \end{array} \right\}$$

looks like this:




LEMMA 4.1.5. *The simplicial complex $X = |V|_{\mathcal{S}}$ associated with a vertex scheme $(V, \mathcal{S})$ is a Hausdorff space.*

PROOF. Let $f$ and $g$ be distinct elements of $X$. Keep in mind that $f$ and $g$ are functions from $V$ to $[0,1]$. Choose $v_0 \in V$ such that $f(v_0) \neq g(v_0)$. Let $\varepsilon = |f(v_0) - g(v_0)|$. Let $U_f$ be the set of all $h \in X$ such that $|h(v_0) - f(v_0)| < \varepsilon/2$. Let $U_g$ be the set of all $h \in X$ such that $|h(v_0) - g(v_0)| < \varepsilon/2$. From the definition of the topology on $X$, the sets $U_f$ and $U_g$ are open. They are also disjoint, for if $h \in U_f \cap U_g$ then $|f(v_0) - g(v_0)| \leq |f(v_0) - h(v_0)| + |h(v_0) - g(v_0)| < \varepsilon$, contradiction. Therefore $f$ and $g$ have disjoint neighborhoods in $X$. $\square$

LEMMA 4.1.6. *Let $(V, \mathcal{S})$ be a vertex scheme and $(W, \mathcal{T})$ a vertex sub-scheme, that is, $W \subset V$ and $\mathcal{T} \subset \mathcal{S} \cap \mathcal{P}(W)$. Then the evident map $\iota \colon |W|_{\mathcal{T}} \to |V|_{\mathcal{S}}$ is a closed, continuous and injective map and therefore a homeomorphism onto its image.*

PROOF. The map $\iota$ is obtained by viewing functions from $W$ to $[0,1]$ as functions from $V$ to $[0,1]$ by defining the values on elements of $V \smallsetminus W$ to be $0$. A subset $A$ of $|V|_{\mathcal{S}}$ is closed if and only if $A \cap \Delta(T)$ is closed for the standard topology on $\Delta(T)$, for every $T \in \mathcal{S}$. Therefore, if $A$ is a closed subset of $|V|_{\mathcal{S}}$, then $\iota^{-1}(A)$ is a closed subset of $|W|_{\mathcal{T}}$; and if $C$ is a closed subset of $|W|_{\mathcal{S}}$, then $\iota(C)$ is closed in $|V|_{\mathcal{S}}$. $\square$

Remark 4.1.7. The notion of a simplicial complex is old. Related vocabulary comes in many dialects. I have taken the expression *vertex scheme* from Dold's book *Lectures on algebraic topology* with only a small change (for me, $\emptyset \notin \mathcal{S}$). It is in my opinion a good choice of words, but the traditional expression for that appears to be *abstract simplicial complex*. Most authors agree that a *simplicial complex* (non-abstract) is a topological space with additional data. For me, a simplicial complex is a space of the form $|V|_{\mathcal{S}}$ for some vertex scheme $(V, \mathcal{S})$; other authors prefer to write, in so many formulations, that a simplicial complex is a topological space $X$ together with a homeomorphism $|V|_{\mathcal{S}} \to X$, for some vertex scheme $(V, \mathcal{S})$.

## 4.2. Semi-simplicial sets and their geometric realizations

Semi-simplicial sets are closely related to vertex schemes. A semi-simplicial set has a *geometric realization*, which is a topological space; this is similar to the way in which a vertex scheme determines a simplicial complex.

Definition 4.2.1. A semi-simplicial set $Y$ consists of a sequence of sets

$$(Y_0, Y_1, Y_2, Y_3, ...)$$

(each $Y_k$ is a set) and, for each injective order-preserving map

$$f\colon \{0, 1, 2, \dots, k\} \longrightarrow \{0, 1, 2, \dots, \ell\}$$

where $k, \ell \geq 0$, a map $f^*\colon Y_\ell \to Y_k$. The maps $f^*$ are called *face operators* and they are subject to conditions:

- if $f$ is the identity map from $\{0, 1, 2, \dots, k\}$ to $\{0, 1, 2, \dots, k\}$ then $f^*$ is the identity map from $Y_k$ to $Y_k$.
- $(g \circ f)^* = f^* \circ g^*$ when $g \circ f$ is defined (so $f\colon \{0, 1, \dots, k\} \to \{0, 1, \dots, \ell\}$ and $g\colon \{0, 1, \dots, \ell\} \to \{0, 1, \dots, m\}$).

Elements of $Y_k$ are often called $k$-*simplices* of $Y$. If $x \in Y_k$ has the form $f^*(y)$ for some $y \in Y_\ell$, then we may say that $x$ is a *face* of $y$ corresponding to face operator $f^*$.

Remark 4.2.2. The definition of a semi-simplicial set can be reformulated in category language as follows. There is a category $\mathcal{C}$ whose objects are the sets $[n] = \{0, 1, \dots, n\}$, where $n$ can be any non-negative integer. A morphism in $\mathcal{C}$ from $[m]$ to $[n]$ is an order-preserving injective map from the set $[m]$ to the set $[n]$. Composition of morphisms is, by definition, composition of such order-preserving injective maps.
A semi-simplicial set is a contravariant functor $Y$ from $\mathcal{C}$ to the category of sets. We like to write $Y_n$ when we ought to write $Y([n])$. We like to write $f^*\colon Y_n \to Y_m$ when we ought to write $Y(f)\colon Y([n]) \to Y([m])$, for a morphism $f\colon [m] \to [n]$ in $\mathcal{C}$.
Nota bene: if you wish to define (invent) a semi-simplicial set $Y$, you need to invent sets $Y_0, Y_1, Y_2, \dots$ (one set $Y_n$ for each integer $n \geq 0$) *and* you need to invent maps $f^*\colon Y_n \to Y_m$, one for each order-preserving injective map $f\colon [m] \to [n]$. *Then* you need to convince yourself that $(g \circ f)^* = f^* \circ g^*$ whenever $f\colon [k] \to [\ell]$ and $g\colon [\ell] \to [m]$ are order-preserving injective maps.

Example 4.2.3. Let $(V, \mathcal{S})$ be a vertex scheme as in the preceding (sub)section. Choose a total ordering of $V$. From these data we can make a semi-simplicial set $Y$ as follows.

- $Y_n$ is the set of all order-preserving injective maps $\beta$ from $\{0, 1, \dots, n\}$ to $V$ such that $\mathrm{im}(\beta) \in \mathcal{S}$. Note that for each $T \in \mathcal{S}$ of cardinality $n+1$, there is exactly one such $\beta$.

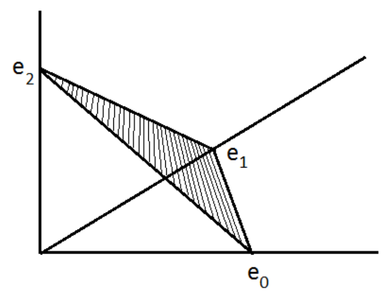
- For an order-preserving injective $f\colon \{0,1,\dots,m\} \to \{0,1,\dots,n\}$ and $\beta \in Y_n$, define $f^*(\beta) = \beta \circ f \in Y_m$.

The category $\mathcal{C}$ is more officially denoted $\Delta$ (which can be confusing since we use the symbol $\Delta$ in so many other closely related situations).

In order to warm up for geometric realization, we introduce a (covariant) functor from the category $\mathcal{C}$ in remark 4.2.2 to the category of topological spaces. On objects, the functor is given by

$$\{0,1,2,\dots,m\} \mapsto \Delta^m$$

where $\Delta^m$ is the space of functions $u$ from $\{0,1,\dots,m\}$ to $\mathbb{R}$ which satisfy the condition $\sum_{j=0}^m u(j) = 1$. (As usual we view this as a subspace of the finite-dimensional real vector space of all functions from $\{0,1,\dots,n\}$ to $\mathbb{R}$. It is often convenient to think of $u \in \Delta^n$ as a vector, $(u_0,u_1,\dots,u_m)$, where all coordinates are $\geq 0$ and their sum is $1$.) Here is a picture of $\Delta^2$ as a subspace of $\mathbb{R}^3$ (with basis vectors $e_0,e_1,e_2$):




For a morphism $f$, meaning an order-preserving injective map

$$f\colon \{0,1,2,\dots,m\} \longrightarrow \{0,1,2,\dots,n\},$$

we want to see an induced map

$$f_*\colon \Delta^m \to \Delta^n.$$

This is easy: for $u = (u_0,u_1,\dots,u_m) \in \Delta^m$ we define

$$f_*(u) = v = (v_0,v_1,\dots,v_n) \in \Delta^n$$

where $v_j = u_i$ if $j = f(i)$ and $v_j = 0$ if $j \notin \mathrm{im}(f)$.
(Keep the following conventions in mind. For a covariant functor $G$ from a category $\mathcal{A}$ to a category $\mathcal{B}$, and a morphism $f\colon x \to y$ in $\mathcal{A}$, we often write $f_*\colon G(x) \to G(y)$ instead of $G(f)\colon G(x) \to G(y)$. For a contravariant functor $G$ from a category $\mathcal{A}$ to a category $\mathcal{B}$, and a morphism $f\colon x \to y$ in $\mathcal{A}$, we often write $f^*\colon G(y) \to G(x)$ instead of $G(f)\colon G(y) \to G(x)$.)

The geometric realization $|Y|$ of a semi-simplicial set $Y$ is a topological space defined as follows. Our goal is to have, for each $n \geq 0$ and $y \in Y_n$, a preferred continuous map

$$c_y\colon \Delta^n \to |Y|$$

(the *characteristic map* associated with the simplex $y \in Y_n$). These maps should match in the sense that whenever we have an injective order-preserving

$$f \colon \{0,1,\dots,m\} \to \{0,1,\dots,n\}$$

and $y \in Y_n$, so that $f^*y \in Y_m$, then the diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "c_y"] & {|Y|} \\ \Delta^m \arrow[u, "f_*"] \arrow[r, "c_{f^*y}"] & {|Y|} \arrow[u, "="'] \end{tikzcd}$$

is commutative. There is a "most efficient" way to achieve this. As a set, let $|Y|$ be the set of all pairs $(y,u)$ where $y \in Y_n$ for some $n \geq 0$ and $u \in \Delta^n$, modulo the relations[1]

$$(y, f_*(u)) \sim (f^*(y), u)$$

(notation and assumptions as in that diagram). This ensures that we have maps $c_y$ from $\Delta^n$ to $|Y|$, for each $y \in Y_n$, given in the best tautological manner by

$$c_y(u) := \text{ equivalence class of } (y,u)\ .$$

Also, those little squares which we wanted to be commutative are now commutative because we enforced it. Finally, we say that a subset $U$ of $|Y|$ shall be *open* (definition coming) if and only if $c_y^{-1}(U)$ is open in $\Delta^n$ for each characteristic map $c_y \colon \Delta^n \to |Y|$.

A faster way to say the same thing is as follows:

$$|Y| := \left( \coprod_{n \geq 0} Y_n \times \Delta^n \right) \Big/ \sim$$

where $\sim$ is a certain equivalence relation on $\coprod_n Y_n \times \Delta^n$. It is the smallest equivalence relation which has $(y, f_*(u))$ equivalent to $(f^*(y), u)$ whenever $f \colon \{0,1,\dots,m\} \to \{0,1,\dots,n\}$ is injective order-preserving and $y \in Y_n$, $u \in \Delta^m$. Note that, where it says $Y_n \times \Delta^n$, the set $Y_n$ is regarded as a topological space with the discrete topology, so that $Y_n \times \Delta^n$ has meaning; we could also have written $\coprod_{y \in Y_n} \Delta^n$ instead of $Y_n \times \Delta^n$.

This new formula for $|Y|$ emphasizes the fact that $|Y|$ is a *quotient space* of a topological disjoint union of many standard simplices $\Delta^n$ (one simplex for every pair $(n,y)$ where $y \in Y_n$). Go ye forth and look up *quotient space* or *identification topology* in your favorite book on point set topology.—

EXAMPLE 4.2.4. Fix an integer $n \geq 0$. We might like to invent a semi-simplicial set

$$Y = \underline{\Delta}^n$$

such that $|Y|$ is homeomorphic to $\Delta^n$. The easiest way to achieve that is as follows. Define $Y_k$ to be the set of all order-preserving injective maps from $\{0,1,\dots,k\}$ to $\{0,1,\dots,n\}$. So $Y_k$ has $\binom{n+1}{k+1}$ elements (which implies $Y_k = \emptyset$ if $k > n$). For an injective order-preserving map

$$g \colon \{0,1,\dots,k\} \to \{0,1,\dots,\ell\},$$

define the face operator $g^* \colon Y_\ell \to Y_k$ by $g^*(f) = f \circ g$. This makes sense because $f \in Y_\ell$ is an order-preserving injective map from $\{0,1,\dots,\ell\}$ to $\{0,1,\dots,n\}$. There is a unique

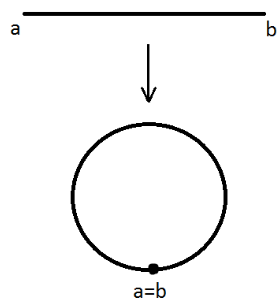
[1] *Modulo the relations* is short for the following process: form the smallest equivalence relation on the set of all those pairs $(y,u)$ which contains the stated relation. Then pass to the set of equivalence classes for that equivalence relation. That set of equivalence classes is $|Y|$.

element $y \in Y_n$, corresponding to the identity map of $\{0, 1, \ldots, n\}$. It is an exercise to verify that the characteristic map $c_y : \Delta^n \to |Y|$ is a homeomorphism.

EXAMPLE 4.2.5. Up to relabeling there is a unique semi-simplicial set $Y$ such that $Y_0$ has exactly one element, $Y_1$ has exactly one element, and $Y_n = \emptyset$ for $n > 1$. Then $|Y|$ is homeomorphic to $S^1$. More precisely, let $z \in Y_1$ be the unique element; then the characteristic map

$$c_z : \Delta^1 \longrightarrow |Y|$$

is an identification map. (Translation: it is surjective and a subset of the target is open in the target if and only if its preimage is open in the source.) The only identification taking place is $c_z(a) = c_z(b)$, where $a$ and $b$ are the two boundary points of $\Delta^1$.




## 4.3. Technical remarks concerning the geometric realization

PROPOSITION 4.3.1. *Let $(V, \mathcal{S})$ be a vertex scheme, with a total ordering on $V$, and let $Y$ be the associated semi-simplicial set, as in example 4.2.3. The geometric realization $|Y|$ is homeomorphic to the simplicial complex $|V|_{\mathcal{S}}$.*

PROOF. An element of $Y_n$ is an order-preserving injective map from $[n] = \{0, 1, \ldots, n\}$ to $V$. This is determined by its image $T$, a distinguished subset of $V$ (where *distinguished* means that $T \in \mathcal{S}$). We make a continuous map

$$\bar{\alpha}\colon \coprod_{n \geq 0} Y_n \times \Delta^n \longrightarrow |V|_{\mathcal{S}}$$

by taking a pair $(y, b)$ with $y \in Y_n$ and $b \in \Delta^n$ to the function $f\colon V \to [0, 1]$ which has $f(y(t)) = b_t$ for $t \in [n]$ and $f(v) = 0$ if $v \in V$ is not in the image of $y\colon [n] \to V$. This is clearly onto, and continuous. It is easy to see that $\bar{\alpha}((y, b)) = \bar{\alpha}((x, a))$ if and only if $(y, b)$ and $(x, y)$ are *equivalent* in the sense that they have the same image in $|Y|$. Therefore the map $\bar{\alpha}$ determines a bijective continuous map $\alpha\colon |Y| \to |V|_{\mathcal{S}}$. The inverse is continuous when we restrict to a subset of the form $\Delta(T) \subset |V|_{\mathcal{S}}$ (where $T \in \mathcal{S}$) since it is essentially $c_y : \Delta^n \to |Y|$ for the unique $y : [n] \to V$ which has image $T$ (up to an identification of $\Delta^n$ with $\Delta(T)$). Therefore the inverse is continuous. $\square$

LEMMA 4.3.2. *Let $Y$ be any semi-simplicial set. For every element $a$ of $|Y|$ there exist unique $m \geq 0$ and $(z, w) \in Y_m \times \Delta^m$ such that $a = c_z(w)$ and $w$ is in the "interior" of $\Delta^m$, that is, the coordinates $w_0, w_1, \ldots, w_m$ are all strictly positive.*

*Furthermore, if* $a = c_x(u)$ *for some* $(x, u) \in Y_k \times \Delta^k$*, then there is a unique order-preserving injective* $f\colon \{0, 1, \ldots, m\} \to \{0, 1, 2, \ldots, k\}$ *such that* $f^*(x) = z$ *and* $f_*(w) = u$*, for the above-mentioned* $(z, w) \in Y_m \times \Delta^m$ *with* $w_0, w_1, \ldots, w_m > 0$*.*

Proof. Let us call such a pair $(z, w)$ with $a = c_z(w)$ a *reduced presentation* of $a$; the condition is that all coordinates of $w$ must be positive. More generally we say that $(x, u)$ is a *presentation* of $a$ if $(x, u) \in Y_k \times \Delta^k$ for some $k \geq 0$ and $a = c_x(u)$. First we show that $a$ admits a reduced presentation and then we show uniqueness.
We know that $a = c_x(u)$ for some $(x, u) \in Y_k \times \Delta^k$. Some of the coordinates $u_0, \ldots, u_k$ can be zero (not all, since their sum is $1$). Suppose that $m + 1$ of them are nonzero. Let $f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, k\}$ be the unique order-preserving map such that $u_{f(j)} \neq 0$ for $j = 0, 1, 2, \ldots, m$. Then $a = c_z(w)$ where $z = f^*(x)$ and $w \in \Delta^m$ with coordinates $w_j = u_{f(j)}$. (Note that $f_*(w) = u$.) So $(z, w)$ is a reduced presentation of $a$.
We have also shown that any presentation $(x, u)$ of $a$ (whether reduced or not) determines a reduced presentation. Namely, there exist unique $m$, $f$ and $w \in \Delta^m$ such that $v = f_*(w)$ for some $w \in \Delta^m$ with all $w_i > 0$; then $(f^*(x), w)$ is a reduced presentation of $a$.
It remains to show that if $a$ has two presentations, say $(x, u) \in Y_k \times \Delta^k$ and $(y, v) \in Y_\ell \times \Delta^\ell$, then they determine the *same* reduced representation of $a$. We are assuming that $(x, u)$ and $(y, v)$ are equivalent, and so (recalling how that equivalence relation was defined) we find that there is no loss of generality in assuming that $x = g^*(y)$ and $v = g_*(u)$ for some order-preserving injective $g\colon \{0, 1, \ldots, k\} \to \{0, 1, \ldots, \ell\}$. Now determine the unique $m$ and order-preserving injective $f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, k\}$ such that $u = f_*(w)$ where $w \in \Delta^m$ and all $w_i > 0$. Then we also have $v = g_*(u) = g_*(f_*(w)) = (g \circ f)_*(w)$ and it follows that we get the same reduced presentation,

$$(f^*(x), w) = ((g \circ f)^*(y), w),$$

in both cases. □

Corollary 4.3.3. *The space* $|Y|$ *is a Hausdorff space.*

Proof. For $a \in |Y|$ with reduced presentation $(z, w) \in Y_m \times \Delta^m$ and $\varepsilon > 0$, define $N(a, \varepsilon) \subset |Y|$ as follows. It consists of all $b \in |Y|$ with a presentation $(x, u) \in Y_k \times \Delta^k$ (which does not have to be reduced) such that there exists an order-preserving injective $f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, k\}$ for which

- $f^*(x) = z$
- $f_*(w)$ is $\varepsilon$-close to $u$, that is, the numbers $|w_j - u_{f(j)}|$ for $j \in [m]$ and the numbers $u_i$ for $i \notin \mathrm{im}(f)$ are all $< \varepsilon$;
- $u_{f(j)} > 0$ for all $j \in [m]$.

Then from the definitions, $N(a, \varepsilon)$ is open in $|Y|$; so it is a neighborhood of $a$.
Let $a' \in |Y|$ be another element, with reduced presentation $(y, v) \in Y_n \times \Delta^n$. Suppose that $N(a', \varepsilon) \cap N(a, \varepsilon)$ is nonempty. Then we know, first of all, that there exists $(x, u) \in Y_k \times \Delta^k$ (reduced presentation) and order-preserving injective maps $f\colon [m] \to [k]$ and $g\colon [n] \to [k]$ such that $f^*(x) = z$ and $g^*(x) = y$ and $f_*(w)$ is $\varepsilon$-close to $u$ and $g_*(v)$ is $\varepsilon$-close to $u$. Then $f_*(w)$ and $g_*(v)$ are $2\varepsilon$-close to each other in $\Delta^k$. If $2\varepsilon$ is less than the minimum of the (barycentric) coordinates of $v$ and $w$, then we can deduce that $f = g$ and $m = n$, and $w$ is $2\varepsilon$-close to $v$ in $\Delta^m = \Delta^n$. Therefore $N(a', \varepsilon) \cap N(a, \varepsilon) \neq \emptyset$ can only happen if $2\varepsilon$ is at least as large as the the minimum of the (barycentric) coordinates of $v$ and $w$, or if $m = n$ and $v$ is $2\varepsilon$-close to $w$ in $\Delta^m = \Delta^n$. So if $a \neq a'$ and $\varepsilon > 0$ is small enough, it will not happen. □

Remark 4.3.4. In the proof above, and in a similar proof in the previous section, arguments involving distances make an appearance, suggesting that we have a metrizable situation. To explain what is going on let me return to the situation of a vertex scheme $(V, \mathcal{S})$ with simplicial complex $|V|_{\mathcal{S}}$, which is easier to understand. A metric on the set $|V|_{\mathcal{S}}$ can be introduced for example by $d(f,g) = \sum_v |f(v)-g(v)|$. Here we insist/remember that elements of $|V|_{\mathcal{S}}$ are functions $f, g, \ldots : V \to [0,1]$ subject to some conditions. The sums in the formulas for $d(f,g)$ are finite, even though $V$ might not be a finite set. However the topology on $|V|_{\mathcal{S}}$ that we have previously decreed (let me call it the *weak* topology) is not in all cases the same as the topology determined by that metric. Every subset of $|V|_{\mathcal{S}}$ which is open in the metric topology is also open in the weak topology. But the weak topology can have more open sets. Since the topology determined by the metric is certainly Hausdorff, the weak topology is *a fortiori* a Hausdorff topology. — In the case where $V$ is finite, weak topology and metric topology on $|V|_{\mathcal{S}}$ coincide. (Exercise.)

## 4.4. A shorter but less conceptual definition of semi-simplicial set

Every injective order-preserving map from $[k] = \{0,1,\ldots,k\}$ to $[\ell] = \{0,1,\ldots,\ell\}$ is a composition of $\ell - k$ injective order preserving maps

$$[m-1] \longrightarrow [m]$$

where $k < m \le \ell$. It is easy to list the injective order-preserving maps from $[m-1]$ to $[m]$; there is one such map $f_i$ for every $i \in [m]$, characterized by the property that the image of $f_i$ is

$$[m] \smallsetminus \{i\} .$$

(This $f_i$ really depends on two parameters, $m$ and $i$. Perhaps we ought to write $f_{m,i}$ , but it is often practical to suppress the $m$ subscript.) We have the important relations

$$\text{(I)} \qquad f_i f_j = f_j f_{i-1} \quad \text{if } j < i$$

(You are allowed to read this from left to right or from right to left! It is therefore a formal consequence that $f_i f_j = f_{j+1} f_i$ when $j \ge i$.) These *generators and relations* suffice to describe the category $\mathcal{C}$ of remark 4.2.2 (also denoted $\Delta$). In other words, the structure of $\mathcal{C}$ alias $\Delta$ as a category is pinned down if we say that it has objects $[k]$ for $k \ge 0$ and that, for every $k > 0$ and $i \in \{0,1,\ldots,k\}$, there are certain morphisms $f_i \colon [k-1] \to [k]$ which, under composition when it is applicable, satisfy the relations (I). Prove it!
Consequently a semi-simplicial set $Y$, which is a contravariant functor from $\mathcal{C}$ to spaces, can also be described as a sequence of sets $Y_0, Y_1, Y_2, \ldots$ and maps

$$d_i \colon Y_k \to Y_{k-1}$$

which are subject to the relations

$$\text{(II)} \qquad d_j d_i = d_{i-1} d_j \quad \text{if } j < i$$

Here $d_i \colon Y_k \to Y_{k-1}$ denotes the map induced by $f_i \colon [k-1] \to [k]$, whenever $0 \le i \le k$. Because of contravariance, we had to reverse the order of composition in translating relations (I) to obtain relations (II).

## 4.5. The singular semi-simplicial set of a space

Let $X$ be a topological space.

DEFINITION 4.5.1. The *singular semi-simplicial set of* $X$ is the semi-simplicial set $\mathrm{sing}(X)$ defined as follows. An $n$-simplex of $\mathrm{sing}(X)$ is a continuous map from $\Delta^n$ to $X$; in other words, $\mathrm{sing}(X)_n$ is the set of all continuous maps from $\Delta^n$ to $X$. For a monotone injective $f\colon [m] \to [n]$, the induced map of sets $f^*\colon \mathrm{sing}(X)_n \to \mathrm{sing}(X)_m$ is given by pre-composition with $f_*\colon \Delta^m \to \Delta^n$.

There are curious historical explanations for the appearance of the word *singular* in the expression *the singular semi-simplicial set of* $X$. Somebody wanted to emphasize that we allow *all* continuous maps from standard simplices $\Delta^n$ to $X$ when we define $\mathrm{sing}(X)$. Older variants might have asked for additional conditions to be satisfied, such as injectivity, or other conditions which make sense for some $X$ but not others.

There is a comparison map
$$\kappa\colon |\mathrm{sing}(X)| \longrightarrow X\,.$$
It is defined in such a way that the composition
$$\coprod_{n\ge 0} \mathrm{sing}(X)_n \times \Delta^n \longrightarrow |\mathrm{sing}(X)| \longrightarrow X$$
agrees with the *evaluation map*, $(y,v) \mapsto y(v)$, on $\mathrm{sing}(X)_n \times \Delta^n$. Equivalently, we can define $\kappa$ by saying that $\kappa \circ c_y$ equals $y$, for every $y \in \mathrm{sing}(X)_n$. (What does this mean? Remember that $y \in \mathrm{sing}(X)_n$ is a continuous map from $\Delta^n$ to $X$; and $c_y\colon \Delta^n \to |\mathrm{sing}(X)|$ is the characteristic map for $y$; so the formula makes sense.)

Clearly, sing is a functor from the category $\mathcal{T}\mathrm{op}$ to the category of semi-simplicial sets. There is an important relationship between this functor and the functor *geometric realization*, which is a functor from the category of semi-simplicial sets to the category $\mathcal{T}\mathrm{op}$.

PROPOSITION 4.5.2. *Let* $X$ *be a topological space and let* $Y$ *be a semi-simplicial set. There is a* (bi-)natural *bijection*
$$\mathrm{mor}_{\mathcal{T}\mathrm{op}}(|Y|, X) \longrightarrow \mathrm{mor}_{\mathrm{ss}\mathcal{S}\mathrm{ets}}(Y, \mathrm{sing}(X))$$
*where* ss$\mathcal{S}$ets *denotes the category of semi-simplicial sets.*

The bijection is rather obvious. If $f\colon |Y| \to X$ is a continuous map, and $y \in Y_n$ , then we can make a continuous map $\Delta^n \to X$ by composing $f$ with the characteristic map $c_y\colon \Delta^n \to |Y|$. with $f$. Now $Y_n \ni y \mapsto f \circ c_y \in \mathrm{sing}(X)_n$ is a morphism from $Y$ to $\mathrm{sing}(X)$ in ss$\mathcal{S}$ets. Conversely, if we have a morphism $g\colon Y \to \mathrm{sing}(X)$, then the composition of $|g|\colon |Y| \to |\mathrm{sing}(X)|$ with the above map $\kappa\colon |\mathrm{sing}(X)| \to X$ is a continuous map $\kappa \circ |g|\colon |Y| \to X$. The naturality properties are also obvious as soon as we state them. But let us do this at a more abstract level.

DEFINITION 4.5.3. Suppose given categories $\mathcal{C}$, $\mathcal{D}$ and functors $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$. We say that $F$ is *left adjoint to* $G$ (or synonymously that $G$ is right adjoint to $F$) if there exists a bi-natural bijection
$$\mathrm{mor}_{\mathcal{D}}(F(c), d) \longrightarrow \mathrm{mor}_{\mathcal{C}}(c, G(d))$$
for objects $c$ in $\mathcal{C}$ and $d$ in $\mathcal{D}$. (The left-hand side is a functor from $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ to $\mathcal{S}$ets and the right-hand side is also a functor from $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ to $\mathcal{S}$ets. We are asking for an invertible natural transformation between these two.)

Adjoints are unique. The following lemma makes this precise for right adjoints; there is an analogue for left adjoints (mutatis mutandis).

LEMMA 4.5.4. *Suppose that* $\mathsf{F}\colon \mathcal{C} \to \mathcal{D}$ *has two right adjoints,* $\mathsf{G}\colon \mathcal{D} \to \mathcal{C}$ *and* $\mathsf{G}'\colon \mathcal{D} \to \mathcal{C}$*. Then there exists a natural isomorphism from* $\mathsf{G}$ *to* $\mathsf{G}'$*.*

PROOF. We suppose that we have bi-natural bijections

$$\alpha\colon \mathrm{mor}_{\mathcal{D}}(\mathsf{F}(c), d) \longrightarrow \mathrm{mor}_{\mathcal{C}}(c, \mathsf{G}(d)), \qquad \beta\colon \mathrm{mor}_{\mathcal{D}}(\mathsf{F}(c), d) \longrightarrow \mathrm{mor}_{\mathcal{C}}(c, \mathsf{G}'(d)).$$

Fix an object $d$ in $\mathcal{D}$. Put $c := \mathsf{G}(d)$. Then $\mathrm{id}_c \in \mathrm{mor}_{\mathcal{C}}(c, \mathsf{G}(d))$ and so

$$\beta\alpha^{-1}(\mathrm{id}_c) \in \mathrm{mor}_{\mathcal{C}}(c, \mathsf{G}'(d)) = \mathrm{mor}_{\mathcal{C}}(\mathsf{G}(d), \mathsf{G}'(d)).$$

This means that we have selected a morphism $\mathsf{G}(d) \to \mathsf{G}'(d)$, for every $d$. The conditions on $\alpha$ and $\beta$ imply that this was a natural selection, i.e., a natural transformation $u$ from $\mathsf{G}$ to $\mathsf{G}'$. In the same way, we obtain a preferred natural transformation $v$ from $\mathsf{G}'$ to $\mathsf{G}$ (using $\alpha\beta^{-1}$ instead of $\beta\alpha^{-1}$). It is clear from the constructions that $uv = \mathrm{id}_{\mathsf{G}'}$ and $vu = \mathrm{id}_{\mathsf{G}}$. □

Therefore we can say that *geometric realization* (of semi-simplicial sets) is the (essentially unique) left adjoint to *singular simplicial set* (of topological spaces), and *singular simplicial set* is the right adjoint to *geometric realization.*

REMARK 4.5.5. Adjoint functors are unique up to natural isomorphism if they exist, but they don't always exist. How should we search for the left adjoint (for example) of a functor $\mathsf{G}\colon \mathcal{A} \to \mathcal{B}$ if we suspect that it exists? To generate some ideas, let us first suppose that a left adjoint $\mathsf{F}$ does exist. Take some object $b$ in $\mathcal{B}$. There is a distinguished morphism

$$u_b\colon b \longrightarrow \mathsf{G}(\mathsf{F}(b))$$

in $\mathcal{B}$. This is the morphism which corresponds to $\mathrm{id}_{\mathsf{F}(b)} \in \mathrm{mor}_{\mathcal{A}}(\mathsf{F}(b), \mathsf{F}(b))$ under the adjunction $\mathrm{mor}_{\mathcal{A}}(\mathsf{F}(b), \mathsf{F}(b)) \leftrightarrow \mathrm{mor}_{\mathcal{B}}(b, \mathsf{G}(\mathsf{F}(b)))$. The morphism $u_b$ is called the *unit* morphism of the adjunction (for the object $b$). It has the following universal property. *Given any object* $a$ *in* $\mathcal{A}$ *and a morphism* $v\colon b \to \mathsf{G}(a)$ *in* $\mathcal{B}$*, there exists a unique morphism* $w\colon \mathsf{F}(b) \to a$ *such that* $\mathsf{G}(w) \circ u_b = v$. In fact $w$ corresponds to $v$ under the adjunction

$$\mathrm{mor}_{\mathcal{A}}(\mathsf{F}(b), a) \leftrightarrow \mathrm{mor}_{\mathcal{B}}(b, \mathsf{G}(a)).$$

What is being said here is that the adjunction can be written in the form $w \mapsto \mathsf{G}(w) \circ u_b$. (Reader, prove it.)

Therefore, if for an object $b$ in $\mathcal{B}$, we can find an object $a_b$ of $\mathcal{A}$ and a morphism $u\colon b \to \mathsf{G}(a_b)$ which has this universal property (... *for any object* $a$ *in* $\mathcal{A}$ *and morphism* $v\colon b \to \mathsf{G}(a)$ *in* $\mathcal{B}$*, there exists a unique morphism* $w\colon a_b \to a$ *such that* $\mathsf{G}(w) \circ u = v$), then $a_b$ is an excellent candidate for $\mathsf{F}(b)$. (A more precise statement can be made: up to isomorphism it is the only possible candidate.)

Here is a standard illustration of this principle. Let $\mathsf{G}$ be the forgetful functor from the category of abelian groups to the category of sets. For a set $S$, let $\mathbb{Z}[S]$ be the free abelian group with generating set $S$. (Alternative description: $\bigoplus_{s \in S} \mathbb{Z}$.) Then there is an obvious inclusion map $S \to \mathbb{Z}[S]$ (of sets), and this has the well-known universal property: every map from $S$ to an abelian group $A$ has a unique extension to a homomorphism from $\mathbb{Z}[S]$ to $A$. Therefore if $\mathsf{G}$ has a left adjoint $\mathsf{F}$, then $\mathbb{Z}[S]$ is our candidate for $\mathsf{F}(S)$, and the inclusion $S \to \mathbb{Z}[S]$ is our candidate for the unit morphism of the adjunction. And this works ...

This illustration suggests that left adjoints have something to do with "free" constructions. Indeed this is not bad as a guiding principle if the functor $\mathsf{G}$ whose left adjoint we are trying to find looks enough like a forgetful functor. As a student, long ago, I participated in a seminar where, on one occasion, the professor made the following observation. *You can try to describe free constructions in the style of Arthur Schopenhauer, by asking for the necessary generators and the absolute minimum imaginable of relations ... but as an alternative you can describe them using the language of categories, specifically the concept of adjunction.* That was roughly what he said. I think he was not trying to say that one description is worse or better than the other. I am sure he was trying to be funny, but there was a lot of wisdom in his words.

CHAPTER 5

# Chain complexes

## 5.1. The category of chain complexes

DEFINITION 5.1.1. A *chain complex* is a collection of abelian groups $C_r$ indexed by the integers $r \in \mathbb{Z}$, together with homomorphisms

$$d_r \colon C_r \to C_{r-1}$$

which satisfy the condition

$$d_r \circ d_{r+1} = 0$$

for all $r \in \mathbb{Z}$. The homomorphisms $d_r$ taken together are often called *the differential*, and denoted simply by $d$. (It is customary to use the letter $d$ for the differential in any chain complex whatsoever.)

$$\cdots \longleftarrow C_{-2} \longleftarrow C_{-1} \longleftarrow C_0 \longleftarrow C_1 \longleftarrow C_2 \longleftarrow C_3 \longleftarrow \cdots$$

We may write $C$ for the entire chain complex $((C_r)_{r\in\mathbb{Z}}, (d_r)_{r\in\mathbb{Z}})$.

DEFINITION 5.1.2. A *chain map* from a chain complex $B$ to a chain complex $C$ is a collection of homomorphisms

$$f_r \colon B_r \to C_r$$

which satisfy the conditions $d_r f_r = f_{r-1} d_r$ , so that the following diagram is commutative:

$$\begin{tikzcd}
\cdots & B_{-2} \arrow[l] \arrow[d, "f_{-2}"] & B_{-1} \arrow[l] \arrow[d, "f_{-1}"] & B_0 \arrow[l] \arrow[d, "f_0"] & B_1 \arrow[l] \arrow[d, "f_1"] & B_2 \arrow[l] \arrow[d, "f_2"] & B_3 \arrow[l] \arrow[d, "f_3"] & \cdots \arrow[l] \\
\cdots & C_{-2} \arrow[l] & C_{-1} \arrow[l] & C_0 \arrow[l] & C_1 \arrow[l] & C_2 \arrow[l] & C_3 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

We may write $f \colon B \to C$ for the chain map, instead of $(f_r \colon B_r \to C_r)_{r\in\mathbb{Z}}$.

It is clear from these definitions that chain complexes are the objects of a category $\mathcal{C}$. A morphism from chain complex $C$ to chain complex $D$ is, by definition, a chain map from $C$ to $D$. Composition is obvious.
This category $\mathcal{C}$ has some additional structure. The set of morphisms from $C$ to $D$ comes equipped with the structure of an abelian group (i.e., chain maps from $C$ to $D$ can be added, subtracted, etc.) Composition of chain maps is bilinear (more precisely, bi-homomorphic).

EXAMPLE 5.1.3. This is our most important example of a chain complex. A semi-simplicial set $Y$ determines a chain complex $C(Y)$ in the following way. For $r \geq 0$ let $C(Y)_r$ be the free abelian group generated by the set $Y_r$. (In other words, $C(Y)_r$ is the set of formal linear combinations, with integer coefficients, of elements in $Y_r$.) For $r < 0$ let $C(Y)_r := 0$. The differential

$$d \colon C(Y)_r \to C(Y)_{r-1}$$

is defined for $r > 0$ by the formula

$$d(z) = \sum_{i=0}^{r} (-1)^i f_i^*(z)$$

where $z$ is an element of $Y_r$ , viewed as one of the generators of $C(Y)_r$ , and $f_i\colon [r{-}1] \to [r]$ is the unique monotone injective map whose image is $[r] \smallsetminus \{i\}$. (Exercise: show that $dd$ from $C(Y)_r$ to $C(Y)_{r-2}$ is the zero homomorphism.)

EXAMPLE 5.1.4. A morphism $f\colon X \to Y$ of semi-simplicial sets determines a chain map $C(X) \to C(Y)$ in the obvious way, that is, the chain map takes $x \in X_r$ , viewed as one of the generators of $C(X)_r$ , to $f(x) \in Y_r$ , viewed as one of the generators of $C(Y)_r$. (Exercise: show that this is indeed a chain map.) Therefore $Y \mapsto C(Y)$ is a *functor* from the category of semi-simplicial sets to the category of chain complexes.

## 5.2. Chain homotopies and homology groups

The category of chain complexes has some features in common with the category $\mathcal{T}op$ of topological spaces. In particular there is a concept of *chain homotopy*, analogous to the concept of homotopy between continuous maps.

DEFINITION 5.2.1. Let $B$ and $C$ be chain complexes. Let $f$ and $g$ be chain maps from $C$ to $D$. A *chain homotopy* $h$ from $f$ to $g$ is a collection of group homomorphisms

$$h_r\colon B_r \longrightarrow C_{r+1}$$

such that $d_{r+1}h_r + h_{r-1}d_r = g_r - f_r$ for all $r \in \mathbb{Z}$. If such a chain homotopy exists, then we say that $f$ and $g$ are *chain homotopic*. Notation: $f \simeq g$.

Let's make a few remarks on this relation, *chain homotopic*.

- It is a transitive relation: if $e, f, g$ are chain maps from $B$ to $C$, and $e$ is homotopic to $f$, and $f$ is homotopic to $g$, then $e$ is homotopic to $g$. Proof: let $h$ be a homotopy from $e$ to $f$ and let $k$ be a homotopy from $f$ to $g$. Then $h{+}k$ is a homotopy from $e$ to $g$.
- It is reflexive (obvious) and symmetric (obvious).
- It is a congruence relation, i.e., if $f \simeq g$ and $u \simeq v$, then $f{+}u \simeq g{+}v$, assuming that $f, g, u, v$ are chain maps from $B$ to $C$.
- If $h$ is a homotopy from $f\colon B \to C$ to $g\colon B \to C$ and $u\colon C \to D$ is a chain map, then $uh$ is meaningful and it is a homotopy from $uf$ to $ug$.
- If $h$ is a homotopy from $f\colon B \to C$ to $g\colon B \to C$ and $v\colon A \to B$ is a chain map, then $hv$ is meaningful and it is a homotopy from $fv$ to $gv$.

It follows in the usual manner that we can make a category $\mathcal{HC}$ where the objects are the chain complexes, as before, but the morphisms are chain homotopy classes of chain maps. We write $[C, D]$ for the abelian group of chain homotopy classes of chain maps from chain complex $C$ to chain complex $D$. We write $[f] \in [C, D]$ for the chain homotopy class of a chain map $f\colon C \to D$. Composition is (well) defined by

$$[D, E] \times [C, D] \longrightarrow [C, E]\ ;\ ([g], [f]) \mapsto [gf]$$

and it is bi-homomorphic. We say that $f\colon C \to D$ is a *chain homotopy equivalence* if $[f] \in [C, D]$ is invertible, that is, if there exists a chain map $g\colon D \to C$ such that $gf$ is chain homotopic to $\mathrm{id}_C$ and $fg$ is chain homotopic to $\mathrm{id}_D$. It is acceptable to say *homotopic* etc. instead of *chain homotopic* (if confusion is unlikely).

DEFINITION 5.2.2. Let $C$ be a chain complex. The subgroup $\ker(d_n \colon C_n \to C_{n-1})$ of $C_n$ is often called the group of $n$*-cycles*. The subgroup $\operatorname{im}(d_{n+1} \colon C_{n+1} \to C_n)$ is often called the group of $n$*-boundaries*. The group of $n$-boundaries is contained in the group of $n$-cycles since $dd = 0$. The $n$-th homology group of a chain complex $C$ is the abelian group

$$H_n(C) := \frac{\ker(d_n \colon C_n \to C_{n-1})}{\operatorname{im}(d_{n+1} \colon C_{n+1} \to C_n)}$$

called the $n$-th *homology group* of $C$. The element of $H_n(C)$ represented by an $n$-cycle $x$ can be called the *homology class of* $x$.

The $n$-th homology group $H_n(C)$ can also be defined as follows. We introduce a special chain complex denoted $(\mathbb{Z}, n)$; this has $(\mathbb{Z}, n)_n = \mathbb{Z}$ and $(\mathbb{Z}, n)_k = 0$ for $k \in \mathbb{Z}$, $k \neq n$. The differentials in $(\mathbb{Z}, n)$ are necessarily all zero. Now observe that a chain map $f$ from $(\mathbb{Z}, n)$ to $C$ is determined by $f(1) \in C_n$, which must be an $n$-cycle. Similarly a chain homotopy $h$ from $f \colon (\mathbb{Z}, n) \to C$ to $g \colon (\mathbb{Z}, n) \to C$ is determined by $h(1)$, and this must satisfy

$$d(h(1)) = g(1) - f(1)\,.$$

Therefore the group of chain homotopy classes of chain maps from $(\mathbb{Z}, n)$ to $C$ is identified with the quotient of subgroup of $n$-cycles by subgroup of $n$-boundaries. In a formula,

$$H_n(C) \cong [\,(\mathbb{Z}, n), C\,]\,.$$

As a corollary of this observation we obtain:

PROPOSITION 5.2.3. *A chain map* $f \colon C \to D$ *induces a homomorphism* $H_n(C) \to H_n(D)$ *by the rule*

$$\textit{homology class of } x \mapsto \textit{homology class of } f(x)$$

*for* $n$*-cycles* $x \in C_n$. *Therefore* $H_n$ *is a functor from the category of chain complexes to the category of abelian groups. If* $f$ *and* $g$ *are homotopic chain maps, then they induce the same homomorphism* $H_n(C) \to H_n(D)$.

COROLLARY 5.2.4. *If* $f \colon C \to D$ *is a chain homotopy equivalence, then the homomorphisms* $H_n(C) \to H_n(D)$ *induced by* $f$ *are isomorphisms, for all* $n \in \mathbb{Z}$.

EXAMPLE 5.2.5. Fix an integer $n \geq 0$. Let $f \colon [0] \to [n]$ be the monotone injection taking $0$ to $0$. Let

$$X := \underline{\Delta}^0, \qquad Y := \underline{\Delta}^n$$

(semi-simplicial sets defined in example 4.2.4). There is a unique semi-simplicial map $X \to Y$ taking the unique element in $X_0$ to the element $f$ in $Y_0 = \operatorname{mor}_\Delta([0], [n])$. This induces a chain map $C(X) \to C(Y)$. That chain map is a chain homotopy equivalence. (Exercise.) It follows that $H_j(C(Y)) = 0$ for $j \neq 0$ and $H_0(C(Y)) \cong \mathbb{Z}$.

## 5.3. Long exact sequence of homology groups

A diagram of abelian groups and homomorphisms $A \to B \to C$ is *exact* if the image of the first arrow agrees with the kernel of the second. Similarly, a chain complex $B$ is *exact* if the image of $d_{n+1} \colon B_{n+1} \to B_n$ agrees with the kernel of $d_n \colon B_n \to B_{n-1}$, for all $n$. (The image of $d_{n+1} \colon B_{n+1} \to B_n$ is automatically *contained* in the kernel of $d_n \colon B_n \to B_{n-1}$.) The expression *long exact sequence* is also used for *exact chain complex*.

A diagram of abelian groups and homomorphisms $A \to B \to C$ is *short exact* if it is exact and, moreover, $A \to B$ is injective and $B \to C$ is surjective. (Another way to say that:

$A \to B \to C$ is short exact if and only if, in the diagram $0 \to A \to B \to C \to 0$, each pair of consecutive arrows is exact.)

Similarly, a diagram of chain complexes and chain maps $A \to B \to C$ is *exact* if the diagram of abelian groups $A_n \to B_n \to C_n$ is exact for every $n$. And the diagram $A \to B \to C$ of chain complexes and chain maps is called *short exact* if $A_n \to B_n \to C_n$ is short exact for every $n$. (Here we are not assuming that the chain complexes $A$, $B$ and $C$ are exact in their own right.)

Up to isomorphism, if we see a short exact sequence of chain complexes $A \to B \to C$, then we can always pretend that $A$ is a chain subcomplex of $B$ and $C$ is the quotient chain complex $B/A$. (I did not define chain subcomplex, but the meaning should be clear: we have $A_n \subset B_n$ as an abelian subgroup, for each $n$, and each $d_n \colon B_n \to B_{n-1}$ satisfies $d_n(A_n) \subset A_{n-1}$.)

PROPOSITION 5.3.1. *Let* $A \xrightarrow{j} B \xrightarrow{p} C$ *be a short exact sequence of chain complexes. For each* $n \in \mathbb{Z}$ *there is a homomorphism* $\partial \colon H_n(C) \longrightarrow H_{n-1}(A)$*, well defined by*

$$\partial[z] := [d(z^!)],$$

*where* $z \in C_n$ *denotes a cycle and* $z^! \in B_n$ *satisfies* $p(z^!) = z$*.*

PROOF. We can assume that $A$ is a chain subcomplex of $B$ and $C = B/A$. First we verify that the formula for $\partial$ makes sense. If $z$ is a cycle in $B_n/A_n$ , then we can certainly choose $z^!$ in $B_n$ mapping to $z$ under the projection $p_n \colon B_n \to B_n/A_n$. We are assuming that $d(z) = 0$, but it does not follow that $d(z^!) = 0$. It does follow that

$$d(z^!) \in A_{n-1} \subset B_{n-1}$$

and clearly $dd(z^!) = 0$ in $A_{n-2} \subset B_{n-2}$, because $dd$ is zero as a homomorphism from $B_n$ to $B_{n-2}$. Therefore $d(z^!)$ is an $(n-1)$-cycle for the chain complex $A$, and we may form its homology class.

That was a long explanation. Now we need to verify that the definition of $\partial$ is unambiguous. There are two choices that we made, but we can wrap them up as one: we started with an element of $H_n(B/A)$ and we chose $z^! \in B_n$ such that $p(z^!) \in B_n/A_n$ is an $n$-cycle in $B/A$ representing that element of $H_n(B/A)$. So suppose that $z^{!!}$ is another element of $B_n$ such that $p(z^{!!})$ is an $n$-cycle in $B/A$ which represents the same element of $H_n(B/A)$. Then

$$p(z^! - z^{!!}) = p(z^!) - p(z^{!!}) = d(p(y)) = p(d(y)) \quad \text{in } B/A$$

for some $y \in B_{n+1}$. Therefore $p(z^! - z^{!!} - d(y)) = 0$ in $B/A$ and therefore $z^! - z^{!!} - d(y)$ belongs to $A_n$ (calculation in $B$) and so

$$d(z^!) - d(z^{!!}) = d(z^!) - d(z^{!!}) - dd(y) = d(z^! - z^{!!} - dy)$$

is an $(n-1)$-boundary in $A$, so that it represents zero in $H_{n-1}(A)$. This means that $\partial$ is well defined. Finally, it is clear from the definition that $\partial$ is a homomorphism. □

THEOREM 5.3.2. *In the situation of proposition* 5.3.1*, the diagram*

$$\cdots \longrightarrow H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{j_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{j_*} H_{n-1}(B) \longrightarrow \cdots$$

(*where* $n \in \mathbb{Z}$) *is exact.*

PROOF. To show: (i) $j_*\partial = 0$, (ii) $p_*j_* = 0$, (iii) $\partial p_* = 0$, (iv) $\operatorname{im}(j_*) \supset \ker(p_*)$, (v) $\operatorname{im}(\partial) \supset \ker(j_*)$, (vi) $\operatorname{im}(p_*) \supset \ker(\partial)$. We can assume $A \subset B$ and $C = B/A$ as before.

(i) For an $(n+1)$-cycle $z$ in $B/A$ choose $z^! \in B_{n+1}$ which represents $z$. Then $\partial[z] = [d(z^!)]$. But $d(z^!)$ is a boundary in $B_n$ , so that $j(d(z^!))$ represents zero in $H_n(B)$.

(ii) Obvious.

(iii) If $y$ is an $n$-cycle in $B$ and $z = p(y)$ in $B/A$, then we can take $z^! = y$ in the definition of $\partial[z]$ and we get $d(z^!) = 0$, therefore $\partial[z] = \partial[p(y)] = 0$.

(iv) Suppose that $y$ is an $n$-cycle in $B$ and $p(y)$ is an $n$-boundary in $B/A$. Choose $x \in B_{n+1}$ such that $d(p(x)) = p(y)$. Then $d(x) - y \in A_n$ and $d(d(x) - y) = dd(x) - d(y) = -d(y) = 0$. Therefore $d(x) - y$ represents an element of $H_n(A)$ and the image of that in $H_n(B)$ agrees with $[y]$.

(v) Suppose that $y$ is an $n$-cycle in $A$ which becomes an $n$-boundary in $B$. Choose $x \in B_{n+1}$ such that $d(x) = y$ in $B_n$. Then $p(x) \in B_{n+1}/A_{n+1}$ is a cycle and we calculate $\partial[p(x)] = [d(x)] = [y]$ in $H_n(A)$.

(vi) Suppose that $z$ is an $n$-cycle in $B/A$ and $\partial[z] = 0$. We can write $z = p(z^!)$ where $z^! \in B_n$. Then $d(z^!)$ is a boundary in $A$ by assumption. Choose $x \in A_n$ such that $d(x) = d(z^!)$ in $A$. Then $x - z^!$ is an $n$-cycle in $B$. Clearly $p_*[x - z^!] = [p(x) - p(z^!)] = [z]$. □

COROLLARY 5.3.3. *Let* $E$ *be a chain complex with chain subcomplexes* $K$ *and* $L$ *such that* $K + L = E$*; we are not assuming* $K \cap L = 0$. *Then there is an exact sequence of homology groups*

$$\cdots \longrightarrow H_{n+1}(E) \longrightarrow H_n(K \cap L) \longrightarrow H_n(K) \oplus H_n(L) \longrightarrow H_n(E) \longrightarrow H_{n-1}(K \cap L) \longrightarrow \cdots$$

PROOF. Let us write $j_K \colon K \to E$, $j_L \colon L \to E$, $g_K \colon K \cap L \to K$, $g_L \colon K \cap L \to L$ for the various inclusions. Then there is a short exact sequence of chain complexes

$$K \cap L \xrightarrow{(g_K, -g_L)} K \oplus L \xrightarrow{j_K \oplus j_L} E$$

so that we are in the situation of theorem 5.3.2. — This reasoning adds more precision to the statement. Namely, the homomorphism $H_n(K \cap L) \to H_n(K) \oplus H_n(L)$ in the exact sequence is $((g_K)_*, -(g_L)_*)$. The homomorphism $H_n(K) \oplus H_n(L) \to H_n(E)$ in the exact sequence is $(j_K)_* \oplus (j_L)_*$. □

The long exact sequence of corollary 5.3.3 is associated with the names Mayer and Vietoris; *Mayer-Vietoris sequence.*

## 5.4. Euler characteristic

The *rank* of a finitely generated abelian group $A$ is the dimension of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ as a vector space over the field $\mathbb{Q}$. To put it differently, the classification theorem for finitely generated abelian groups says that $A \cong \mathbb{Z}^k \oplus B$ where $B$ is a finite abelian group; the integer $k$ is the rank of $A$. Notation: $\mathrm{rk}(A)$.

DEFINITION 5.4.1. Let $C$ be a chain complex. Suppose that the homology groups $H_n(C)$ are all finitely generated, and only finitely many of them are nonzero. Then the *Euler characteristic* of $C$ is defined; it is the integer

$$\chi(C) := \sum_{n \in \mathbb{Z}} (-1)^n \mathrm{rk}(H_n(C)) .$$

Proposition 5.4.2. *Let* $\mathsf{C}$ *be a chain complex; suppose that the groups* $\mathsf{C}_n$ *are all finitely generated, and only finitely many of them are nonzero. Then the Euler characteristic* $\chi(\mathsf{C})$ *is defined and it is equal to*

$$\chi(\mathsf{C}) = \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{rk}(\mathsf{C}_n)\,.$$

Proof. It is clear that the homology groups of $\mathsf{C}$ are finitely generated, and that only finitely many of them can be nonzero. Therefore $\chi(\mathsf{C})$ is defined. Next, each abelian group $\mathsf{C}_n$ has two distinguished subgroups: the group of cycles $\mathsf{Z}_n = \ker[\, \mathsf{d}\colon \mathsf{C}_n \to \mathsf{C}_{n-1}\,]$ and the group of boundaries $\mathsf{B}_n = \mathrm{im}[\, \mathsf{d}\colon \mathsf{C}_{n+1} \to \mathsf{C}_n\,]$. Therefore

$$\begin{aligned}\mathrm{rk}(\mathsf{C}_n) &= \mathrm{rk}(\mathsf{C}_n/\mathsf{Z}_n) + \mathrm{rk}(\mathsf{Z}_n/\mathsf{B}_n) + \mathrm{rk}(\mathsf{B}_n) \\ &= \mathrm{rk}(\mathsf{B}_{n-1}) + \mathrm{rk}(\mathsf{H}_n(\mathsf{C})) + \mathrm{rk}(\mathsf{B}_n)\end{aligned}$$

(where we have used $\mathsf{C}_n/\mathsf{Z}_n \cong \mathsf{B}_{n-1}$, special case of the Nöther isomorphism theorem). Substituting $\mathrm{rk}(\mathsf{B}_{n-1}) + \mathrm{rk}(\mathsf{H}_n(\mathsf{C})) + \mathrm{rk}(\mathsf{B}_n)$ for $\mathrm{rk}(\mathsf{C}_n)$ in the expression

$$\sum_{n\in\mathbb{Z}} (-1)^n \mathrm{rk}(\mathsf{C}_n)$$

and making the obvious cancellations, we obtain $\chi(\mathsf{C})$. $\square$

CHAPTER 6

# The singular chain complex of a space

## 6.1. A functor from spaces to chain complexes

A topological space $X$ determines a semi-simplicial set $\mathrm{sing}(X)$ and a semi-simplicial set $Y$ determines a chain complex $C(Y)$. More precisely, there is a functor sing which to a space $X$ associates $\mathrm{sing}(X)$ and to a continuous map $g\colon X \to X'$ a semi-simplicial map $\mathrm{sing}(X) \to \mathrm{sing}(Y)$. And there is a functor which to a semi-simplicial set $Y$ associates the chain complex $C(Y)$ and to a semi-simplicial map $Y \to Y'$ a chain map from $C(Y)$ to $C(Y')$. These are two functors which we have already seen. Now we compose them:

$$X \mapsto \mathrm{sing}(X) \mapsto C(\mathrm{sing}(X)).$$

Instead of $C(\mathrm{sing}(X))$ I may also write $sC(X)$. (This is not exactly standard notation. Other people write $C$ or $S$ or $S_*$ instead of $sC$.) The chain complex $sC(X)$ is called the *singular chain complex* of $X$. Therefore *singular chain complex* is a functor $sC$ from $\mathcal{T}op$ to the category of chain complexes (and chain maps).
The singular chain complex $sC(X)$ is typically gigantic. Let us write out the definition once again: $sC(X)_n$ is the free abelian group generated by the set of all continuous maps from the standard simplex $\Delta^n$ to $X$. The differential $sC(X)_n \to sC(X)_{n-1}$ is defined, on the generators corresponding to continuous maps $\sigma\colon \Delta^n \to X$, by

$$d(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ \varphi_i$$

where $\varphi_i\colon \Delta^{n-1} \to \Delta^n$ is the continuous map given, in barycentric coordinates, by inserting a $0$ in position $i$. (For example, if $n = 5$ and $i = 2$, we get $\varphi_2(x_0, x_1, x_2, x_3, x_4) = (x_0, x_1, 0, x_2, x_3, x_4) \in \Delta^5$.) In the notation of section 4.4, the map $\varphi_i$ could also be described as $(f_i)_*$ where $f_i\colon [n-1] \to [n]$ is the unique monotone injection whose image does not contain $i$.

EXAMPLE 6.1.1. Suppose that $X$ is a one-point space. This is one of the few cases where we can get a good idea of $sC(X)$ by inspection. Namely, $sC(X)_n$ for $n \geq 0$ is an infinite cyclic group, freely generated by the unique continuous map from $\Delta^n$ to $X$.

$$\cdots \longleftarrow 0 \longleftarrow 0 \longleftarrow \mathbb{Z} \xleftarrow{d_1} \mathbb{Z} \xleftarrow{d_2} \mathbb{Z} \xleftarrow{d_3} \mathbb{Z} \longleftarrow \cdots$$

The differential $d_n$ is multiplication with the integer $\sum_{i=0}^{n} (-1)^i$, in other words with $0$ if $n$ is odd and with $1$ if $n$ is even $(n > 0)$. Therefore the homology groups of $sC(X)$, for this $X$, are $H_0(sC(X)) = \mathbb{Z}$ and $H_j(sC(X)) = 0$ for all $j \neq 0$.

## 6.2. Homotopy invariance of the singular chain complex

THEOREM 6.2.1. *Let* $g_0\colon X \to X \times [0,1]$ *and* $g_1\colon X \to X \times [0,1]$ *be the two continuous maps given by* $g_0(x) = (x,0)$ *and* $g_1(x) = (x,1)$. *Then the chain maps*

$$sC(g_0)\colon sC(X) \to sC(X \times [0,1]), \qquad sC(g_1)\colon sC(X) \to sC(X \times [0,1])$$

*are chain homotopic.*

PROOF. For integers $n \ge 0$ and $k \in \{0,1,2,\dots,n\}$ let $q_k\colon \Delta^{n+1} \to \Delta^n \times [0,1]$ be defined as follows. On the vertices $e_i$ of $\Delta^{n+1}$ (where $0 \le i \le n+1$) we want to have $q_k(e_i) = (e_i, 0)$ if $i \le k$ and $q_k(e_i) = (e_{i-1}, 1)$ if $i > k$. Extend this linearly. In barycentric coordinates this means

$$q_k(x_0, x_1, \dots, x_n, x_{n+1}) = \Big((x_0, x_1, \dots, x_{k-1}, x_k + x_{k+1}, x_{k+2}, \dots, x_{n+1}),\ \sum_{i=k+1}^{n+1} x_i\Big).$$

*Warning*: we write $q_k$ although the thing depends on $n$ and $k$; the correct $n$ must be guessed from the context (if possible). — The maps $q_k$ satisfy the equations

$$q_0\varphi_0 = (\mathrm{id}, 1),\ q_0\varphi_1 = q_1\varphi_1,\ q_1\varphi_2 = q_2\varphi_2,\ \dots,\ q_{n-1}\varphi_n = q_n\varphi_n,\ q_n\varphi_{n+1} = (\mathrm{id}, 0)$$

and also (for $n > 0$) clearly

$$q_k\varphi_i = (\varphi_{i-1} \times [0,1]) \circ q_k$$

if $i > k+1$, as well as

$$q_k\varphi_i = (\varphi_i \times [0,1]) \circ q_{k-1}$$

if $i < k$. (Here the warning applies. The $q_k$ in the left-hand side is a map from $\Delta^{n+1}$ to $\Delta^n \times [0,1]$ and the $q_k$ or $q_{k-1}$ in the right-hand side is a map from $\Delta^n$ to $\Delta^{n-1} \times [0,1]$. Similarly the $\varphi_i$ in the left-hand side is a map from $\Delta^n$ to $\Delta^{n+1}$ and the $\varphi_i$ or $\varphi_{i-1}$ in the right-hand side is a map from $\Delta^{n-1}$ to $\Delta^n$.) A chain homotopy $h = (h_n)_{n\in\mathbb{Z}}$ from $sC(g_0)$ to $sC(g_1)$ can be defined by

$$h_n\colon sC(X)_n \to sC(X \times [0,1])_{n+1}\ ;\ \sigma \mapsto \sum_{k=0}^{n} (-1)^k (\sigma \times [0,1]) \circ q_k$$

where $\sigma\colon \Delta^n \to X$ is a continuous map. Indeed,

$$d_{n+1}h_n(\sigma) = \sum_{k=0}^{n}\sum_{i=0}^{n+1} (-1)^{k+i} (\sigma \times [0,1]) \circ q_k\varphi_i\,.$$

If $n > 0$, the terms where $i = k$ and $i = k+1$, except $i = k = 0$ and $i = k+1 = n+1$, cancel out. The terms where $i \notin \{k, k+1\}$ can be rewritten using $q_k\varphi_i = (\varphi_{i-1} \times [0,1]) \circ q_k$ if $i > k+1$ and $q_k\varphi_i = (\varphi_i \times [0,1]) \circ q_{k-1}$ if $i < k$. Their total contribution is therefore equal to $-h_{n-1}d_n(\sigma)$. We must still account for the terms corresponding to indices $i, k$ where $i = k = 0$ and $i = k+1 = n+1$. They are equal to

$$(-1)^0(\sigma \times [0,1]) \circ q_0\varphi_0 = (\sigma \times [0,1]) \circ (\mathrm{id}, 1) = (g_1)_*(\sigma)$$

and

$$(-1)^{n+1+n}(\sigma \times [0,1]) \circ q_n\varphi_{n+1} = -(\sigma \times [0,1]) \circ (\mathrm{id}, 0) = -(g_0)_*(\sigma),$$

respectively. This proves $d_{n+1}h_n(\sigma) + h_{n-1}d_n(\sigma) = (g_1)_*(\sigma) - (g_0)_*(\sigma)$ provided $n > 0$. Finally, in the important case $n = 0$ the map $\sigma$ is a map from $\Delta^0$ to $X$. Then it is easy to verify directly that $d_{n+1}h_n(\sigma)$ is $(g_1)_*(\sigma) - (g_0)_*(\sigma)$, since the double sum has exactly these two terms. ∎

COROLLARY 6.2.2. *Let* $f_0, f_1 : X \to Y$ *be maps of spaces. If they are homotopic, then* $(f_0)_*$ *and* $(f_1)_*$ *are chain homotopic chain maps from* $sC(X)$ *to* $sC(Y)$.

PROOF. Choose a homotopy $h : X \times [0,1] \to Y$ from $f_0$ to $f_1$. Then $f_0 = hg_0$ and $f_1 = hg_1$ where $g_0, g_1 : X \to X \times [0,1]$ are as in the theorem. Therefore $(f_0)_* = h_*(g_0)_*$ is chain homotopic to $(f_1)_* = h_*(g_1)_*$, since already $(g_0)_*$ is chain homotopic to $(g_1)_*$. □

COROLLARY 6.2.3. *If* $f : X \to Y$ *is a homotopy equivalence, then* $f_* : sC(X) \to sC(Y)$ *is a chain homotopy equivalence.* □

COROLLARY 6.2.4. *If* $X$ *is contractible, then* $H_n(sC(X)) \cong \mathbb{Z}$ *for* $n = 0$ *and* $H_n(sC(X)) = 0$ *for* $n \neq 0$.

PROOF. See example 6.1.1. □

## 6.3. Barycentric subdivision

For every permutation $\lambda$ of $[n] = \{0, 1, \ldots, n\}$ there is a map $u_\lambda : \Delta^n \to \Delta^n$ defined as follows. We want the map to be linear,

$$u_\lambda(x_0, x_1, \ldots, x_n) = \sum_{i=0}^{n} x_i u_\lambda(e_i)$$

and for the vertices $e_i \in \Delta^n$ we let

$$u_\lambda(e_i) = \text{average of the } e_{\lambda(j)} \text{ for } j \leq i.$$

The right-hand side of this formula is the *barycenter* of the face of $\Delta^n$ spanned by the vertices $e_{\lambda(j)}$ for $j \leq i$. (That face is the subset or subspace of $\Delta^n$ consisting of all $(y_0, y_1, \ldots, y_n)$ such that $y_{\lambda(j)} = 0$ if $j > i$.) So $u_\lambda(e_0)$ is a vertex of $\Delta^n$, $u_\lambda(e_1)$ is the midpoint of an edge spanned by that vertex and another, $u_\lambda(e_2)$ is the midpoint of a triangle spanned by these two vertices and a third, and so on. An explicit formula for $u_\lambda$ is therefore

$$u_\lambda(x_0, x_1, \ldots, x_n) = \left( \sum_{i \geq \lambda^{-1}(0)} \frac{x_i}{i+1}, \sum_{i \geq \lambda^{-1}(1)} \frac{x_i}{i+1}, \ldots, \sum_{i \geq \lambda^{-1}(n)} \frac{x_i}{i+1} \right).$$

But it is better to forget that immediately.

PROPOSITION 6.3.1. *Let* $X$ *be a topological space. The formula*

$$\sigma \;\mapsto \sum_{\lambda \text{ perm. of } [n]} \operatorname{sgn}(\lambda) \cdot \sigma \circ u_\lambda$$

*(for* $n \geq 0$ *and continuous* $\sigma : \Delta^n \to X$*, viewed as a generator of* $sC(X)_n$*) defines a natural chain map* $\beta : sC(X) \to sC(X)$.

PROOF. For a permutation $\lambda$ of $[n]$ let $\lambda_?$ be the permutation of $[n-1]$ obtained from $\lambda$ by deleting $n$ in the source and $\lambda(n)$ in the target, and renumbering to close the gaps. Then we have

$$u_\lambda \varphi_n = \varphi_{\lambda(n)} u_{\lambda_?} : \Delta^{n-1} \longrightarrow \Delta^n.$$

For $i$ such that $0 \leq i < n$ we have

$$u_{\lambda\tau} \varphi_i = u_\lambda \varphi_i$$

where $\tau$ is the transposition interchanging $i$ and $i+1$. Therefore if $\sigma\colon \Delta^n \to X$ is any continuous map, then we have, very briefly,

$$\begin{aligned}
d\beta(\sigma) &= (-1)^n \sum_{\lambda \text{ per. of } [n]} \mathrm{sgn}(\lambda)\cdot \sigma u_\lambda \varphi_n + \sum_{i\in[n-1]} \Big( (-1)^i \sum_{\lambda \text{ per. of } [n]} \mathrm{sgn}(\lambda)\cdot \sigma u_\lambda \varphi_i \Big) \\
&= (-1)^n \sum_{\lambda \text{ per. of } [n]} \mathrm{sgn}(\lambda)\cdot \sigma u_\lambda \varphi_n \quad + \quad 0 \\
&= (-1)^n \sum_{\lambda \text{ per. of } [n]} \mathrm{sgn}(\lambda)\cdot \sigma \varphi_{\lambda(n)} u_{\lambda_?} \\
&= \sum_{j\in[n]} (-1)^j \Big( \sum_{\rho \text{ per. of } [n-1]} \mathrm{sgn}(\rho)\cdot \sigma\varphi_j u_\rho \Big)
\end{aligned}$$

and that agrees with $\beta d(\sigma)$. The last equality sign was obtained by writing $\rho$ for $\lambda_?$ and $j$ for $\lambda(n)$. We have used $(-1)^n \mathrm{sgn}(\lambda) = (-1)^j \mathrm{sgn}(\rho)$. — The naturality of $\beta$ is obvious. □

### 6.4. The method of acyclic models

THEOREM 6.4.1. *Let* $\alpha\colon sC(X) \to sC(X)$ *be a natural chain map. If* $\alpha\colon sC(*)_0 \to sC(*)_0$ *is the zero homomorphism, then* $\alpha$ *admits a natural chain homotopy to zero.*

PROOF. We use the method of *acyclic models* (due to Eilenberg and MacLane). These words stand for two ideas.

(i) If we wish to construct a *natural* chain maps from $sC(X)$ to $sC(X)$, or natural chain homotopies between such, then the cases where $X$ is $\Delta^n$ for some $n$ deserve special attention. (The geometric simplices $\Delta^n$ are the *models*.)

(ii) If $X$ is $\Delta^n$, then it is contractible and so corollary 6.2.4 is applicable. (The models are *acyclic*. The word *acyclic* is often used informally in connection with chain complexes $C$ whose homology groups $H_j(C)$ are all zero, except perhaps for a specfic $j$ such as $j=0$.)

In the spirit of (i) we make the following observation. Let $\iota_m \in sC(\Delta^m)_m$ be the identity map of $\Delta^m$, viewed as one of the generators of the free abelian group $sC(\Delta^m)_m$. *A natural homomorphism* $g\colon sC(X)_m \to sC(X)_n$ *is determined by its value on* $\iota_m \in sC(\Delta^m)_m$, *which is an element of* $sC(\Delta^m)_n$*; and this can be selected arbitrarily.* Proof: if we know $g(\iota_m)$, then we know $g(\sigma)$ for every continuous $\sigma\colon \Delta^m \to X$ (viewed as a generator of $sC(X)_m$) since $g(\sigma) = g(\sigma_*(\iota_m)) = \sigma_*(g(\iota_m))$. The last equality sign uses the naturality of $g$. The formula $g(\sigma) = \sigma_*(g(\iota_m))$ can also be taken as a definition of $g$ in terms of the element $g(\iota_m)$.

$$\begin{tikzcd}
sC(\Delta^m)_m \arrow[r, "g"] \arrow[d, "\sigma_*"] & sC(\Delta^m)_n \arrow[d, "\sigma_*"] \\
sC(X)_m \arrow[r, "g"] & sC(X)_m
\end{tikzcd}$$

We apply this observation first in the case $m=n$ and $g=\alpha_m$. So let $a_m = \alpha_m(\iota_m) \in sC(\Delta^m)_m$. Then we know that $a_m$ determines $\alpha_m$. The fact that the $\alpha_m$ together form a chain map can be expressed by equations relating the $a_m$ for different $m$:

$$(\mathrm{I}_m) \qquad d(a_m) = \sum_{i=0}^{m} (-1)^i (\varphi_i)_*(a_{m-1})$$

where $\varphi_i\colon \Delta^{m-1} \to \Delta^m$ is the usual map which omits vertex $i$. Proof of this:

$$\begin{aligned} d(a_m) = d\alpha_m(\iota_m) = \alpha_{m-1}d(\iota_m) = \textstyle\sum_{i=0}^m (-1)^i \alpha_{m-1}\varphi_i \\ = \textstyle\sum_{i=0}^m (-1)^i (\varphi_i)_* \alpha_{m-1}(\iota_{m-1}) = \sum_{i=0}^m (-1)^i (\varphi_i)_*(a_{m-1})\,. \end{aligned}$$

Next we apply similar ideas to the natural homomorphisms $h_m\colon sC(X) \to sC(X)_{m+1}$ in a (still hypothetical) natural chain homotopy $h$ from $0$ to $\alpha$. Write $b_m \in sC(\Delta^m)_{m+1}$ for $h_m(\iota_m)$. Since we want

$$\alpha_m = dh_m + h_{m-1}d$$

we must have

$$\begin{aligned} a_m = \alpha_m(\iota_m) = (dh_m + h_{m-1}d)(\iota_m) = d(b_m) + \textstyle\sum_{i=0}^m (-1)^i h_{m-1}(\varphi_i) \\ = d(b_m) + \textstyle\sum_{i=0}^m (-1)^i (\varphi_i)_* h_{m-1}(\iota_{m-1}) = d(b_m) + \sum_{i=0}^m (-1)^i (\varphi_i)_*(b_{m-1}), \end{aligned}$$

therefore

$$(\mathrm{II}_m) \qquad d(b_m) = a_m \; - \sum_{i=0}^m (-1)^i (\varphi_i)_*(b_{m-1})$$

for all $m \geq 0$. Now we try to solve $(\mathrm{II}_m)$ by induction on $m$. For $m = 0$ we have $a_m = 0$ and $b_{m-1} = 0$ inevitably, so that the right-hand side of $(\mathrm{II}_0)$ is zero. Therefore we can choose $b_0 = 0$. Next, suppose that $(\mathrm{II}_{m-1})$ is satisfied, where $m > 0$. We apply the differential $d$ to the right-hand side of $(\mathrm{II}_m)$. This gives

$$\begin{aligned} d(a_m) \; - \textstyle\sum_{i=0}^m (-1)^i (\varphi_i)_*(d(b_{m-1})) = \sum_{i=0}^m (-1)^i (\varphi_i)_*\big(a_{m-1} - d(b_{m-1})\big) \\ = \textstyle\sum_{i=0}^m (-1)^i (\varphi_i)_*\big(\sum_{j=0}^{m-1} (-1)^j (\varphi_j)_*(b_{m-2})\big) \\ = \textstyle\sum_{i=0}^m \sum_{j=0}^{m-1} (-1)^{i+j} (\varphi_i\varphi_j)_*(b_{m-2}) = 0 \end{aligned}$$

where we have used equation $(\mathrm{I}_m)$, too. Therefore the right-hand side of $(\mathrm{II}_m)$ is an $m$-cycle in $sC(\Delta^m)$. By corollary 6.2.4, it follows that it is an $m$-boundary; so it can be written as $d(b_m)$ for some $b_m$ in $sC(\Delta^m)_{m+1}$, which we simply choose. This completes the induction step. □

Corollary 6.4.2. *The natural chain map* $\beta\colon sC(X) \to sC(X)$ *in proposition* 6.3.1 *is naturally chain homotopic to the identity.*

Proof. Apply theorem 6.4.1 using $\alpha := \beta - \mathrm{id}$. □

## 6.5. Small singular chains

Let $X$ be a topological space and let $\mathcal{U}$ be a cover of $X$. To be more specific, $\mathcal{U}$ is a family $(U_j)_{j\in J}$ of subsets of $X$; the $U_j$ are *not* required to be open subsets of $X$, but instead we impose the condition

$$\bigcup_{j\in J} \mathrm{int}(U_j) = X.$$

(So the sets $\mathrm{int}(U_j)$ for $j \in J$ do form an open cover of $X$.) A continuous map $\sigma\colon \Delta^n \to X$ will be called $\mathcal{U}$*-small* if its image is contained in one of the sets $U_j$ of the cover $\mathcal{U}$. Let

$$sC(X,\mathcal{U}) \subset sC(X)$$

be the chain subcomplex defined as follows: $sC(X,\mathcal{U})_n \subset sC(X)_n$ is the free abelian group generated by the set of all continuous maps $\sigma\colon \Delta^n \to X$ which are $\mathcal{U}$-small. (Whereas $sC(X)_n$ itself is the free abelian group generated by the set of all continuous maps $\sigma$ from $\Delta^n$ to $X$.)

The following important theorem can be viewed as a "user-friendly" summary of statements that we have proved in the last few sections.

THEOREM 6.5.1. *The homomorphism* $H_n(sC(X,\mathcal{U})) \longrightarrow H_n(sC(X))$ *determined by the inclusion of* $sC(X;\mathcal{U})$ *in* $sC(X)$ *is an isomorphism, for all* $n$.

PROOF. Let us begin by observing that the natural chain map $\beta\colon sC(X) \to sC(X)$ of proposition 6.3.1 maps the chain subcomplex $sC(X,\mathcal{U})$ to itself. This is obvious from the construction: if $\sigma\colon \Delta^n \to X$ is $\mathcal{U}$-small, then $\beta(\sigma) \in sC(X)_n$ is a formal linear combination of terms $\sigma u_\lambda$ which are again $\mathcal{U}$-small.
Slightly less obvious: the natural chain homotopy $h$ from $\beta$ to id which we get from corollary 6.4.2 also maps the chain subcomplex $sC(X,\mathcal{U})$ to itself. This is a consequence of naturality. If $\sigma\colon \Delta^n \to X$ is $\mathcal{U}$-small, and if we view it as an element of $sC(X)_n$, then we can write it in the form $\sigma_*(\iota_n)$ where $\iota_n \in sC(\Delta^n)_n$ is the special element which we know from the proof of theorem 6.4.1. Therefore $h_n(\sigma) = h_n(\sigma_*(\iota_n)) = \sigma_*(h_n(\iota_n))$, and this is clearly an element of $sC(X,\mathcal{U})_{n+1}$ since $\sigma_*\colon sC(\Delta^n) \to sC(X)$ has image contained in $sC(X,\mathcal{U})$.
An element of $H_n(sC(X))$ can be represented by some $n$-cycle $z \in sC(X)_n$. For sufficiently large $k$, the $n$-cycle $\beta^k(z)$ belongs to $sC(X,\mathcal{U})$. But in $sC(X)$, the difference $z - \beta^k(z)$ is an $n$-boundary since $\beta^k$ is chain homotopic to the identity. Therefore $[\beta^k(z)] \in H_n(sC(X,\mathcal{U}))$ maps to $[z] \in H_n(sC(X))$. This proves surjectivity.
If $y$ is an $n$-cycle in $sC(X,\mathcal{U})$ which is an $n$-boundary in $sC(X)$, then we can choose $w \in sC(X)_{n+1}$ such that $d(w) = y$ in $sC(X)$. For sufficiently large $k$, we have $\beta^k(w) \in sC(X,\mathcal{U})$, and we still have $d(\beta^k(w)) = \beta^k(y)$. Therefore the class $[\beta^k(y)] \in H_n(sC(X,\mathcal{U}))$ is zero. But since $\beta$ as a chain map from $sC(X,\mathcal{U})$ to $sC(X,\mathcal{U})$ is chain homotopic to the identity, this implies that $[y]$ itself was zero to begin with. This proves injectivity. $\square$

CHAPTER 7

# Homology of spaces

## 7.1. Generalities

The *singular homology groups* $H_n(X)$ of a topological space $X$ are defined as the homology groups of the singular chain complex $sC(X)$:

$$H_n(X) := H_n(sC(X)) = H_n(C(\mathrm{sing}(X)))$$

for $n \in \mathbb{Z}$. (As a rule we just say: the homology groups of $X$.) To be more precise, $H_n$ is a functor from $\mathcal{T}\mathrm{op}$ to the category of abelian groups. Namely, a continuous map $f\colon X \to Y$ determines a chain map $sC(X) \to sC(Y)$ and the chain map determines a homomorphism $H_n(sC(X)) \to H_n(sC(Y))$ which we can also write in the form

$$f_*\colon H_n(X) \to H_n(Y).$$

The plan for this chapter is modest: the results on $sC(X)$ in the last chapter have consequences for the groups $H_n(X)$ which we want to spell out.

Proposition 7.1.1. *If $X$ is a one-point space, then $H_n(X) \cong \mathbb{Z}$ for $n = 0$ and $H_n(X) = 0$ for $n \neq 0$. If $X = \emptyset$, then $H_n(X) = 0$ for all $n$.*

Proof. The calculation for the one-point space was done in example 6.1.1. For the case $X = \emptyset$, we observe that $sC(\emptyset)_n = 0$ for all $n \in \mathbb{Z}$ since there are no maps $\Delta^n \to \emptyset$ for $n \geq 0$. □

Theorem 7.1.2. *For each $n \in \mathbb{Z}$, the functor $H_n$ from $\mathcal{T}\mathrm{op}$ to the category of abelian groups is* homotopy invariant. *That is to say, if $f, g\colon X \to Y$ are continuous maps, and if $f$ is homotopic to $g$, then $f_* = g_*\colon H_n(X) \to H_n(Y)$ for all $n$.*

Proof. This follows from corollary 6.2.2 and proposition 5.2.3. □

Theorem 7.1.3. *Let $U$ and $V$ be subsets of a space $X$ such that* $\mathrm{int}(U) \cup \mathrm{int}(V) = X$. *Then there is a natural long exact sequence of homology groups* (Mayer-Vietoris sequence)

$$\cdots \longrightarrow H_{n+1}(X) \xrightarrow{\partial} H_n(U \cap V) \longrightarrow H_n(U) \oplus H_n(V) \longrightarrow H_n(X) \xrightarrow{\partial} H_{n-1}(U \cap V) \longrightarrow \cdots$$

Proof. In the chain complex $sC(X)$, we have the chain subcomplexes $sC(U)$ and $sC(V)$. For the chain complex $sC(U) + sC(V) \subset sC(X)$ (internal sum, not direct sum), we have corollary 5.3.3. That is, we can take $E = sC(U) + sC(V) \subset sC(X)$ and $K = sC(U)$ and $L = sC(V)$ in corollary 5.3.3. Then we get a long exact sequence of homology groups. Furthermore, by theorem 6.5.1 the inclusion of $sC(U) + sC(V)$ in $sC(X)$ is a chain homotopy equivalence. So we can substitute $H_n(X) = H_n(sC(X))$ for $H_n(sC(U) + sC(V))$ in the long exact sequence which we already have. □

Nota bene: the unlabeled homomorphisms in the long exact sequence of the theorem are

$$((j_U)_*, -(j_V)_*)\colon H_n(U \cap V) \longrightarrow H_n(U) \oplus H_n(V)$$

(where $j_U: U \cap V \to U$ and $j_V: U \cap V \to V$ are the inclusions), and

$$(g_U)_* \oplus (g_V)_*: H_n(U) \oplus H_n(V) \longrightarrow H_n(X)$$

where $g_U: U \to X$ and $g_V: V \to X$ are the inclusions.

*Naturality* in theorem 7.1.3 means the following. Suppose that we have a continuous map $f: X_0 \to X_1$ , open subsets $U_0, V_0$ of $X_0$ and open subsets $U_1$, $V_1$ of $X_1$ such that $f(U_0) \subset U_1)$ and $f(V_0) \subset V_1$, and such that $U_0: V_0 = X_0$ and $U_1 \cup V_1 = X_1$. Then we get a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X_0) \arrow[r, "\partial"] \arrow[d] & H_n(U_0 \cap V_0) \arrow[r] \arrow[d] & H_n(U_0) \oplus H_n(V_0) \arrow[r] \arrow[d] & H_n(X_0) \arrow[r, "\partial"] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X_1) \arrow[r, "\partial"] & H_n(U_1 \cap V_1) \arrow[r] & H_n(U_1) \oplus H_n(V_1) \arrow[r] & H_n(X_1) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

where the rows are Mayer-Vietoris sequences as in the theorem and the vertical arrows are induced by $f$ (and appropriate restrictions of $f$).

Remark 7.1.4. From the construction, the homomorphisms $\partial$ in theorem 7.1.3 depend on the order on which we list $U$ and $V$ (here $U$ first, $V$ second). If we interchange this order, that is if we list $V$ before $U$, then $U \cap V$ does not change (it is the same as $V \cap U$), but the new

$$\partial: H_n(X) \longrightarrow H_{n-1}(V \cap U)$$

is equal to $-1$ times the old $\partial: H_n(X) \longrightarrow H_{n-1}(U \cap V)$. The verification is left to the gentle reader.

The theorems 7.1.2 and 7.1.3 together with proposition 7.1.1 can be viewed as a set of *axioms* for homology. That is to say, they should be enough to determine the functors $H_n$ for all $n$ as long as we restrict attention to spaces $X, Y, ...$ which are not too complicated: for example, compact simplicial complexes or geometric realizations of finite semi-simplicial sets. Remark 7.1.4 is probably superfluous, that is to say, it can probably be deduced from the other axioms listed, but since this is somewhat tedious I prefer to think of this remark as fine print to be included in the Mayer-Vietoris axiom, 7.1.3.

The idea of writing out axioms for homology goes back to Eilenberg and Steenrod; therefore we speak of the *Eilenberg-Steenrod* axioms. In fact Eilenberg and Steenrod preferred a slightly different setup which relies on the notion *homology of pairs*. We come to this later. I am not planning to present a formal proof that axioms so-and-so determine the homology functors on such-and-such spaces. But there are many situations where computations of homology groups of spaces can be deduced from only a few axiom-like statements, and when that is happening I will try to alert the reader to it.

## 7.2. Homology of spheres

Proposition 7.2.1. *The homology groups of* $S^1$ *are* $H_0(S^1) \cong \mathbb{Z}$, $H_1(S^1) \cong \mathbb{Z}$ *and* $H_k(S^1) = 0$ *for all* $k \neq 0, 1$.

Proof. Choose two distinct points $p$ and $q$ in $S^1$. Let $V \subset S^1$ be the complement of $p$ and let $W \subset S^1$ be the complement of $q$. Then $V \cup W = S^1$. Clearly $V$ is homotopy equivalent to a point, $W$ is homotopy equivalent to a point and $V \cap W$ is homotopy equivalent to a discrete space with two points. Therefore $H_k(V) \cong H_k(W) \cong \mathbb{Z}$ for $k = 0$ and $H_k(V) \cong H_k(W) = 0$ for all $k \neq 0$. Similarly $H_k(V \cap W) \cong \mathbb{Z} \oplus \mathbb{Z}$ for $k = 0$ and

$H_k(V \cap W) = 0$ for all $k \neq 0$. The exactness of the Mayer-Vietoris sequence associated with the open covering of $S^1$ by $V$ and $W$ implies immediately that $H_k(S^1) = 0$ for $k \neq 0, 1$. The part of the Mayer-Vietoris sequence which remains interesting after this observation is

$$0 \longrightarrow H_1(S^1) \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(S^1) \longrightarrow 0$$

The homomorphism in the middle is not a mystery, since all spaces that go into it are contractible. It is given in matrix form by

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Therefore its kernel and cokernel are both isomorphic to $\mathbb{Z}$. This means that $H_1(S^1) \cong \mathbb{Z}$ and $H_0(S^1) \cong \mathbb{Z}$. □

THEOREM 7.2.2. *The homology groups of* $S^n$ *(for* $n > 0$*) are*

$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \\ \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. We proceed by induction on $n$. The induction beginning is the case $n = 1$ which we have already dealt with separately in proposition 7.2.1. For the induction step, suppose that $n > 1$. We use the Mayer-Vietoris sequence for $S^n$ and the open covering $\{V, W\}$ with $V = S^n \smallsetminus \{p\}$ and $W = S^n \smallsetminus \{q\}$ where $p, q \in S^n$ are the north and south pole, respectively. We will also use the homotopy invariance of homology. This gives us

$$H_k(V) \cong H_k(W) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

because $V$ and $W$ are homotopy equivalent to a point. Also we get

$$H_k(V \cap W) \cong \begin{cases} \mathbb{Z} & \text{if } k = n-1 \\ \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

by the induction hypothesis, since $V \cap W$ is homotopy equivalent to $S^{n-1}$. Furthermore it is clear what the inclusion maps $V \cap W \to V$ and $V \cap W \to W$ induce in homology: an isomorphism in $H_0$ (see remark 7.2.3 below) and (necessarily) the zero map in $H_k$ for all $k \neq 0$. Thus the homomorphism

$$H_k(V \cap W) \longrightarrow H_k(V) \oplus H_k(W)$$

from the Mayer-Vietoris sequence takes the form

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}\,;\ 1 \mapsto 1 \oplus -1$$

when $k = 0$, and

$$\mathbb{Z} \longrightarrow 0$$

when $k = n - 1$. In all other cases, its source and target are both zero. Therefore the exactness of the Mayer-Vietoris sequence implies that $H_0(S^n)$ and $H_n(S^n)$ are isomorphic to $\mathbb{Z}$, while $H_k(S^n) = 0$ for all other $k \in \mathbb{Z}$. □

REMARK 7.2.3. In the above calculation we used the following:

(i) if $X$ is a space such that $H_0(X) \cong \mathbb{Z}$, then the unique map $p\colon X \to \star$ induces an isomorphism $p_*\colon H_0(\mathbb{Z}) \to H_0(\star)$;

(ii) if $X$ and $Y$ are spaces such that $H_0(X) \cong \mathbb{Z}$ and $H_0(Y) \cong \mathbb{Z}$, then any continuous map $f\colon X \to Y$ induces an isomorphism $f_*\colon H_0(X) \to H_0(\mathbb{Z})$.

Proof of (i): if $H_0(X) \cong \mathbb{Z}$, then $X \neq \emptyset$ and we can choose a map $j\colon \star \to X$. There is a unique map $p\colon X \to *$. Since $pj$ is the identity of $\star$, the composition

$$H_0(\star) \xrightarrow{j_*} H_0(X) \xrightarrow{p_*} \to H_0(\star)$$

is the identity of $H_0(\star) \cong \mathbb{Z}$. Therefore $p_*\colon H_0(X) \to H_0(\star)$ is onto, and therefore it is an isomorphism. — Proof of (ii): We have the maps $p_X\colon X \to \star$ and $p_Y\colon Y \to \star$ and we know $p_Y f = p_X$. Therefore $(p_Y)_* f_*\colon H_0(X) \to H_0(\star)$ agrees with $(p_X)_*$. Using (i), we deduce that $f_*\colon H_0(X) \to H_0(Y)$ is an isomorphism.
Later we will come to a more detailed and illuminating description of $H_0(X)$ for arbitrary spaces. This could also be used in the calculation of theorem 7.2.2, but I prefer the simple-minded argument just given because it uses only the axioms.

THEOREM 7.2.4. *Let $f\colon S^n \to S^n$ be the antipodal map ($f(z) = -z$ for all $z$). The induced homomorphism $f_*\colon H_n(S^n) \to H_n(S^n)$ is multiplication by $(-1)^{n+1}$.*

PROOF. We proceed by induction again. For the induction beginning, we take $n = 1$. The antipodal map $f : S^1 \to S^1$ is homotopic to the identity, so that $f^* : H_1(S^1) \to H_1(S^1)$ has to be the identity, too. For the induction step, we use the setup and notation from the previous proof. Exactness of the Mayer-Vietoris sequence for $S^n$ and the open covering $\{V, W\}$ shows that

$$\partial : H_n(S^n) \longrightarrow H_{n-1}(V \cap W)$$

is an isomorphism. The diagram

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{\partial} & H_{n-1}(V \cap W) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ H_n(S^n) & \xrightarrow{\partial} & H_{n-1}(W \cap V) \end{array}$$

is meaningful because $f$ takes $V \cap W$ to $V \cap W = W \cap V$. But the diagram is not commutative (i.e., it is not true that $f_* \circ \partial$ equals $\partial \circ f_*$). The reason is that $f$ interchanges $V$ and $W$, and it does matter in the Mayer-Vietoris sequence which of the two comes first, as pointed out in remark 7.1.4. Therefore we have instead

$$f_* \circ \partial = -\partial \circ f_*$$

in the above square. By the inductive hypothesis, the $f_*$ in the left-hand column of the square is multiplication by $(-1)^n$, and therefore the $f^*$ in the right-hand column of the square must be multiplication by $(-1)^{n+1}$. □

## 7.3. The usual applications

THEOREM 7.3.1. (Brouwer's fixed point theorem). *Let $f : D^n \to D^n$ be a continuous map, where $n \geq 1$. Then $f$ has a fixed point, i.e., there exists $y \in D^n$ such that $f(y) = y$.*

PROOF. Suppose for a contradiction that $f$ does not have a fixed point. For $x \in D^n$, let $g(x)$ be the point where the ray (half-line) from $f(x)$ to $x$ intersects the boundary $S^{n-1}$ of the disk $D^n$. Then $g$ is a smooth map from $D^n$ to $S^{n-1}$, and we have $g|S^{n-1} = \mathrm{id}_{S^{n-1}}$. Summarizing, we have

$$S^{n-1} \xrightarrow{\;j\;} D^n \xrightarrow{\;g\;} S^{n-1}$$

where $j$ is the inclusion, $g \circ j = \mathrm{id}_{S^{n-1}}$. Therefore we get

$$H_{n-1}(S^{n-1}) \xrightarrow{j_*} H_{n-1}(D^n) \xrightarrow{g_*} H_{n-1}(S^{n-1})$$

where $g_* j_* = \mathrm{id}$. Thus the abelian group $H_{n-1}(S^{n-1})$ is isomorphic to a direct summand of $H_{n-1}(D^n)$. But from our calculations above, we know that this is not true. If $n > 1$ we have $H_{n-1}(D^n) = 0$ while $H_{n-1}(S^{n-1})$ is not trivial. If $n = 1$ we have $H_{n-1}(D^n) \cong \mathbb{Z}$ while $H_{n-1}(S^{n-1}) \cong \mathbb{Z} \oplus \mathbb{Z}$. □

Let $f\colon S^n \to S^n$ be any continuous map, $n > 0$. The induced homomorphism $f_*$ from $H_n(S^n)$ to $H_n(S^n)$ is multiplication by some number $n_f \in \mathbb{Z}$, since $H_n(S^n)$ is isomorphic to $\mathbb{Z}$.

DEFINITION 7.3.2. The number $n_f$ is the *degree* of $f$. We may write $\deg(f)$ for it.

*Remark.* The degree of $f\colon S^n \to S^n$ is clearly an invariant of the homotopy class of $f$.

*Remark.* In the case $n = 1$, the definition of $\deg(f)$ as given just above agrees with the definition of $\deg(f)$ given in section 1. See exercises.

EXAMPLE 7.3.3. According to theorem 7.2.4, the degree of the antipodal map $S^n \to S^n$ is $(-1)^{n+1}$.

PROPOSITION 7.3.4. *Let $f\colon S^n \to S^n$ be a continuous map. If $f(x) \neq x$ for all $x \in S^n$, then $f$ is homotopic to the antipodal map, and so has degree $(-1)^{n+1}$. If $f(x) \neq -x$ for all $x \in S^n$, then $f$ is homotopic to the identity map, and so has degree $1$.*

PROOF. Let $g\colon S^n \to S^n$ be the antipodal map, $g(x) = -x$ for all $x$. Assuming that $f(x) \neq x$ for all $x$, we show that $f$ is homotopic to $g$. We think of $S^n$ as the unit sphere in $\mathbb{R}^{n+1}$, with the usual notion of distance. We can make a homotopy $(h_t : S^n \to S^n)_{t \in [0,1]}$ from $f$ to $g$ by "sliding" along the unique minimal geodesic arc from $f(x)$ to $g(x)$, for every $x \in S^n$. In other words, $h_t(x) \in S^n$ is situated $t \cdot 100$ percent of the way from $f(x)$ to $g(x)$ along the minimal geodesic arc from $f(x)$ to $g(x)$. (The important thing here is that $f(x)$ and $g(x)$ are not antipodes of each other, by our assumptions. Therefore that minimal geodesic arc is unique.)
Next, assume $f(x) \neq -x$ for all $x \in S^n$. Then, for every $x$, there is a unique minimal geodesic from $x$ to $f(x)$, and we can use that to make a homotopy from the identity map to $f$. □

COROLLARY 7.3.5. (Hairy ball theorem). *Let $\xi$ be a tangent vector field* (explanations follow) *on $S^n$. If $\xi(z) \neq 0$ for every $z \in S^n$, then $n$ is odd.*

*Comments.* A tangent vector field on $S^n \subset \mathbb{R}^{n+1}$ can be defined as a continuous map $\xi$ from $S^n$ to the vector space $\mathbb{R}^{n+1}$ such that $\xi(x)$ is perpendicular to (the position vector of) $x$, for every $x \in S^n$. We say that vectors in $\mathbb{R}^{n+1}$ which are perpendicular to $x \in S^n$ are *tangent* to $S^n$ at $x$ because they are the velocity vectors of smooth curves in $S^n \subset \mathbb{R}^n$ as they pass through $x$.

PROOF. Define $f : S^n \to S^n$ by $f(x) = \xi(x)/\|\xi(x)\|$. Then $f(x) \neq x$ and $f(x) \neq -x$ for all $x \in S^n$, since $f(x)$ is always perpendicular to $x$. Therefore $f$ is homotopic to the antipodal map, and also homotopic to the identity. It follows that the antipodal map is homotopic to the identity. Therefore $n$ is odd by theorem 7.2.4. □

Remark 7.3.6. Theorem 7.2.4 has an easy generalization which says that the degree of the map $f\colon S^n \to S^n$ given by

$$(x_1, x_2, \dots, x_{n+1}) \mapsto (x_1, \dots, x_k, -x_{k+1}, \dots, -x_{n+1})$$

is $(-1)^{n+1-k}$. Here we assume $n \geq 1$ as usual. The proof can be given by induction on $n+1-k$. The induction step is now routine, but the induction beginning must cover all cases where $n = 1$. This leaves the three possibilities $k = 0, 1, 2$. One of these gives the identity map $S^1 \to S^1$, and another gives the antipodal map $S^1 \to S^1$ which is homotopic to the identity. The interesting case which remains is the map $f\colon S^1 \to S^1$ given by $f(x_1, x_2) = (x_1, -x_2)$. We need to show that it has degree $-1$, in the sense of definition 7.3.2. One way to do this is to use the following diagram

$$\begin{tikzcd} H_1(S^1) \arrow[r, "f_*"] \arrow[d, "\partial"] & H_1(S^1) \arrow[d, "\partial"] \\ H_0(V \cap W) \arrow[r, "f_*"] & H_0(W \cap V) \end{tikzcd}$$

where $V = S^1 \smallsetminus \{(0,1)\}$ and $W = S^1 \smallsetminus \{(0,-1)\}$. We know from the previous chapter that it commutes up to a factor $(-1)$. In the lower row, we have the identity homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. The vertical arrows are injective (seen earlier in the proof of proposition 7.2.1). Therefore the upper horizontal arrow is multiplication by $-1$.

To state this result in a more satisfying manner, let us note that the orthogonal group $O(n+1)$ (the group of orthogonal $(n+1) \times (n+1)$-matrices with real entries) is a topological group which has two path components. The two path components are the preimages of $+1$ and $-1$ under the homomorphism

$$\det\colon O(n+1) \to \{-1, +1\}.$$

Let $f\colon S^n \to S^n$ be given by $f(z) = Az$ for some $A \in O(n+1)$. Because $\deg(f)$ depends only on the homotopy class of $f$, it follows that $\deg(f)$ depends only on the path component of $A$ in $O(n+1)$, and hence only on $\det(A)$. What we have just shown means that $\deg(f)$ is *equal* to $\det(A)$.

Remark 7.3.7. The path components of a space $X$ ... you should know this, but they are the equivalence classes of an equivalence relation on the set $X$ which is defined as follows: $x_0 \in X$ is equivalent to $x_1 \in X$ if there exists a continuous $\gamma\colon [0,1] \to X$ (a path) such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

To show that $O(n+1)$ has exactly two path components (if $n \geq 0$): the case $n = 0$ is easy and for $n \geq 1$ we begin with the observation that the map

$$q\colon O(n+1) \longrightarrow S^n$$

given by $q(A) = Ae_{n+1}$ (where $e_{n+1} = (0, 0, \dots, 0, 1) \in \mathbb{R}^{n+1}$) is a fiber bundle projection. (The details of that are left to the gentle reader, once again.) Therefore $q$ has the HLP, homotopy lifting property. Given $A \in O(n+1)$, we can choose a path $\gamma\colon [0,1] \to S^n$ from $Ae_{n+1}$ to $e_{n+1}$ (since $S^n$ is path connected, since $n \geq 1$). By the HLP, we can then find a path $\bar{\gamma}\colon [0,1] \to O(n+1)$ such that $\bar{\gamma}(0) = A$ and $q\bar{\gamma} = \gamma$. This implies that $B := \bar{\gamma}(1) \in O(n+1)$ satisfies $Be_{n+1} = e_{n+1}$. Therefore the matrix $B$ satisfies $B_{11} = 1$ and $B_{i1} = 0$, $B_{1j} = 0$ for all $i, j \in \{2, 3, \dots, n+1\}$. That is to say, $B$ comes from $O(n)$. In other words, since $B$ is in the same path component as $A$ by construction, and $A$ was arbitrary, we have shown that the standard inclusion $O(n) \to O(n+1)$ induces a surjection of the sets of path components. Since it is clear that $O(1)$ has exactly two path components, it follows that $O(n+1)$ has at most two path components. But the continuous map $\det\colon O(n+1) \to \{1,-1\}$ is onto. So there must be at least two path components.

## 7.4. Homology of pairs: Generalities again

DEFINITION 7.4.1. A *pair* of topological spaces $(X,Y)$ consists of a space $X$ and a subspace $Y$ of $X$. Pairs of topological spaces are the objects of a category: a *morphism* from a pair $(X_0,Y_0)$ to a pair $(X_1,Y_1)$ is a continuous map $f\colon X_0 \to X_1$ such that $f(Y_0) \subset Y_1$. It is permitted to write

$$f\colon (X_0,Y_0) \longrightarrow (X_1,Y_1)$$

for such a morphism. We may write $\mathcal{T}\mathrm{op}^{(2)}$ for the category. Two morphisms

$$f,g\colon (X_0,Y_0) \longrightarrow (X_1,Y_1)$$

are considered *homotopic* if there exists a homotopy $(h_t)_{t\in[0,1]}$ from $f\colon X_0 \to X_1$ to $g\colon X_0 \to X_1$ such that $h_t(Y_0) \subset Y_1$ for all $t \in [0,1]$.
The *singular chain complex* of the pair $(X,Y)$ is the quotient chain complex $sC(X)/sC(Y)$. I may occasionally write $sC(X,Y)$ for this. The *homology groups* of the pair $(X,Y)$ are the groups $H_n(X,Y) := H_n(sC(X)/sC(Y)) = H_n(\mathcal{C}(X,Y))$ where $n \in \mathbb{Z}$. Homology $H_n$ can therefore be viewed as a functor from the category of pairs of spaces to the category of abelian groups.

Homology of pairs is a generalization of homology of spaces: we can write and we will write, without lying very hard, $H_n(X) = H_n(X,\emptyset)$. From a slightly axiomatic point of view, let us say that $H_n(X)$ is just an *abbreviation* for $H_n(X,\emptyset)$.
Now let's list *axioms* for homology of pairs.

PROPOSITION 7.4.2. *If $X$ is a one-point space, then $H_n(X) \cong \mathbb{Z}$ for $n = 0$ and $H_n(X) = 0$ for $n \neq 0$.*

This is just a part proposition 7.1.1, repeated here for bureaucratic reasons. □

PROPOSITION 7.4.3. *For pairs of spaces $(X,Y)$ there is a natural long exact sequence*

$$\cdots \longrightarrow H_{n+1}(X,Y) \xrightarrow{\partial} H_n(Y) \longrightarrow H_n(X) \longrightarrow H_n(X,Y) \longrightarrow H_{n-1}(Y) \xrightarrow{\partial} \cdots$$

PROOF. The short exact sequence $sC(Y) \to sC(X) \to sC(X)/sC(Y)$ of chain complexes determines a long exact sequence of homology groups as in theorem 5.3.2. □

THEOREM 7.4.4. *For each $n \in \mathbb{Z}$, the functor $H_n$ from $\mathcal{T}\mathrm{op}^{(2)}$ to the category of abelian groups is* homotopy invariant. *That is to say, if $f,g\colon (X_0,Y_0) \to (X_1,Y_1)$ are morphisms in $\mathcal{T}\mathrm{op}^{(2)}$, and if they are homotopic as such, then $f_* = g_*\colon H_n(X_0,Y_0) \to H_n(X_1,Y_1)$ for all $n$.*

PROOF. This can be proved like 7.1.2. We require a version of theorem 6.2.1 for a pair $(X,Y)$ of spaces. This would state that the chain maps $sC(X,Y) \to sC(X\times[0,1], Y\times[0,1])$ induced by

$$g_0\colon (X,Y) \to (X\times[0,1], Y\times[0,1]), \qquad g_1\colon (X,Y) \to (X\times[0,1], Y\times[0,1])$$

(where $g_0(x) = (x,0)$ and $g_1(x) = (x,1)$), respectively, are chain homotopic. The proof is the same for arbitrary $Y \subset X$ as in the case $Y = \emptyset$. (As an alternative, one could point out that the chain homotopy constructed in the proof of theorem 6.2.1 is natural, and exploit this naturality in the case of the inclusion $Y \to X$.) □

THEOREM 7.4.5. (Excision axiom.) *Let* $(X, Y)$ *be a pair of spaces and let* $Z$ *be a subspace of* $X$ *such that the closure of* $Z$ *is contained in the interior of* $Y$. *Then the inclusion of pairs* $(X \smallsetminus Z, Y \smallsetminus Z) \to (X, Y)$ *induces an isomorphism* $H_n(X \smallsetminus Z, Y \smallsetminus Z) \to H_n(X, Y)$ *for all* $n \in \mathbb{Z}$.

For the proof of this, we need:

LEMMA 7.4.6. (The five lemma) *Suppose that, in a commutative diagram of abelian groups and homomorphisms*

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "(1)"] & b \arrow[r] \arrow[d, "(2)"] & c \arrow[r] \arrow[d, "(3)"] & d \arrow[r, "g"] \arrow[d, "(4)"] & e \arrow[d, "(5)"] \\ a' \arrow[r, "f'"] & b' \arrow[r] & c' \arrow[r] & d' \arrow[r, "g'"] & e' \end{tikzcd}$$

*the rows are exact and the arrows labeled* $(1)$, $(2)$, $(4)$ *and* $(5)$ *are isomorphisms. Then the arrow* $(3)$ *is an isomorphism.*

PROOF. We can quickly reduce to the situation where $a = a' = 0$ and $e = e' = 0$. (Replace $b$ by $b/\mathrm{im}(f) = \mathrm{coker}(f)$ and $b'$ by $\mathrm{coker}(f')$, and $d$ by $\ker(g)$, and $d'$ by $\ker(g')$.) Then the rows are short exact sequences:

$$\begin{tikzcd} b \arrow[r] \arrow[d, "(2)", "\cong"'] & c \arrow[r] \arrow[d, "(3)"] & d \arrow[d, "(4)", "\cong"'] \\ b' \arrow[r] & c' \arrow[r] & d' \end{tikzcd}$$

It is easy to see that any element in the kernel of $(3)$ must come from $b$. But then it is zero since the map from $b$ to $c'$ is injective by assumption. Therefore (3) is injective. Next, given any $y \in c'$ we can find an element of $c$ having the same image as $y$ in $d'$, since the map from $c$ to $d'$ is surjective. Therefore $y = y_1 + y_2$ where $y_1$ is in the image (3) and $y_2$ comes from $b'$. But the image of $b' \to c'$ is contained in the image of $(3)$; so $y$ is in the image of $(3)$. □

REMARK 7.4.7. The five lemma has the following consequence. If

$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & D \arrow[d] \\ B' \arrow[r] & C' \arrow[r] & D' \end{tikzcd}$$

is a commutative diagram of chain complexes and chain maps with short exact rows, and if two of the three vertical arrows induce isomorphisms of the homology groups $H_n$ for all $n$, then the same can be said of the remaining vertical arrow. To see this, form the long exact homology group sequences and arrange them in a ladder-shaped commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & H_n(B) \arrow[r] \arrow[d] & H_n(C) \arrow[r] \arrow[d] & H_n(D) \arrow[r] \arrow[d] & H_{n-1}(B) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_n(B') \arrow[r] & H_n(C') \arrow[r] & H_n(D') \arrow[r] & H_{n-1}(B') \arrow[r] & \cdots \end{tikzcd}$$

Now two out of any three adjacent vertical arrows are isomorphisms. The five lemma then implies that all vertical arrows are isomorphisms. □

PROOF OF THEOREM 7.4.5. Since $\bar{Z} \subset \mathrm{int}(Y)$, we have a cover $\mathcal{U}$ of $X$ by subsets $X \smallsetminus Z$ and $Y$ such that $\mathrm{int}(X \smallsetminus Z) \cup \mathrm{int}(Y) = X$. By theorem 6.5.1, the inclusion

$$sC(X, \mathcal{U}) \longrightarrow sC(X)$$

is a chain homotopy equivalence. Therefore by remark 7.4.7, the inclusion

$$sC(X, \mathcal{U})/sC(Y) \to sC(X)/sC(Y)$$

induces an isomorphism in homology groups. But there is another inclusion-induced map

$$sC(X \smallsetminus Z)/sC(Y \smallsetminus Z) \longrightarrow sC(X, \mathcal{U})/sC(Y)$$

which is an isomorphism, by inspection. Therefore the composition of these two,

$$sC(X \smallsetminus Z)/sC(Y \smallsetminus Z) \longrightarrow sC(X)/sC(Y)$$

induces an isomorphism in homology groups. □

## 7.5. Homology of spheres again

To illustrate how homology of pairs works, let us calculate the homology of spheres again. The following more or less abstract observation will be useful.

LEMMA 7.5.1. Let $f\colon (X_0, Y_0) \to (X_1, Y_1)$ be a map of pairs of spaces such that the underlying maps $X_0 \to X_1$ and $Y_0 \to Y_1$ are homotopy equivalences. Then

$$f_*\colon H_m(X_0, Y_0) \longrightarrow H_m(X_1, Y_1)$$

is in isomorphism for all $m$.

PROOF. Exercise. (This should preferably be deduced from the axiom-like statements in section 7.4; of course it is allowed to use the five lemma, too.)

LEMMA 7.5.2. $H_m(\emptyset) = 0$ *for all* $m \in \mathbb{Z}$.

This is obvious from the definitions, and I emphasized it before, but it can also be deduced from the axiom-like statements in section 7.4. And that is an exercise. Remember that $H_m(X)$ is an abbreviation for $H_m(X, \emptyset)$.

LEMMA 7.5.3. *For all* $m \geq 0$ *and all* $k \in \mathbb{Z}$ *and any choice of base point* $* \in S^m$*, we have* $H_k(S^m, *) \cong H_k(D^m, S^{m-1})$.

PROOF. Let $z \in S^m$ be the point opposite to $* \in S^m$ and let $V = S^m \smallsetminus \{z\}$. Then the inclusion $(S^m, *) \to (S^m, V)$ induces isomorphisms

$$H_k(S^m, *) \xrightarrow{\cong} H^k(S^m, V)$$

by lemma 7.5.1. By excision,

$$H_k(S^m, V) \cong H_k(Q^m, Q^m \cap V)$$

where $Q^m$ is the closed upper hemisphere of $S^m$ (consisting of all $y \in S^m$ whose scalar product with $z$ is $\geq 0$). By lemma 7.5.1 again, we have

$$H_k(Q^m, Q^m \cap V) \cong H_k(Q^m, \partial Q^m)$$

where $\partial Q^m$ is the boundary of $Q^m$ (consisting of all $y \in S^m$ whose scalar product with $z$ is $= 0$). But clearly $(Q^m, \partial Q^m)$ is homeomorphic to $(D^m, S^{m-1})$. □

COROLLARY 7.5.4. $H_k(S^m, *) \cong H_{k-1}(S^{m-1}, *)$ *for* $m > 0$ *and arbitrary* $k \in \mathbb{Z}$.

*Proof.* In the long exact homology sequence for the pair $(D^m, S^{m-1})$, the homomorphisms $H_k(S^{m-1}) \to H_k(D^m)$ induced by the inclusion $j\colon S^{m-1} \to D^m$ are always onto (since there exists a map $g\colon D^m \to S^{m-1}$ such that $jg$ is homotopic to the identity of $D^m$). Therefore the homomorphisms $\partial\colon H_k(D^m, S^{m-1}) \to H_{k-1}(S^{m-1})$ in the long exact sequence are injective, and by exactness $H_k(D^m, S^{m-1})$ is isomorphic to the kernel of $H_{k-1}(S^{m-1}) \to H_{k-1}(D^m)$. That kernel is isomorphic to the kernel of $H_{k-1}(S^{m-1}) \to H_{k-1}(*)$, homomorphism induced by the unique map $S^{m-1} \to *$, and also to the cokernel of $H_{k-1}(*) \longrightarrow H_{k-1}(S^{m-1})$, homomorphism induced by inclusion of the base point (since the composition of $* \to S^{m-1}$ and $S^{m-1} \to *$ is an identity map). Finally that cokernel is isomorphic to

$$H_{k-1}(S^{m-1}, *)$$

as we can see from the long exact homology sequence of the pair $(S^{m-1}, *)$. □

This is nearly the end of the calculation, for now we can say that

$$H_k(S^0, *) \cong \begin{cases} 0 & \text{if} \quad k \neq 0 \\ \mathbb{Z} & \text{if} \quad k = 0 \end{cases}$$

by using the long exact homology sequence of the pair $(S^0, *)$. Then we deduce

$$H_k(S^m, *) \cong \begin{cases} 0 & \text{if} \quad k \neq m \\ \mathbb{Z} & \text{if} \quad k = m \end{cases}$$

using corollary 7.5.4. To that we can add the observation

$$H_k(S^m) \cong H_k(S^m, *) \oplus H_k(*)$$

which follows from the long exact homology sequence of the pair $(S^m, *)$.

CHAPTER 8

# Special properties of singular homology

## 8.1. Path components

Let $X$ be a topological space. The set of path components of $X$ is denoted $\pi_0(X)$. I allow myself to write things like $\alpha \in \pi_0(X)$ and $X_\alpha \subset X$ for the corresponding nonempty subspace of $X$ (that path component), although strictly speaking it is wrong to make a distinction between $\alpha$ and $X_\alpha$. In any case $X$, as a set, is the disjoint union of the subsets $X_\alpha$ for $\alpha \in \pi_0(X)$. But as a space $X$ is not always the (topological) disjoint union of the $X_\alpha$. Also, the path components of $X$ should not be confused with the connected components of $X$. (Each connected component of $X$ is a union of path components of $X$.)

EXAMPLE 8.1.1. Let $X$ be the closure (in $\mathbb{R}^2$ with the euclidean metric) of

$$Y := \{ (x, \sin(1/x)) \mid x > 0 \}.$$

Then $X$ has exactly two path components: $Y$ and $X \smallsetminus Y$. So $\pi_0(X)$ has two elements. But $X$ has only one connected component. It follows that $X$ is not the topological disjoint union of its path components. Indeed, it is clear that one of the path components of $X$ is not a closed subset of $X$.

PROPOSITION 8.1.2. *For every $X$ and every $k \in \mathbb{Z}$, the inclusions $j_\alpha \colon X_\alpha \to X$ of the path components determine an isomorphism*

$$\oplus_\alpha (j_\alpha)_* \colon \bigoplus_{\alpha \in \pi_0(X)} H_k(X_\alpha) \longrightarrow H_k(X).$$

PROOF. It is enough to show that the singular chain complex $sC(X)$ splits as a direct sum of chain subcomplexes $sC(X_\alpha)$. For that it is enough to show that, for $n \geq 0$, the free abelian group $sC(X)_n$ is a direct sum

$$\bigoplus_\alpha sC(X_\alpha)_n$$

and that the differential $d \colon sC(X)_n \to sC(X)_{n-1}$ respects the splitting (for $n > 0$). These statements are nearly obvious. Namely, $sC(X)_n$ is freely generated by the set of continuous maps $\sigma \colon \Delta^n \to X$. Since $\Delta^n$ is always path connected, each $\sigma$ has image contained in a unique path component $X_\alpha$. Moreover, if $n > 0$ and $\sigma \colon \Delta^n \to X$ has image contained in $X_\alpha$, then

$$d(\sigma) = (-1)^i \sigma \circ \varphi_i$$

is a formal linear combination of terms $\sigma \circ \varphi_i \colon \Delta^{n-1} \to X$ whose image is again contained in the same path component $X_\alpha$. □

PROPOSITION 8.1.3. *If $X$ is nonempty and path connected, and $z \in X$, then the inclusion $\{z\} \to X$ induces an isomorphism*

$$\mathbb{Z} \cong H_0(\{z\}) \longrightarrow H_0(X).$$

Proof. From the definition of $H_0(X)$ we have a surjective map $sC(X)_0 \to H_0(X)$; therefore $H_0(X)$ is generated by the homology classes $[\sigma]$ where $\sigma\colon \Delta^0 \to X$ can be any map. To put it differently, $H_0(X)$ is generated by the images of $\sigma_*\colon H_0(\Delta^0) \to H_0(X)$. But these maps $\sigma\colon \Delta^0 \to X$ are all homotopic to each other (since $X$ is path connected), so that, by the homotopy axiom, the image of $\sigma_*\colon H_0(\Delta^0) \to H_0(X)$ is always the same. We may choose one of the $\sigma$, for example the one which hits $z \in X$, and we may conclude that $\sigma_*\colon H_0(\Delta^0) \to H_0(X)$ is surjective for this $\sigma$. It is also injective since we can make a map $\tau\colon X \to \Delta^0$ such that $\tau\sigma = \mathrm{id}\colon \Delta^0 \to \Delta^0$. □

Corollary 8.1.4. $H_0(X) \cong \bigoplus_{\alpha \in \pi_0(X)} \mathbb{Z}$.

## 8.2. Compact subspaces

Here we want to make the following idea precise. For every $k \in \mathbb{Z}$, the functor $H_k$ (from $\mathcal{T}op$ to ab$\mathcal{G}$roups) is determined by its behavior on *compact* spaces and continuous maps between compact spaces.

Proposition 8.2.1. *Let $X$ be any topological space and let $k \in \mathbb{Z}$. For every $z \in H_k(X)$ there exists a compact subspace $L \subset X$ and an element $z_L \in H_k(L)$ such that the homomorphism $H_k(L) \to H_k(X)$ induced by the inclusion $L \to X$ takes $z_L$ to $z$.*
*If $L$ and $M$ are two compact subspaces of $X$, and $z_L \in H_k(L)$, $z_M \in H_k(M)$ have the same image $z$ in $H_k(X)$, then there exists a compact subspace $N$ of $X$ such that $L \cup M \subset N$ and $z_L$ , $z_M$ have the same image in $H_k(N)$.*

Proof. First part: given $z \in H_k(X)$, choose a $k$-cycle $\bar{z}$ in $sC(X)$ representing $z$. This may be written in the form

$$\bar{z} = \sum_{i=1}^{r} a_i \sigma_i \quad \in sC(X)_k$$

where the $\sigma_i$ are continuous maps from $\Delta^k$ to $X$ and that $a_i$ are integers. Let

$$L := \bigcup_{i=1}^{r} \mathrm{im}(\sigma_i)\,.$$

This is a compact subspace of $X$ since $\mathrm{im}(\sigma_i)$ is compact for each $i$ (the image of a compact space under a continuous map is compact). Now we can think of $\bar{z}$ as a $k$-cycle in $sC(L) \subset sC(X)$, and we can write $z_L \in H_k(L)$ for its homology class.
Second part: Suppose that $z_L \in H_k(L)$ and $z_M \in H_k(M)$ are represented by $k$-cycles

$$\bar{z}_L \in sC(L) \subset sC(X), \qquad \bar{z}_M \in sC(M) \subset sC(X)\,.$$

By assumption, $\bar{z}_M - \bar{z}_L$ is a boundary in $sC(X)$; in other words, there exists $y \in sC(X)_{k+1}$ such that

$$d(y) = \bar{z}_M - \bar{z}_L\,.$$

Again we can write

$$y = \sum_{j=1}^{t} b_j \tau_j \quad \in sC(X)_{k+1}$$

where the $\tau_j$ are continuous maps from $\Delta^{k+1}$ to $X$. Let

$$N := L \cup M \cup \bigcup_{j=1}^{t} \mathrm{im}(\tau_j)$$

This is compact and the equation $d(y) = \bar{z}_M - \bar{z}_L$ can now be viewed as an equation in $sC(N)$. Therefore $z_L$ , $z_M$ have the same image in $H_k(N)$. □

## 8.3. Homology of realizations of semi-simplicial sets

Let $X$ be a semi-simplicial set. With that we associated a chain complex $C(X)$ where

$$C(X)_n = \text{free abelian group generated by the set } X_n .$$

The differential was defined, on a generator $z \in X_n \subset C(X)_n$ , by

$$d(x) := \sum_{i=1}^{n} (-1)^i f_i^* x$$

where $f_i\colon [n-1] \to [n]$ is the monotone injection which has image $[n] - \{i\}$. This chain complex $C(X)$ is in many cases a rather small and computable chain complex. For example, if $X_n$ is finite for all $n$, then $C(X)_n$ is a finitely generated free abelian group and the differential $d\colon C(X)_n \to C(X)_{n-1}$ can be described as a matrix with integer coefficients. (The matrix format is $a_{n-1} \times a_n$ where $a_n$ is the number of elements of $X_n$.)
By contrast the singular chain complex of the geometric realization $|X|$ is typically gigantic; if $X_k$ is nonempty for some $k > 0$, then $sC(|X|)_n$ is a free abelian group with an uncountable set of generators, for each $n \geq 0$. Unraveling the definitions, we can write

$$sC(|X|) = C(\mathrm{sing}(|X|))$$

which reminds us that the gigantic nature of this chain complex is due to the gigantic nature of the semi-simplicial set $\mathrm{sing}(|X|)$. But there is an important (natural) morphism of semi-simplicial sets

$$u\colon X \longrightarrow \mathrm{sing}(|X|).$$

This is the unit morphism of the adjunction

$$\text{geometric realization} : \text{ss}\mathcal{S}\text{ets} \rightleftharpoons \mathcal{T}\text{op} : \text{sing}$$

(See proposition 4.5.2 and definition 4.5.3.) In other words, $u \in \mathrm{mor}_{\mathrm{ss}\mathcal{S}\mathrm{ets}}(X, \mathrm{sing}(|X|))$ corresponds to $\mathrm{id} \in \mathrm{mor}_{\mathcal{T}\mathrm{op}}(|X|, |X|)$ under the natural bijection

$$\mathrm{mor}_{\mathrm{ss}\mathcal{S}\mathrm{ets}}(X, \mathrm{sing}(|X|)) \longrightarrow \mathrm{mor}_{\mathcal{T}\mathrm{op}}(|X|, |X|).$$

It is easy to give a formula for $u$:

$$u(z) = c_z$$

for $z \in X_n$ , where $c_z\colon \Delta^n \to |X|$ is the characteristic map associated with $z$. (If you feel like it, verify that this is a morphism of simplicial sets.)

THEOREM 8.3.1. *The map of chain complexes*

$$u_*\colon C(X) \longrightarrow C(\mathrm{sing}(|X|)) = sC(|X|)$$

*induces an isomorphism of homology groups* $H_n$ *, for all* $n \in \mathbb{Z}$*. Therefore* $H_n(|X|)$*, the* $n$*-th homology group of the space* $|X|$*, is isomorphic to the* $n$*-th homology group of the chain complex* $C(X)$*.*

*Slightly more generally, if* $Y$ *is a semi-simplicial subset of* $X$*, then*

$$u_*\colon C(X)/C(Y) \longrightarrow C(\mathrm{sing}(|X|))/C(\mathrm{sing}(|Y|)) = sC(|X|,|Y|)$$

*induces an isomorphism of homology groups* $H_n$ *for all* $n\in\mathbb{Z}$*. Therefore* $H_n(|X|,|Y|)$ *is isomorphic to* $H_n$ *of the chain complex* $C(X)/C(Y)$*.*

EXAMPLE 8.3.2. Let $X$ be the semi-simplicial set such that $X_0$ and $X_1$ have exactly one element, while all other $X_n$ are empty. Then $|X|$ is homeomorphic to $S^1$. The chain complex $C(X)$ has the following form: $C(X)_n$ is isomorphic to $\mathbb{Z}$ if $n=0$ or $1$, and is zero otherwise. The differential from $C(X)_1$ to $C(X)_0$ is zero (small calculation). Therefore it is immediately clear that $H_n(C(X))$ is isomorphic to $\mathbb{Z}$ if $n=0$ or $1$, and $H_n(C(X))=0$ otherwise. By the theorem, this computes the homology of $|X|\cong S^1$; that is, we get $H_1(S^1)\cong\mathbb{Z}\cong H_0(S^1)$ and $H_n(S^1)=0$ for all other values of $n$. Of course this only confirms what we already know.

EXAMPLE 8.3.3. Fix $n\ge 0$, let $X=\underline{\Delta}^n$ and let $Y=\partial\underline{\Delta}^n\subset X$ be the semi-simplicial subset such that $Y_k=X_k$ for $k<n$ but $Y_n=\emptyset$ (in contrast to $X_n$, which has one element). We can say without lying very hard that $|X|$ is $\Delta^n$ and that $|Y|$ is $\partial\Delta^n$ (the subspace of $\Delta^n$ consisting of all $(x_0,x_1,\dots,x_n)$, in barycentric coordinates, for which at least one $x_i$ is zero). — Then $C(X)/C(Y)$ is a very small chain complex: it has $(C(X)/C(Y))_n=\mathbb{Z}$ and $(C(X)/C(Y))_k=0$ for $k\neq n$. The preferred generator of $(C(X)/C(Y))_n=\mathbb{Z}$ maps to the element represented by

$$\mathrm{id}\colon \Delta^n\to|X|$$

under $u_*\colon C(X)/C(Y)\to sC(|X|,|Y|)$ (remember that $|X|=\Delta^n$). Therefore we have shown that the $n$-cycle $\mathrm{id}\colon\Delta^n\to\Delta^n$ in

$$sC(\Delta^n,\partial\Delta^n)$$

represents a generator of $H_n(\Delta^n,\partial\Delta^n)\cong\mathbb{Z}$. This is quite useful.

One important ingredient in the proof of theorem 8.3.1 is lemma 8.3.4 just below. In the lemma we assume that $X$ is a semi-simplicial set and that $n$ is a positive integer. We assume that $Y\subset X$ is a semi-simplicial subset, and that $X_n\smallsetminus Y_n$ has exactly one element $z$, whereas $X_k\smallsetminus Y_k$ is empty for all $k\neq n$. The characteristic map

$$c_z\colon\Delta^n\longrightarrow|X|$$

associated with $z\in X_n$ can then be viewed as a map of pairs:

$$c_z\colon(\Delta^n,\partial\Delta^n)\longrightarrow(|X|,|Y|)\,.$$

Here, as before, $\partial\Delta^n\subset\Delta^n$ is the subspace consisting of all $(x_0,x_1,\dots,x_n)$ (in barycentric coordinates) for which at least one of the $x_i$ is zero.

LEMMA 8.3.4. *The homomorphisms in homology*

$$H_k(\Delta^n,\partial\Delta^n)\longrightarrow H_k(|X|,|Y|)$$

*induced by* $c_z$ *are isomorphisms.*

PROOF. Let $V=\Delta^n\smallsetminus b_n$ where $b_n$ is the barycenter of $\Delta^n$. Let $W=|X|\smallsetminus c_z(b_n)$. Then $V$ is open in $\Delta^n$ and $W$ is open in $|X|$. We have

$$\partial\Delta^n\subset V\subset\Delta^n,\qquad |Y|\subset W\subset|X|\,.$$

It is easy to see that the inclusions $\partial\Delta^n \to V$ and $|Y| \to W$ are homotopy equivalences. Now we have a commutative diagram

$$\begin{array}{ccc}
H_k(\Delta^n, \partial\Delta^n) & \longrightarrow & H_k(|X|, |Y|) \\
\downarrow & & \downarrow \\
H_k(\Delta^n, V) & \longrightarrow & H_k(|X|, W) \\
\uparrow & & \uparrow \\
H_k(\Delta^n \smallsetminus \partial\Delta^n, V \smallsetminus \partial\Delta^n) & \longrightarrow & H_k(|X| \smallsetminus |Y|, W \smallsetminus |Y|)
\end{array}$$

where the upper vertical arrows are isomorphisms by lemma 7.5.1 and the lower vertical arrows are isomorphisms by excision. (All vertical arrows are induced by inclusion maps.) The lower horizontal arrow is an isomorphism because the map of pairs

$$(\Delta^n \smallsetminus \partial\Delta^n, V \smallsetminus \partial\Delta^n) \longrightarrow (|X| \smallsetminus |Y|, W \smallsetminus |Y|)$$

which induces it is a homeomorphism. $\square$

Proof of theorem 8.3.1 for finite $X$. Here we assume that the semi-simplicial set $X$ is finite, that is to say, each of the sets $X_k$ is finite and there exists $n \geq -1$ such that $X_k = \emptyset$ for $k > n$. The minimal $n$ with this property is called the *dimension* of $X$. (In particular, if $X_k = \emptyset$ for all $n \geq 0$, we say that the dimension of $X$ is $-1$.)
(i) *Induction beginning*: If $X$ has dimension $-1$, then $C(X)$ is the zero chain complex and $|X| = \emptyset$ and $sC(|X|)$ is also the zero chain complex. Therefore $u_*\colon C(X) \longrightarrow sC(|X|)$ is an isomorphism of chain complexes.
(ii) *Not an induction step*: let us verify that the theorem is true for $X = \underline{\Delta}^n$, any $n \geq 0$. In this case $|X| = \Delta^n$ and we know that $H_0(\Delta^n) \cong \mathbb{Z}$ and $H_k(\Delta^n) = 0$ for $k \neq 0$. Therefore it is nearly enough to show that $H_0(C(X)) \cong \mathbb{Z}$ and $H_k(C(X)) = 0$ for $k \neq 0$. This was an exercise (a few weeks ago) and I do not want to spoil the exercise. But we still have to show that $u_*\colon C(X) \to sC(|X|)$ induces an isomorphism

$$H_0(C(X)) \longrightarrow H_0(sC(|X|)) = H_0(|X|) = H_0(\Delta^n).$$

Since $\Delta^n$ is path connected, we know (by ...) that any map $\sigma\colon \Delta^0 \to \Delta^n$ is a $0$-cycle in $sC(\Delta^n)$ and as such represents a generator of $H_0(\Delta^n) \cong \mathbb{Z}$. If we choose $\sigma$ so that it hits one of the vertices of $\Delta^n$, for example $(1, 0, 0, \ldots, 0)$, then it is in the image of the chain map $u_*\colon C(X) \to sC(|X|)$. Therefore this homomorphism $H_0(C(X)) \longrightarrow H_0(sC(|X|))$ is surjective, and so it is an isomorphism (because any surjective homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ is injective).
(iii) *Induction step*: Suppose that $X$ has dimension $n$ and that $X_n$ has $r$ elements, where $r > 0$. Choose $z \in X_n$ and let $Y \subset X$ be the simplicial subset which has $Y_k = X_k$ for $k < n$ and $Y_n = X_n \smallsetminus \{z\}$. Now we want to prove the theorem for $X$ and we may assume that it holds for all semi-simplical sets $Z$ which have either dimension $< n$, or dimension $= n$ and fewer than $r$ elements in $Z_n$. In particular, we may assume that the theorem holds for $Y$. Now we have the following commutative diagram of chain complexes

$$(*) \qquad \begin{array}{ccccc}
C(Y) & \longrightarrow & C(X) & \longrightarrow & C(X)/C(Y) \\
\downarrow & & \downarrow & & \downarrow \\
sC(|Y|) & \longrightarrow & sC(|X|) & \longrightarrow & sC(|X|)/sC(|Y|)
\end{array}$$

where the rows are short exact. We know that the left-hand vertical arrow induces an isomorphism in homology groups. We want to know that the middle vertical arrow induces an isomorphism in homology groups. Therefore (by remark 7.4.7) it suffices to show that the right-hand vertical arrow induces an isomorphism in homology groups. In order to deal with this, we set up another commutative diagram

$$(**) \qquad \begin{array}{ccc} C(X)/C(Y) & \longleftarrow & C(\underline{\Delta}^n)/C(\partial\underline{\Delta}^n) \\ \downarrow & & \downarrow \\ sC(|X|)/sC(|Y|) & \longleftarrow & sC(\Delta^n)/sC(\partial\Delta^n) \end{array}$$

where the vertical arrows are of type $u_*$ and the horizontal arrows are induced by the unique morphism $\underline{\Delta}^n \to X$ which takes the unique element in $(\underline{\Delta}^n)_n$ to $z \in X_n$. By lemma 8.3.4, the lower horizontal arrow induces an isomorphism in homology groups. By inspection, the upper horizontal arrow induces an isomorphism in homology groups. (Indeed the underlying chain map is an isomorphism; the two chain complexes involved are zero in all dimensions except dimension $n$, where $C(X)_n/C(Y)_n \cong \mathbb{Z}$ and similarly $C(\underline{\Delta}^n)_n/C(\partial\underline{\Delta}^n)_n \cong \mathbb{Z}$.) Therefore it is enough to show that the vertical arrow on the right in $(**)$ induces an isomorphism in homology. In order to show this, we return to diagram $(*)$, taking $X = \underline{\Delta}^n$ and $Y = \partial\underline{\Delta}^n$. Now we want to show that the right-hand vertical arrow in $(*)$ induces an isomorphism in homology. But we know that the middle vertical arrow induces an isomorphism in homology (by step (ii) of this proof, which was neither an induction beginning nor an induction step) and we know that the vertical arrow on the left induces an isomorphism in homology (by inductive assumption, since $\partial\underline{\Delta}^n$ is $(n-1)$-dimensional. This is enough by remark 7.4.7. □

To complete the proof of theorem 8.3.1 in general, i.e., to remove the condition that $X$ is finite, we need mainly proposition 8.1.3. But we also need an observation concerning the topology of $|X|$. This could have been mentioned earlier.

LEMMA 8.3.5. *Let $X$ be a semi-simplicial set. If $L \subset |X|$ is a compact subset, then there is a finite semi-simplicial subset $Y \subset X$ such that $L \subset |Y| \subset |X|$.*

PROOF. Suppose there is no finite semi-simplicial subset $Y$ of $X$ such that $L \subset |Y|$. Then $L$ is nonempty, and we can choose some $a(0) \in L$. As in lemma 4.3.2, the element can be written in a unique way as $c_z(w)$ for some $z \in X_r$ (and some $r$) and some $w \in \Delta^r$ whose barycentric coordinates are all $> 0$. Here $c_z \colon \Delta^r \to |X|$ is the characteristic map associated with $z$. For the bookkeeping, we write $z(0)$ instead of $z$ and $r(0)$ instead of $r$ and $w(0)$ instead of $w$, so that

$$a(0) = c_{z(0)}(w(0))$$

where $z(0) \in X_{r(0)}$. Next, we choose $a(1) \in L$ which is not contained in the smallest semi-simplicial subset of $X$ that contains $z(0) \in X_{r(0)}$. Then $a(1)$ can be written in a unique way as

$$a(1) = c_{z(1)}(w(1))$$

for some $z(1) \in X_{r(1)}$ and $w(1) \in \Delta^{r(1)}$ whose barycentric coordinates are all $> 0$. Next, we choose $a(2) \in L$ which is not contained in the smallest semi-simplicial subset of $X$ that contains $z(0) \in X_{r(0)}$ and $z(1) \in X_{r(1)}$. Then $a(2)$ can be written in a unique way as

$$a(2) = c_{z(2)}(w(2))$$

for some $z(2) \in X_{r(2}$ and $w(2) \in \Delta^{r(2)}$. And so on. We have constructed an infinite subset

$$S = \{a(0), a(1), a(2), \dots\} \subset L\,.$$

Claim: $S$ is a closed subset of $X$ and every subset of $S$ is also closed in $X$. To prove this for $S$, we ask: what is the pre-image $q^{-1}(S)$ of $S$ under the quotient map

$$q\colon \coprod_{r\geq 0}\coprod_{z\in X_r} \Delta^r \longrightarrow |X|.$$

Whatever it is exactly, its intersection with any $\Delta^r$ corresponding to some $r \geq 0$ and $z \in X_r$ is a finite subeset of $\Delta^r$ and as such a closed subset of $\Delta^r$. Therefore $q^{-1}(S)$ is closed and therefore $S$ is closed in $|X|$, by definition of the topology on $|X|$. The same argument works for any subset of $S$.

Now it is easy to finish the argument. Since $S$ is a closed subset of the compact Hausdorff space $L$, it is compact in its own right. But we saw that the subspace topology on $S$ is discrete (every subset of $S$ is closed in $S$, and so every subset of $S$ is open in $S$). It follows that $S$ is finite. Contradiction. □

Proof of theorem 8.3.1 for general $X$. The aim is to show that

$$(*) \qquad H_k(C(X)) \longrightarrow H_k(sC(|X|)) = H_k(|X|)$$

induced by $u\colon X \to \mathrm{sing}(|X|)$ is bijective. The case where $X$ is finite has been taken care of. To show surjectivity of $(*)$ in general, we choose some $z \in H_k(X)$. By proposition 8.2.1, there exists a compact subspace $L$ of $|X|$ such that $z$ is in the image of $H_k(L) \to H_k(|X|)$, homomorphism induced by the inclusion. By lemma 8.3.5, there exists a finite simplicial subset $Y$ of $X$ such that $L \subset |Y| \subset |X|$. Therefore $z$ is in the image of $H_k(|Y|) \to H_k(|X|)$, homomorphism induced by the inclusion. The commutative diagram

$$\begin{array}{ccc} H_k(C(Y)) & \xrightarrow{\cong} & H_k(|Y|) \\ \downarrow & & \downarrow \\ H_k(C(X)) & \longrightarrow & H_k(|X|) \end{array}$$

(horizontal arrows induced by $u$, vertical arrows by inclusions) allows us to conclude that $z$ comes from $H_k(C(X))$. This proves surjectivity.

To prove injectivity, suppose that $w \in H_k(C(X))$ is in the kernel of $(*)$. By definition of $C(X)$, we can find a finite semi-simplicial subset $Y$ of $X$ such that $w$ comes from an element $w_Y \in H_k(C(Y))$. By proposition 8.2.1, there exists a compact subspace $M$ of $|X|$, containing $|Y|$, such that $w_Y$ is taken to $0$ in $H_k(M)$. By lemma 8.3.5, there exists a finite simplicial subset $Y'$ of $X$ such that $M \subset |Y'| \subset |X|$. Then $w_Y$ is taken to zero in $H_k(|Y'|)$. Therefore $w_Y$ is taken to zero in $H_k(C(Y'))$, and also to zero in $H_k(C(X))$, and this means $w = 0$. □

Proof of theorem 8.3.1 for pairs $(X, Y)$. Let $X$ be a semi-simplicial set and let $Y \subset X$ be a semi-simplicial subset. Then we have a commutative diagram of chain complexes

$$\begin{array}{ccccc} C(Y) & \longrightarrow & C(X) & \longrightarrow & C(X)/C(Y) \\ \downarrow & & \downarrow & & \downarrow \\ sC(Y) & \longrightarrow & sC(X) & \longrightarrow & sC(X)/sC(Y) \end{array}$$

with short exact rows. The vertical arrows are of type $u_*$. We know already that the left-hand vertical arrow and the middle vertical arrow induce isomorphisms in $H_k$ for all $k \in \mathbb{Z}$. Therefore, by the five lemma, the right-hand vertical arrow induces an isomorphism in $H_k$ for all $k$. (The five lemma is applicable once we write out the long exact homology sequences for the two short exact rows, and arrange them in a ladder-shaped diagram.) ☐

Corollary 8.3.6. *Let $X$ be a finite semi-simplicial set, i.e., the disjoint union of the sets $X_n$ for $n \geq 0$ is finite. Let $a_n$ be the number of elements of $X_n$. Then the Euler characteristic of $|X|$ is $\sum_{n\geq 0}(-1)^n a_n$.*

Proof. Since $C(X)_n$ is the free abelian group generated by the set $X_n$, we have $\mathrm{rk}(C(X)_n) = a_n$. By proposition 5.4.2, the Euler characteristic $\chi(C(X))$ of the chain complex $C(X)$ is

$$\sum_{n\geq 0}(-1)^n \mathrm{rk}(H_n(C(X)) = \sum_{n\geq 0}(-1)^n \mathrm{rk}(C(X)_n) = \sum_{n\geq 0}(-1)^n a_n .$$

But $H_n(C(X)) \cong H_n(|X|)$ by theorem 8.3.1. It follows that the Euler characteristic of $|X|$ is defined and equal to $\sum_{n\geq 0}(-1)^n a_n$. ☐

Example 8.3.7. Let $X$ be any finite semi-simplicial set with the property that $|X|$ is *homotopy equivalent* to $S^3$. Writing $a_n$ for the number of elements of $X_n$ as before, we must have

$$\sum_{n\geq 0}(-1)^n a_n = 0$$

because the Euler characteristic of $S^3$ is $0$. (Compute this directly from the definition: $\sum_{n\geq 0}(-1)^n \mathrm{rk}(H_n(S^3))$, using theorem 7.2.2.)
In the exercises we constructed a finite semi-simplicial set $X$ with the property that $|X|$ is *homeomorphic* to $S^3$. In that construction we had $a_0 = 2$, $a_1 = 3$, $a_2 = 2$ and $a_3 = 1$. The alternating sum of these is indeed $2 - 3 + 2 - 1 = 0$.

## 8.4. Jordan curve theorem, Schönflies theorem and related matters

The Jordan curve theorem in the easiest imaginable form states that if $f\colon S^1 \to \mathbb{R}^2$ is an injective continuous map, then $\mathbb{R}^2 \smallsetminus f(S^1)$ has two connected components. The Schönflies theorem states that $f$ can be extended to a continuous injective map $F\colon D^2 \to \mathbb{R}^2$ (so that $F$ restricted to $S^1 \subset D^2$ agrees with $f$).
The Schönflies statement does not have a straightforward generalization to higher dimensions. There is an example of a continuous injective map $S^2 \to \mathbb{R}^3$ wich does not extend to a continuous injective $F\colon D^3 \to \mathbb{R}^3$.
But there is a version of the Jordan curve theorem for arbitrary dimensions. The statement is as follows: if $f\colon S^{m-1} \to \mathbb{R}^m$ is an *injective* continuous map, then $\mathbb{R}^m \smallsetminus f(S^{m-1})$ has two connected components.
Let us reformulate this statement in homological language. First of all, it does not make a great difference whether we use $\mathbb{R}^m$ or $S^m$ as the target space, since we can think of $\mathbb{R}^m$ as $S^m$ minus a point. But it turns out that we get prettier results if we use $S^m$. Suppose then that $f\colon S^{m-1} \to S^m$ is continuous and injective. Since $f(S^{m-1})$ is compact, $S^m \smallsetminus f(S^{m-1})$ is an open subset of $S^m$. For an open subset of $S^m$, the connected components are path connected (exercise), and so they are also the path components. Therefore the "generalized Jordan curve statement" is saying that $S^m \smallsetminus f(S^{m-1})$ has two path components. This is equivalent to saying that

$$H_0(S^m \smallsetminus f(S^{m-1})) \cong \mathbb{Z} \oplus \mathbb{Z} .$$

And this is also equivalent to saying that $H_0(S^m \smallsetminus f(S^{m-1})) \cong H_0(S^0)$. But it turns out that we can prove more.

THEOREM 8.4.1. *Let* $f\colon S^k \to S^m$ *be an injective continuous map, where* $k < m$. *Then*

$$H_j(S^m \smallsetminus f(S^k)) \cong H_j(S^{m-k-1})$$

*for all* $j \in \mathbb{Z}$.

In particular, if $k = m-1$ then $S^m \smallsetminus f(S^k)$ has the same homology as $S^0$, meaning that it has two connected components and each of these has the homology of a point. This goes some way towards a Schönflies statement (but not all the way — for example, we are not allowed to conclude that each of these connected components is contractible).
We can allow $k = -1$ in theorem 8.4.1. This is not a very interesting case, since $S^{-1} = \emptyset$. But we will see that it is useful as an induction beginning.

In proving theorem 8.4.1, and also theorem 8.5.1 below, I follow E. Spanier (his book, *Algebraic Topology*, McGraw-Hill 1966, ch.4 §7). It is a very fine book, hard to beat in the selection of topics, although the style has something old-fashioned to it. I feel guilty for not recommending this earlier.

PROPOSITION 8.4.2. *Let* $I = [0, 1]$ *and let* $g\colon I^k \to S^m$ *be an injective continuous map, where* $0 \le k \le m$. *Then* $H_j(S^m \smallsetminus g(I^k)) \cong H_j(*)$ *for all* $r \in \mathbb{Z}$.

PROOF. We proceed by induction on $k$. Induction beginning: if $k = 0$ then $S^m \smallsetminus g(I^k)$ is homeomorphic to $\mathbb{R}^m$ since $g(I^k)$ is a single point. Therefore $H_r(S^m \smallsetminus g(I^k))$ is isomorphic to $H_r(\mathbb{R}^m) \cong H_r(*)$ in the case $k = 0$.
For the induction step we introduce some notation and terminology. The *reduced homology* $\tilde{H}_j(X)$ of a nonempty space $X$ is the kernel of the (surjective) homomorphism

$$H_j(X) \to H_j(*)$$

induced by the unique map $X \to *$. Clearly the tilde makes a difference only if $j = 0$. Nevertheless this is useful notation in situations where we prefer to be unspecific about $j$. If $X$ is the union of two subsets $V$ and $W$ such that $\mathrm{int}(V) \cup \mathrm{int}(W) = X$, then we have the long exact Mayer-Vietoris sequence in homology involving $H_*(X)$ and $H_*(V) \oplus H_*(W)$, $H_*(V \cap W)$. If moreover $V$, $W$ and $V \cap W$ are nonempty, then we can replace $H_j$ by $\tilde{H}_j$ in that sequence (where applicable) and it will still be a long exact sequence (easy diagram chase). This is the long exact *Mayer-Vietoris sequence in reduced homology.*
In that connection, let us note that $g\colon I^k \to S^m$ is not surjective, for otherwise it would be a homeomorphism (a continuous bijective map between compact Hausdorff spaces is a homeomorphism). It cannot be a homeomorphism because $I^k$ and $S^m$ can be distinguished by their homology groups. It follows that $S^m \smallsetminus g(I^k) \neq \emptyset$.
More notation: for a closed interval $J \subset I$, let $V(J) = S^m \smallsetminus g(I^{k-1} \times J)$. (We allow cases where $J$ is a single point.) This is a nonempty open subset of $S^m$. In particular for $J = I$ we get $V(I) = S^m \smallsetminus g(I^k)$. In reduced homology language, what we have to show is that

$$\tilde{H}_r(V(I)) = 0 \qquad \text{for all } r.$$

Now assume that this statement is *false* for a specific $k > 0$ and for our choice of $g$, but correct in all cases for the previous integer, also known as $k-1$. We need to generate a contradiction from that. Since we are assuming that $\tilde{H}_j(V(I))$ is not zero for all $j$, we can choose $r \ge 0$ and a nonzero element

$$z_0 \in \tilde{H}_r(V(J)).$$

As a first step towards the contradiction which we must generate, we show:

(∗) *there exists a closed interval* $J \subset I$, *of length* $1/2$, *such that the image of* $z_0$ *under the homomorphism* $H_r(V(I)) \to H_r(V(J))$ *induced by the inclusion* $V(I) \to V(J)$ *is still nonzero.*

To prove this we write $I = J' \cup J''$ where $J' = [0, 1/2]$ and $J'' = [1/2, 1]$. Then

$$V(I) = V(J' \cup J'') = V(J') \cap V(J''), \qquad V(J' \cap J'') = V(J') \cup V(J'').$$

Since $J' \cap J''$ is a point, we know (by inductive assumption) that the reduced homology of $V(J') \cup V(J'') = V(J' \cap J'') = S^m \smallsetminus g(I^{k-1} \times \{1/2\})$ is zero in all dimensions. From the exactness of the Mayer-Vietoris sequence in reduced homology (for the open covering of $V(J') \cup V(J'')$ by $V(J')$ and $V(J'')$), we obtain therefore that the homomorphism

$$\tilde{H}_r(V(I)) \longrightarrow \tilde{H}_r(V(J')) \oplus \tilde{H}_r(V(J''))$$

in the sequence (induced by the inclusions $V(I) \to V(J')$ and $V(I) \to V(J'')$ , up to a sign which is not important here) is an isomorphism. Therefore one of these inclusion-induced homomorphisms

$$\tilde{H}_r(V(I)) \to \tilde{H}_r(V(J')), \qquad \tilde{H}_r(V(I) \to \tilde{H}_r(V(J''))$$

must take $z_0$ to a nonzero element. If the first one does that, we take $J := J'$; if not, then $J := J''$. This proves (∗).

Iterating this construction, we obtain a descending infinite sequence of closed intervals

$$I = J_0 \supset J_1 \supset J_2 \supset J_3 \supset \dots$$

such that $J_s$ has length $2^{-s}$ and such that the image $z_s$ of $z_0$ under the inclusion-induced homomorphism

$$\tilde{H}_r(V(I)) \longrightarrow \tilde{H}_r(V(J_s))$$

is nonzero, for $s = 1, 2, 3, \dots$ . Now we can make good use of proposition 8.1.3. We have an increasing sequence of open subsets

$$V(I) = V(J_0) \subset V(J_1) \subset V(J_2) \subset V(J_3) \subset \dots$$

Let $W$ be the union, $W = \bigcup_{s \geq 0} V(J_s)$. Let $z_\infty \in \tilde{H}_r(W)$ be the image of $z_0$ under

$$\tilde{H}_r(V(I)) \to \tilde{H}_r(W)$$

induced by the inclusion $V(I) \to W$. We want to show $z_\infty \neq 0$ (as part of our struggle to generate a contradiction). In any case we can find a compact subset $K \subset V(I)$ such that $z_0$ comes from an element

$$\bar{z}_0 \in \tilde{H}_r(K).$$

If $z_0$ maps to zero in $\tilde{H}_r(W)$, then so does $\bar{z}_0$ and so there exists compact $L \subset W$ such that $K \subset L$ and $\bar{z}_0$ maps to zero already in $\tilde{H}_r(L)$. But $L$ must be contained in $V(J_s)$ for some $s$. Therefore $\bar{z}_0$ and consequently also $z_0$ map to zero in

$$\tilde{H}_r(V(J_s))$$

for this $s$. This contradicts what we know: $z_s \neq 0$. Therefore $z_\infty \neq 0$ as claimed. In particular we get $\tilde{H}_r(W) \neq 0$. But this contradicts our inductive assumption, since

$$W = V(\bigcap_{s \geq 0} J_s) = S^m \smallsetminus g(I^{k-1} \times \{p\})$$

where $p$ is the unique element of $\bigcap_{s \geq 0} J_s$. $\square$

Proof of theorem 8.4.1. Again we proceed by induction on $k$. And again we use reduced homology; so the statement to be proved is that $S^m \smallsetminus f(S^k)$ is nonempty and

$$\tilde{H}_j(S^m \smallsetminus f(S^k)) \cong \begin{cases} \mathbb{Z} & \text{if } j = m-k-1 \\ 0 & \text{otherwise} \end{cases}$$

A good induction beginning is the case $k = -1$; as pointed out earlier, an obvious case. For the induction step we assume $k \geq 0$ and $k < m$ and we write $S^k = A \cup B$ where $A$ is the closed upper hemisphere and $B$ is the closed lower hemisphere. Then $A \cap B$ is the equator, $A \cap B = S^{k-1}$. The open set $V = S^m \smallsetminus f(A \cap B)$ is the union of open sets $W_0 = S^m \smallsetminus f(A)$ and $W_1 = S^m \smallsetminus f(B)$. By inductive assumption, $V$ is nonempty and we have

$$\tilde{H}_j(V) \cong \begin{cases} \mathbb{Z} & \text{if } j = m-k \\ 0 & \text{otherwise} \end{cases}$$

since $V = S^m \smallsetminus f(A \cap B) = S^m \smallsetminus f(S^{k-1})$. By proposition 8.4.2 the reduced homology groups of $W_0$ and $W_1$ are all zero. It follows that $W_0 \cap W_1$ is nonempty; otherwise $V = W_0 \cup W_1$ has two connected components, which contradicts $\tilde{H}_0(V) = 0$. — Therefore we may use the Mayer-Vietoris sequence in reduced homology (for the open cover of $V$ by $W_0$ and $W_1$). Exactness implies

$$\tilde{H}_{j+1}(V) \cong \tilde{H}_j(W_0 \cap W_1)$$

for all $j$. But this is exactly what we need since $W_0 \cap W_1 = S^m \smallsetminus f(A \cup B) = S^m \smallsetminus f(S^k)$. □

## 8.5. Invariance of domain

Theorem 8.5.1. *Let $V$ and $W$ be subsets of $\mathbb{R}^m$. If $V$ is homeomorphic to $W$ and $V$ is open in $\mathbb{R}^m$, then $W$ is open in $\mathbb{R}^m$.*

Proof. Let $f\colon V \to W$ be a homeomorphism. Let $g\colon V \to S^m$ be the composition of $f\colon V \to W$ and the inclusion $W \to \mathbb{R}^m \subset \mathbb{R}^m \cup \infty = S^m$. It suffices to show that $g(V) = W$ is open in $S^m$. Let $x \in V$ and choose a compact neighborhood $K$ of $x$ in $V$ which is a small disk (radius $\varepsilon$ and center $x$). This has a boundary sphere $\partial K \cong S^{m-1}$. By the Schönflies theorem, $S^m \smallsetminus g(\partial K)$ has two connected components. Now $S^m \smallsetminus g(\partial K)$ is the disjoint union (as a set, disregarding the topology) of $g(K \smallsetminus \partial K)$ and $S^m \smallsetminus g(K)$. Both of these subsets are connected; the first because it is the image of a connected set under a continuous map, and the other by proposition 8.4.2. Therefore these two subsets must be *the* connected components of $S^m \smallsetminus g(\partial K)$. As such they are open in $S^m$, because $S^m \smallsetminus g(\partial K)$ is clearly open in $S^m$. In particular $g(K \smallsetminus \partial K)$ is open in $S^m$. It also contains $g(x)$. Therefore $g(V)$ is a neighborhood of $g(x)$. Since this holds for all $x \in V$, or for all $g(x) \in g(V)$, it follows that $g(V)$ is open in $S^m$. □

Corollary 8.5.2. *Let $V$ be a nonempty open subset of $\mathbb{R}^m$ and let $g\colon V \to \mathbb{R}^n$ be a continuous map. If $g$ is injective, then $n \geq m$.*

Proof. Suppose for a contradiction that $n < m$. Choose a linear injective map $e\colon \mathbb{R}^n \to \mathbb{R}^m$. Choose $x \in V$ and choose a compact neighborhood $K$ of $x \in V$. Now $e \circ g$ gives a homeomorphism from $K$ to $e(g(K))$, and from $\mathrm{int}(K)$ to $e(g(\mathrm{int}(K)))$. But $\mathrm{int}(K)$ is open in $\mathbb{R}^m$ whereas $e(g(\mathrm{int}(K)))$ is not open in $\mathbb{R}^m$ (because it is contained in the linear subspace $e(\mathbb{R}^n)$ of $\mathbb{R}^m$). This contradicts theorem 8.5.1. □

CHAPTER 9

# Review

In a (mathematical) lecture course with clear goals, the lecturer may not find the time to point out all the pitfalls, to be generous in giving examples or counterexamples, and to point out important analogies which may explain a new definition in terms of older ones. In the last two weeks, which were set aside for a review, I wanted to take the time to do just that. Time to make "associations".

## 9.1. Mo 15.1. and Thu 18.1.

*Notion of homotopy.* We define *homotopies* before we define the meaning of *homotopy equivalences* and *homotopy equivalent.*
Structure/syntax: a *homotopy* between maps $f, g\colon X \to Y$ is a map $h\colon X \times [0,1] \longrightarrow Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x$. If such an $h$ exists, we say that $f$ and $g$ are *homotopic*. (Continuity is assumed where applicable.)
A map $p\colon Y \to Z$ is a *homotopy equivalence* if there exists a map $q\colon Z \to Y$ such that $qp$ is homotopic to $\mathrm{id}_Y$ and $pq$ is homotopic to $\mathrm{id}_Z$. If such a homotopy equivalence exists, we say that $Y$ and $Z$ are *homotopy equivalent.*
Moral: we deform maps (as in a homotopy), but as a rule we do not try to *deform spaces* in order to prove that they are homotopy equivalent.

*Example.* Take integers $\ell > k > 0$. Then we have the usual inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^\ell$ and also $S^{k-1} \hookrightarrow S^{\ell-1}$.

$$\begin{tikzcd} S^{k-1} \arrow[r, "\text{incl.}"] \arrow[d, "\text{incl.}"] & S^{\ell-1} \arrow[d, "\text{incl.}"] \\ \mathbb{R}^k \arrow[r, "\text{incl.}"] & \mathbb{R}^\ell \end{tikzcd}$$

Let us show that $B = S^{\ell-1} \smallsetminus S^{k-1}$ is homotopy equivalent to $S^{\ell-k-1}$. More precisely, let

$$A = \big\{(x_1, x_2, \dots, x_\ell) \in S^{\ell-1} \subset \mathbb{R}^\ell \;\big|\; x_1, x_2, \dots, x_k = 0\big\}.$$

This is the unit sphere in the linear subspace of $\mathbb{R}^\ell$ which is the orthogonal complement of $\mathbb{R}^k$. Let us show that the inclusion $f\colon A \to B$ is a homotopy equivalence. Then we need, first of all, a (continuous) map $g\colon B \to A$. I define this by the unsurprising formula

$$g(x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_\ell) := \frac{1}{\|(0,0,\dots,0,x_{k+1},\dots,x_\ell)\|}(0,0,\dots,0,x_{k+1},\dots,x_\ell).$$

Then $gf$ is *equal* to the $\mathrm{id}_A$ (so we don't have to work hard to invent a homotopy from $gf$ to $\mathrm{id}_A$). But we need a homotopy from $fg$ to $\mathrm{id}$. This can be done by the unsurprising formula

$$h_t(x_1, \dots, x_k, x_{k+1}, \dots, x_\ell) := \frac{1}{\|(tx_1, \dots, tx_k, x_{k+1}, \dots, x_\ell)\|}(tx_1, \dots, tx_k, x_{k+1}, \dots, x_\ell).$$

Then $h_0$ is $fg$ and $h_1$ is $\mathrm{id}_B$. Finished.

The following situation could be regarded as an interesting exception to the rule *we do not deform spaces in order to show that one space is homotopy equivalent to another.* Suppose that $p\colon E \to B$ is a fibration. (See definition 2.5.1.) Let $x, y$ be two points in $B$. We might want to show that $p^{-1}(x)$ is homotopy equivalent to $p^{-1}(y)$. If there is a path $\gamma\colon [0,1] \to B$ from $x$ to $y$ in $B$, then this is true. (And yikes, this was not proved in lectures officially, although it is easy. A dreadful omission on my part and a very good exercise for you on the theme of fibrations.) We might loosely say that the spaces $p^{-1}(\gamma(t))$ for $t \in [0,1]$ constitute a *deformation* of spaces starting with $p^{-1}(x) = p^{-1}(\gamma(0))$ and ending with $p^{-1}(y) = p^{-1}(\gamma(1))$. But saying so does not constitute a *proof* that $p^{-1}(x)$ and $p^{-1}(y)$ are homotopy equivalent (although they are).

We have developed good methods for showing (sometimes) that two maps $f, g\colon X \to Y$ are not homotopic. *Example*: if $f, g\colon X \to Y$ induce different homomorphisms from $H_j(X)$ to $H_j(Y)$ for some $j \in \mathbb{Z}$, then they are not homotopic. This follows from the important theorem 7.1.2. *Example of example*: let $f\colon S^2 \to S^2$ be the identity map and let $g\colon S^2 \to S^2$ be the antipodal map, $g(x) = -x$. Then $f$ and $g$ induce different homomorphisms from $H_2(S^2) \cong \mathbb{Z}$ to $H_2(S^2) \cong \mathbb{Z}$. Therefore they are not homotopic. See theorem 7.2.4.
We have also developed good methods for showing (sometimes) that two spaces are not homotopy equivalent. *Example*: if $X$ and $Y$ are spaces and $H_j(X)$ is not isomorphic to $H_j(Y)$, for some $j \in \mathbb{Z}$, then $X$ is not homotopy equivalent to $Y$. (Again this follows from the important theorem7.1.2.) For example, we found that $H_3(S^3) \cong \mathbb{Z}$ whereas $H_3(S^2) = 0$. Therefore $S^3$ is not homotopy equivalent to $S^2$. Similarly, $S^3$ is not homotopy equivalent to a point $*$, since $H_3(*) = 0$.
So far we have hardly developed any methods for showing that two maps $f, g\colon X \to Y$ *are* homotopic, or that two spaces $X, Y$ *are* homotopy equivalent. Some exceptions: very early in the course we showed that every map $S^1 \to S^1$ is homotopic to one of the maps given by $z \mapsto z^n$ (fixed $n$, complex number notation). The HLP, homotopy lifting property, is a condition on a map which says something about existence of homotopies. See definition 2.5.1.

There was an "iffy" example with the Hopf map $p\colon S^3 \to S^2$. We showed that the Hopf map is a fibration (by showing that it is a fiber bundle). This led to an argument as follows: *if* there is a homotopy from $p$ to a constant map, *then* there is a homotopy from $\mathrm{id}\colon S^3 \to S^3$ to a constant map. [Later we learned, using homology, that there is no homotopy from $\mathrm{id}\colon S^3 \to S^3$ to a constant map; and so we were able to conclude that there is no homotopy from $p\colon S^3 \to S^2$ to a constant map.]

I repeated the definitions of: *Category, functor, natural transformation.* There is no point in writing these out here; see chapter 3. [But let me just emphasize again that the homotopy category is an interesting example of a category. The objects are the topological spaces. A morphism from $X$ to $Y$ is a homotopy class of maps from $X$ to $Y$. Composition of morphisms is given by $[g] \circ [f] := [g \circ f]$, where $f$ might be a map from $X$ to $Y$ and $g$ might be a map from $Y$ to $Z$, and $[f]$ denotes the homotopy class of $f$. It is of course neccessary to show that this composition is well defined, since we gave the definition using representatives. Also, before it comes to that, one should show that the homotopy relation is an equivalence relation. We did all that long ago.]

In a category $\mathcal{C}$, we often write $f\colon a \to b$ to mean: $f$ is a morphism from $a$ to $b$, in other words $f \in \mathrm{mor}_{\mathcal{C}}(a,b)$. (It does not have to mean that $a,b$ are sets and $f$ is a map between sets.)

The notions of *opposite category* and *contravariant functor* are closely related notions. The opposite category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ is defined as follows: the objects of $\mathcal{C}^{\mathrm{op}}$ are the objects of $\mathcal{C}$, but a morphism from $a$ to $b$ in $\mathcal{C}^{\mathrm{op}}$ is the same thing as a morphism in $\mathcal{C}$ from $b$ to $a$. (Fill in the remaining details.) A *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$ is the same thing as a (covariant) functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. Alternatively, it could also be defined as a (covariant) functor from $\mathcal{C}$ to $\mathcal{D}^{\mathrm{op}}$.

An example of a category which we use a lot for combinatorial purposes: the category $\Delta$ (a very unfortunate choice of notation, but it is not my fault). The objects of $\Delta$ are the sets $[n] = \{0,1,\dots,n\}$, where $n$ can be any non-negative integer. A morphism in $\Delta$ from $[m]$ to $[n]$ is an order-preserving injective map. Composition of morphisms is defined to be ordinary composition of such maps. (I often write or say: *monotone injective* instead of order-preserving injective.)

A *semi-simplicial set* $Y$ is defined to be a contravariant functor from $\Delta$ to the category of sets. We normally write $Y_n$ instead of $Y([n])$ and we write $f^*\colon Y_n \to Y_m$ for the map $Y(f)$ of sets induced by a morphism $f\colon [m] \to [n]$ in $\Delta$.

We use semi-simplicial sets as combinatorial models for spaces. From this point of view, the definition might be a little hard to understand, and I will try something less abstract. Suppose that we try to make spaces out of little pieces of the form $\Delta^n$. Here $\Delta^n$ is the geometric $n$-simplex, a subspace of $\mathbb{R}^{n+1}$ (not a linear subspace!) defined as

$$\Delta^n := \big\{(x_0,x_1,\dots,x_n) \in \mathbb{R}^{n+1} \;\big|\; x_i \ge 0 \;\text{ for all } i,\; \Sigma_i x_i = 1\big\}.$$

(Note also that this is the convex hull of the set of basis elements $e_0,e_1,\dots,e_n$ where $e_0 = (1,0,0,\dots)$, $e_1 = (0,1,0,\dots)$, $e_2 = (0,0,1,0,\dots)$ and so on.) A morphism $f\colon [m] \to [n]$ in $\Delta$ determines a (continuous) map

$$f_*\colon \Delta^m \longrightarrow \Delta^n$$

in an unsurprising manner: $f_*(x_0,x_1,\dots,x_m) = (y_0,y_1,\dots,y_n)$ where $y_{f(i)} = x_i$ for $i = 0,1,\dots,m$ and $y_j = 0$ in case $j$ is not of the form $f(i)$. Now let us try to make spaces out of little pieces of the form $\Delta^n$, using maps like $f_*\colon \Delta^m \to \Delta^n$ above to do some gluing. In more detail: we start with a disjoint union of such pieces, so we start with

$$\coprod_{n\ge 0} Y_n \times \Delta^n$$

where $Y_n$ is a set which we select (for each $n \ge 0$, in order to say how many pieces of the form $\Delta^n$ we want to use). The space that we want is a quotient space of that, so we imagine it in the form

$$\Big(\coprod_{n\ge 0} Y_n \times \Delta^n\Big)\Big/\sim$$

where $\sim$ is an equivalence relation. The equivalence relation will be described as a collection of gluing instructions, of the form: for fixed $x \in Y_m$ and $y \in Y_n$ and arbitrary $z \in \Delta^m$, the element $(x,z) \in Y_m \times \Delta^m$ shall be glued to $(y, f_*(z)) \in Y_n \times \Delta^n$. We encode this instruction by writing

$$x = f^*(y).$$

This manner of writing may seem a little biased, since it suggests that $x$ is determined by $f$ and $y$, although we didn't ask for that. But: if we have $x = f^*(y)$ and also

$x' = f^*(y)$, then we must glue $(x, z) \in Y_m \times \Delta^m$ to $(y, f_*(z)) \in Y_n \times \Delta^n$, and we must glue $(x', z) \in Y_m \times \Delta^m$ to $(y, f_*(z)) \in Y_n \times \Delta^n$, and when this is done we cannot fail to note that we have glued $(x, z) \in Y_m \times \Delta^m$ to $(x', z) \in Y_m \times \Delta^m$ (for all $z \in \Delta^m$), since the gluing process is transitive. Therefore there was no need to list $x$ and $x'$ separately (the gluing makes one of them superfluous). Consequently there is no loss of generality in assuming that $x$ is indeed determined by $f$ and $y$ if we express our gluing intentions by writing $x = f^*(y)$.

Next, suppose that we have expressed certain gluing intentions by writing $x = f^*(y)$ and also $w = g^*(x)$ (where $y \in Y_n$ and $x \in Y_m$ and $w \in Y_\ell$, and $f\colon [m] \to [n]$ is a morphism in $\Delta$, and $g\colon [\ell] \to [m]$ is another morphism in $\Delta$). Then we must glue $(w, z) \in \Delta^\ell \times Y_\ell$ to $(x, g_*(z)) \in \Delta^m \times Y_m$ and we must glue $(x, g_*(z))$ to $(y, f_*(g_*(z))) = (y, (f \circ g)_*(z)) \in \Delta^n \times Y_n$. Therefore by transitivity of gluing, we have implicitly or otherwise given the gluing instruction which we must express as

$$w = (f \circ g)^* y\,.$$

Tempted by the notation we have chosen, we may also want to write

$$w = g^*(x) = g^*(f^*(y)).$$

This is consistent with $w = (f \circ g)^* y$ if we agree that $(f \circ g)^*$ means the same as $g^* \circ f^*$. Furthermore, if $m = n$ and $f\colon [m] \to [n]$ is the identity, then there is no harm in adding the gluing instruction $x = f^*(x)$ for $x \in Y_m$, since this just means that we are "gluing" points $(x, z) \in Y_m \times \Delta^m$ to themselves. Therefore we begin to see that we have a contravariant functor: for every $f\colon [m] \to [n]$, morphism in $\Delta$, we get a map $f^*\colon Y_n \to Y_m$ which expresses part of our plans for gluing. Moreover, if $e = f \circ g$, then $e^* = g^* \circ f^*$, and if $f$ is an identity morphism, then $f^*$ is an identity map.

By unraveling the definition of *semi-simplicial set* in this way, we have almost automatically given a definition of *geometric realization* of a simplicial set. Namely, the geometric realization $|Y|$ of a semi-simplicial set $Y$ is the quotient space obtained from

$$\coprod Y_n \times \Delta^n$$

by introducing the relations $(y, f_*(z)) \sim (f^*(y), z)$ for $y \in Y_n$, $z \in \Delta^m$ and morphisms $f\colon [m] \to [n]$ in $\Delta$. (*Introducing the relations ...* means: form the smallest equivalence relation containig this one, and pass to equivalence classes. These equivalence classes are therefore the elements of $|Y|$. And make sure you know what a quotient space is! Quotient topology ... I preached it many times.

One reason for me to emphasize semi-simplicial sets in the lectures was the following: they are halfway between *topological spaces* and chain complexes. A semi-simplicial set $Y$ determines a space $|Y|$, and a chain complex $C(Y)$. Without that (or something similar), the emergence of chain complexes in algebraic topology is a little difficult to explain. (In any case it *remains* difficult to explain, but I try it with semi-simplicial sets.)

So let me try to explain what the combinatorial chain complex $C(Y)$ of a semi-simplicial set $Y$ is, pretending that we do not have a definition of *chain complex*. So the definition of chain complex should "emerge" in the process.

For a fixed $n \geq 0$ and elements $y \in Y_n$ and $x \in Y_{n-1}$, we want to define the *incidence number* $J(x, y)$, an integer. Roughly this counts how many relations of type $x = f^*(y)$ we have for the chosen $x$ and $y$. In other words we are asking: how many morphisms $f\colon [n-1] \to [n]$ in $\Delta$ are there such that $x = f^*(y)$ ? First of all, we describe the possible morphisms $[n-1] \to [n]$ as $f_i$ where $i = 0, 1, \ldots, n$ and $f_i$ is the unique monotone injection from $[n-1]$ to $[n]$ whose image is $[n] \smallsetminus \{i\}$. But we want to count these with signs: therefore

$$J(x,y) := \sum_{i\in[n]:\ x=(f_i)^*(y)} (-1)^i .$$

(The signs have something to do with orientations; an attempt to explain this will be made later.) It is convenient to organize these integers $J(x,y)$ in a matrix with integer entries; let me write this matrix as

$$D(n) := (J(x,y))_{x\in Y_{n-1},\ y\in Y_n} .$$

(The rows of this matrix are indexed by the elements of the set $Y_{n-1}$ and the columns are indexed by the elements of the set $Y_n$. The sets $Y_{n-1}$ and/or $Y_n$ are not equipped with any ordering, as a rule, and we must live with that.) Now we have an interesting little "theorem" stating that the matrix product

$$D(n-1)D(n)$$

is a zero matrix (if $n \geq 2$). This is really all. What does it mean? The product matrix $D(n-1)D(n)$ has one entry for each selection $(w,y)$ where $w \in Y_{n-2}$ and $y \in Y_n$. That entry is

$$\sum_{x\in Y_{n-1}} J(w,x)\cdot J(x,y)$$

(The sum might look infinite, but it is really finite since there are only finitely many $x$ such that $J(x,y) \neq 0$ for the fixed $y$.) We are saying that all these entries are zero. (The proof is an exercise; it is not deep.)

If we had to define the concept of *chain complex* on the basis of this observation about incidence numbers, it might come out like this: a chain complex is a sequence of matrices $D(n)$, where $n = 1,2,3,\dots$, such that all matrix products $D(n-1)D(n)$ are defined and equal to zero. (More details would be required. From which ring should the entries of these matrices be taken? What can be said about the format of the matrices ... etc.) This is not bad as a definition. But the generally accepted definition of *chain complex* is slightly more abstract. We remember (from linear algebra) that a matrix is a convenient way to describe a *linear map* between vector spaces. In our case, the matrices have integer coefficients, so that vector spaces are not quite the appropriate concept. Instead we might say: each matrix $D(n)$ describes a homomorphism between free abelian groups (and each of the free abelian groups is equipped with a *basis*, although not an ordered one). The equation $D(n-1)D(n) = 0$ means that certain compositions (of homomorphisms) are zero. In this way, we arrive at definition 5.1.1. Note that, for greater generality, we do not insist on *free* abelian groups, with or without basis; we just ask for abelian groups. Note that $d_n$ in definition 5.1.1 corresponds to the matrix $D(n)$ in the above "experiment".

### 9.2. Mo 22.1. and Thu 25.1.

I tried to say something about *orientations*. Strictly speaking we did not need or use orientations anywhere, but knowing what they are can help us to understand some formulas better. (*Warning*: this discussion of orientations seems to generate an awful lot of writing for very little purpose.)

First a definition of *orientation* of a finite dimensional real vector space $V$. An orientation of $V$ is a function $\omega$ which for every ordered basis of $V$ consisting of vectors $v_1, v_2, \dots, v_n$ selects an element

$$\omega(v_1,\dots,v_n) \in \{+1,-1\}.$$

Condition: if $u_1, u_2, \ldots, u_n$ is one ordered basis of $V$ and $v_1, v_2, \ldots, v_n$ is another, then

$$\frac{\omega(u_1, u_2, \ldots, u_n)}{\omega(v_1, \ldots, v_n)} = \frac{\det(M)}{|\det(M)|}$$

where $M$ is the matrix describing the base change. (There are obviously *two* matrices which we can set up to describe the base change, but it does not matter which of the two we choose for this.) *Consequence*: such an $\omega$ is fully determined by its value on a particular ordered basis of $V$, and we can choose that value (to be $=1$ or $-1$ as we wish).
*Important example.* Suppose that $V$ has dimension 0. Then there is exactly one ordered basis of $V$. Therefore $V$ has two orientations: the $\omega$ which assigns $+1$ to the unique ordered basis and the $\omega$ which assigns $-1$ to the unique ordered basis.
Every finite dimensional real vector space $V$ has exactly two orientations. The following slightly incorrect definition of *orientation* comes to mind easily: "*an orientation of $V$ is an equivalence class of ordered bases for $V$, where two bases of $V$ are considered to be equivalent if the base change matrix has positive determinant*". This is actually correct in all cases *except* in the case where $V$ has dimension 0. (If $V$ has dimension 0, then there is only one ordered basis and so there can only be one equivalence class of such; therefore, according to this proposed new definition, a 0-dimensional real vector space would only have one orientation.) But orientations of 0-dimensional vector spaces are very important in topology, and it is important to insist that a 0-dimensional vector space has two possible orientations.
Here is a key idea about orientations. Suppose that $\psi \colon V \to W$ is an injective linear map between finite dimensional real vector spaces. *A choice of orientations for $V$ and $W$ determines an orientation on the quotient vector space $W/\psi(V)$.* (That's more a definition than a theorem.) Let $\omega$ be the chosen orientation on $W$ and let $\omega'$ be the chosen orientation of $V$. We plan to define $\omega''$, an orientation on $W/\psi(V)$. So let $w_1, \ldots, w_r$ be an ordered basis for $W/\psi(V)$. Choose vectors $\bar{w}_1, \ldots, \bar{w}_r$ in $W$ such that the projection $W \to W/\psi(V)$ takes $\bar{w}_j$ to $w_j$ for $j = 1, 2, \ldots, r$. Choose any ordered basis $v_1, \ldots, v_m$ for $V$. Then $\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_r, v_1, \ldots, v_m$ constitute an ordered basis for $W$. We define

$$\omega''(w_1, \ldots, w_r) := \frac{\omega(\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_r, v_1, \ldots, v_m)}{\omega'(v_1, \ldots, v_m)}.$$

(Reader: verify that $\omega''$ is well defined as a function on ordered bases and satisfies the conditions for an orientation on $W/\psi(V)$.)
*Important special case*: suppose that the dimension of $W/\psi(V)$ is $1$. An orientation of $W/\psi(V)$ amounts to a choice of nonzero vector in $W/\psi(V)$, up to multiplication by a *positive* real number. This is the same as a choice of one of the two connected components of $W \smallsetminus \psi(V)$. Therefore we can summarize: if $\psi \colon V \to W$ is an injective linear map where $W$ has dimension $n$ and $V$ has dimension $n-1$, then a choice of orientations for $V$ and $W$ determines a choice of one of the two connected components of $W \smallsetminus \psi(V)$. I like to call this the *outer* component, and the other one is of course the *inner* component. *Example*: if $W = \mathbb{R}^2$ with standard ordered basis $(1,0), (0,1)$ (and corresponding orientation) and $V = \mathbb{R}$ with the standard basis and the standard orientation, and $\psi$ is given by $\psi(1) = (1,0)$, then the outer connected component of $W \smallsetminus \psi(V)$ is the lower half plane (it consists of all vectors $(x_1, x_2)$ where $x_2 < 0$).
Unfortunately we need a generalization of this to affine spaces. Let us say that an affine subspace of a (finite dimensional real) vector space $W$ is a subset $V$ of $W$ which is of the form $V^\lambda + w = \{v + w \mid v \in V^\lambda\}$, where $V^\lambda$ is a linear subspace of $W$ and $w$ is some (fixed)

element of $W$. Then we say that an orientation of $V$ is the same thing as an orientation of the vector space $V^\lambda$. (Note that $V^\lambda$ can be determined from $V$ as follows: it is the set of all $v - v' \in W$ where $v, v' \in V$.)
Now let's apply this to the maps $\varphi_i\colon \Delta^{n-1} \to \Delta^n$, induced by $f_i\colon [n-1] \to [n]$. Roughly, we want to decide whether $\varphi_i$ is orientation preserving or not. This question does not make sense for a number of reasons; the strongest reason is that $\Delta^n$ and $\Delta^n$ are neither vector spaces nor affine spaces. However, $\Delta^n$ is contained in the affine subspace

$$V_n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \Sigma_j x_j = 1\}$$

of $\mathbb{R}^{n+1}$. The corresponding linear subspace $V_n^\lambda \subset \mathbb{R}^{n+1}$ has dimension $n$ and our preferred ordered basis for it is

$$e_1 - e_0, e_2 - e_1, \ldots, e_n - e_{n-1}.$$

Similarly $\Delta^{n-1}$ is contained in the affine subspace $V_{n-1} \subset \mathbb{R}^n$. The map $\varphi_i\colon \Delta^{n-1} \to \Delta^n$ extends to an affine map $V_{n-1} \to V_n$ which we still call $\varphi_i$. After adding/subtracting constants it can be viewed as a a linear map $\varphi_i^\lambda\colon V_{n-1}^\lambda \to V_n^\lambda$. This linear map is given on the basis vectors by

$$\varphi_i^\lambda(e_k - e_{k-1}) = e_{f_i(k)} - e_{f_i(k-1)} = \begin{cases} e_k - e_{k-1} & \text{if} \quad k < i \\ (e_{k+1} - e_k) + (e_k - e_{k-1}) & \text{if} \quad k = i \\ e_{k+1} - e_k & \text{if} \quad k > i \end{cases}$$

where $k \in \{1, 2, \ldots, n-1\}$. Since $V_{n-1}$ and $V_n$ are *oriented* affine spaces, and $\varphi_i$ is an injective affine map, we can decide in principle which connected component of

$$V_n \smallsetminus \varphi_i(V_{n-1})$$

should be called the *outer* component. (To be precise ... we should work with the corresponding linear spaces and the corresponding linear map ... but in the end we can make that decision.) We now ask whether this outer component is the connected component which we would intuitively call *outer* because it has empty intersection with the subset $\Delta^n$ of $V_n$. (One of the two components has empty intersection with $\Delta^n$ and the other contains all of $\Delta^n \smallsetminus \varphi_i(\Delta^{n-1})$.) If the answer is yes, then we can say that $\varphi_i$ is compatible with the standard orientations. If the answer is no, then we say that $\varphi_i$ is not compatible with the standard orientations.

*Exercise for myself*: Do the calculation, for each $n \geq 0$ and $i \in [n]$, and show that $\varphi_i$ is compatible with the orientations if and only if $i$ is even. *Answer*: We want to choose a vector $w$ in

$$V_n^\lambda \smallsetminus \varphi_i^\lambda(V_{n-1}^\lambda)$$

such that the matrix of the linear map

$$\mathbb{R} \oplus V_{n-1}^\lambda \longrightarrow V_n^\lambda$$

given by $(t, v) \mapsto tw + \varphi_i^\lambda(v)$ for $t \in \mathbb{R}$ and $v \in V_{n-1}^\lambda$ has positive determinant (where we must use the preferred ordered bases). Suppose first $i > 0$. If we try $w = \pm(e_i - e_{i-1})$ then the matrix is

$$\begin{bmatrix} \pm c_i & c_1 & c_2 & \ldots & c_{i-1} & c_i + c_{i+1} & c_{i+2} & \ldots & c_n \end{bmatrix}$$

where $c_k$ denotes a column vector of length $n$ having an entry $1$ in position $k$, all other entries $0$. The determinant is positive if we choose the sign $\pm$ to be $(-1)^{i-1}$. So a correct solution is

$$w = (-1)^{i-1}(e_i - e_{i-1}) = (-1)^i(e_{i-1} - e_i).$$

If $i$ is even, this selects the connected component of $V_n \smallsetminus \varphi_i(V_{n-1})$ which has empty intersection with $\Delta^n$. If $i$ is odd, it selects the other component. Very good. Now suppose $i = 0$. (This is an even integer.) Then we try $w = e_1 - e_0$. The matrix is an $n \times n$ identity matrix. It has positive determinant. So $w = e_1 - e_0$ is a correct solution in this case. This selects the connected component of $V_n \smallsetminus \varphi_i(V_{n-1})$ which has empty intersection with $\Delta^n$. Very good. □

This long essay on orientations had the modest purpose of explaining why we define the differential in the singular chain complex of a topological space $X$ by

$$d(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ \varphi_i$$

where $\sigma \colon \Delta^n \to Y$ is a continuous map (also known as *singular* $n$*-simplex* in $Y$).

Now back to chain complexes and chain homotopies. There is no need to repeat the definitions here. I wanted to make one remark about chain homotopies. (This should help when it comes to calculating sets of chain homotopy classes of chain maps.) Let $C$ and $D$ be chain complexes and let $f \colon C \to D$ be a chain map. Suppose that $h_n \colon C_n \to D_{n+1}$ are some homomorphisms (for $n \in \mathbb{Z}$). Define

$$g_n := d_{n+1}h_n + h_{n-1}d_n + f_n \colon C_n \to D_n\,.$$

Then $g = (g_n)_{n \in \mathbb{Z}}$ is again a chain map from $C$ to $D$. Proof:

$$dg - gd = d(dh + hd + f) - (dh + hd + f)d = dhd + df - dhd - fd = df - fd = 0.$$

And it is clear from the definition that $h$ is now a chain homotopy from $f$ to $g$ since we have $dh + hd = g - f$. In other words: any sequence of homomorphisms $h_n \colon C_n \to D_{n+1}$ qualifies as a chain homotopy from $f \colon C \to D$ to *some* other chain map.

There are ways to make the concept of chain homotopy look analogous to the concept of homotopy between continuous maps. Remember that the combinatorial chain complex $C(\underline{\Delta}^1)$ of the semi-simplicial set $\underline{\Delta}^1$ looks like this:

$$\cdots \longleftarrow 0 \longleftarrow \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{1 \mapsto (-1,1)} \mathbb{Z} \longleftarrow 0 \longleftarrow \cdots$$

(where $\mathbb{Z} \oplus \mathbb{Z}$ sits in degree $0$). Let us view $C(\underline{\Delta}^1)$ as an algebraic analogue of the unit interval $[0, 1]$. Then we need an algebraic analogue of *product* (of topological spaces). Although the product of two chain complexes, also known as direct sum of two chain complexes, is an accepted concept with the obvious meaning, it is not a good analogue of the product of topological spaces. (Instead it is a good analogue of the *disjoint union* of topological spaces.) The correct analogue of the product of two spaces is the *tensor product* of two chain complexes. I do not want to define this in general, but here is a definition of the chain complex

$$T \;= E \otimes C(\underline{\Delta}^1)$$

for an arbitrary chain complex $E$. We take $T_n := E_n \oplus E_{n-1} \oplus E_n$. The differential $d^T \colon T_n \to T_{n-1}$ is therefore best described as a $3 \times 3$ matrix and as such it is

$$d^T = \begin{bmatrix} d^E & -\mathrm{id} & 0 \\ 0 & -d^E & 0 \\ 0 & \mathrm{id} & d^E \end{bmatrix}.$$

It is a differential because

$$\begin{bmatrix} d^E & -\mathrm{id} & 0 \\ 0 & -d^E & 0 \\ 0 & \mathrm{id} & d^E \end{bmatrix} \begin{bmatrix} d^E & -\mathrm{id} & 0 \\ 0 & -d^E & 0 \\ 0 & \mathrm{id} & d^E \end{bmatrix} = \begin{bmatrix} d^E d^E & d^E - d^E & 0 \\ 0 & d^E d^E & 0 \\ 0 & d^E - d^E & d^E d^E \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There are two rather obvious chain maps $f^u \colon E \to T$ and $g^u \colon E \to T$. These are given by the inclusion of $E_n$ as the first summand of $T_n = E_n \oplus E_{n-1} \oplus E_n$ and as the third summand, respectively (for all $n$). There is also chain homotopy $h^u$ from $f^u$ to $g^u$. This is given by the inclusion of $E_n$ as the second summand of $T_{n+1} = E_{n+1} \oplus E_n \oplus E_{n+1}$, for all $n$. Here is the computation to show that it is a homotopy from $f^u$ to $g^u$:

$$d^T h^u(x) = (-x, -d^E(x), x), \qquad h^u d(x) = (0, d^E(x), 0)$$

so that $(dh^u + h^u d)(x) = (-x, 0, x) = g^u(x) - f^u(x)$. I chose the "u" superscript to indicate something like *universal*. Indeed it looks as if $h^u$ is the mother of all chain homotopies (between chain maps from $E$ to another chain complex). We can make this precise as follows. Chain maps $\Phi \colon T \to E'$ (for some other chain complex $E'$) correspond to triples $(f, g, h)$ where $f, g \colon E \to E'$ are chain maps and $h$ is a homotopy from $f$ to $g$. The correspondence is given by $f := \Phi \circ f^u$, $g = \Phi \circ g^u$ and $h = \Phi \circ h^u$. (Here $T$ is still an abbreviation for $C(\underline{\Delta}^1) \otimes E$, so it depends very much on $E$. Note that it is easy to recover $\Phi \colon T \to E'$ if $f$, $g$ and $h$ are given.)
(I said I did not define the tensor product $E \otimes F$ of two arbitrary chain complexes $E$ and $F$, but perhaps it does not do any harm to reveal that $(E \otimes F)_n$ is the direct sum of the $E_p \otimes F_q$ where $p + q = n$.)

New topic: subdivision. In the important and difficult chapter 6, we used a few formulas involving subdvision of simplices. I want to point out that these are inspired by constructions on semi-simplicial sets. If I had to do this in full generality, then I would define, for every semi-simplicial set $X$, another semi-simplicial set $X'$ which could be called the barycentric subdivision of $X$; and for any two semi-simplicial sets $X$ and $Y$, another semi-simplicial set $X \boxtimes Y$ which (under mild conditions) satisfies $|X \boxtimes Y| \cong |X| \times |Y|$. But let's not do it in full generality; instead let's assume that $X$ and $Y$ are of the form $X = \underline{\Delta}^m$, $Y = \underline{\Delta}^m$.
Therefore remember that $X = \underline{\Delta}^m$ is the semi-simplicial set which has $X_q = \mathrm{mor}_\Delta([q], [m])$, where $\mathrm{mor}_\Delta([q], [m])$ is the set of order-preserving injective maps from $[q]$ to $[m]$. The map $X_q \to X_p$ induced by an injective monotone $f \colon [p] \to [q]$ is given by pre-composition with $f$.
For a set $S$, let $\mathcal{P}_*(S)$ be the set of all *nonempty* subsets of $S$. This is (partially) ordered by inclusion; so for nonempty subsets $T, T'$ of $S$ we agree to write $T \le T'$ if and only if $T \subset T'$. We define

$$X'_q = \text{set of order preserving injective maps from } [q] \text{ to } \mathcal{P}_*([m]).$$

(*Example*: if $q > m$ then $X'_q = \emptyset$ and if $q = m$ then $X'_q$ has exactly $(m+1)!$ elements.) An order preserving map $f$ from $[p]$ to $[q]$ induces a map $X'_q \to X'_p$ by (pre-)composition with $f$. In this way, $X'$ is a semi-simplicial set. There is a "good" homeomorphism

$$\Phi \colon |X'| \longrightarrow \Delta^m$$

defined as follows. For $g\colon [q] \to \mathcal{P}_*([m])$, an element of $X'_q$ , we have the characteristic map $c_g\colon \Delta^q \to |X'|$ and we want

$$\Phi(c_g(x_0, x_1, \dots, x_q)) := \sum_{i=0}^{q} x_i \cdot b_{g(i)}$$

where $b_{g(i)}$ is the barycenter of the face of $\Delta^m$ spanned by the vertices $e_j$ where $j \in g(i)$. (More directly: $b_{g(i)}$ is the *average* of the $e_j$ for $j \in g(i)$. Remember that $g(i)$ is a nonempty subset of $[m] = \{0, 1, \dots, m\}$.) These conditions are enough to determine $\Phi$ as a continuous map. It is also easy to show that they are consistent, i.e., that $\Phi$ with these properties exists. But it is not quite so easy to show that $\Phi$ is a homeomorphism. (I omit this.)

Next, let $X = \underline{\Delta}^m$ and $Y = \underline{\Delta}^n$. We make a new semi-simplicial set $Z = X \boxtimes Y$ by

$$Z_q = \text{ set of order-preserving injective maps from } [q] \text{ to } [m] \times [n].$$

(Here $[m] \times [n]$ has the product ordering; therefore an order-preserving map from $[q]$ to $[m] \times [n]$ is as good as two order-preserving maps $[q] \to [m]$ and $[q] \to [n]$. But the injectivity condition comes on top of that. *Example:* if $m = n = 1$ then $Z_2$ has two elements and $Z_1$ has five elements.) An order preserving map $f\colon [p] \to [q]$ induces a map $Z_q \to Z_p$ by (pre-)composition with $f$. In this way, $Z = X \boxtimes Y$ is a semi-simplicial set. There is a "good" homeomorphism

$$\Psi\colon |Z| \longrightarrow \Delta^m \times \Delta^n$$

defined as follows. For $g = (g_1, g_2)\colon [q] \to [m] \times [n]$, an element of $Z_q$ , we have the characteristic map $c_g\colon \Delta^q \to |Z|$ and we want

$$\Psi(c_g(x_0, x_1, \dots, x_q)) := ((x'_0, x'_1, \dots, x'_m), (x''_0, x''_1, \dots, x''_n)) \in \Delta^m \times \Delta^n$$

where $x'_i$ is the sum of the $x_j$ such that $g_1(j) = i$, and $x''_i$ is the sum of the $x_j$ such that $g_2(j) = i$. These conditions are enough to determine $\Psi$ as a continuous map. It is also easy to show that they are consistent, i.e., that $\Psi$ with these properties exists. But it is not quite so easy to show that $\Psi$ is a homeomorphism. (Again I omit this.)

*Question/exercise*: I described $\Phi$ and $\Psi$ as *good* homeomorphisms; is there a better word for this?

Finally, let's review how the singular chain complex of a space was introduced. We took the view that semi-simplicial sets are half-way between spaces and chain complexes. More precisely, we have two functors:

$$\begin{array}{c} \mathcal{T}op = \text{category of topological spaces} \\ \Big\uparrow {\scriptstyle \text{geometric realization}} \\ \text{category of semi-simplicial sets} \\ \Big\downarrow {\scriptstyle \text{combinatorial chain complex}} \\ \text{category of chain complexes} \end{array}$$

One of these (the geometric realization) seems to go in the wrong direction for our purposes. It does not have an inverse, but it has a *right adjoint*, which is nearly as good. (Section 4.5 has some information on adjoint functors, including a definition of the concept.) The right adjoint is the functor "sing" from $\mathcal{T}op$ to the category of semi-simplicial sets. For a space $X$, the semi-simplicial set $\mathrm{sing}(X)$ has

$$\mathrm{sing}(X)_n = \text{set of continuous maps from } \Delta^n \text{ to } X\,.$$

The map $\mathrm{sing}(X)_n \to \mathrm{sing}(X)_m$ induced by a monotone injective $f\colon [m] \to [n]$ is given by pre-composition with $f_*\colon \Delta^m \to \Delta^n$. Therefore we obtain a functor from $\mathcal{T}op$ to the category of chain complexes by composing sing with the combinatorial chain complex functor:

$$X \mapsto C(\mathrm{sing}(X)) := sC(X)$$

for topological spaces $X$. (This can be made more explicit ... but there is no need to write it out here.) We then defined $H_j(X) := H_j(sC(X))$ and proved a number of difficult and important theorems about these groups.

This point of view emphasizes the following. For a semi-simplicial set $Y$ there is a so-called *unit* morphism $Y \to \mathrm{sing}(|Y|)$. It is the morphism which corresponds to $\mathrm{id}\colon |Y| \to |Y|$ under the adjunction

$$\mathrm{mor}(Y, \mathrm{sing}(|Y|) \leftrightarrow \mathrm{mor}(|Y|, |Y|)$$

(since $|...|$ is left adjoint to sing). The unit morphism induces a chain map

$$C(Y) \to C(\mathrm{sing}(|Y|)) = sC(|Y|)$$

which has many uses. (In particular we proved that it induces an isomorphism in the homology groups; if $Y$ is not too complicated, this can be viewed as a way to calculate the homology groups $H_j(|Y|)$ for all $j$.)

The two important theorems that we proved about $sC(X)$ were *homotopy invariance* and the theorem about small simplices. The first of these has many equivalent formulations, but one of them states that the two standard inclusions $X \to X \times [0,1]$ given by $x \mapsto (x,0)$ and $x \mapsto (x,1)$ induce chain homotopic chain maps

$$sC(X) \longrightarrow sC(X \times [0,1]).$$

There is no need to repeat all of that, but I want to explain how we can "guess" a chain homotopy $h$ for this. Given a continuous map $\sigma\colon \Delta^m \to X$ (which we view as a special element of $sC(X)_m$), we get a commutative diagram

$$\begin{tikzcd}
C(\underline{\Delta}^m) \arrow[r] \arrow[d] & C(\underline{\Delta}^m \boxtimes \underline{\Delta}^1) \arrow[d] & C(\underline{\Delta}^m) \arrow[l] \arrow[d] \\
sC(X) \arrow[r] & sC(X \times [0,1]) & sC(X) \arrow[l]
\end{tikzcd}$$

Here the left-hand and right-hand vertical arrows are both the chain map obtained by composing

$$C(\underline{\Delta}^m) \xrightarrow{\text{induced by unit map}} sC(|\underline{\Delta}^m|) \cong sC(\Delta^m) \xrightarrow{\text{induced by } \sigma} sC(X)$$

and the middle vertical arrow is defined similarly, using $|\Delta^m \boxtimes \underline{\Delta}^1| \cong \Delta^m \times \Delta^1$. The lower horizontal arrows are induced by $x \mapsto (x,0)$ and $x \mapsto (x,1)$, respectively. The upper horizontal arrows are defined by analogy with the lower ones, so that the diagram commutes. And we use $\Delta^1 \cong [0,1]$. Now we define $h(\sigma) \in sC(X \times [0,1])_{m+1}$ in such a way that it is the image (under the middle vertical arrow) of a carefully chosen element in $C(\underline{\Delta}^m \boxtimes \underline{\Delta}^1)_{m+1}$ (which depends only on $m$, not on $\sigma$). The point is that $C(\underline{\Delta}^m \boxtimes \underline{\Delta}^1)_{m+1}$

is not huge. It is a free abelian group of rank $m+1$. Our choice is limited. This is an advantage. Another advantage is that, if we construct $h$ in this way, it will be a *natural* chain homotopy.

In connection with “small simplices” we can do something similar. A continuous map $\sigma\colon \Delta^n \to X$ determines a chain map $C(\underline{\Delta}^n) \to sC(X)$ as before. We want to define the chain map $\beta\colon sC(X) \to sC(X)$ (barycentric subdivision) in such a way that

$$\begin{tikzcd} C(\underline{\Delta}^n) \arrow[r, dotted] \arrow[d, "\text{induced by } \sigma"'] & C((\underline{\Delta}^n)') \arrow[d, "\text{induced by } \sigma"] \\ sC(X) \arrow[r, "\beta"] & sC(X) \end{tikzcd}$$

commutes, where the upper horizontal arrow is a chain map to be defined. In particular $\beta(\sigma)$ is then the image of a carefully chosen element in $C((\underline{\Delta}^n)')_n$ under the right-hand vertical arrow. (The right-hand vertical arrow uses the homeomorphism $|(\underline{\Delta}^n)'| \cong \Delta^n$ sketched above.) Again, because the chain complexes in the upper row are not huge, this looks like a manageable task. (But a solution must be found which has good naturality properties.)

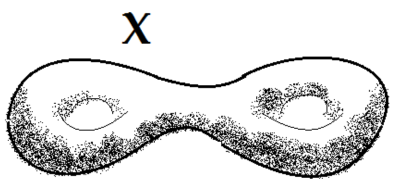
Once $\beta\colon sC(X) \to sC(X)$ is defined, it should be possible to carry on in the same manner to show that $\beta$ is homotopic to the identity. But I must admit that I found this too tedious. So I gave a less explicit, more abstract argument for that, using the Eilenberg-Zilber method of acyclic models.

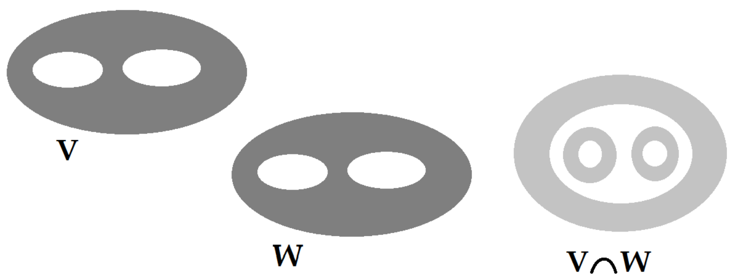
The theorem on barycentric subdivision leads easily to the theorem on small simplices. The most important special case for us is the following. Suppose that the topological space $X$ is the union of subsets $V$ and $W$, where $\mathrm{int}(V) \cup \mathrm{int}(W) = X$. Then we are told that the inclusion $sC(V) + sC(W) \to sC(X)$ is a chain homotopy equivalence. *Very important*: the sum $sC(V) + sC(W)$ is an internal sum (not a direct sum) taken inside $sC(X)$. In other words $sC(V)_n + sC(W)_n = (sC(V) + sC(W))_n$ is the subgroup of $sC(X)_n$ consisting of all elements $c$ which can somehow be written in the form $c = a + b$ where $a$ belongs to $sC(V)_n \subset sC(X)_n$ and $b$ belongs to $sC(W)_n \subset sC(X)_n$. This manner of writing $c = a+b$ is typically not unique, and this absence of uniqueness, or the distinction between $sC(V) + sC(W)$ and $sC(V) \oplus sC(W)$, is the key to the Mayer-Vietoris sequence. Namely, there is a short exact sequence of chain complexes and chain maps

$$sC(V \cap W) \longrightarrow sC(V) \oplus sC(W) \longrightarrow sC(V) + sC(W).$$

This leads to a long exact sequence relating the homology groups of the three chain complexes. (Use corollary 5.3.3, taking $E$ to be $sC(V) + sC(W)$, chain subcomplex of $sC(X)$.) If if we use the chain homotopy equivalence $sC(V) + sC(W) \to sC(X)$, then we can write this as a long exact sequence relating the homology groups of $V, W, V \cap W$ and $X = V \cup W$.

Last not least, I tried to illustrate the Mayer-Vietoris sequence by using it to calculate the homology groups of $X$, a surface of genus 2 (surface of a pretzel). The pictures below show $X$ and two open subsets $V$ and $W$ of $X$ such that $V \cup W = X$. They also show $V \cap W$.

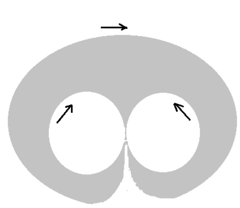



It is not very difficult to determine the homology groups of $V$ and $W$, and of $V \cap W$. We get $H_1(V) \cong \mathbb{Z}^2 \cong H_1(W)$ and $H_1(V \cap W) \cong \mathbb{Z}^3$. (Here $\mathbb{Z}^3$ means the same as $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ or as $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$; do not misread as $\mathbb{Z}/3$.) For $q > 1$, the groups $H_q(V)$, $H_q(W)$ and $H_q(V \cap W)$ are all zero. The interesting part of the MV sequence is therefore

$$H_2(X) \longrightarrow H_1(V \cap W) \longrightarrow H_1(V) \oplus H_1(W) \longrightarrow H_1(X) \longrightarrow H_0(V \cap W) \longrightarrow H_0(V) \oplus H_0(W)$$

(the arrow on the left, out of $H_2(X)$, must be injective because of the exactness). It is a little more difficult to determine the homomorphisms in homology induced by the inclusions $V \cap W \to V$ and $V \cap W \to W$. Here one has to improvise. My suggestion is the following: we note that $V$ is homotopy equivalent to $|Z|$ where $Z$ is the semi-simplicial set which has $Z_n = \emptyset$ for $n > 2$, $Z_2 = \{u\}$ (one element), $Z_1 = \{f_0^*u, f_1^*u, f_2^*u\}$ (three elements) and $Z_0 = \{v\}$ (one element). The geometric realization $|Z|$ looks like this:



We can use the combinatorial chain complex $C(Z)$ (and theorem 8.3.1) to calculate the homology of $|Z|$. Again we get $H_1(|Z|) \cong \mathbb{Z}^2$; more precisely $C(Z)$ looks like

$$\cdots \longleftarrow 0 \longleftarrow \mathbb{Z} \xleftarrow{\;0\;} \mathbb{Z}^3 \xleftarrow{\;\nu\;} \mathbb{Z} \longleftarrow 0 \longleftarrow \cdots$$

where $\nu$ is given by $1 \mapsto (1,-1,1)$. Therefore $H_1(|Z|)$ is the free abelian group on generators $f_0^*u$ and $f_2^*u$ (these are the "inside" boundary curves in the picture), but now we also see that $f_1^*u$ (the outside boundary curve) is in the same homology class as $f_0^*u + f_2^*u$. Good! This means that the homomorphism

$$\mathbb{Z}^3 \cong H_1(V \cap W) \longrightarrow H_1(V) \cong \mathbb{Z}^2$$

induced by the inclusion is given by $(a,b,c) \mapsto (a+c, b+c)$. (This is on the understanding that the three $\mathbb{Z}$ summands in $\mathbb{Z}^3$ correspond to $H_1$ of the left-hand inner annulus, right-hand inner annulus and large outside annulus in the picture of $V \cap W$, in this order.) Then it is easy to finish the calculation. We find that

$$H_2(X) \cong \mathbb{Z}$$

and we find a short exact sequence

$$\mathbb{Z}^2 \longrightarrow H_1(X) \longrightarrow \mathbb{Z}^2$$

(where the $\mathbb{Z}^2$ on the left is the cokernel of $H_1(V \cap W) \longrightarrow H_1(V) \oplus H_1(W)$ and the $\mathbb{Z}^2$ on the right is the kernel of $H_0(V \cap W) \longrightarrow H_0(V) \oplus H_0(W)$, both from the MV sequence). The short exact sequence implies somewhat automatically that

$$H_1(X) \cong \mathbb{Z}^4$$

(because a surjective homomorphism $p$ from an abelian group, here $H_1(X)$, to a *free* abelian group, here $\mathbb{Z}^2$, always admits a right inverse homomorphism $q$, meaning that $p \circ q = \mathrm{id}$). Exactness of the MV sequence also implies that $H_q(X) = 0$ for $q > 2$. Finally $H_0(X) \cong \mathbb{Z}$ is clear because $X$ is path connected.