Lecture notes chapter 2, WS 2015-2016 (Weiss):
Vector bundles, J-homomorphism & Adams conjecture

This chapter can be regarded as a digression. The purpose of the digression is to explain why spherical fibrations and the Adams conjecture matter in the classification theory of manifolds.

2.1. Poincaré duality spaces

Definition 2.1.1. A compact 1-connected CW-space $X$ is a Poincaré duality space of formal dimension $n$ if there exists an element $\varphi \in H_n(X;\mathbb{Z})$ such that the homomorphisms

$$H^k(X;\mathbb{Z}) \to H_{n-k}(X;\mathbb{Z}) ; \ a \mapsto a \cap \varphi$$

(cap product with $\varphi$) are isomorphisms for all $k$. The element $\varphi$ is called a fundamental class for $X$.

Example 2.1.2. Every 1-connected compact orientable $n$-manifold (without boundary) is a Poincaré duality space. This follows from the Poincaré duality theorem.

Remark 2.1.3. Let $X$ be a PD space as in definition 2.1.1, and suppose in addition that it is connected. Then $H^0(X;\mathbb{Z}) \cong \mathbb{Z}$ and we deduce $H_n(X;\mathbb{Z}) \cong \mathbb{Z}$ by Poincaré duality. It is clear that $\varphi$ must be a generator of the group $H_n(X;\mathbb{Z}) \cong \mathbb{Z}$. Therefore there are exactly two choices for a fundamental class $\varphi$. (A choice of fundamental class can also be called an orientation.)

A more general definition of Poincaré duality space is available. We will not need this, but it is worth knowing anyway. The standard version is as follows: A compact CW-space $X$ (which need not be 1-connected) is an orientable Poincaré duality space of formal dimension $n$ if there exists an element $\varphi \in H_n(X;\mathbb{Z})$ such that the homomorphisms

$$H^k(X;J) \to H_{n-k}(X;J) ; \ a \mapsto a \cap \varphi$$

are isomorphisms for all $k$ and every local coefficient system $J$ on $X$. — To explain what a local coefficient system is, let me assume that $X$ is a connected and based CW-space (no Poincaré duality whatsoever required here), so that we have a universal covering

$$\tilde{X} \to X$$

and $\pi_1 := \pi_1(X)$ acts on the left of $\tilde{X}$ by deck transformations. Then the local coefficient system $J$ is nothing but a $\pi_1$-module, in other words an abelian group with a left action of $\pi_1$ which respects the addition (so $g(x + y) =$
\( gx + gy \) for \( x, y \in J \) and \( g \in \pi_1 \). We can define \( H^k(X; J) \) as \( H^k \) of the cochain complex \( \text{hom}_{\pi_1}(C(\hat{X}), J) \) and \( H_\ell(X; J) \) as the \( \ell \)-th homology of the chain complex \( C(\hat{X}) \otimes_{\pi_1} J \). Here \( C(\cdot) \) denotes the singular or cellular chain complex (it does not matter which). There is a slight subtlety in the definition of \( C(\hat{X}) \otimes_{\pi_1} J \).

Since \( C(\hat{X}) \) is a chain complex of left \( \pi_1 \)-modules and \( J \) is also a left \( \pi_1 \)-module, the construction \( \otimes_{\pi_1} \) is to be interpreted in such a way that we enforce the relations \( a \otimes b \sim ga \otimes gb \) in the ordinary tensor product \( \otimes \) for all \( g \in \pi_1 \). Equivalently, we can make left \( \pi_1 \)-modules into right \( \pi_1 \)-modules by defining \( ag := g^{-1}a \), in which case \( a \otimes b \sim ga \otimes gb \) turns into \( a \otimes b \sim ag^{-1} \otimes gb \) which may look more familiar (to algebraists). With these definitions, there is a “refined” cap product which takes the form of a bi-additive (essentially bilinear) map \( H^k(X; J) \times H_\ell(X; Z) \to H^k_\ell(X; J) \).

This form of Poincaré duality, with local coefficient systems \( J \), still holds for compact orientable manifolds without boundary. The standard proof is actually not very different from the standard proof of Poincaré duality for ordinary coefficients \( Z \). (Note that \( Z \) can be viewed as a \( \pi_1 \)-module with the trivial action of \( \pi_1 = \pi_1(X) \).) Also, it should be mentioned that if \( X \) is 1-connected, then all local coefficient systems on \( X \) are just “coefficients” and it is easy to show that Poincaré duality for coefficients \( Z \) implies Poincaré duality for all coefficients \( J \) in such a case.

**Exercise 2.1.4.** Let \( X \) be a connected based CW-space and write \( \pi_1 := \pi_1(X) \). Write \( \hat{X} \) for the universal cover.

(i) Take \( J = \text{map}(\pi_1, Z) \), the abelian group of all functions from \( \pi_1 \) to \( Z \). There is a nearly-obvious left action of \( \pi_1 \) on \( J \) by translation: for \( g \in \pi_1 \) and \( f \in J \) let \( g \cdot f \) be defined by \( (g \cdot f)(h) = f(hg^{-1}) \). So \( J \) is a \( \pi_1 \)-module. Show that \( H^k(X; J) \cong H^k(\hat{X}; Z) \) for all \( k \).

(ii) Take \( J = \bigoplus_{g \in \pi_1} Z \), with the left action of \( \pi_1 \) by translation. (Details as in (i); this \( J \) here is a \( \pi_1 \)-submodule of the \( J \) in (i).) Show that \( H_k(X; J) \cong H_k(\hat{X}; Z) \).

(iii) Taking \( J \) as in (ii), show that \( H^0(X; J) = 0 \) if \( \pi_1 \) is an infinite group.
2.2. Normal bundles and Spivak normal fibrations

Let $M$ be a smooth compact manifold of dimension $n$, without boundary, embedded smoothly in $\mathbb{R}^k$ for some $k$, possibly quite large. Then $M$ has a normal disk bundle $E \to M$ of fiber dimension $k - n$.

In more detail, without too much differential topology jargon: for each $x \in M$ we have the tangent space $T_xM$ which can be viewed as a linear subspace (!) of $\mathbb{R}^k$. The orthogonal complement $T^\perp_xM$ of $T_xM$ in $\mathbb{R}^k$ is the fiber of the normal bundle of $M$ at $x$, another vector bundle on $M$. The map

$$TM \to \mathbb{R}^k$$

given by $T_xM \ni v \mapsto x + v$ is far from being an embedding (make a drawing, taking for example $M = S^1$ and $\mathbb{R}^k = \mathbb{R}^2$). The map

$$T^\perp M \to \mathbb{R}^k$$

given by $T^\perp_xM \ni v \mapsto x + v$ is usually still far from being an embedding, but if we restrict it by allowing only vectors $v$ of norm $\leq \varepsilon$ (for small enough $\varepsilon$), then it is a smooth embedding. So we think of $E \to M$ as the disk bundle of fiber radius $\varepsilon$ associated with the normal bundle $T^\perp M \to M$, and then we have a canonical smooth embedding $E \hookrightarrow \mathbb{R}^k$ by the formula just given. Let $\partial E \to M$ be the boundary sphere bundle (with fibers $\cong S^{k-n-1}$). Clearly $(E, \partial E)$ is a smooth manifold with boundary, of dimension $k$ and contained in $\mathbb{R}^k$ as a compact codimension 0 submanifold (with boundary).

The Pontryagin collapse map

$$c: S^k \cong \mathbb{R}^k \cup \infty \to E/\partial E$$

is defined by $c(z) = z$ if $z \in E \setminus \partial E \subset \mathbb{R}^k$ and $c(z) = \partial E/\partial E$ otherwise (also when $z = \infty$). Note that it is continuous! It is easy to see that $c$ takes the fundamental class in $H_k(S^k; \mathbb{Z})$ to a fundamental class for the manifold-with-boundary $(E, \partial E)$.

Exercise 2.2.1. Prove this “easy” statement about fundamental classes.

We can formulate this observation as follows. Recall that we have $M \subset \mathbb{R}^k$ with normal vector bundle $T^\perp M \to M$ and associated disk bundle $E \to M$. Then $E$ is a compact manifold with boundary $\partial E$, no surprise here; but remarkably, the fundamental class $[c] \in H_k(E, \partial E; \mathbb{Z})$ is in the image of the Hurewicz homomorphism from $\pi_k(E, \partial E; \mathbb{Z})$ to $H_k(E, \partial E; \mathbb{Z})$. Indeed it is the image of the element $[c] \in \pi_k(E/\partial E)$. It turns out that something similar is true for Poincaré duality spaces. In this situation we should not be looking for a vector bundle playing the role of normal bundle, but for a spherical fibration.
So let \( X \) be a 1-connected Poincaré duality space of formal dimension \( n \). For simplicity we assume that \( X \) is a compact simplicial complex (not really an additional condition, since every compact CW-space is homotopy equivalent to a simplicial complex). Then we can always find an embedding
\[
X \hookrightarrow \mathbb{R}^k
\]
(for some \( k \gg 0 \)) which is linear on each simplex of \( X \). Let’s use this to think of \( X \) as a simplicial subcomplex of \( \mathbb{R}^k \) (in some triangulation of \( \mathbb{R}^k \)). Then \( X \subset \mathbb{R}^k \) admits a regular neighborhood \( E \) which can also be described as a compact simplicial subcomplex in \( \mathbb{R}^k \). I am not planning to give many details; I think it is customary and safe to define \( E \) as the union of all simplices in the two-fold barycentric subdivision of (the given triangulation of) \( \mathbb{R}^k \) which have nonempty intersection with \( X \).

(i) \( E \) is a compact \( k \)-dimensional manifold with boundary \( \partial E \).

(ii) There is a preferred projection \( r : E \to X \) (continuous, at least) which is a homotopy equivalence. The restriction of \( r \) to \( X \) is the identity \( \text{id}_X \). We write \( r_\partial : \partial E \to X \) for the restriction of \( r \) to \( \partial E \).

Note that we have a Pontryagin collapse map
\[
c : S^k \cong \mathbb{R}^k \cup \{0\} \to E/\partial E
\]
defined much as before; and again, this takes fundamental class to fundamental class. Now we would like to say that \((E, \partial E)\) behaves like the total space (or total pair) of a disk bundle.

**Theorem 2.2.2.** Each homotopy fiber of \( r_\partial : \partial E \to X \) has the homology of a sphere of dimension \( k-n-1 \).

This is due to M Spivak (his Princeton PhD thesis, supervised by J Milnor) and it is therefore customary to say Spivak normal fibration of \( X \) for the fibration associated with \( \partial E \to X \). As a rule we are not averse to stabilization (taking fiberwise join with \( S^0 \), several times if required) and in that sense we can say that the Spivak normal fibration is a spherical fibration. See the following remark.

**Remark 2.2.3.** Replacing the inclusion \( X \hookrightarrow \mathbb{R}^k \) by the composition
\[
X \hookrightarrow \mathbb{R}^k \cong \mathbb{R}^k \times \{0\} \hookrightarrow \mathbb{R}^{k+1},
\]
a new regular neighborhood is \( E \times D^1 \), and for the new retraction we may take the composition
\[
E \times D^1 \xrightarrow{\text{proj}} E \xrightarrow{r} X.
\]
Restricting that to \( \partial(E \times D^1) \) we have a new map \( \partial(E \times D^1) \to X \). As an application of the “cube theorem” we get
\[
\text{hofiber}_x[\partial(E \times D^1) \to X] \cong (\text{hofiber}_x[r_\partial : \partial E \to X]) \ast S^0
\]
where hofiberₓ[...] is short for homotopy fiber of “...” over x ∈ X. If hofiberₓ[r₃: ∂E → X] has the homology of Sᵏ−n−₁ as claimed in theorem 2.2.2, then the join of it with S⁰ is homotopy equivalent to Sᵏ−n. (See the exercise which follows.)

Exercise 2.2.4. Let F be a space (≃ CW-space) which has the homology of a sphere Sˡ. Show that F ∗ S⁰ ≃ Sˡ+₁. (Use fundamental theorems of homotopy theory: W Hurewicz and G Whitehead).

The proof of theorem 2.2.2 reduces easily to the following statement.

Proposition 2.2.5. Let p: A → B be a map of spaces (≃ CW-spaces) and let B♮ be the mapping cylinder of p, so that there is a pair (B♮, A). Let R be any commutative ring (with 1). Suppose that H∗(B♮, A; R) is free on one generator u ∈ H⁽⁰⁾(B♮, A; R) as a module over the ring H∗(B; R). If B is 1-connected, then the homotopy fibers of p have the cohomology (with coefficients R) of S⁽⁰⁾⁻¹.

Reduction of theorem 2.2.2 to proposition 2.2.5. Apply the proposition with p = r₃, so that A = ∂E and B = X. Then we can identify (B♮, A) with (E, ∂E), by a homotopy equivalence of pairs. Poincaré duality for the oriented manifold pair (E, ∂E) gives an isomorphism

H⁽⁺⁾(E, ∂E) ≃ H⁽⁻ⁿ⁾(E)

of graded H⁽⁺⁾(E)-modules; cohomology taken with coefficients in any commutative ring R. But H⁽⁻ⁿ⁾(E) is free on one generator (in degree k−n = n) as an H⁽⁺⁾(E)-module, since E ≃ X and X is a Poincaré duality space. Therefore H⁽⁺⁾(E, ∂E) is also free on one generator u as an H⁽⁺⁾(E)-module. This u lives in degree k−n; so we take j = k − n.

Now the proposition implies that the homotopy fibers of p = r₃ have the cohomology of a sphere S⁽⁻ⁿ⁾⁻¹, for any choice of coefficient ring R. It follows that they have the Z-homology of a sphere. (See exercise just below.) □

Exercise 2.2.6. Show that if a space Y satisfies H⁽⁺⁾(Y; R) ≃ H⁽⁺⁾(S⁽⁻ⁿ⁾⁻¹; R) for any commutative ring R, then it satisfies H⁽⁺⁾(Y; Z) ≃ H⁽⁺⁾(S⁽⁻ⁿ⁾⁻¹; Z). (Hint: reduce as fast as possible to a statement about chain complexes of free abelian groups. Hint: Exercise 5 in §VI.6 of Dold’s book Lectures on algebraic topology is close to this one and comes with helpful instructions.)

Proof of proposition 2.2.5. Without loss of generality, B is a CW-space and p: A → B is a fibration. (If not, we can use the Serre construction to turn it into one.) Without loss of generality, and comes with a chosen base point. The cylinder projection (B⁎, A) → B, which we should strictly speaking write in the form (B⁎, A) → (B, B), is a fibration pair. Let (K, ∂K) be the fiber pair over the base point of B.
Note that $K$ is contractible, being the (homotopy) fiber of $B^2 \to B$. We want to show that $\partial K$ has the cohomology (with coefficients $R$) of $S^{j-1}$. Equivalently, we want to show that $H^*(K, \partial K) = R$ if $* = j$ and $H^*(K, \partial K) = 0$ if $* \neq j$.

Let us now use the cohomology Serre spectral sequence (with coefficients $R$ throughout) for the fibration pair $(B^2, A) \to B$. It has the form

$$E_2^{s,t} = H^s(B; H^t(K, \partial K)) \Rightarrow H^{s+t}(B^2, A).$$

The spectral sequence comes with cup products. Using these gives me a guilty conscience, because the cohomology Serre spectral sequence with cup products is hard to set up (and I did not do it convincingly in my topology course of years ago). But here we only need cup products in the following sense: we want to regard the spectral sequence as a spectral sequence of graded modules over the graded ring $H^*(B)$. This is much easier to set up. (Consider it done, therefore.)

The differentials in $E_2^{s,t}$ go from position $(s, t)$ to $(s + 2, t - 1)$; in $E_3^{s,t}$, from $(s, t)$ to $(s + 3, t - 2)$; in $E_4^{s,t}$, from $(s, t)$ to $(s + 4, t - 3)$; and so on.

If $H^{t_0}(K, \partial K)$ is nontrivial for some $t_0 < j$, then we can choose this minimal and the spectral sequence shows us that the corresponding term

$$E_2^{0,t_0} = H^0(B; H^{t_0}(K, \partial K)) \cong H^{t_0}(K, \partial K)$$

survives to the infinity page, i.e., maps injectively to $H^{t_0}(B^2, A)$. But since $t_0 < j$, that cohomology group is zero by assumption; contradiction.

It follows that the term $E_2^{0,j} = H^j(K, \partial K)$ survives unharmed to the infinity page, and by our assumption must map isomorphically to $H^j(B^2, A) \cong \mathbb{Z}$. The $H^*(B)$ module structure, along with our assumption, now implies that all terms

$$E_2^{s,j} = H^s(B; H^j(K, \partial K))$$

survive to the infinity page and map isomorphically to the corresponding groups $H^{s+j}(B^2, A)$.

If $H^{t_1}(K, \partial K)$ is nonzero for some $t_1 > j$, then we can take this minimal again, and we find that the corresponding term

$$E_2^{0,t_1} = H^0(B; H^{t_1}(K, \partial K)) \cong H^{t_1}(K, \partial K)$$

survives unharmed to the infinity page. This contradicts the fact that we have already exhausted $H^*(B^2, A)$ with the terms coming from row $j$ of the $E_2^{s,t}$-page. \hspace{1em} \square

Now I want to indicate briefly how theorem 2.2.2 can still be proved if we drop the assumption that $X$ be 1-connected. Let’s assume nevertheless that $X$ is connected (= path connected, since $X$ is a CW-space) and equipped with a base point. Let $X \to \tilde{X}$ be the universal covering.
We use proposition 2.2.5 again, but this time we choose $B := \tilde{X}$ (which is 1-connected) and for $A$ we take the pullback of

$$\begin{array}{ccc}
\tilde{X} & \to & X \\
\downarrow & & \downarrow \\
\partial E & \longrightarrow & X
\end{array}$$

so that we have a projection $A \to \partial E$ which is again a covering space (fiber bundle with discrete fibers). In order to use the proposition, we need to know that $H^*(B^2, A; \mathbb{R})$ is free on one generator as a graded module over the graded ring $H^*(B; \mathbb{R})$. To that end we write

$$H^*(B^2, A; \mathbb{R}) = H^*(E, \partial E; J)$$

where $J = \text{map}(\pi_1(X), \mathbb{R})$, viewed as a left module over $\pi_1(X)$ in the usual manner. (Use exercise 2.1.4.) Then we have

$$H^*(E, \partial E; J) \cong (a) H_{k-*}(E; J)$$

$$\cong (b) H_k(X; J)$$

$$\cong (c) H^{n-k++}(\tilde{X}; R)$$

$$\cong (c) H^{n-k++}(B; R).$$

(The isomorphism with label (a) is Poincaré duality for $(E, \partial E)$; the one with label (b) is Poincaré duality for $X$, which is quite a different thing; and the one labelled (c) comes from exercise 2.1.4.) So we see that $H^*(E, \partial E; J)$ is isomorphic to $H^{n-k++}(B; \mathbb{R})$, and it is straightforward to verify that this isomorphism (with a shift by $n - k$) is one of graded modules over $H^{n-k++}(B; \mathbb{R})$. □