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Lecture Notes, week 9
Topology WS 2013/14 (Weiss)

7.3. Homology of spheres

Proposition 7.7. *The homology groups of* S^1 *are* $H_0(S^1) \cong \mathbb{Z}$, $H_1(S^1) \cong \mathbb{Z}$ *and* $H_k(S^1) = 0$ *for all* $k \neq 0, 1$.

Proof. Choose two distinct points p and q in S^1. Let $V \subset S^1$ be the complement of p and let $W \subset S^1$ be the complement of q. Then $V \cup W = S^1$. Clearly V is homotopy equivalent to a point, W is homotopy equivalent to a point and $V \cap W$ is homotopy equivalent to a discrete space with two points. Therefore $H_k(V) \cong H_k(W) \cong \mathbb{Z}$ for $k = 0$ and $H_k(V) \cong H_k(W) = 0$ for all $k \neq 0$. Similarly $H_k(V \cap W) \cong \mathbb{Z} \oplus \mathbb{Z}$ for $k = 0$ and $H_k(V \cap W) = 0$ for all $k \neq 0$. The exactness of the Mayer-Vietoris sequence associated with the open covering of S^1 by V and W implies immediately that $H_k(S^1) = 0$ for $k \neq 0, 1$. The part of the Mayer-Vietoris sequence which remains interesting after this observation is

$$0 \longrightarrow H_1(S^1) \xrightarrow{\partial} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(S^1) \longrightarrow 0$$

Since S^1 is path-connected, the group $H_0(S^1)$ is isomorphic to $\mathbb{Z}$. The homomorphism from $\mathbb{Z} \oplus \mathbb{Z}$ to $H_0(S^1)$ is onto by exactness, so its kernel is isomorphic to $\mathbb{Z}$. Hence the image of the homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to $\mathbb{Z}$, so its kernel is again isomorphic to $\mathbb{Z}$. Now exactness at $H_1(S^1)$ leads to the conclusion that $H_1(S^1) \cong \mathbb{Z}$. ☐

Theorem 7.8. *The homology groups of* S^n *(for* $n > 0$*) are*

$$H_k(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = n \\ \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We proceed by induction on n. The induction beginning is the case $n = 1$ which we have already dealt with separately in proposition 7.7. For the induction step, suppose that $n > 1$. We use the Mayer-Vietoris sequence for S^n and the open covering $\{V, W\}$ with $V = S^n \smallsetminus \{p\}$ and $W = S^n \smallsetminus \{q\}$ where $p, q \in S^n$ are the north and south pole, respectively. We will also use the homotopy invariance of homology. This gives us

$$H_k(V) \cong H_k(W) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

because V and W are homotopy equivalent to a point. Also we get

$$H_k(V \cap W) \cong \begin{cases} \mathbb{Z} & \text{if } k = n-1 \\ \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

by the induction hypothesis, since $V \cap W$ is homotopy equivalent to S^{n-1}. Furthermore it is clear what the inclusion maps $V \cap W \to V$ and $V \cap W \to W$ induce in homology: an isomorphism in H_0 and (necessarily) the zero map in H_k for all $k \neq 0$. Thus the homomorphism

$$H_k(V \cap W) \longrightarrow H_k(V) \oplus H_k(W)$$

from the Mayer-Vietoris sequence takes the form

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

when $k = 0$, and

$$\mathbb{Z} \longrightarrow 0$$

when $k = n - 1$. In all other cases, its source and target are both zero. Therefore the exactness of the Mayer-Vietoris sequence implies that $H_0(S^n)$ and $H_n(S^n)$ are isomorphic to $\mathbb{Z}$, while $H_k(S^n) = 0$ for all other $k \in \mathbb{Z}$. $\square$

Theorem 7.9. *Let* $f\colon S^n \to S^n$ *be the antipodal map. The induced homomorphism* $f_*\colon H_n(S^n) \to H_n(S^n)$ *is multiplication by* $(-1)^{n+1}$.

Proof. We proceed by induction again. For the induction beginning, we take $n = 1$. The antipodal map $f : S^1 \to S^1$ is homotopic to the identity, so that $f^* : H_1(S^1) \to H_1(S^1)$ has to be the identity, too. For the induction step, we use the setup and notation from the previous proof. Exactness of the Mayer-Vietoris sequence for S^n and the open covering $\{V, W\}$ shows that

$$\partial : H_n(S^n) \longrightarrow H_{n-1}(V \cap W)$$

is an isomorphism. The diagram

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{\partial} & H_{n-1}(V \cap W) \\ {\scriptstyle f_*}\big\downarrow & & {\scriptstyle f_*}\big\downarrow \\ H_n(S^n) & \xrightarrow{\partial} & H_{n-1}(W \cap V) \end{array}$$

is meaningful because f takes $V \cap W$ to $V \cap W = W \cap V$. But the diagram is not commutative (i.e., it is not true that $f_* \circ \partial$ equals $\partial \circ f_*$). The reason is that f interchanges V and W, and it does matter in the Mayer-Vietoris sequence which of the two comes first. Therefore we have instead

$$f_* \circ \partial = -\partial \circ f_*$$

in the above square. By the inductive hypothesis, the f_* in the left-hand column of the square is multiplication by $(-1)^n$, and therefore the f^* in the right-hand column of the square must be multiplication by $(-1)^{n+1}$. □

7.4. The usual applications

Theorem 7.10. (Brouwer's fixed point theorem). *Let* $f : D^n \to D^n$ *be a continuous map, where* $n \geq 1$*. Then* f *has a fixed point, i.e., there exists* $y \in D^n$ *such that* $f(y) = y$*.*

Proof. Suppose for a contradiction that f does not have a fixed point. For $x \in D^n$, let $g(x)$ be the point where the ray (half-line) from $f(x)$ to x intersects the boundary S^{n-1} of the disk D^n. Then g is a smooth map from D^n to S^{n-1}, and we have $g|S^{n-1} = \mathrm{id}_{S^{n-1}}$. Summarizing, we have

$$S^{n-1} \xrightarrow{\ j\ } D^n \xrightarrow{\ g\ } S^{n-1}$$

where j is the inclusion, $g \circ j = \mathrm{id}_{S^{n-1}}$. Therefore we get

$$H_{n-1}(S^{n-1}) \xrightarrow{\ j_*\ } H_{n-1}(D^n) \xrightarrow{\ g_*\ } H_{n-1}(S^{n-1})$$

where $g_* j_* = \mathrm{id}$. Thus the abelian group $H_{n-1}(S^{n-1})$ is isomorphic to a direct summand of $H_{n-1}(D^n)$. But from our calculations above, we know that this is not true. If $n > 1$ we have $H_{n-1}(D^n) = 0$ while $H_{n-1}(S^{n-1})$ is not trivial. If $n = 1$ we have $H_{n-1}(D^n) \cong \mathbb{Z}$ while $H_{n-1}(S^{n-1}) \cong \mathbb{Z} \oplus \mathbb{Z}$. □

Let $f \colon S^n \to S^n$ be any continuous map, $n > 0$. The induced homomorphism $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by some number $n_f \in \mathbb{Z}$, since $H_n(S^n)$ is isomorphic to $\mathbb{Z}$.

Definition 7.11. The number n_f is the *degree* of f.

Remark. The degree n_f of $f \colon S^n \to S^n$ is clearly an invariant of the homotopy class of f.

Remark. In the case $n = 1$, the definition of degree as given just above agrees with the definition of degree given in section 1. See exercises.

Example 7.12. According to theorem 7.9, the degree of the antipodal map $S^n \to S^n$ is $(-1)^{n+1}$.

Proposition 7.13. *Let* $f \colon S^n \to S^n$ *be a continuous map. If* $f(x) \neq x$ *for all* $x \in S^n$*, then* f *is homotopic to the antipodal map, and so has degree* $(-1)^{n+1}$*. If* $f(x) \neq -x$ *for all* $x \in S^n$*, then* f *is homotopic to the identity map, and so has degree* 1*.*

Proof. Let $g \colon S^n \to S^n$ be the antipodal map, $g(x) = -x$ for all x. Assuming that $f(x) \neq x$ for all x, we show that f is homotopic to g. We think of S^n as the unit sphere in $\mathbb{R}^{n+1}$, with the usual notion of distance. We can make

a homotopy $(h_t : S^n \to S^n)_{t\in[0,1]}$ from f to g by "sliding" along the unique minimal geodesic arc from $f(x)$ to $g(x)$, for every $x \in S^n$. In other words, $h_t(x) \in S^n$ is situated $t \cdot 100$ percent of the way from $f(x)$ to $g(x)$ along the minimal geodesic arc from $f(x)$ to $g(x)$. (The important thing here is that $f(x)$ and $g(x)$ are not antipodes of each other, by our assumptions. Therefore that minimal geodesic arc is unique.)
Next, assume $f(x) \neq -x$ for all $x \in S^n$. Then, for every x, there is a unique minimal geodesic from x to $f(x)$, and we can use that to make a homotopy from the identity map to f. □

Corollary 7.14. (Hairy ball theorem). *Let ξ be a tangent vector field* (explanations follow) *on S^n. If $\xi(z) \neq 0$ for every $z \in S^n$, then n is odd.*

Comments. A tangent vector field on $S^n \subset \mathbb{R}^{n+1}$ can be defined as a continuous map ξ from S^n to the vector space $\mathbb{R}^{n+1}$ such that $\xi(x)$ is perpendicular to (the position vector of) x, for every $x \in S^n$. We say that vectors in $\mathbb{R}^{n+1}$ which are perpendicular to $x \in S^n$ are *tangent* to S^n at x because they are the velocity vectors of smooth curves in $S^n \subset \mathbb{R}^n$ as the pass through x.

Proof. Define $f : S^n \to S^n$ by $f(x) = \xi(x)/\|\xi(x)\|$. Then $f(x) \neq x$ and $f(x) \neq -x$ for all $x \in S^n$, since $f(x)$ is always perpendicular to x. Therefore f is homotopic to the antipodal map, and also homotopic to the identity. It follows that the antipodal map is homotopic to the identity. Therefore n is odd by theorem 7.9. □

Remark 7.15. Theorem 7.9 has an easy generalization which says that the degree of the map $f\colon S^n \to S^n$ given by

$$(x_1, x_2, \dots, x_{n+1}) \mapsto (x_1, \dots, x_k, -x_{k+1}, \dots, -x_{n+1})$$

is $(-1)^{n+1-k}$. Here we assume $n \geq 1$ as usual. The proof can be given by induction on $n+1-k$. The induction step is now routine, but the induction beginning must cover all cases where $n = 1$. This leaves the three possibilities $k = 0, 1, 2$. One of these gives the identity map $S^1 \to S^1$, and another gives the antipodal map $S^1 \to S^1$ which is homotopic to the identity. The interesting case which remains is the map $f\colon S^1 \to S^1$ given by $f(x_1, x_2) = (x_1, -x_2)$. We need to show that it has degree -1, in the sense of definition 7.11. One way to do this is to use the following diagram

$$\begin{tikzcd} H_1(S^1) \arrow[r, "f_*"] \arrow[d, "\partial"] & H_1(S^1) \arrow[d, "\partial"] \\ H_0(V \cap W) \arrow[r, "f_*"] & H_0(W \cap V) \end{tikzcd}$$

where $V = S^1 \smallsetminus \{(0,1)\}$ and $W = S^1 \smallsetminus \{(0,-1)\}$. We know from the previous chapter that it commutes up to a factor (-1). In the lower row, we have the identity homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. The vertical arrows are injective (seen earlier in the proof of proposition 7.7). Therefore the upper horizontal arrow is multiplication by -1.
To state this result in a more satisfying manner, let us note that the orthogonal group $\mathrm{O}(n+1)$ (the group of orthogonal $(n+1) \times (n+1)$-matrices with real entries) is a topological group which has two path components. The two path components are the preimages of $+1$ and -1 under the homomorphism

$$\det\colon \mathrm{O}(n+1) \to \{-1,+1\}.$$

Let $f\colon S^n \to S^n$ be given by $f(z) = Az$ for some $A \in \mathrm{O}(n+1)$. Because $\deg(f)$ depends only on the homotopy class of f, it follows that $\deg(f)$ depends only on the path component of A in $\mathrm{O}(n+1)$, and hence only on $\det(A)$. What we have just shown means that $\deg(f)$ is *equal* to $\det(A)$.