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Lecture Notes, week 8
Topology WS 2013/14 (Weiss)

7.1. The homotopy decomposition theorem

Notation for the following theorem and the corollary: X and Y are topological spaces, V and W are open subsets of Y such that $V \cup W = Y$, and C is a closed subset of X. We assume that X is *paracompact*.

Theorem 7.1. *Let $\gamma : X \times [0,1] \to Y$ be a mapping cycle which restricts to zero on an open neighborhood of $X \times \{0\}$. Then there exists a decomposition*

$$\gamma = \gamma^V + \gamma^W ,$$

where $\gamma^V \colon X \times [0,1] \to V$ and $\gamma^W \colon X \times [0,1] \to W$ are mapping cycles, both zero on an open neighborhood of $X \times \{0\}$. If γ is zero on some neighborhood of $C \times [0,1]$, then it can be arranged that γ_V and γ_W are zero on a neighborhood of $C \times [0,1]$.

The proof of this is hard. We postpone it.

Corollary 7.2. *Let $a \in [[X,V]]$ and $b \in [[X,W]]$ be such that the images of a and b in $[[X,Y]]$ agree. Then there exists $c \in [[X, V \cap W]]$ whose image in $[[X,V]]$ is a and whose image in $[[X,W]]$ is b.*

Proof. Let α be a mapping cycle which represents a and let β be a mapping cycle which represents b. Choose a mapping cycle $\gamma \colon X \times [0,1] \to Y$ which is a homotopy from 0 to $\beta - \alpha$. It is easy to arrange this in such a way that γ is zero on a neighborhood of $X \times \{0\}$. Use the theorem to obtain a decomposition $\gamma = \gamma^V + \gamma^W$. Let γ_1^V and γ_1^W be the restrictions of γ^V and γ^W to $X \times \{1\}$. Then α and $\alpha + \gamma_1^V$ are homotopic as mapping cycles $X \to V$, by the homotopy $\alpha \circ p + \gamma^V$, where p is the projection $X \times [0,1] \to X$. Similarly $\beta = \alpha + \gamma_1^V + \gamma_1^W$ and $\alpha + \gamma_1^V$ are homotopic as mapping cycles $X \to W$. Finally, $\alpha + \gamma_1^V = \beta - \gamma_1^W$ lands in $V \cap W$ by construction. □

7.2. Mayer-Vietoris sequence in homology

A sequence of abelian groups $(A_n)_{n \in \mathbb{Z}}$ together with homomorphisms

$$f_n \colon A_n \to A_{n-1}$$

for all $n \in \mathbb{Z}$ is called an *exact sequence of abelian groups* if the kernel of f_n is equal to the image of f_{n+1} , for all $n \in \mathbb{Z}$. More generally, we sometimes have to deal with diagrams of abelian groups and homomorphisms in the shape of a string

$$A_n \to A_{n-1} \to A_{n-2} \to \cdots \to A_{n-k} .$$

Such a diagram is *exact* if the kernel of each homomorphism in the string is equal to the image of the preceding one, if there is a preceding one.

Definition 7.3. *Alternative definition of homology*: Write $I = [0, 1]$. For a space Y, and $n \geq 0$, re-define $H_n(Y)$ as the abelian group of homotopy classes of mapping cycles $I^n \to Y$ which vanish on some open neighborhood of ∂I^n.

Comment. In this definition, we regard two mapping cycles $I^n \to Y$ which vanish on some neighborhood of ∂I^n as *homotopic* if they are related by a homotopy $I^n \times I \to Y$ which vanishes on some neighborhood of $\partial I^n \times I$. Such a homotopy will be called (informally) a *homotopy relative to* ∂I^n or a *homotopy rel* ∂I^n.

To relate the old definition of $H_n(Y)$ to the new one, we make a few observations. Given a mapping cycle $\alpha \colon I^n \to Y$ which vanishes on some neighborhood of ∂I^n, we immediately obtain a mapping cycle from the quotient $I^n/\partial I^n$ to Y. To view this as a mapping cycle $\beta \colon S^n \to Y$, we pretend that $S^n = \mathbb{R}^n \cup \infty$ (one-point compactification of $\mathbb{R}^n$) and specify a homeomorphism $u \colon I^n/\partial I^n \to \mathbb{R}^n \cup \infty$ taking base point to base point. (Note that $I^n/\partial I^n$ has a preferred base point, the point represented by all elements of ∂I^n.) We are specific enough if we say that u is smooth and orientation preserving on $I^n \smallsetminus \partial I^n$ (i.e., the Jacobian determinant is everywhere positive). Conversely, given a mapping cycle $\beta \colon S^n \to Y$ representing an element of $H_n(Y)$ according to the old definition, we may subtract a suitable constant to arrange that β is zero when restricted to the base point of S^n. We can also assume that β is zero on a neighborhood of the base point; if not, compose with a continuous map $S^n \to S^n$ which is homotopic to the identity and takes a neighborhood of the base point to the base point. Then $\beta \circ u$ is a mapping cycle $I^n/\partial \to Y$ which can also be viewed as a mapping cycle $I^n \to Y$ vanishing on a neighborhood of ∂I^n.

Definition 7.4. Suppose that Y comes with two open subspaces V and W such that $V \cup W = Y$. The *boundary homomorphism*

$$\partial : H_n(Y) \to H_{n-1}(V \cap W)$$

is defined as follows, using the alternative definition of H_n. Let $x \in H_n(Y)$ be represented by a mapping cycle $\gamma : I^n \to Y$ which is zero on an open neighborhood of ∂I^n. Think of γ as a homotopy, $\gamma : I^{n-1} \times I \to Y$. Choose a decomposition $\gamma = \gamma^V + \gamma^W$ as in theorem 7.1. The theorem guarantees that γ^V and γ^W can be arranged to vanish on a neighborhood of $\partial I^{n-1} \times I$. Let $\partial(x)$ be the class of the mapping cycle

$$\gamma_1^W : I^{n-1} \to V \cap W ,$$

composition of γ^W with the map $\iota_1\colon I^{n-1} \to I^{n-1} \times I$ defined by $\iota_1(x) = (x, 1)$. (Then γ_1^W vanishes on some open neighborhood of ∂I^{n-1}.)

We must show that this is well defined. There were two choices involved: the choice of representative γ, and the choice of decomposition $\gamma = \gamma^V + \gamma^W$. For the moment, keep γ fixed, and let us see what happens if we try another decomposition of γ. Any other decomposition will have the form

$$(\gamma^V + \eta) + (\gamma^W - \eta)$$

where $\eta : I^{n-1} \times I \to V \cap W$ is a mapping cycle which vanishes on an open neighborhood of $\partial I^{n-1} \times I$ and on an open neighborhood of $I^{n-1} \times \{0\}$. We need to show that $\gamma_1^W - \eta_1$ is homotopic (rel boundary of I^{n-1}) to γ_1^W. But η_1 is homotopic to zero by the homotopy η.
Next we worry about the choice of representative γ. Let φ be another representative of the same class x, and let $\lambda : I \times I^n \to Y$ be a homotopy from φ to γ. (Writing the factor I on the left might help us to avoid confusion.) We can think of λ as a homotopy in a different way:

$$(I \times I^{n-1}) \times I \longrightarrow Y.$$

Then we can apply the homotopy decomposition theorem and choose a decomposition $\lambda = \lambda^V + \lambda^W$ where λ^V and λ^W vanish on a neighborhood of $I \times \partial I^{n-1} \times I$. We then find that λ_1^W is a mapping cycle from $X = I \times I^{n-1}$ to $V \cap W$ which we may regard as a homotopy (now with parameters written on the left). The homotopy is between γ_1^W and φ_1^W, provided the decompositions $\gamma = \gamma^V + \gamma^W$ and $\varphi = \varphi^V + \varphi^W$ are the ones obtained by restricting the decomposition $\lambda = \lambda^V + \lambda^W$. □

The boundary homomorphisms ∂ can be used to make a sequence of abelian groups and homomorphisms

$$\begin{array}{ccccc} \cdots \longrightarrow H_{n+1}(Y) & & & & \\ \downarrow \partial & & & & \\ H_n(V \cap W) & \longrightarrow & H_n(V) \oplus H_n(W) & \longrightarrow & H_n(Y) \\ & & & & \downarrow \partial \\ & & & & H_{n-1}(V \cap W) \longrightarrow \cdots \end{array}$$

where $n \in \mathbb{Z}$. (Set $H_n(X) = 0$ for $n < 0$ and any space X. The unlabelled homomorphisms in the sequence are as follows: $H_n(V) \oplus H_n(W) \to H_n(Y)$ is $j_{V*} + j_{W*}$, the sum of the two maps given by composition with the inclusions $j_V\colon V \to Y$ and $j_W\colon W \to Y$, and $H_n(V \cap W) \to H_n(V) \oplus H_n(W)$ is

$(e_{V*}, -e_{W*})$, where e_{V*} and e_{W*} are given by composition with the inclusions $e_V\colon V \cap W \to V$ and $e_W\colon V \cap W \to W$.) The sequence is called the homology *Mayer-Vietoris* sequence of Y and V, W.

Theorem 7.5. *The homology Mayer-Vietoris sequence of* Y *and* V, W *is exact.*[1]

Terminology for the proof. Let X and Q be topological spaces and let $h\colon X \times I \to Q$ be a map or mapping cycle (which we think of as a homotopy). Let $p\colon X \times I \to X$ be the projection and let $\iota_0, \iota_1\colon X \to X \times I$ be the maps given by $x \mapsto (x, 0)$ and $x \mapsto (x, 1)$, respectively. We say that h is *stationary* near $X \times \{0, 1\}$ if there exist open neighborhoods U_0 and U_1 of $X \times \{0\}$ and $X \times \{1\}$, respectively, in $X \times I$ such that h agrees with $h \circ \iota_0 \circ p$ on U_0 and with $h \circ \iota_1 \circ p$ on U_1.

Proof. (i) Exactness of the pieces $H_n(V \cap W) \to H_n(V) \oplus H_n(W) \to H_n(Y)$ follows from corollary 7.2, for all $n \in \mathbb{Z}$. (It is more convenient to use the standard definition of H_n at this point.) More precisely, we have exactness of

$$[[S^n, V \cap W]] \to [[S^n, V]] \oplus [[S^n, W]] \to [[S^n, Y]]$$

by corollary 7.2, and we have exactness of

$$[[\star, V \cap W]] \to [[\star, V]] \oplus [[\star, W]] \to [[\star, Y]]$$

by corollary 7.2. Note also that $[[\star, V]] \oplus [[\star, W]] \to [[\star, Y]]$ is surjective. Then it follows easily that

$$\frac{[[S^n, V \cap W]]}{[[\star, V \cap W]]} \to \frac{[[S^n, V]] \oplus [[S^n, W]]}{[[\star, V]] \oplus [[\star, W]]} \to \frac{[[S^n, Y]]}{[[\star, Y]]}$$

is exact.

(ii) Next we look at pieces of the form

$$H_n(V) \oplus H_n(W) \longrightarrow H_n(Y) \xrightarrow{\ \partial\ } H_{n-1}(V \cap W)\ .$$

The cases $n < 0$ are trivial. In the case $n = 0$, the claim is that the homomorphism $H_0(V) \oplus H_0(W) \to H_0(Y)$ is surjective. This is a pleasant exercise. Now assume $n > 0$. It is clear from the definition of ∂ that the composition of the two homomorphisms is zero. Suppose then that $[\gamma] \in H_n(Y)$ is in the kernel of ∂, where $\gamma : I^n \to Y$ vanishes on a neighborhood of ∂I^n. We must show that $[\gamma]$ is in the image of $H_n(V) \oplus H_n(W) \to H_n(Y)$. As above, we think of γ as a homotopy, $I^{n-1} \times I \to Y$, which we decompose, $\gamma = \gamma^V + \gamma^W$ as in theorem 7.1, where γ^V and γ^W vanish on a neighborhood

[1]If you wish, view this as a sequence of abelian groups and homomorphisms indexed by the integers, by setting for example $A_{3n} = H_n(Y)$ for $n \geq 0$, $A_{3n+1} = H_n(V) \oplus H_n(W)$ for $n \geq 0$, $A_{3n+2} = H_n(V \cap W)$ for $n \geq 0$, and $A_m = 0$ for all $m \leq 0$.

of $\partial I^{n-1} \times I$. We can also arrange that the homotopies γ^V and γ^W are stationary near $I^{n-1} \times \{0,1\}$. The assumption $\partial[\gamma] = 0$ then means that the zero map

$$I^{n-1} \to V \cap W$$

is homotopic to γ_1^W by a homotopy $\lambda : I^{n-1} \times I \to V \cap W$ which vanishes on a neighborhood of $\partial I^{n-1} \times I$. We can arrange that λ is stationary near $I^{n-1} \times \{0,1\}$. Then $\gamma^V + \lambda$ and $\gamma^W - \lambda$ are mapping cycles from $I^{n-1} \times I = I^n$ to V and W, respectively. Both vanish on a neighborhood of ∂I^n. Hence they represent elements in $H_n(V)$ and $H_n(W)$ whose images in $H_n(Y)$ add up to $[\gamma]$.
(iii) We show that the composition

$$H_{n+1}(Y) \xrightarrow{\ \partial\ } H_n(V \cap W) \longrightarrow H_n(V) \oplus H_n(W) .$$

is zero. We can assume $n \geq 0$. Represent an element in $H_n(Y)$ by a mapping cycle $\gamma : I^n \times I \to Y$, vanishing on a neighborhood of the entire boundary; decompose as usual, and obtain $\partial[\gamma] = [\gamma_1^W]$. Now $\gamma_1^W = -\gamma_1^V$ viewed as a mapping cycle $I^n \to V$ is homotopic to zero by the homotopy $-\gamma^V$ vanishing on a neighborhood of $\partial I^{n-1} \times I$. Therefore $\partial[\gamma]$ maps to zero in $H_n(V)$. A similar calculation shows that it maps to zero in $H_n(W)$.
(iv) Finally let $\varphi : I^n \to V \cap W$ be a mapping cycle which vanishes on a neighborhood of ∂I^n, and suppose that $[\varphi] \in H_n(V \cap W)$ is in the kernel of the homomorphism $H_n(V \cap W) \to H_n(V) \oplus H_n(W)$. Choose a homotopy $\gamma^V : I^n \times I \to V$ from zero to $-\varphi$, and another homotopy $\gamma^W : I^n \times I \to W$ from zero to φ, both vanishing on a neighborhood of $\partial I^n \times I$, and both stationary near $I^n \times \{0,1\}$. Then $\gamma := \gamma^V + \gamma^W$ vanishes on the entire boundary of $I^n \times I$, hence represents a class $[\gamma] \in H_{n+1}(Y)$. It is clear that $\partial[\gamma] = [\varphi]$. □

Remark 7.6. The Mayer-Vietoris sequence has a naturality property. The statement is complicated. Suppose that Y and Y' are topological spaces, $g \colon Y \to Y'$ is a continuous map, $Y = V \cup W$ where V and W are open subsets, $Y' = V' \cup W'$ where V'' and W' are open subsets, $g(V) \subset V'$ and $g(W) \subset W'$. Then the Mayer-Vietoris sequences for Y, V, W and Y', V', W'

can be arranged in a ladder-shaped diagram

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H_{n+1}(Y) & \xrightarrow{g_*} & H_{n+1}(Y') \\
\downarrow \partial & & \downarrow \partial \\
H_n(V \cap W) & \xrightarrow{g_*} & H_n(V' \cap W') \\
\downarrow & & \downarrow \\
H_n(V) \oplus H_n(W) & \xrightarrow{g_*} & H_n(V') \oplus H_n(W') \\
\downarrow & & \downarrow \\
H_n(Y) & \xrightarrow{g_*} & H_n(Y') \\
\downarrow \partial & & \downarrow \partial \\
H_{n-1}(V \cap W) & \xrightarrow{g_*} & H_{n-1}(V' \cap W') \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
$$

This diagram is *commutative*; that is the naturality statement. The proof is not complicated (it is by inspection).

Often this can be usefully combined with the following observation: if, in the Mayer-Vietoris sequence for Y and V, W we interchange the roles (order) of V and W, then the homomorphisms ∂ and $H_n(V \cap W) \to H_n(V) \oplus H_n(W)$ change sign. To be more precise, we set up a diagram

$$
\begin{array}{ccc}
H_{n+1}(Y) & \xrightarrow{=} & H_{n+1}(Y) \\
\downarrow \partial & & \downarrow \partial \\
H_n(V \cap W) & \xrightarrow{=} & H_n(W \cap V) \\
\downarrow & & \downarrow \\
H_n(V) \oplus H_n(W) & \xrightarrow{\cong} & H_n(W) \oplus H_n(V)
\end{array}
$$

where the columns are bits from the Mayer-Vietoris sequence of Y, V, W and Y, W, V, respectively. The diagram is *not* (always) commutative; instead each of the small squares in it commutes up to a factor (-1). The proof is by inspection.