17.11.2013/03.12.2013

Lecture Notes, week 5
Topology WS 2013/14 (Weiss)

4.1. Presheaves and sheaves on topological spaces

Definition 4.1. A *presheaf* on a topological space X is a rule $\mathcal{F}$ which to every open subset U of X assigns a set $\mathcal{F}(U)$, and to every pair of nested open sets $U \subset V \subset X$ a map

$$\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$$

which satisfies the following conditions.

- For open sets $U \subset V \subset W$ in X we have $\mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$ (an equality of maps from $\mathcal{F}(W)$ to $\mathcal{F}(U)$).
- $\mathrm{res}_{V,V} = \mathrm{id} \colon \mathcal{F}(V) \to \mathcal{F}(V)$ for every open V in X.

Example 4.2. An important and obvious example for us is the following. Fix X as above and let Y be another topological space. For open U in X let $\mathcal{F}(U)$ be the *set* of all continuous maps from U to Y. Note that we make no attempt here to define a topology on $\mathcal{F}(U)$; we just take it as a set. For open sets $U \subset V \subset X$ there is an obvious restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$. That is, a continuous map from V to Y determines by restriction a continuous map from U to Y. The conditions for a presheaf are clearly satisfied.

Example 4.3. Let $p \colon Y \to X$ be any continuous map. We can use this to make a presheaf $\mathcal{F}$ on X as follows. For an open set U in X, let $\mathcal{F}(U)$ be the set of continuous maps $g \colon U \to Y$ such that $p \circ g = \mathrm{id}_U$. For open sets $U \subset V \subset X$ let $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ be given by restriction in the usual sense. Namely, if $f \in \mathcal{F}(V)$, then $f \colon V \to Y$ is a continuous map which satisfies $p \circ f = \mathrm{id}_V$, and so the restriction $f_{|U}$ is a continuous map $U \to Y$ which satisfies $p \circ f_{|U} = \mathrm{id}_U$.

Example 4.4. Suppose that X happens to be a differentiable (smooth) manifold (in which case it is also a topological space). For open U in X, let $\mathcal{F}(U)$ be the set of smooth functions from U to $\mathbb{R}$. For open subsets $U \subset V \subset X$, let $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ be given by restriction in the usual sense. The conditions for a presheaf are clearly satisfied by $\mathcal{F}$.

Example 4.5. Given a topological space X and a set S, define $\mathcal{F}(U) = S$ for every open U in X. For open sets $U \subset V \subset X$, let

$$\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$$

be the identity map of S. The conditions for a presheaf are clearly satisfied.

Example 4.6. Fix X as above and let Y be another topological space. For open U in X put $\mathcal{F}(U) = [U, Y]$, the set of homotopy classes of continuous maps from U to Y. For open sets $U \subset V \subset X$ there is an obvious restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$. That is, a homotopy class of continuous maps from V to Y determines by restriction a homotopy class of continuous maps from U to Y. The conditions for a presheaf are clearly satisfied.
This example looks as if it might become very important in this course, since it connects presheaves and the concept of homotopy. But it will not become very important except as a source of homework problems and counterexamples.

Example 4.7. Fix X as above and let Y be another topological space. For an open subset U of X let $\mathcal{F}(U)$ be the set of *formal linear combinations* (with integer coefficients) of continuous maps from U to Y. So an element of $\mathcal{F}(U)$ might look like $5f - 3g + 9h$ where f, g and h are continuous maps from U to Y. We do not insist that f, g, h in this expression are distinct, but if for example f and g are equal, then we take the view that $5f - 3g + 9h$ and $2f + 9h$ define the same element of $\mathcal{F}(U)$. This remark is important when we define the restriction map

$$\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$$

This is of course determined by restriction of continuous maps. So for example, if $3a - 6b + 10c - d$ is an element of $\mathcal{F}(V)$, and here we may as well assume that the continuous maps $a, b, c, d \colon V \to Y$ are distinct (because we can simplify the expression if not), then $\mathrm{res}_{V,U}$ takes that element to $3(a_{|U}) - 6(b_{|U}) + 10(c_{|U}) - d_{|U} \in \mathcal{F}(U)$. And here we can not assume that the continuous maps $a_{|U}, b_{|U}, c_{|U}, d_{|U} \colon U \to Y$ are all distinct. In any case the conditions for a presheaf are clearly satisfied.
This example looks silly and unimportant, but it is not silly and it will become very important in this course. Let's also note that there are more grown-up ways to describe $\mathcal{F}(U)$ for this presheaf $\mathcal{F}$. Instead of saying *the set of formal linear combinations with integer coefficients of continuous maps from* U *to* Y, we can say: the free abelian group generated by the set of continuous maps from U to Y. Or we can say: the free $\mathbb{Z}$-module generated by the set of continuous maps from U to Y. (See also subsection 4.4 for some clarifications.)

With a view to the next definition we introduce some practical notation. Let X be a space, let $\mathcal{F}$ be a presheaf on X, and suppose that U, V are open subsets of X such that $U \subset V$. Then we have the restriction map $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$. Let $s \in \mathcal{F}(V)$. Instead of writing $\mathrm{res}_{V,U}(s) \in \mathcal{F}(U)$, we sometimes write $s_{|U} \in \mathcal{F}(U)$.

Definition 4.8. A presheaf $\mathcal{F}$ on a topological space X is called a *sheaf* on X if it has the following additional properties. For every collection of open subsets $(W_i)_{i\in\Lambda}$ of X, and every collection

$$(s_i \in \mathcal{F}(W_i))_{i\in\Lambda}$$

with the property $s_{i|W_i\cap W_j} = s_{j|W_i\cap W_j} \in \mathcal{F}(W_i \cap W_j)$, there exists a unique

$$s \in \mathcal{F}(\bigcup_{i\in\Lambda} W_i)$$

such that $s_{|W_i} = s_i$ for all $i \in \Lambda$. In particular, $\mathcal{F}(\emptyset)$ has exactly one element.

In a slightly more wordy formulation: if we have elements $s_i \in \mathcal{F}(W_i)$ for all $i \in \Lambda$, and we have agreement of s_i and s_j on $W_i \cap W_j$ for all $i, j \in \Lambda$, then there is a unique $s \in \mathcal{F}(\bigcup_i W_i)$ which agrees with s_i on each W_i.
To silence a particularly nagging and persistent type of critic, including the critic within myself, let me explain in detail why this implies that $\mathcal{F}(\emptyset)$ has exactly one element. Put $\Lambda = \emptyset$. For each $i \in \Lambda$, select an open subset W_i. (Easy, because there is no $i \in \Lambda$.) For each $i \in \Lambda$, select an element $s_i \in \mathcal{F}(W_i)$. (Easy.) Verify that, for each i and j in Λ, we have $s_{i|U_i\cap U_j} = s_{j|U_i\cap U_j}$. (Easy.) Conclude that there exists a *unique*

$$s \in \mathcal{F}(\bigcup_{i\in\Lambda} W_i)$$

such that $s_{|W_i} = s_i$ for every $i \in \Lambda$. Now note that $\bigcup_{i\in\Lambda} W_i = \emptyset$ and verify that *every* $t \in \mathcal{F}(\emptyset)$ satisfies the condition $t_{|W_i} = s_i$ for every $i \in \Lambda$. (Easy.) Therefore *every* element t of $\mathcal{F}(\emptyset)$ must be equal to that distinguished element s which we have already spotted.

Obviously it is now our duty to scan the list of the examples above and decide for each of these presheaves $\mathcal{F}$ whether it is a sheaf. It is a good idea to ask first in each case whether $\mathcal{F}(\emptyset)$ has exactly one element. If that is not the case, then it is not a sheaf. It looks like a mean reason to refuse sheaf status to a presheaf. But often when $\mathcal{F}(\emptyset)$ does not have exactly one element, the presheaf $\mathcal{F}$ turns out to have other properties which prevent us from promoting it to sheaf status e.g. by simply redefining $\mathcal{F}(\emptyset)$. — The following lemma is also a good tool in testing for the sheaf property.

Lemma 4.9. *Let $\mathcal{F}$ be a sheaf on X and let $(W_i)_{i\in\Lambda}$ be a collection of pairwise disjoint open subsets of X. Then the formula $s \mapsto (s_{|W_i})_{i\in\Lambda}$ determines a bijection*

$$\mathcal{F}(\bigcup_{i\in\Lambda} W_i) \longrightarrow \prod_{i\in\Lambda} \mathcal{F}(W_i).$$

Proof. Take an element in $\prod_{i\in\Lambda}\mathcal{F}(W_i)$ and denote it by $(s_i)_{i\in\Lambda}$, so that $s_i \in \mathcal{F}(W_i)$. Since $W_i \cap W_j = \emptyset$ and $\mathcal{F}(\emptyset)$ has exactly one element, the matching condition

$$s_{i|W_i\cap W_j} = s_{j|W_i\cap W_j}$$

is vacuously satisfied for all $i, j \in \Lambda$. Hence by the sheaf property, there is a unique $s \in \mathcal{F}(\bigcup_{i\in\Lambda} W_i)$ such that $s_{|W_i} = s_i$ for all $i \in \Lambda$. This means precisely that $s \mapsto (s_{|W_i})_{i\in\Lambda}$ is a bijection. (The surjectivity is expressed in *there is* and the injectivity in the word *unique*.) □

Discussion of example 4.2. This is a sheaf. What is being said here is that if we have open $W_i \subset X$ for each $i \in \Lambda$, and continuous maps $f_i\colon W_i \to Y$ for each i such that f_i and f_j agree on $W_i \cap W_j$ for all $i, j \in \Lambda$, then we have a unique continuous map f from $\bigcup W_i$ to Y which agrees with f_i on W_i for each $i \in \Lambda$.

Discussion of example 4.3. This is a sheaf. We can reason as in the case of example 4.2.

Discussion of example 4.4. This is a sheaf. What is being said here is that if X is a smooth manifold, and we have open $W_i \subset X$ for each $i \in \Lambda$, and smooth functions $f_i\colon W_i \to \mathbb{R}$ for each i such that f_i and f_j agree on $W_i \cap W_j$ for all $i, j \in \Lambda$, then we have a unique smooth $f\colon \bigcup W_i \to Y$ which agrees with f_i on W_i for each $i \in \Lambda$. An interesting aspect of this example is that, in contrast to examples 4.2 and 4.3, it seems to express something which is not part of the world of topological spaces, something "differentiable". So I am suggesting that the notion of *smooth manifold* could be redefined along the following lines: a smooth manifold is a topological Hausdorff space X together with a sheaf $\mathcal{F}$ ... which we would call the sheaf of *smooth* functions (on open subsets of X) and which would presumably have to be a subsheaf (notion yet to be defined) of the sheaf of *continuous* functions on open subsets of X. That would be an alternative to defining smooth manifolds using charts and atlases. Of course this has been noticed and has been done by the ancients, but we are getting ahead of ourselves.

Discussion of example 4.5. Here we have to make a case distinction. If S has exactly one element, then this presheaf $\mathcal{F}$ is a sheaf, and the verification is easy. If S has more than one element, or is empty, then $\mathcal{F}$ is not a sheaf because $\mathcal{F}(\emptyset)$ does not have exactly one element.
Can we fix this by redefining $\mathcal{F}(\emptyset)$ to have exactly one element? Let us try. So let $\mathcal{G}$ be the presheaf on X defined by $\mathcal{G}(U) = S$ when U is nonempty, and $\mathcal{G}(\emptyset) = \{*\}$, a set with a single element $*$. It is a presheaf as follows: for open subsets $U \subset V$ of X we let $\mathrm{res}_{V,U}\colon \mathcal{G}(V) \to \mathcal{G}(U)$ be the identity map of S if $U \neq \emptyset$; otherwise it is the unique map of sets from $\mathcal{G}(V)$ to $\{*\}$.

Is this presheaf $\mathcal{G}$ a sheaf? The answer depends a little on X, and on S. Suppose that X has disjoint open nonempty subsets U_1 and U_2. By lemma 4.9, the diagonal map from $S = \mathcal{G}(U_1 \cup U_2)$ to $S \times S = \mathcal{G}(U_1) \times \mathcal{G}(U_2)$ is bijective. We have a problem with that if S has more than one element. The case where S has exactly one element was excluded, so only the possibility $S = \emptyset$ remains. And indeed, if S is empty, we don't have a problem: $\mathcal{G}$ is a sheaf. Also, if X does not have any disjoint nonempty open subsets U_1 and U_2, we don't have a problem: $\mathcal{G}$ is a sheaf, no matter what S is.

Discussion of example 4.6. In general, this is not a sheaf, although it responds nicely to the two standard tests. (One standard test is to ask: what is $\mathcal{F}(\emptyset)$? Here we get the set of homotopy classes of maps from $\emptyset$ to Y, and that set has exactly one element, as it should have if $\mathcal{F}$ were a sheaf. The other standard test comes from lemma 4.9. If $(W_i)_{i\in\Lambda}$ is a collection of disjoint open subsets of X, then

$$\mathcal{F}(\bigcup_i W_i) = [\bigcup_i W_i, Y]$$

which is in bijection with $\prod_{i\in\Lambda}[W_i, Y]$ by composition with the inclusions $W_j \to \bigcup_{i\in\Lambda} W_i$ for each $j \in \Lambda$.) For a counterexample, let $X = Y = S^1$. In X we have the open sets U_1 and U_2 where $U_1 = S^1 - \{1\}$ and $U_2 = S^1 \smallsetminus \{-1\}$, using complex number notation. Since U_1 and U_2 are contractible and Y is path connected, both $\mathcal{F}(U_1)$ and $\mathcal{F}(U_2)$ have exactly one element. Since $U_1 \cap U_2$ is the disjoint union of two contractible open sets V_1 and V_2, we get

$$\mathcal{F}(U_1 \cap U_2) = \mathcal{F}(V_1 \cup V_2)$$

which is in bijection with $\mathcal{F}(V_1)\times\mathcal{F}(V_2)$, which again has exactly one element. If $\mathcal{F}$ were a sheaf, it would follow from these little calculations that $\mathcal{F}(U_1\cup U_2)$ has exactly one element. But $\mathcal{F}(U_1 \cup U_2) = \mathcal{F}(X) = [X, Y] = [S^1, S^1]$, and we know that this has infinitely many elements.

Discussion of example 4.7. This is obviously not a sheaf because $\mathcal{F}(\emptyset)$ has more than one element. Indeed, there is exactly one continuous map from $\emptyset$ to Y. So $\mathcal{F}(\emptyset)$ is the free $\mathbb{Z}$-module one one generator, which means that it is isomorphic to $\mathbb{Z}$.

It might seem pointless to look for further reasons to deny sheaf status to $\mathcal{F}$. It is like kicking somebody who is already down. Nevertheless, because this is an important example, it will be instructive for us to know more about it, and we could argue that by showing interest we are showing some patience and kindness. Also, there is a new aspect here: the sets $\mathcal{F}(U)$ always always carry the structure of abelian groups alias $\mathbb{Z}$-modules, and the maps $\mathrm{res}_{V,U}$ are always homomorphisms.

Suppose that $X = \{1, 2, 3, 4, 5, 6\}$ with the discrete topology (every subset of X is declared to be open). Let $Y = \{a, b, \}$, a set with two elements, also with the discrete topology. We note that X is the disjoint union of six open subsets U_i , where $i = 1, 2, 3, 4, 5, 6$ and $U_i = \{i\}$. We have $\mathcal{F}(U_i) = \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^2$ (free $\mathbb{Z}$-module on two generators) because each U_i has exactly two continuous maps to Y. We have $\mathcal{F}(\bigcup_i U_i) = \mathcal{F}(X) = \mathbb{Z}^{64}$ (free $\mathbb{Z}$-module on 64 generators) because there are 64 continuous maps from X to Y. It follows that the map

$$\mathcal{F}(\textstyle\bigcup_i U_i) \longrightarrow \prod_{i=1}^{6} \mathcal{F}(U_i)$$

of lemma 4.9 (which in the present circumstances is a $\mathbb{Z}$-module homomorphism) cannot be bijective, because that would make it a $\mathbb{Z}$-module isomorphism between $\mathbb{Z}^{64}$ and $\mathbb{Z}^{12}$. (For an abstract interpretation of what is happening, the notion of *tensor product* is useful. Namely, $\mathcal{F}(\bigcup_i U_i) \cong \mathbb{Z}^{64}$ is isomorphic to the tensor product

$$\mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \otimes \cdots \otimes \mathcal{F}(U_6).$$

It is unsurprising that this is *not* isomorphic to the product $\prod_{i=1}^{6} \mathcal{F}(U_i)$. So it emerges that $\mathcal{F}$ fails to have the sheaf property because it has another respectable property.)

Next, re-define X and Y in such a way that X and Y are two topological spaces related by a covering map $p \colon Y \to X$ with finite fibers. In other words, p is a fiber bundle whose fibers are finite sets (viewed as topological spaces with the discrete topology). For simplicity, suppose also that X is connected. Choose an open covering $(W_j)_{j \in \Lambda}$ of X such that p admits a bundle chart over W_j for each j:

$$h_j \colon p^{-1}(W_j) \to W_j \times F$$

where F is a finite set (with the discrete topology). There is no loss of generality in asking for the same F in all cases, independent of j, because X is connected.[1] For $j \in \Lambda$ and $z \in F$ there is a continuous map $\sigma_{j,z} \colon W_j \to Y$ given by $\sigma_{j,z}(x) = h_j^{-1}(x, z)$ for $x \in W_j$. Define

$$s_j = \sum_{z \in F} \sigma_{j,z} .$$

This is a formal linear combination of continuous maps from W_j to Y which has meaning as an element $\mathcal{F}(W_j)$. So we can write $s_j \in \mathcal{F}(W_j)$. The matching condition

$$s_{i|W_i \cap W_j} = s_{j|W_i \cap W_j}$$

is satisfied. However it seems to be hard or impossible to produce $s \in \mathcal{F}(X) = \mathcal{F}(\bigcup_j W_j)$ such that $s_{|W_i} = s_i$ for all $i \in \Lambda$. This indicates another violation of the sheaf property. (Unfortunately, showing that in many cases such an s does not exist is also hard; we may return to this when we are wiser.)

[1] Lecture notes week 2, prop. 2.3.

4.2. Categories, functors and natural transformations

The concept of a *category* and the related notions *functor* and *natural transformation* emerged in the middle of the 20th century (Eilenberg-MacLane, 1945) and were immediately used to re-organize algebraic topology (Eilenberg-Steenrod, 1952). Later these notions became very important in many other branches of mathematics, especially algebraic geometry. Category theory has many definitions of great depth, I think, but in the foundations very few theorems and fewer proofs of any depth. Among those who love difficult proofs, it has a reputation of shallowness, boring-ness; for many of the theorizers who appreciate good definitions, it is an ever-ongoing revelation. Young mathematicians tend to like it better than old mathematicians ... probably because it helps them to see some order in a multitude of mathematical facts.

Definition 4.10. A *category* $\mathcal{C}$ consists of a class $\mathrm{Ob}(\mathcal{C})$ whose elements are called the *objects of* $\mathcal{C}$ and the following additional data.

- For any two objects c and d of $\mathcal{C}$, a set $\mathrm{mor}_{\mathcal{C}}(c,d)$ whose elements are called the *morphisms* from c to d.
- For any object c in $\mathcal{C}$, a distinguished element $\mathrm{id}_c \in \mathrm{mor}_{\mathcal{C}}(c,c)$, called the *identity morphism* of c.
- For any three objects b,c,d of $\mathcal{C}$, a map from $\mathrm{mor}_{\mathcal{C}}(c,d)\times\mathrm{mor}_{\mathcal{C}}(b,c)$ to $\mathrm{mor}_{\mathcal{C}}(b,d)$ called *composition* and denoted by $(f,g)\mapsto f\circ g$.

These data are subject to certain conditions, namely:

- Composition of morphisms is associative.
- The identity morphisms act as two-sided neutral elements for the composition.

The associativity condition, written out in detail, means that

$$(f\circ g)\circ h = f\circ(g\circ h)$$

whenever a,b,c,d are objects of $\mathcal{C}$ and $f\in\mathrm{mor}_{\mathcal{C}}(c,d)$, $g\in\mathrm{mor}_{\mathcal{C}}(b,c)$, $h\in\mathrm{mor}_{\mathcal{C}}(a,b)$. The condition on identity morphisms means that $f\circ\mathrm{id}_c = f = \mathrm{id}_d\circ f$ whenever c and d are objects in $\mathcal{C}$ and $f\in\mathrm{mor}_{\mathcal{C}}(c,d)$. Saying that $\mathrm{Ob}(\mathcal{C})$ is a *class*, rather than a *set*, is a subterfuge to avoid problems which are likely to arise if, for example, we talk about *the set of all sets* (Russell's paradox). If the object class is a set, which sometimes happens, we speak of a *small category*.

Notation: we shall often write $\mathrm{mor}(c,d)$ instead of $\mathrm{mor}_{\mathcal{C}}(c,d)$ if it is obvious that the category in question is $\mathcal{C}$. Morphisms are often denoted by arrows, as in $f\colon c\to d$ when $f\in\mathrm{mor}(c,d)$. It is customary to say in such a case that c is the *source* or *domain* of f, and d is the *target* or *codomain* of f. A morphism $f\colon c\to d$ in a category $\mathcal{C}$ is said to be an *isomorphism* if there exists a morphism $g\colon d \to c$ in $\mathcal{C}$ such that $g \circ f = \mathrm{id}_c \in \mathrm{mor}_{\mathcal{C}}(c, c)$ and $f \circ g = \mathrm{id}_d \in \mathrm{mor}_{\mathcal{C}}(d, d)$.

Example 4.11. The prototype is $\mathcal{S}$ets, the category of sets. The objects of that are the sets. For two sets S and T, the set of morphisms $\mathrm{mor}(S, T)$ is the set of all maps from S to T. Composition is composition of maps as we know it and the identity morphisms are the identity maps as we know them. Another very important example for us is $\mathcal{T}$op, the category of topological spaces. The objects are the topological spaces. For topological spaces $X = (X, \mathcal{O}_X)$ and $Y = (Y, \mathcal{O}_Y)$, the set of morphisms $\mathrm{mor}(X, Y)$ is the set of continuous maps from X to Y. Composition is composition of continuous maps as we know it and the identity morphisms are the identity maps as we know them.
Another very important example for us is $\mathcal{H}$o$\mathcal{T}$op, the homotopy category of topological spaces. The objects are the topological spaces, as in $\mathcal{T}$op. But the set of morphisms from $X = (X, \mathcal{O}_X)$ to $Y = (Y, \mathcal{O}_Y)$ is $[X, Y]$, the set of *homotopy classes* of continuous maps from X to Y. Composition $\circ$ is defined by the formula

$$[f] \circ [g] = [f \circ g]$$

for $[f] \in [Y, Z]$ and $[g] \in [X, Y]$. Here $f\colon Y \to Z$ and $g\colon X \to Y$ are continuous maps representing certain elements of $[Y, Z]$ and $[X, Y]$, and $f \circ g\colon X \to Z$ is their composition. There is an issue of well-defined-ness here, but fortunately we settled this long ago in chapter 1. As a result, associativity of composition is not in doubt because it is a consequence of associativity of composition in $\mathcal{T}$op. The identity morphisms in $\mathcal{H}$o$\mathcal{T}$op are the homotopy classes of the identity maps.
Another popular example is $\mathcal{G}$roups, the category of groups. The objects are the groups. For groups G and H, the set of morphisms $\mathrm{mor}(G, H)$ is the set of group homomorphisms from G to H. Composition of morphisms is composition of group homomorphisms.
The definition of a category as above permits some examples which are rather strange. One type of strange example which will be important for us soon is as follows. Let $(P, \le)$ be a partially ordered set, alias poset. That is to say, P is a set and $\le$ is a relation on P which is transitive ($x \le y$ and $y \le z$ forces $x \le z$), reflexive ($x \le x$ holds for all x) and antisymmetric (in the sense that $x \le y$ and $y \le x$ together implies $x = y$). We turn this setup into a small category (nameless) such that the object set is P. We decree that, for $x, y \in P$, the set $\mathrm{mor}(x, y)$ shall be empty if x is not $\le y$, and shall contain exactly one element, denoted $*$, if $x \le y$. Composition

$$\circ : \mathrm{mor}(y, z) \times \mathrm{mor}(x, y) \longrightarrow \mathrm{mor}(x, z)$$

is defined as follows. If y is not $\leq z$, then $\mathrm{mor}(y, z)$ is empty and so $\mathrm{mor}(y, z) \times \mathrm{mor}(x, y)$ is empty, too. There is exactly one map from the empty set to $\mathrm{mor}(x, z)$ and we take that. If x is not $\leq y$, then $\mathrm{mor}(y, z) \times \mathrm{mor}(x, y)$ is empty, and we have only one choice for our composition map, and we take that. The remaining case is the one where $x \leq y$ and $y \leq z$. Then $x \leq z$ by transitivity. Therefore $\mathrm{mor}(y, z) \times \mathrm{mor}(x, y)$ has exactly one element, but more importantly, $\mathrm{mor}(x, z)$ has also exactly one element. Therefore, once again, there is exactly one map from $\mathrm{mor}(y, z) \times \mathrm{mor}(x, y)$ to $\mathrm{mor}(x, z)$ and we take that.
Another type of strange example (less important for us but still instructive) can be constructed by starting with a specific group G, with multiplication map $\mu\colon G \times G \to G$. From that we construct a small category (nameless) whose object set has exactly one element, denoted $*$. We let $\mathrm{mor}(*, *) = G$. The composition map

$$\mathrm{mor}(*, *) \times \mathrm{mor}(*, *) \to \mathrm{mor}(*, *)$$

now has to be a map from $G \times G$ to G, and for that we choose μ, the multiplication of G. Since μ has an associativity property, we can be certain that composition of morphisms is associative. For the identity morphism $\mathrm{id}_* \in \mathrm{mor}(*, *)$ we take the neutral element of G.
There are also some easy ways to make new categories out of old ones. One important example: let $\mathcal{C}$ be any category. We make a new category $\mathcal{C}^{\mathrm{op}}$, the *opposite* category of $\mathcal{C}$. It has the same objects as $\mathcal{C}$, but we let

$$\mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(c, d) := \mathrm{mor}_{\mathcal{C}}(d, c)$$

when c and d are objects of $\mathcal{C}$, or equivalently, objects of $\mathcal{C}^{\mathrm{op}}$. The identity morphism of an object c in $\mathcal{C}^{\mathrm{op}}$ is the identity morphism of c in $\mathcal{C}$. Composition

$$\mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(c, d) \times \mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(b, c) \longrightarrow \mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(b, d)$$

is defined by noting $\mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(c, d) \times \mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(b, c) = \mathrm{mor}_{\mathcal{C}}(d, c) \times \mathrm{mor}_{\mathcal{C}}(c, b)$ and going from there to $\mathrm{mor}_{\mathcal{C}}(c, b) \times \mathrm{mor}_{\mathcal{C}}(d, c)$ by an obvious bijection, and from there to $\mathrm{mor}_{\mathcal{C}}(d, b) = \mathrm{mor}_{\mathcal{C}^{\mathrm{op}}}(b, d)$ using composition of morphisms in the category $\mathcal{C}$.

It turns out that there is something like a *category of all categories*. Let us not try to make that very precise because there are some small difficulties and complications in that. In any case there is a concept of morphism between categories, and the name of that is *functor*.

Definition 4.12. A *functor* from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a rule F which to every object c of $\mathcal{C}$ assigns an object $F(c)$ of $\mathcal{D}$, and to every morphism $g\colon b \to c$ in $\mathcal{C}$ a morphism $F(g)\colon F(b) \to F(c)$ in $\mathcal{D}$, subject to the following conditions.

- For any object c in $\mathcal{C}$ with identity morphism $\mathrm{id}_c\colon c \to c$, we have $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.
- Whenever a, b, c are objects in $\mathcal{C}$ and $h \in \mathrm{mor}_{\mathcal{C}}(a, b)$, $g \in \mathrm{mor}_{\mathcal{C}}(b, c)$, we have $F(g \circ h) = F(g) \circ F(h)$ in $\mathrm{mor}_{\mathcal{D}}(F(a), F(c))$.

Example 4.13. A functor F from the category $\mathcal{T}op$ to the category $\mathcal{S}ets$ can be defined as follows. For a topological space X let $F(X)$ be the set of path components of X. A continuous map $g\colon X \to Y$ determines a map $F(g)\colon F(X) \to F(Y)$ like this: $F(g)$ applied to a path component C of X is the unique path component of Y which contains $g(C)$.
Fix a positive integer n. Let $\mathcal{R}ings$ be the category of rings and ring homomorphisms. (For me, a ring does not have to be commutative, but it should have certain elements 0 and 1 and these are required to be distinct.) A functor F from $\mathcal{R}ings$ to $\mathcal{G}roups$ can be defined by $F(R) = \mathrm{GL}_n(R)$, where $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices with entries in R. A ring homomorphism $g\colon R_1 \to R_2$ determines a group homomorphism $F(g)\colon F(R_1) \to F(R_2)$. Namely, in an invertible $n \times n$-matrix with entries in R_1 , apply g to each entry to obtain an invertible $n \times n$-matrix with entries in R_2.
Let G be a group which comes with an action on a set S. In example 4.11 we constructed from G a category with one object $*$ and $\mathrm{mor}(*, *) = G$. A functor F from that category to $\mathcal{S}ets$ can now be defined by $F(*) = S$, and $F(g) =$ translation by g, for $g \in \mathrm{mor}(*, *) = G$. More precisely, to $g \in G = \mathrm{mor}(*, *)$ we associate the map $F(g)$ from $S = F(*)$ to $S = F(*)$ given by $x \mapsto g \cdot x$ (which has a meaning because we are assuming an action of G on S).
Let $\mathcal{C}$ be any category and let x be any object of $\mathcal{C}$. A functor F_x from $\mathcal{C}$ to $\mathcal{S}ets$ can be defined as follows. Let $F_x(c) = \mathrm{mor}_{\mathcal{C}}(x, c)$. For a morphism $g\colon c \to d$ in $\mathcal{C}$ define $F_x(g)\colon F_x(c) \to F_x(d)$ by $F_x(g)(h) = g \circ h$. In more detail, we are assuming $h \in F_x(c) = \mathrm{mor}_{\mathcal{C}}(x, c)$ and $g \in \mathrm{mor}_{\mathcal{C}}(c, d)$, so that $g \circ h \in \mathrm{mor}_{\mathcal{C}}(x, d) = F_x(d)$.

The functors of definition 4.12 are also called *covariant functors* for more precision. There is a related concept of *contravariant functor*. A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is simply a (covariant) functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$ (see example 4.11). If we write this out, it looks like this. A contravariant functor F from $\mathcal{C}$ to $\mathcal{D}$ is a rule which to every object c of $\mathcal{C}$ assigns an object $F(c)$ of $\mathcal{D}$, and to every morphism $g\colon c \to d$ in $\mathcal{C}$ a morphism $F(g)\colon F(d) \to F(c)$; note that the source of $F(g)$ is $F(d)$, and the target is $F(c)$. And so on.

Example 4.14. Let $\mathcal{C}$ be any category and let x be any object of $\mathcal{C}$. A contravariant functor F^x from $\mathcal{C}$ to $\mathcal{S}ets$ can be defined as follows. Let

$F^x(c) = \mathrm{mor}_{\mathcal{C}}(c, x)$. For a morphism $g\colon c \to d$ in $\mathcal{C}$ define

$$F^x(g)\colon F^x(d) \to F^x(c)$$

by $F^x(g)(h) = h \circ g$. In more detail, we are assuming $h \in F^x(d) = \mathrm{mor}_{\mathcal{C}}(d, x)$ and $g \in \mathrm{mor}_{\mathcal{C}}(c, d)$, so that $h \circ g \in \mathrm{mor}_{\mathcal{C}}(c, x) = F^x(c)$.
There is a contravariant functor P from $\mathcal{S}ets$ to $\mathcal{S}ets$ given by $P(S) =$ power set of S, for a set S. In more detail, a morphism $g\colon S \to T$ in $\mathcal{S}ets$ determines a map $P(g)\colon P(T) \to P(S)$ by "preimage". That is, $P(g)$ applied to a subset U of T is $g^{-1}(U)$, a subset of S. (You may have noticed that this example of a contravariant functor is not very different from a special case of the preceding one; we will return to this point later.)
Next, let $\mathcal{M}an$ be the category of smooth manifolds. The objects are the smooth manifolds (of any dimension). The morphisms from a smooth manifold M to a smooth manifold N are the smooth maps from M to N. For any fixed integer $k \geq 0$ the rule which assigns to a smooth manifold M the real vector space $\Omega^k(M)$ of smooth differential k-forms is a contravariant functor from $\mathcal{M}an$ to the category $\mathcal{V}ect$ of real vector spaces (with linear maps as morphisms). Namely, a smooth map $f\colon M \to N$ determines a linear map $f^*\colon \Omega^k(N) \to \Omega^k(M)$. (You must have seen the details if you know anything about differential forms.)
A presheaf $\mathcal{F}$ on a topological space X is nothing but a contravariant functor from the poset of open subsets of X to $\mathcal{S}ets$. In more detail, write $\mathcal{O}$ for the topology on X, the set of open subsets of X. We can regard $\mathcal{O}$ as a partially ordered set (poset) in the following way: for $U, V \in \mathcal{O}$ we decree that $U \leq V$ if and only if $U \subset V$. A partially ordered set is a small category, as explained in example 4.11; therefore $\mathcal{O}$ is (the object set of) a small category. For $U, V \in \mathcal{O}$, there is exactly one morphism from U to V if $U \subset V$, and none if U is not contained in V. To that one morphism (if $U \subset V$) the presheaf $\mathcal{F}$ assigns a map $\mathrm{res}_{V,U}\colon \mathcal{F}(V) \to \mathcal{F}(U)$. The conditions on $\mathcal{F}$ in definition 4.1 are special cases of the conditions on a contravariant functor.

The story does not end there. The functors from a category $\mathcal{C}$ to a category $\mathcal{D}$ also form something like a category. There is a concept of morphism between functors from $\mathcal{C}$ to $\mathcal{D}$, and the name of that is *natural transformation.*

Definition 4.15. Let F and G be functors, both from a category $\mathcal{C}$ to a category $\mathcal{D}$. A *natural transformation* from F to G is a rule ν which for every object c in $\mathcal{C}$ selects a morphism $\nu_c\colon F(c) \to G(c)$ in $\mathcal{D}$, subject to the following condition. Whenever $u\colon c \to d$ is a morphism in $\mathcal{C}$, the square of morphisms

$$\begin{array}{ccc} F(c) & \xrightarrow{\nu_c} & G(c) \\ \downarrow{\scriptstyle F(u)} & & \downarrow{\scriptstyle G(u)} \\ F(d) & \xrightarrow{\nu_d} & G(d) \end{array}$$

in $\mathcal{D}$ commutes; that is, the equation $G(u) \circ \nu_c = \nu_d \circ F(u)$ holds in $\mathrm{mor}_{\mathcal{D}}(F(c), G(d))$.

Example 4.16. MacLane (in his book *Categories for the working mathematician*) gives the following pretty example. For a fixed integer $n \geq 1$ the rule which to a ring R assigns the group $\mathrm{GL}_n(R)$ can be viewed as a functor GL_n from the category of rings to the category of groups, as was shown earlier. There we allowed non-commutative rings, but here we need commutative rings, so we shall view GL_n as a functor from the category c$\mathcal{R}$ings of commutative rings to $\mathcal{G}$roups. Note that $\mathrm{GL}_1(R)$ is essentially the group of units of the ring R. The group homomorphisms

$$\det\colon \mathrm{GL}_n(R) \to \mathrm{GL}_1(R)$$

(one for every commutative ring R) make up a natural transformation from $\mathrm{GL}_n\colon \mathrm{c}\mathcal{R}\mathrm{ings} \to \mathcal{G}\mathrm{roups}$ to $\mathrm{GL}_1\colon \mathrm{c}\mathcal{R}\mathrm{ings} \to \mathcal{G}\mathrm{roups}$.
Returning to smooth manifolds and differential forms: we saw that for any fixed $k \geq 0$ the assignment $M \mapsto \Omega^k(M)$ can be viewed as a contravariant functor from $\mathcal{M}$an to $\mathcal{V}$ect. The exterior derivative maps

$$d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

(one for each object M of $\mathcal{M}$an) make up a natural transformation from the contravariant functor Ω^k to the contravariant functor Ω^{k+1}.

Notation: let F and G be functors from $\mathcal{C}$ to $\mathcal{D}$. Sometimes we describe a natural transformation ν from F to G by a strong arrow, as in $\nu\colon F \Rightarrow G$.

Remark: one reason for being a little cautious in saying *category of categories* etc. is that the functors from one big category (such as $\mathcal{T}$op for example) to another big category (such as $\mathcal{S}$ets for example) do not obviously form a set. Of course, some people would not exercise that kind of caution and would instead say that the definition of category as given in 4.10 is not bold enough. In any case, it must be permitted to say *the category of small categories.*

4.3. The category of presheaves on a space

Let $X = (X, \mathcal{O})$ be a topological space. We have seen that a presheaf $\mathcal{F}$ on X is the same thing a contravariant functor from the poset $\mathcal{O}$ (partially ordered by inclusion, and then viewed as a category) to $\mathcal{S}$ets. Therefore it is not surprising that we define a *morphism* from a presheaf $\mathcal{F}$ on X to a presheaf $\mathcal{G}$ on X to be a natural transformation between contravariant functors from $\mathcal{O}$ to $\mathcal{S}ets$. Writing this out in detail, we obtain the following definition.

Definition 4.17. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on the topological space X. A *morphism* or *map* of presheaves from $\mathcal{F}$ to $\mathcal{G}$ is a rule which for every open set U in X selects a map $\lambda_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$, subject to the following condition. Whenever U and V are open subsets of X and $U \subset V$, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\lambda_U} & \mathcal{G}(U) \\ {\scriptstyle \mathrm{res}_{V,U}} \uparrow & & \uparrow {\scriptstyle \mathrm{res}_{V,U}} \\ \mathcal{F}(V) & \xrightarrow{\lambda_V} & \mathcal{G}(V) \end{array}$$

in $\mathcal{S}ets$ commutes; that is, the maps $\mathrm{res}_{V,U} \circ \lambda_V$ and $\lambda_U \circ \mathrm{res}_{V,U}$ from $\mathcal{F}(V)$ to $\mathcal{G}(U)$ agree.

With this definition of morphism, it is clear that there is a category of presheaves on X. It is a small category.

Example 4.18. Let X be a topological space. Let $\mathcal{F}$ be the presheaf on X such that $\mathcal{F}(U)$, for open $U \subset X$, is the set of continuous maps from U to $\mathbb{R}$, and such that $\mathrm{res}_{V,U} \colon \mathcal{F}(V) \to \mathcal{F}(U)$ is given by restriction of functions. Let $\mathcal{G}$ be the presheaf on X such that $\mathcal{G}(U)$, for open $U \subset X$, is the set of all open subsets of X which are contained in U. More precisely $\mathcal{G}$ is a presheaf because in the situation $U \subset V$ we define

$$\mathrm{res}_{V,U} \colon \mathcal{G}(V) \to \mathcal{G}(U)$$

by $W \mapsto W \cap U$ for an open subset W of X contained in V. (Then $W \cap U$ is an open subset of X contained in U.) A morphism α from presheaf $\mathcal{F}$ to presheaf $\mathcal{G}$ is defined by

$$\alpha_U(g) = g^{-1}(]0, \infty[)$$

for $g \in \mathcal{F}(U)$. In a more wordy formulation: to an element g of $\mathcal{F}(U)$, alias continuous function $g \colon U \to \mathbb{R}$, the morphism $\alpha \colon \mathcal{F} \to \mathcal{G}$ assigns an element of $\mathcal{G}(U)$, alias open set of X contained in U, by taking the preimage of $]0, \infty[$ under g.

4.4. (Appendix): Abelian group vocabulary

It is customary to describe the binary operation in an abelian group by a $+$ sign, if there is no danger of confusion. Thus, if A is an abelian group and $a, b \in A$, we like to write $a + b$ instead of ab or $a \cdot b$; also $-b$ instead of b^{-1} and 0 instead of 1 for the neutral element.

The expression *abelian group* is synonymous with $\mathbb{Z}$*-module*. The name $\mathbb{Z}$*-module* is a reminder that there is some interaction between the ring $\mathbb{Z}$ and the elements of any abelian group A. This looks a lot like the multiplication of vectors by scalars in a vector space. Namely, let A be an abelian group (written with $+$ etc.), let a be an element of A and $z \in \mathbb{Z}$. Then we can define

$$z \cdot a \in A$$

as follows: if $z \geq 0$ we mean $a + a + \cdots + a$ (there are z summands in the sum); if $z \leq 0$ then we know already what $(-z) \cdot a$ means and $z \cdot a$ should be the inverse, $z \cdot a = -((-z) \cdot a)$. This "scalar multiplication" has an associativity property:

$$(wz) \cdot a = w \cdot (z \cdot a)$$

and also two distributivity properties, $(w + z) \cdot a = w \cdot a + z \cdot a$ as well as $z \cdot (a + b) = z \cdot a + z \cdot b$. Furthermore, $1 \cdot a = a$ for all $a \in A$ and $z \cdot 0 = 0$ for all $z \in \mathbb{Z}$. We might feel tempted to say that A is a vector space over the field $\mathbb{Z}$, but there is the objection that $\mathbb{Z}$ is not a field.

(Of course there is a more general concept of R*-module*, where R can be any ring. An R-module is an abelian group A with a map $R \times A \to A$ which we write in the form $(r, a) \mapsto r \cdot a$. That map is subject to many conditions, such as $(rs) \cdot a = r \cdot (s \cdot a)$ and $r \cdot (a + b) = r \cdot a + r \cdot b$, for all $r \in R$ and $a, b \in A$, and a few more. Look it up in any algebra book.)

Definition 4.19. Let S be a set. The *free abelian group* generated by S is the set A_S of all functions $f \colon S \to \mathbb{Z}$ such that $\{s \in S \mid f(s) \neq 0\}$ is a *finite* subset of S. It is an abelian group by pointwise addition; that is, for $f, g \in A_S$ we define $f + g \in A_S$ by $(f + g)(s) = f(s) + g(s) \in \mathbb{Z}$.

Notation. Elements of the free abelian group A_S generated by S can also be thought of as *formal linear combinations*, with integer coefficients, of elements of S. In other words, we may write

$$\sum_{s \in S} a_s \cdot s$$

where $a_s \in \mathbb{Z}$ for all $s \in \mathbb{Z}$, and we mean the function $f \in A_S$ such that $f(s) = a_s$ for all $s \in S$. Now it is important to insist that the sum have only finitely many (nonzero) summands, $a_s \neq 0$ for only finitely many $s \in S$.
My notation A_S for the free abelian group generated by S is meant to be temporary. I can't think of any convincing standard notation for it.

An important property of the free abelian group generated by S. The group A_S comes with a distinguished map $u \colon S \to A_S$ so that $u(s)$ is the function from S to $\mathbb{Z}$ taking s to 1 and all other elements of S to 0. Together, the abelian group A_S and the map (of sets) $u\colon S \to A_S$ have the following property. *Given any abelian group* B *and map* $v\colon S \to B$*, there exists a unique homomorphism of abelian groups* $q_v\colon A_S \to B$ *such that* $q_v \circ u = v$*.* Diagrammatic statement:

$$\begin{array}{ccc} S & \xrightarrow{u} & A_S \\ & \searrow^{v} & \vdots\, q_v \\ & & B \end{array}$$

The proof is easy. Every element a of A_S can be written uniquely in the form

$$\sum_{s \in S} a_s \cdot u(s)$$

with $a_s \in \mathbb{Z}$, with only finitely many nonzero a_s. Therefore

$$q_v(a) = q_v\Big(\sum_{s\in S} a_s \cdot u(s)\Big) = \sum_{s\in S} q_v\big(a_s \cdot u(s)\big) = \sum_{s\in S} a_s \cdot q_v(u(s)) = \sum_{s\in S} a_s \cdot v(s)\,.$$

(The following complaint can be made: *Just a minute ago you said that we can write elements* a *of* A_S *in the form* $\sum_{s\in S} a_s \cdot s$ *, but now it is* $\sum_{s\in S} a_s \cdot u(s)$*, or what?* The complaint is justified: $\sum_{s\in S} a_s \cdot s$ is a short and imprecise form of $\sum_{s\in S} a_s \cdot u(s)$.)

4.5. (Appendix): Preview

If our main interest is in understanding notions like homotopy and classifying topological spaces up to homotopy equivalence, why should we learn something about presheaves and sheaves? In this appendix I try to give an answer, very much from the point of view of category theory.

Summarizing the experience of the first few weeks in category language, we might agree on the following. In the category $\mathcal{T}\mathrm{op}$ of topological spaces (and continuous maps), we introduced the homotopy relation $\simeq$ on morphisms. This led to a new category $\mathcal{H}\mathrm{o}\mathcal{T}\mathrm{op}$ with the same objects as $\mathcal{T}\mathrm{op}$, where a morphism from X to Y is a homotopy class of continuous maps from X to Y. We made some attempts to understand sets of homotopy classes $[X, Y] = \mathrm{mor}_{\mathcal{H}\mathrm{o}\mathcal{T}\mathrm{op}}(X, Y)$ in some cases; for example we understood $[S^1, S^1]$ and we showed that $[S^3, S^2]$ has more than one element. A vague impression of computability may have taken hold, but nothing very systematic emerged.

Here is a very simple-minded attempt to make things easier by introducing some algebra into topology. We can make a new category $\mathbb{Z}\mathcal{T}\mathrm{op}$ where the objects are still the topological spaces and where the set of morphisms from X to Y is the *free abelian group* generated by the set of continuous maps from X to Y. In other words, a morphism from X to Y in $\mathbb{Z}\mathcal{T}\mathrm{op}$ is a formal linear combination (with integer coefficents) of continuous maps from X to Y, such as $4f-3g+7u+1v$, where $f, g, u, v\colon X \to Y$ are continuous maps. Note that *formal* is *formal*; we make no attempt to simplify such expressions, except by allowing $4f - 3g + 7u + 1v = 4f + 4u + 1v$ if we happen to know that $g = u$, and the like. How do we compose morphisms in $\mathbb{Z}\mathcal{T}\mathrm{op}$? We use the composition of morphisms in $\mathcal{T}\mathrm{op}$ and enforce a distributive law, so we say for example that the composition of the morphism $4f - 3g + 7u$ from X to Y with the morphism $-2p + 5q$ from Y to Z is

$$-8(p \circ f) + 6(p \circ g) - 14(p \circ u) + 20(q \circ f) - 15(q \circ g) + 35(q \circ u),$$

a morphism from X to Z. In many ways $\mathbb{Z}\mathcal{T}\mathrm{op}$ is a fine category, and perhaps better than $\mathcal{T}\mathrm{op}$; the morphism sets are abelian groups and composition of morphisms

$$\mathrm{mor}_{\mathbb{Z}\mathcal{T}\mathrm{op}}(Y, Z) \times \mathrm{mor}_{\mathbb{Z}\mathcal{T}\mathrm{op}}(X, Y) \longrightarrow \mathrm{mor}_{\mathbb{Z}\mathcal{T}\mathrm{op}}(X, Z)$$

is bilinear. That is, post-composition with a fixed element of $\mathrm{mor}_{\mathbb{Z}\mathcal{T}\mathrm{op}}(Y, Z)$ gives a homomorphism of abelian groups $\mathrm{mor}_{\mathbb{Z}\mathcal{T}\mathrm{op}}(X, Y) \to \mathrm{mor}_{\mathbb{Z}\mathcal{T}\mathrm{op}}(X, Z)$ and pre-composition with a fixed element of $\mathrm{mor}_{\mathbb{Z}\mathcal{T}\mathrm{op}}(X, Y)$ gives a homomorphism of abelian groups $\mathrm{mor}_{\mathbb{Z}\mathcal{T}\mathrm{op}}(Y, Z) \to \mathrm{mor}_{\mathbb{Z}\mathcal{T}\mathrm{op}}(X, Z)$. We can relate $\mathcal{T}\mathrm{op}$ to $\mathbb{Z}\mathcal{T}\mathrm{op}$ by a functor

$$\mathcal{T}\mathrm{op} \to \mathbb{Z}\mathcal{T}\mathrm{op}$$

which takes any object to the same object, and a continuous map $f\colon X \to Y$ to the formal linear combination $1f$. And yet, it is hard to believe that any of this will give us new insights into anything.

But let us try to make a well-formulated objection. We have lost something in replacing $\mathcal{T}\mathrm{op}$ by $\mathbb{Z}\mathcal{T}\mathrm{op}$: the sheaf property. More precisely, we know that we can construct a continuous map $f\colon X \to Y$ by specifying an open cover $(U_i)_{i\in\Lambda}$ of X, and for each i a continuous map $f_i\colon U_1 \to Y$, in such a way that

$$f_{i|U_i\cap U_j} = f_{j|U_i\cap U_j}$$

for all $i, j \in \Lambda$. (Then there is a unique continuous map $f\colon X \to Y$ such that $f_{|U_i} = f_i$ for all $i \in \Lambda$.) We could take the view that this is a property of $\mathcal{T}\mathrm{op}$ which is important to us, one that we don't want to sacrifice when we experiment with modifications of $\mathcal{T}\mathrm{op}$. But as we have seen, the sheaf property fails in so many ways in $\mathbb{Z}\mathcal{T}\mathrm{op}$; see example 4.7 and the elaborate discussion of that example. I propose that we regard that as the one great weakness of $\mathbb{Z}\mathcal{T}\mathrm{op}$.

Fortunately, in sheaf theory there is a fundamental construction called *sheafification* by which the sheaf property is enforced. In the following chapters we will apply that construction to $\mathbb{Z}\mathcal{T}\mathrm{op}$ to restore the sheaf property. When that is done, we can once again speak of homotopies and homotopy classes, and it will turn out that we have a very manageable situation.