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Lecture Notes, week 2
Topology WS 2013/14 (Weiss)

2.1. Fiber bundles

Definition 2.1. Let $p\colon E \to B$ be a continuous map between topological spaces an $x \in B$. The subspace $p^{-1}(\{x\})$ is sometimes called the *fiber* of p over x.

Definition 2.2. Let $p\colon E \to B$ be a continuous map between topological spaces. We say that p is a *fiber bundle* if for every $x \in B$ there exist an open neighborhood U of x in B, a topological space F and a homeomorphism $h\colon p^{-1}(U) \to U \times F$ such that h followed by projection to U equals p.

Note that h restricts to a homeomorphism from the fiber of f over x to $\{x\} \times F$. Therefore F must be homeomorphic to the fiber of p over x.

Terminology. Often E is called the *total space* of the fiber bundle and B is called the *base space*. A homeomorphism $h\colon p^{-1}(U) \to U \times F$ as in the definition is called a *bundle chart*.

A fiber bundle $p\colon E \to B$ whose fibers are discrete spaces (intuitively, just sets) is also called a *covering space*. (A *discrete space* is a topological space $(X, \mathcal{O})$ in which $\mathcal{O}$ is the entire power set of X.)

Here is an easy way to make a fiber bundle with base space B. Choose a topological space F, set $E = B \times F$ and let $p\colon E \to B$ be the projection to the first factor. Such a fiber bundle is considered unexciting and is therefore called *trivial*. Slightly more generally, a fiber bundle $p\colon E \to B$ is *trivial* if there exist a topological space F and a homeomorphism $h\colon E \to B \times F$ such that h followed by the projection $B \times F \to B$ agrees with p. Equivalently, the bundle is trivial if it admits a bundle chart $h\colon p^{-1}(U) \to U \times F$ where U is all of B.

Two fiber bundles $p_0\colon E_0 \to B$ and $p_1\colon E_1 \to B$ with the same base space B are considered *isomorphic* if there exists a homeomorphism $g\colon E_0 \to E_1$ such that $p_1 \circ g = p_0$. In that case g is an *isomorphism* of fiber bundles.

According to the definition above a fiber bundle is a *map*, but the expression is often used informally for a space rather than a map (the total space of the fiber bundle).

Proposition 2.3. *Let $p\colon E \to B$ be a fiber bundle where B is a connected space. Let $x_0, y_0 \in B$. Then the fibers of p over x_0 and y_0, respectively, are homeomorphic.*

Proof. For every $x \in B$ choose an open neighborhood U_x of $x \in B$, a space F_x and a bundle chart $h_x \colon p^{-1}(U_x) \to U_x \times F_x$. The open sets U_x for all $x \in B$ form an open cover of B. We make an equivalence relation R on the set B in the following manner: xRy means that there exist elements

$$x_0, x_1, \dots, x_k \in B$$

such that $x_0 = x$, $x_k = y$ and $U_{x_{j-1}} \cap U_{x_j} \neq \emptyset$ for $j = 1, \dots, k$. Clearly xRy implies that F_x is homeomorphic to F_y. Therefore it suffices to show that R has only one equivalence class. Each equivalence class is open, for if $x \in B$ belongs to such an equivalence class, then U_x is contained in the equivalence class. Each equivalence class is closed, since its complement is open, being the union of the other equivalence classes. Since B is connected, this means that there can only be one equivalence class. □

Example 2.4. One example of a fiber bundle is $p \colon \mathbb{R} \to S^1$, where $p(t) = \exp(2\pi i t)$. We saw this in section 1. To show that it is a fiber bundle, select some $z \in S^1$ and some $t \in \mathbb{R}$ such that $p(t) = z$. Let $V =]t - \delta, t + \delta[$ where δ is a positive real number, not greater than $1/2$. Then p restricts to a homeomorphism from $V \subset \mathbb{R}$ to an open neighborhood $U = p(V)$ of z in S^1; let $q \colon U \to V$ be the inverse homeomorphism. Now $p^{-1}(U)$ is the disjoint union of the translates $\ell + V$, where $\ell \in \mathbb{Z}$. This amounts to saying that

$$g \colon U \times \mathbb{Z} \to p^{-1}(U)$$

given by $(y, m) \mapsto m + q(y)$ is a homeomorphism. The inverse h of g is then a bundle chart. Moreover $\mathbb{Z}$ plays the role of a discrete space. Therefore this fiber bundle is a covering space. It is not a trivial fiber bundle because the total space, $\mathbb{R}$, is not homeomorphic to $S^1 \times \mathbb{Z}$.

Example 2.5. The Möbius strip leads to another example of a fiber bundle. Let $E \subset S^1 \times \mathbb{C}$ consist of all pairs (z, w) where $w^2 = c^2 z$ for some $c \in \mathbb{R}$. This is an implementation of the Möbius strip. There is a projection

$$q \colon E \to S^1$$

given by $q(z, w) = z$. Let us look at the fibers of q. For fixed $z \in S^1$, the fiber of q over z is identified with the space of all $w \in \mathbb{C}$ such that $w^2 = c^2 z$ for some real c. This is equivalent to $w = c\sqrt{z}$ where $\sqrt{z}$ is one of the two roots of z in $\mathbb{C}$. In other words, w belongs to the one-dimensional linear *real* subspace of $\mathbb{C}$ spanned by the two square roots of z. In particular, each fiber of q is homeomorphic to $\mathbb{R}$. The fact that all fibers are homeomorphic to each other should be taken as an indication (though not a proof) that q is a fiber bundle. The full proof is left as an exercise, along with another exercise which is slightly harder: show that this fiber bundle is not trivial.

In preparation for the next example I would like to recall the concept of *one-point compactification*. Let $X = (X, \mathcal{O})$ be a locally compact topological space. (That is to say, X is a Hausdorff space in which every element $x \in X$ has a compact neighborhood.) Let $X^c = (X^c, \mathcal{U})$ be the topological space defined as follows. As a set, X^c is the disjoint union of X and a singleton (set with one element, which in this case we call ∞). The topology $\mathcal{U}$ on X^c is defined as follows. A subset V of X^c belongs to $\mathcal{U}$ if and only if

- either $\infty \notin V$ and $V \in \mathcal{O}$;
- or $\infty \in V$ and $X^c \smallsetminus V$ is a *compact* subset of X.

Then X^c is compact Hausdorff and the inclusion $u \colon X \to X^c$ determines a homeomorphism of X with $u(X) = X^c \smallsetminus \{\infty\}$. The space X^c is called the *one-point compactification* of X. The notation X^c is not standard; instead people often write $X \cup \infty$ and the like. The one-point compactification can be characterized by various good properties; see books on point set topology. For use later on let's note the following, which is clear from the definition of the topology on X^c. Let $Y = (Y, \mathcal{W})$ be any topological space. A map $g \colon Y \to X^c$ is continuous if and only if the following hold:

- $g^{-1}(X)$ is open in Y
- the map from $g^{-1}(X)$ to X obtained by restricting g is continuous
- for every compact subset K of X, the preimage $g^{-1}(K)$ is a closed subset of Y (that is, its complement is an element of $\mathcal{W}$).

Example 2.6. A famous example of a fiber bundle which is also a crucial example in homotopy theory is the Hopf map from S^3 to S^2, so named after its inventor Heinz Hopf. (Date of invention: around 1930.) Let's begin with the observation that S^2 is homeomorphic to the one-point compactification $\mathbb{C} \cup \infty$ of $\mathbb{C}$. (The standard homeomorphism from S^2 to $\mathbb{C} \cup \infty$ is called *stereographic projection*.) We use this and therefore describe the Hopf map as a map

$$p \colon S^3 \to \mathbb{C} \cup \infty.$$

Also we like to think of S^3 as the unit sphere in $\mathbb{C}^2$. So elements of S^3 are pairs (z, w) where $z, w \in \mathbb{C}$ and $|z|^2 + |w|^2 = 1$. To such a pair we associate

$$p(z, w) = z/w$$

using complex division. This is the Hopf map. Note that in cases where $w = 0$, we must have $z \neq 0$ as $|z|^2 = |z|^2 + |w|^2 = 1$; therefore z/w can be understood and must be understood as $\infty \in \mathbb{C} \cup \infty$ in such cases. In the remaining cases, $z/w \in \mathbb{C}$.

Again, let us look at the fibers of p before we try anything more ambitious. Let $s \in \mathbb{C} \cup \infty$. If $s = \infty$, the preimage of $\{s\}$ under p consists of all $(z, w) \in S^3$ where $w = 0$. This is a circle. If $s \notin \{0, \infty\}$, the preimage of $\{s\}$

under p consists of all $(z, w) \in S^3$ where $w \neq 0$ and $z/w = s$. So this is the intersection of $S^3 \subset \mathbb{C}^2$ with the one-dimensional complex linear subspace $\{(z, w) \mid z = sw\} \subset \mathbb{C}^2$. It is also a circle! Therefore all the fibers of p are homeomorphic to the same thing, S^1. We take this as an indication (though not a proof) that p is a fiber bundle.
Now we *show* that p is a fiber bundle. First let $U = \mathbb{C}$, which we view as an open subset of $\mathbb{C} \cup \infty$. Then

$$p^{-1}(U) = \{(z, w) \in S^3 \subset \mathbb{C}^2 \mid w \neq 0\} .$$

A homeomorphism h from there to $U \times S^1 = \mathbb{C} \times S^1$ is given by

$$(z, w) \mapsto (z/w, w/|w|).$$

This has the properties that we require from a bundle chart: the first coordinate of $h(z, w)$ is $z/w = p(z, w)$. (The formula $g(y, z) = (yz, z)/\|(yz, z)\|$ defines a homeomorphism g inverse to h.) Next we try $V = (\mathbb{C} \cup \infty) \smallsetminus \{0\}$, again an open subset of $\mathbb{C} \cup \infty$. We have the following commutative diagram

$$\begin{array}{ccc} S^3 & \xrightarrow{\alpha} & S^3 \\ \downarrow p & & \downarrow p \\ \mathbb{C} \cup \infty & \xrightarrow{\zeta} & \mathbb{C} \cup \infty \end{array}$$

where $\alpha(z, w) = (w, z)$ and $\zeta(s) = s^{-1}$. (This amounts to saying that $p \circ \alpha = \zeta \circ p$.) Therefore the composition

$$p^{-1}(V) \xrightarrow{\alpha} p^{-1}(U) \xrightarrow{h} U \times S^1 \xrightarrow{(s,w) \mapsto (s^{-1}, w)} V \times S^1$$

has the properties required of a bundle chart. Since $U \cup V$ is all of $\mathbb{C} \cup \infty$, we have produced enough charts to know that p is a fiber bundle. □

2.2. Restricting fiber bundles

Let $p \colon E \to B$ be a fiber bundle. Let A be a subset of B. Put $E_{|A} = p^{-1}(A)$. This is a subset of E. We want to regard A as a subspace of B (with the subspace topology) and $E_{|A}$ as a subspace of E.

Proposition 2.7. *The map* $p_A \colon E_{|A} \to A$ *obtained by restricting* p *is also a fiber bundle.*

Proof. Let $x \in A$. Choose a bundle chart $h \colon p^{-1}(U) \to U \times F$ for p such that $x \in U$. Let $V = U \cap A$, an open neighborhood of x in A. By restricting h we obtain a bundle chart $h_A \colon p^{-1}(V) \to V \times F$ for p_A. □

Remark. In this proof it is important to remember that a bundle chart as above is not just *any* homeomorphism $h \colon p^{-1}(U) \to U \times F$. There is a condition: for every $y \in p^{-1}(U)$ the U-coordinate of $h(y) \in U \times F$ must be equal to $p(y)$. The following informal point of view is recommended: A bundle chart $h\colon p^{-1}(U) \to U \times F$ for p is just a way to specify, simultaneously and continuously, homeomorphisms h_x from the fibers of p over elements $x \in U$ to F. Explicitly, h determines the h_x and the h_x determine h by means of the equation

$$h(y) = (x, h_x(y)) \in U \times F$$

when $y \in p^{-1}(x)$, that is, $x = p(y)$.

Let $p\colon E \to B$ be any fiber bundle. Then B can be covered by open subsets U_i such that $E_{|U_i}$ is a trivial fiber bundle. This is true by definition: choose the U_i together with bundle charts $h_i\colon p^{-1}(U_i) \to U_i \times F_i$. Rename $p^{-1}(U_i) = E_{|U_i}$ if you must. Then each h_i is a bundle isomorphism of $p_{|U_i}\colon E_{|U_i} \to U_i$ with a trivial fiber bundle $U_i \times F_i \to U_i$.
There are cases where we can say more. One such case merits a detailed discussion because it takes us back to the concept of homotopy.

Lemma 2.8. *Let* B *be any space and let* $q\colon E \to B \times [0,1]$ *be a fiber bundle. Then* B *admits a covering by open subsets* U_i *such that*

$$q_{|U_i \times [0,1]}\colon E_{|U_i \times [0,1]} \longrightarrow U_i \times [0,1]$$

is a trivial fiber bundle.

Proof. We fix $x_0 \in B$ for this proof. We try to construct an open neighborhood U of $\{x_0\}$ in B such that $q_{|U \times [0,1]}\colon E_{|U \times [0,1]} \longrightarrow U \times [0,1]$ is a trivial fiber bundle. This is enough.
To minimize bureaucracy let us set it up as a proof by *analytic induction.* So let J be the set of all $t \in [0,1]$ for which there exist an open $U' \subset B$ and an open subset U'' of $[0,1]$ which is also an interval containing 0, such that $x_0 \in U'$ and $t \in U''$ and such that $q_{|U' \times U''}$ is a trivial fiber bundle. The following should be clear.

- J is an open subset of $[0,1]$.
- J is nonempty since $0 \in J$.
- If $t \in J$ then $[0,t] \subset J$; hence J is an interval.

If $1 \in J$, then we are happy. So we assume $1 \notin J$ for a contradiction. Then $J = [0, \sigma[$ for some σ where $0 < \sigma \leq 1$. Since q is a fiber bundle, the point (x_0, σ) admits an open neighborhood V in $B \times [0,1]$ with a bundle chart $g\colon q^{-1}(V) \to V \times F_V$. Without loss of generality V has the form $V' \times V''$ where $V' \subset B$ is an open neighborhood of x_0 in B and V'' is an interval which is also an open neighborhood of σ in $[0,1]$. There exists $r < \sigma$ such that $V'' \supset [r, \sigma]$. Then $r \in J$ and so there exists $W = W' \times W''$ open in $B \times [0,1]$ with a bundle chart $h\colon q^{-1}(W) \to U \times F_W$ such that $x_0 \in W'$ and $W'' = [0, \tau[$ where $\tau > r$. Without loss of generality, $W' = V'$. Now

$W'' \cup V''$ is an open subset of $[0,1]$ which is an interval (since $r \in W'' \cap V''$). It contains both 0 and σ. Now let $U' = V'$ and $U'' = W'' \cup V''$. If we can show that $q_{|U' \times U''}$ is a trivial fiber bundle, then the proof is complete because $U' \times U''$ contains $\{x_0\} \times [0, \sigma]$, which implies that $\sigma \in J$, which is the contradiction that we need. Indeed we can make a bundle chart

$$k\colon q^{-1}(U' \times U'') \to (U' \times U'') \times F_W$$

as follows. For $(x,t) \in U' \times U''$ with $t \le r$ we take $k_{(x,t)} = h_{(x,t)}$. For $(x,t) \in U' \times U''$ with $t \ge r$ we take $k_{(x,t)} = h_{(x,r)} \circ g^{-1}_{(x,r)} \circ g_{(x,t)}$.
Decoding: remember that $h_{(x,t)}$ is a homeomorphism from the fiber of q over $(x,t) \in W \subset B \times [0,1]$ to F_W. Similarly $g_{(x,t)}$ is a homeomorphism from the fiber of q over $(x,t) \in V \subset B \times [0,1]$ to F_V. Also note that $h_{(x,r)} \circ g^{-1}_{(x,r)}$ is a homeomorphism from F_V to F_W, depending on $x \in V_1 = W_1 \subset B$. $\square$