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Lecture Notes, week 12
Topology WS 2013/14 (Weiss)

9.3. Technical remarks concerning the geometric realization

Let Y be a semi-simplicial set. We reformulate the definition of the geometric realization $|Y|$ once again.

From the semi-simplicial set Y, we make a category $\mathcal{C}_Y$ as follows. An object is a pair (n, z) where n is a nonnegative integer and $z \in Y_n$. A morphism from (m, y) to (n, z) is, by definition, an order-preserving injective map $g\colon \{0, 1, 2, \dots, m\}$ to $\{0, 1, 2, \dots, n\}$ which has the property $g^*(z) = y$ (where $g^*\colon Y_n \to Y_m$ is the face operator determined by g).
We define a covariant functor F_Y from $\mathcal{C}_Y$ to the category of topological spaces as follows. The definition of F_Y on objects is simply

$$F_Y(n, z) = \Delta^n$$

where Δ^n is the standard n-simplex. (Recall that this is the space of all functions u from $\{0, 1, \dots, n\}$ to $[0, 1]$ which satisfy $\sum_j u(j) = 1$, viewed as a subspace of the real vector space of all functions from $\{), 1, \dots, n\}$ to $\mathbb{R}$.)
If we have a morphism from (m, y) to (n, z) given by an an order-preserving injective map $g\colon \{0, 1, 2, \dots, m\}$ to $\{0, 1, 2, \dots, n\}$, then we define

$$F_Y(f) = g_*\colon \Delta^m \to \Delta^n,$$

that is to say, $F_Y(f)(u_1, \dots, u_m) = (v_1, \dots, v_n)$ where $v_i = u_j$ if $i = g(j)$ and $v_i = 0$ if i is not of the form $g(j)$. Note that I have written u_i instead of $u(i)$ etc. ; strictly speaking $u(i)$ is correct because we said that u is a function from $\{0, 1, \dots, m\}$ to $[0, 1]$.

Now the definition of $|Y|$ can be recast as follows:

$$|Y| = \left(\coprod_{(n,z)} F_Y(n, z) \right) \Big/ \sim$$

where $\sim$ is the equivalence relation generated by

$$F_Y(m, y) \ \ni \ (u_1, \dots, u_m) \sim F_Y(g)(u_1, \dots, u_m) \in F_Y(n, z)$$

whenever g is a morphism from (m, y) to (n, z); in other words g is an order-preserving injective map from $\{0, 1, 2, \dots, m\}$ to $\{0, 1, 2, \dots, n\}$ which has $g^*(z) = y$. It may look as if the formula defines $|Y|$ only as a set, but we want to view it as a formula defining a topology on $|Y|$ as well, namely, the *quotient topology*. Therefore, a subset of $|Y|$ is considered to be *open* (definition) if and only if its preimage in $\coprod_{(n,z)} F_Y(n, z)$ is open.

Warning: do not read these $2\frac{1}{2}$ lines unless you are somewhat familiar with category theory. You will notice that $|Y|$ has been defined to be the direct limit (also called colimit) of the functor F_Y.

Example 9.1. Let $(V, \mathcal{S})$ be a vertex scheme, choose a total ordering on V, and let Y be the associated semi-simplicial set, as in lecture notes week 11. We are going to show that the geometric realization $|Y|$ is homeomorphic to the simplicial complex $|V|_{\mathcal{S}}$.
An element of Y_n is an order-preserving injective map from $\{0, 1, \dots, n\}$ to V. This is determined by its image T, a distinguished subset of V (where *distinguished* means that $T \in \mathcal{S}$). So we can pretend that Y_n is simply the set of all distinguished subsets of V that have exactly $n+1$ elements. Furthermore, if $T' \in Y_m$ and $T \in Y_n$, then there exists at most one morphism from T' to T in the category $\mathcal{C}_Y$. It exists if and only if $T' \subset T$. Therefore we may say that $\mathcal{C}_Y$ is the category whose objects are the distinguished subsets $T, T', \dots$ of V, with exactly one morphism from T' to T if $T' \subset T$, and no morphism from T' to T otherwise. In this description, the functor F_Y is given on objects by

$$F_Y(T) = \Delta(T)$$

where $\Delta(T)$ replaces Δ^n (assuming that T has exactly $n+1$ elements) and means: the space of functions u from T to $[0, 1]$ that satisfy $\sum_{j \in T} u(j) = 1$. For $T' \subset T$ we have exactly one morphism from T' to T, and the induced map $F_Y(T') = \Delta(T') \to \Delta(T) = F_Y(T)$ is given by $u \mapsto v$ where $v(t) = u(t)$ if $t \in S'$ and $v(t) = 0$ if $t \in S \smallsetminus S'$. Therefore

$$|Y| = \Big(\coprod_{T \in \mathcal{S}} \Delta(T) \Big) \Big/ \sim$$

where the equivalence relation is generated by $u \in \Delta(T') \sim v \in \Delta(T)$ if $T' \subset T$ and $v(t) = u(t)$ for $t \in T'$, $v(t) = 0$ for $t \in T \smallsetminus T'$.
There is a map of sets

$$\coprod_{T \in \mathcal{S}} \Delta(T) \longrightarrow |V|_{\mathcal{S}}$$

which is equal to the inclusion $\Delta(T) \to |V|_{\mathcal{S}}$ on each $\Delta(T)$. That map clearly determines a *bijective* map

$$|Y| = \Big(\coprod_{T \in \mathcal{S}} \Delta(T) \Big) \Big/ \sim \qquad \longrightarrow \qquad |V|_{\mathcal{S}} \,.$$

By our definition of the topology on $|V|_{\mathcal{S}}$, a subset of $|V|_{\mathcal{S}}$ is open if and only if its preimage in $\coprod_{T \in \mathcal{S}} \Delta(S)$ is open; and by our definition of the topology in $|Y|$, that happens if and only if its preimage in $|Y|$ is open. So that bijective map from $|Y|$ to $|V|_{\mathcal{S}}$ is a homeomorphism.

Lemma 9.2. *Let* Y *be any semi-simplicial set. For every element* a *of* $|Y|$ *there exist unique* $m \geq 0$ *and* $(z, w) \in Y_m \times \Delta^m$ *such that* $a = c_z(w)$ *and* w *is in the "interior" of* Δ^m*, that is, the coordinates* $w_0, w_1, \ldots, w_m$ *are all strictly positive.*
Furthermore, if $a = c_x(u)$ *for some* $(x, u) \in Y_k \times \Delta^k$*, then there is a unique order-preserving injective* $f\colon \{0, 1, \ldots, m\} \to \{0, 1, 2, \ldots, k\}$ *such that* $f^*(x) = z$ *and* $f_*(w) = u$*, for the above-mentioned* $(z, w) \in Y_m \times \Delta^m$ *with* $w_0, w_1, \ldots, w_m > 0$*.*

Proof. Let us call such a pair (z, w) with $a = c_z(w)$ a *reduced presentation* of a; the condition is that all coordinates of w must be positive. More generally we say that (x, u) is a *presentation* of a if $(x, u) \in Y_k \times \Delta^k$ for some $k \geq 0$ and $a = c_x(u)$. First we show that a admits a reduced presentation and then we show uniqueness.
We know that $a = c_x(u)$ for some $(x, u) \in Y_k \times \Delta^k$. Some of the coordinates $u_0, \ldots, u_k$ can be zero (not all, since their sum is 1). Suppose that $m+1$ of them are nonzero. Let $f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, k\}$ be the unique order-preserving map such that $u_{f(j)} \neq 0$ for $j = 0, 1, 2, \ldots, m$. Then $a = c_z(w)$ where $z = f^*(x)$ and $w \in \Delta^m$ with coordinates $w_j = u_{f(j)}$. (Note that $f_*(w) = u$.) So (z, w) is a reduced presentation of a.
We have also shown that any presentation (x, u) of a (whether reduced or not) determines a reduced presentation. Namely, there exist unique m, f and $w \in \Delta^m$ such that $v = f_*(w)$ for some $w \in \Delta^m$ with all $w_i > 0$; then $(f^*(x), w)$ is a reduced presentation of a.
It remains to show that if a has two presentations, say $(x, u) \in Y_k \times \Delta^k$ and $(y, v) \in Y_\ell \times \Delta^\ell$, then they determine the *same* reduced representation of a. If indeed $a = c_x(u) = c_y(v)$ then $\bar{c}_x(u)$ and $\bar{c}_y(v)$ are equivalent, and so (recalling how that equivalence relation was defined) we find that there is no loss of generality in assuming that $x = g^*(y)$ and $v = g_*(u)$ for some order-preserving injective $g\colon \{0, 1, \ldots, k\} \to \{0, 1, \ldots, \ell\}$. Now determine the unique m and order-preserving injective $f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, k\}$ such that $u = f_*(w)$ where $w \in \Delta^m$ and all $w_i > 0$. Then we also have $v = g_*(u) = g_*(f_*(w)) = (g \circ f)_*(w)$ and it follows that we get the same reduced presentation, $(f^*(x), w) = ((g \circ f)^*(y), w)$, in both cases. □

Corollary 9.3. *The space* $|Y|$ *is a Hausdorff space.*

Proof. For $a \in Y$ with reduced presentation $(z, w) \in Y_m \times \Delta^m$ and $\varepsilon > 0$, define $N(a, \varepsilon) \subset |Y|$ as follows. It consists of all $b \in |Y|$ with reduced presentation $(x, u) \in Y_k \times \Delta^k$ such that there exists an order-preserving injective $f\colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, k\}$ for which $f^*(x) = z$ and $f_*(w)$ is ε-close to u, that is, the maximum of the numbers $|w_{f(j)} - u_j|$ is $< \varepsilon$. From the definitions, $N(a, \varepsilon)$ is open in $|Y|$; so it is a neighborhood of a.

Let $a' \in |Y|$ be another element, with reduced presentation $(y, v) \in Y_n \times \Delta^n$. We assume $a \neq a'$ and proceed to show that $N(a', \varepsilon) \cap N(a, \varepsilon) = \emptyset$ if ε is small enough. More precisely, we take ε to be less than half the minimum of the coordinates of v and w; and if it should happen that $m = n$ and $y = z$, then we know $v, w \in \Delta^m$ but $v \neq w$, and we take ε to be less than half the maximum of the $|v_j - w_j|$ as well. Now suppose for a contradiction that $b \in N(a, \varepsilon) \cap N(a', \varepsilon)$ and that b has reduced presentation $(x, u) \in Y_k \times \Delta^k$. Then there exist order-preserving injective $f \colon \{0, 1, \ldots, m\} \to \{0, 1, \ldots, k\}$ and $g \colon \{0, 1, \ldots, n\} \to \{0, 1, \ldots, k\}$ such that $f^*(x) = z$, $g^*(x) = y$ and $f_*(w), g_*(v)$ are both ε-close to u in Δ^k. Then $f_*(w)$ is 2ε-close to $g_*(v)$ in Δ^k, and now we can deduce that $m = n$ and $f = g$. (Otherwise there is some $j \in \{0, 1, \ldots, k\}$ which is in the image of g but not in the image of f, or vice versa, and then the j-th coordinate of $g_*(w)$ differs by more than 2ε from the j-th coordinate of $f_*(v)$.) Therefore $z = f^*(x) = g^*(x) = y$ and so a has reduced presentation (z, w) while a' has reduced presentation (z, v), with $v, w \in \Delta^m$ and the same $z \in Y_m$. It follows that v and w are already 2ε-close in Δ^m. This contradicts our choice of ε. □

Remark 9.4. In the proof above, and in a similar proof in the previous section, arguments involving distances make an appearance, suggesting that we have a metrizable situation. To explain what is going on let me return to the situation of a vertex scheme $(V, \mathcal{S})$ with simplicial complex $|V|_{\mathcal{S}}$, which is easier to understand. A metric on the set $|V|_{\mathcal{S}}$ can be introduced for example by $d(f, g) = (\sum_v (f(v) - g(v))^2)^{1/2}$ or $d(f, g) = \sum_v |f(v) - g(v)|$. Here we insist/remember that elements of $|V|_{\mathcal{S}}$ are functions $f, g, \ldots : V \to [0, 1]$ subject to some conditions. The sums in the formulas for $d(f, g)$ are finite, even though V might not be a finite set. It is not hard to show that the two formulas for $d(f, g)$, although different as metrics, determine the same topology. However the topology on $|V|_{\mathcal{S}}$ that we have previously decreed (let me call it the *weak* topology) is not in all cases the same as that metric topology. Every subset of $|V|_{\mathcal{S}}$ which is open in the metric topology is also open in the weak topology. But the weak topology can have more open sets. (We reasoned that the weak topology is Hausdorff because it has all the open sets that the metric topology has, and perhaps a few more, and the metric topology is certainly Hausdorff.) In the case where V is finite, weak topology and metric topology on $|V|_{\mathcal{S}}$ coincide. (Exercise.)

9.4. A more economical definition of *semi-simplicial set*

Every injective order-preserving map from $[k] = \{0, 1, \ldots, k\}$ to $[\ell] = \{0, 1, \ldots, \ell\}$ is a composition of $\ell - k$ injective order preserving maps

$$[m-1] \longrightarrow [m]$$

where $k < m \leq \ell$. It is easy to list the injective order-preserving maps from $[m-1]$ to $[m]$; there is one such map f_i for every $i \in [m]$, characterized by the property that the image of f_i is

$$[m] \smallsetminus \{i\} \, .$$

(This f_i really depends on two parameters, m and i. Perhaps we ought to write $f_{m,i}$, but it is often practical to suppress the m subscript.) We have the important relations

$$f_i f_j = f_j f_{i-1} \quad \text{if } j < i \tag{9.5}$$

(You are allowed to read this from left to right or from right to left! It is therefore a formal consequence that $f_i f_j = f_{j+1} f_i$ when $j \geq i$.) These *generators and relations* suffice to describe the category $\mathcal{C}$ (lecture notes week 11) whose objects are the sets $[k] = \{0, 1, \ldots, k\}$ for $k \geq 0$ and whose morphisms are the order-preserving injective maps between those sets. In other words, the structure of $\mathcal{C}$ as a category is pinned down if we say that it has objects $[k]$ for $k \geq 0$ and that, for every $k > 0$ and $i \in \{0, 1, \ldots, k\}$, there are certain morphisms $f_i \colon [k-1] \to [k]$ which, under composition when it is applicable, satisfy the relations (9.5). Prove it!
Consequently a semi-simplicial set Y, which is a contravariant functor from $\mathcal{C}$ to spaces, can also be described as a sequence of sets $Y_0, Y_1, Y_2, \ldots$ and maps

$$d_i \colon Y_k \to Y_{k-1}$$

which are subject to the relations

$$d_j d_i = d_{i-1} d_j \quad \text{if } j < i \tag{9.6}$$

Here $d_i \colon Y_k \to Y_{k-1}$ denotes the map induced by $f_i \colon [k-1] \to [k]$, whenever $0 \leq i \leq k$. Because of contravariance, we have had to reverse the order of composition in translating relations (9.5) to obtain relations (9.6).

10.1. Chain complexes and their homology groups

Definition 10.7. A *chain complex* C consists a sequence of abelian groups $C_0, C_1, C_2, \ldots$ and homomorphisms $\partial_n \colon C_n \to C_{n-1}$ for $n > 0$ such that

$$\partial_n \circ \partial_{n-1} = 0$$

for all $n > 1$. We say that C_n is the n-th chain group of the chain complex C. The homomorphisms ∂_n are called *differentials* or (sometimes) *boundary operators.*

Think of a chain complex as a diagram

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} C_3 \xleftarrow{\partial_4} C_4 \xleftarrow{\partial_5} \cdots$$

where the composition of any two consecutive arrows is zero. It is very common to drop the subscript n in ∂_n. So a more standard picture of a chain complex looks like

$$C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} C_3 \xleftarrow{\partial} C_4 \xleftarrow{\partial} \cdots,$$

and we just write $\partial\partial = 0$ instead of writing $\partial_n \circ \partial_{n-1} = 0$ for all $n > 1$.

Unsurprisingly, chain complexes are the objects of a category. The morphisms in that category are called *chain maps*. A chain map from a chain complex C to a chain complex D is a sequence of homomorphisms

$$f_n \colon C_n \to D_n$$

(for $n \geq 0$) making the diagram

$$\begin{array}{ccccccccccc} C_0 & \xleftarrow{\partial} & C_1 & \xleftarrow{\partial} & C_2 & \xleftarrow{\partial} & C_3 & \xleftarrow{\partial} & C_4 & \xleftarrow{\partial} & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \\ D_0 & \xleftarrow{\partial} & D_1 & \xleftarrow{\partial} & D_2 & \xleftarrow{\partial} & D_3 & \xleftarrow{\partial} & D_4 & \xleftarrow{\partial} & \cdots \end{array} \tag{10.8}$$

commutative; in other words $\partial \circ f_n = f_{n-1} \circ \partial$ for all $n > 0$. The preferred shorthand notation for such a morphism is $f \colon C \to D$.

Remark 10.9. Some would say that what has been defined above is a chain complex *graded over the non-negative integers*. There are also chain complexes graded over the integers, which look like

$$\cdots \xleftarrow{\partial_{-2}} C_{-2} \xleftarrow{\partial_{-1}} C_{-1} \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} C_3 \xleftarrow{\partial_4} C_4 \xleftarrow{\partial_5} \cdots$$

If a chain complex C comes along being graded over the non-negative integers, then it is often a good idea to view it as a chain complex graded over the integers by setting $C_n = 0$ for $n < 0$.

Definition 10.10. For $n \geq 0$, the n*-th homology group* $H_n(C)$ of a chain complex C is the (group-theoretic) quotient

$$H_n(C) = \frac{\ker[\partial_n \colon C_n \to C_{n-1}]}{\mathrm{im}[\partial_{n+1} \colon C_{n+1} \to C_n]}.$$

Note that the kernel of ∂_n and the image of ∂_{n+1} are both subgroups of C_n, and the kernel of ∂_n contains the image of ∂_{n+1}.

When $n = 0$, we need to make sense of $\ker[\partial_0 \colon C_0 \to C_{-1}]$. In agreement with the convention that chain complexes graded over the non-negative integers can be viewed as chain complexes graded over the integers, we take the view that this is all of C_0 and so

$$H_0(C) = \frac{C_0}{\mathrm{im}[\partial_1 \colon C_1 \to C_0]}$$

Remark 10.11. For fixed n the rule $C \mapsto H_n(C)$ is a (covariant) functor from the category of chain complexes to the category of abelian groups. Let $f\colon C \to D$ be a chain map, consisting of homomorphisms $f_n\colon C_n \to D_n$ for $n \geq 0$. Then f_n takes $\ker[\partial_n\colon C_n \to C_{n-1}]$ to $\ker[\partial_n\colon D_n \to D_{n-1}]$, and takes $\mathrm{im}[\partial_{n+1}\colon C_{n+1} \to C_n]$ to $\mathrm{im}[\partial_{n+1}\colon C_{n+1} \to C_n]$, and so determines a homomorphism

$$H_n(C) \to H_n(D) .$$

10.2. The combinatorial chain complex of a semi-simplicial set

A semi-simplicial set Y determines a chain complex $C(Y)$, the *combinatorial chain complex of* Y, in the following way.
The chain group $C(Y)_n$ is defined to be the direct sum of copies of $\mathbb{Z}$, one copy for each $z \in Y_n$. We can write

$$C(Y)_n = \bigoplus_{z \in Y_n} \mathbb{Z} .$$

(It is also customary to say that $C(Y)_n$ is the *free abelian group generated by the set* Y_n.) If we agree to denote the element "1" in the summand corresponding to $z \in Y_n$ by $\langle z \rangle$, then we can describe elements of $C(Y)_n$ as linear combinations

$$\sum_{z \in Y_n} a_z \cdot \langle z \rangle$$

where the coefficients a_z are integers (and the sum is understood to be finite, that is, $a_z \neq 0$ for only finitely many $z \in Y_n$.) The differential or boundary operator

$$\partial_n\colon C(Y)_n \longrightarrow C(Y)_{n-1}$$

is defined by

$$\langle z \rangle \quad \mapsto \quad \sum_{j=0}^{n} (-1)^j \langle d_j z \rangle \in C(Y)_{n-1}$$

where $d_j\colon Y_n \to Y_{n-1}$ is the face operator discussed previously, corresponding to the unique monotone injective map from $\{0, 1, \dots, n-1\}$ to $\{0, 1, \dots, n\}$ which has image $\{0, 1, \dots, n\} \smallsetminus \{j\}$.

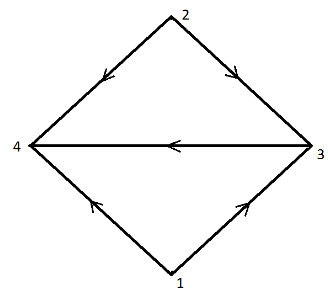
Now we need to show that $\partial_{n-1}\partial_n = 0$ for all $n > 1$. This is a straightforward calculation based on the relations (9.6).

$$\partial_{n-1}(\partial_n(\langle z \rangle)) = \cdots = \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} \langle d_j d_i z \rangle$$

The double sum can be split into two parts, one part comprising the summands $(-1)^{k+\ell} \langle d_k d_\ell z \rangle$ where $k < \ell$ and the other comprising the summands $(-1)^{k+\ell} \langle d_k d_\ell \rangle$ where $k \geq \ell$. Each summand $(-1)^{i+j} \langle d_j d_i z \rangle$ in the first part

part cancels exactly one in the other part, $(-1)^{j+i-1}\langle d_{i-1}d_j z\rangle$, where we are using (9.6).

Example 10.12. The projective plane $\mathbb{R}P^2$ can be described as $|Y|$ for a semi-simplicial set Y. To construct this we start with a simplicial complex or vertex scheme $(V, \mathcal{S})$ describing the upper hemisphere of S^2. Picture:



Therefore $V = \{1, 2, 3, 4\}$ and

$$\mathcal{S} = \Big\{ \{1,3,4\},\{2,3,4\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1\},\{2\},\{3\},\{4\} \Big\}\,.$$

The vertex set V is already ordered and we can immediately pass to a semi-simplicial set X where

$$\begin{aligned} X_2 &= \{(134),(234)\} \\ X_1 &= \{(13),(14),(23),(24),(34)\} \\ X_0 &= \{(1),(2),(3),(4)\}. \end{aligned}$$

(To clarify the improvised notation: X_2 has two elements, X_1 has five elements and X_0 has four elements.) For $n > 2$ we set $X_n = \emptyset$. The operators d_i are defined by omitting the digit in position i (but we label the positions with integers from 0 upwards), so that for example

$$d_0(134) = (34),\ d_1(134) = (14),\ d_2(134) = (13),\ d_0(13) = (3).$$

By a certain proposition we have $|X| = |V|_{\mathcal{S}}$, which we think of as the upper hemisphere of S^2, but now we want to identify opposite points on the boundary (=equator). In the semi-simplicial set code, this means that we enforce

$$(14) \sim (23), \qquad (13) \sim (24), \qquad (1) \sim (2), \qquad (3) \sim (4).$$

(NB: it seems to me that I had to think fairly hard to get the numbering of vertices right, so that by making these identifications we do in fact identify opposite points on the equator when we pass to geometric realizations.) In this way we get a new semi-simplicial set Y where

$$\begin{aligned} Y_2 &= \{(134),(234)\} \\ Y_1 &= \{(13)=(24),(14)=(23),(34)\} \\ Y_0 &= \{(1)=(2),(3)=(4)\}. \end{aligned}$$

(To clarify the very improvised notation: Y_2 has two elements, Y_1 has three elements and Y_0 has two elements.) For $n > 2$ we still have $Y_n \geq 0$. The operators d_i are defined by omitting the digit in position i (but we label the positions with integers from 0 upwards). Now we are ready to follow instructions above to make the chain complex $C(Y)$:

$$\mathbb{Z}^2 \leftarrow \mathbb{Z}^3 \leftarrow \mathbb{Z}^2 \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

The boundary operators can be described as matrices with integer entries: a 2×3 matrix for ∂_1 and a 3×2 matrix for ∂_2. (For the matrix descriptions we need and we have *ordered* bases: so the columns for example in the 2×3 matrix are labeled with the three elements of Y_1, in the order in which they are listed above.) The differential ∂_1 is given by

$$(13) \mapsto (3) - (1), \quad (14) \mapsto (4) - (1) = (3) - (1), \quad (34) \mapsto (4) - (3) = 0$$

which in matrix form is

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

The differential ∂_2 is given by

$$(134) \mapsto (34) - (14) + (13), \qquad (234) \mapsto (34) - (24) + (23) = (34) - (13) + (14)$$

which in matrix form is

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Note that the product of the two matrices (in the correct order) *is* zero, confirming that $\partial_1 \partial_2 = 0$, as it should be. It is easy to see using the matrix description that the image of ∂_1 consists of all elements in $\mathbb{Z}^2$ which have coordinate sum equal to 0, and it follows immediately that

$$H_0(C(Y)) = \operatorname{coker}[\partial_1] \cong \mathbb{Z}\ .$$

Determining $H_1(C(Y))$ is not straightforward! I note that the kernel of ∂_1 consists of all elements of $\mathbb{Z}^3$ which are perpendicular to the row vector $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, and so $\ker[\partial_1]$ has a complement in $\mathbb{Z}^3$, the subgroup A of $\mathbb{Z}^3$ spanned by the element

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$H_1(C(Y) = \frac{\ker[\partial_1]}{\operatorname{im}[\partial_2]} \cong \frac{\mathbb{Z}^3/A}{\operatorname{im}[p \circ \partial_2]}$$

where $p \colon \mathbb{Z}^3 \to \mathbb{Z}^3/A$ is the projection. We can identify $\mathbb{Z}^3/A$ with $\mathbb{Z}^2$ in the obvious manner. Then we can describe $p \circ \partial_2$ as a homomorphism from $\mathbb{Z}^2$ to $\mathbb{Z}^2$ by a square matrix

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

obtained by deleting the top row in the matrix description of ∂_2 . This 2×2 matrix has determinant -2 and so the cokernel of the homomorphism $\mathbb{Z}^2 \to \mathbb{Z}^2$ which it describes has order $|-2| = 2$. It must be cyclic of order 2. Similar reasoning shows that ∂_2 is injective. Indeed $p \circ \partial_2$ is injective, because you can use Kramer's rule and the nonzero determinant to recover elements in the source from their values in the target. Therefore

$$H_1(C(Y)) \cong \mathbb{Z}/2, \qquad H_2(C(Y)) = 0$$

and clearly $H_n(C(Y)) = 0$ for all $n > 2$ as well.
Let's not fail to observe that the groups $H_n(C(Y))$ coincide with the groups $H_n(\mathbb{R}P^2) = H_n(|Y|)$, for every $n \geq 0$. This is not an accident, as we shall see in the next section.