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Lecture Notes, week 10
Topology WS 2013/14 (Weiss)

8.1. The homotopy decomposition theorem: reductions

Here we reduce the proof of the homotopy decomposition theorem to the following lemmas.

Lemma 8.1. *Let Z be a paracompact topological space, Y any topological space. Let $\beta\colon Z\times[0,1]\to Y$ be a mapping cycle. Write $\iota_0,\iota_1\colon Z\to Z\times[0,1]$ for the maps given by $\iota_0(z)=(z,0)$ and $\iota_1(z)=(z,1)$. If there exists a decomposition*

$$\beta\circ\iota_0=\beta_0^V+\beta_0^W$$

where β_0^V and β_0^W are mapping cycles from Z to V and W, respectively, then there exists a decomposition $\beta\circ\iota_1=\beta_1^V+\beta_1^W$.

Lemma 8.2. *In the situation of lemma 8.1, every element of Z has an open neighborhood U such that the restriction $\beta_{U\times[0,1]}$ of β to $U\times[0,1]$ admits a decomposition*

$$\beta_{U\times[0,1]}=\beta^V_{U\times[0,1]}+\beta^W_{U\times[0,1]}$$

where $\beta^V_{U\times[0,1]}$ and $\beta^W_{U\times[0,1]}$ are mapping cycles from $U\times[0,1]$ to V and W, respectively.

Showing that lemma 8.2 implies lemma 8.1. In the situation of lemma 8.1, choose an open cover $(U_k)_{k\in\Lambda}$ such that the restriction $\beta_{[k]}$ of β to $U_k\times[0,1]$ admits a decomposition

$$\beta_{[k]}=\beta^V_{[k]}+\beta^W_{[k]}\ .$$

Such an open cover exists by lemma 8.2. Since Z is paracompact, there is no loss of generality in assuming that the open cover is locally finite. Moreover, there exists a partition of unity $(\varphi_k)_{k\in\Lambda}$ subordinate to the cover $(U_k)_{k\in\Lambda}$. Choose a total ordering of Λ. If Λ is finite, we can proceed as follows. We may assume that Λ is $\{1,2,3,\dots,m\}$ for some m, with the standard ordering. For $k\in\{0,1,\dots,m\}$ let

$$f_k\colon Z\to Z\times[0,1]$$

be the function $z\mapsto(z,\sum_{\ell=1}^k\varphi_\ell)$. Then $f_0=\iota_0$ and $f_m=\iota_1$ in the notation of lemma 8.1. By induction on k we define a decomposition

$$\beta\circ f_k=(\beta\circ f_k)^V+(\beta\circ f_k)^W\ .$$

For $k = 0$ this decomposition (of $\beta \circ f_0 = \beta \circ \iota_0$) is already given to us. If we have constructed the decomposition for $\beta \circ f_{k-1}$, where $0 < k \leq m$, we define it for $\beta \circ f_k$ in such a way that

$$(\beta \circ f_k)^V = (\beta \circ f_{k-1})^V + \beta^V_{[k]} \circ f_k - \beta^V_{[k]} \circ f_{k-1}$$

on $U_k \subset Z$ and $(\beta \circ f_k)^V = (\beta \circ f_{k-1})^V$ outside the support of φ_k. Similarly, define

$$(\beta \circ f_k)^W = (\beta \circ f_{k-1})^W + \beta^W_{[k]} \circ f_k - \beta^W_{[k]} \circ f_{k-1}$$

on U_k and $(\beta \circ f_k)^W = (\beta \circ f_{k-1})^W$ outside the support of φ_k. Then on U_k we have

$$(\beta \circ f_k)^V + (\beta \circ f_k)^W = \beta \circ f_{k-1} + \beta \circ f_k - \beta \circ f_{k-1} = \beta \circ f_k$$

and outside the support of φ_k we have

$$(\beta \circ f_k)^V + (\beta \circ f_k)^W = (\beta \circ f_{k-1})^V + (\beta \circ f_{k-1})^W = \beta \circ f_{k-1} = \beta \circ f_k \ .$$

Therefore $(\beta \circ f_k)^V + (\beta \circ f_k)^W = \beta \circ f_k$ as required. The case $k = m$ is the decomposition of $\beta \circ \iota_1 = \beta \circ f_m$ that we are after.
If Λ is not finite, we can proceed as follows. Choose $z \in Z$ and an open neighborhood Q of z in Z such that the set

$$J = \{ k \in \Lambda \mid Q \cap U_k \neq \emptyset \}$$

is finite. Now J is a finite set with a total ordering, and the φ_j where $j \in J$ constitute a partition of unity for Q, subordinate to the open cover $(U_k \cap Q)_{k \in J}$ of Q. Use this as above to find a decomposition of $\beta \circ \iota_1$, restricted to Q, into summands which are mapping cycles from Q to V and W, respectively. Do this for every z and open neighborhood Q. The decompositions obtained match on overlaps, and so define a decomposition of $\beta \circ \iota_1$ of the required sort. □

Showing that lemma 8.1 implies the homotopy decomposition theorem. Given X, Y and a mapping cycle $\gamma \colon X \times [0,1] \to Y$, we look for a decomposition $\gamma = \gamma^V + \gamma^W$ where γ^V and γ^W are mapping cycles from $X \times [0,1]$ to V and W, respectively. There is an additional condition to be satisfied. Namely, γ is zero on an open neighborhood U of $(X \times \{0\}) \cup (C \times [0,1])$ in $X \times [0,1]$, and we want γ^V, γ^W to be zero on some (perhaps smaller) open neighborhood U' of $(X \times \{0\}) \cup (C \times [0,1])$ in $X \times [0,1]$.
Put $Z = X \times [0,1]$. Since X was assumed to be paracompact, Z is also paracompact; it is a general topology fact that the product of a paracompact space with a compact Hausdorff space is paracompact. We have a map

$$h \colon Z \times [0,1] \to Z$$

defined by $h((x, s), t)) = (x, st)$ for $(x, t) \in X \times [0, 1] = Z$ and $t \in [0, 1]$. Now $\beta := \gamma \circ h$ is a mapping cycle from $Z \times [0, 1]$ to Y. In the notation of lemma 8.1, we have

$$\beta \circ \iota_1 = \gamma, \qquad \beta \circ \iota_0 \equiv 0 \ .$$

There exists a decomposition $\beta_0 = \beta_0^V + \beta_0^W$ because we can take $\beta_0^V \equiv 0$ and $\beta_0^W \equiv 0$. Therefore, by lemma 8.1, there exists a decomposition $\beta \circ \iota_1 = \beta_1^V + \beta_1^W$, and we can write that in the form

$$\gamma = \beta_1^V + \beta_1^W \ .$$

This is a decomposition of the kind that we are looking for. Unfortunately there is no reason to expect that β_1^V, β_1^W are zero on $(X \times \{0\}) \cup (C \times [0, 1]\,)$, or on a neighborhood of that in $X \times [0, 1]$.
But it is easy to construct a continuous map $\psi\colon X \times [0, 1] \to X \times [0, 1]$ such that $\psi(X \times [0, 1]\,)$ is contained in the open set U specified above, and such that ψ agrees with the identity on some open neighborhood U' of $(X \times \{0\}) \cup (C \times [0, 1]\,)$ in $X \times [0, 1]$. Then obviously $U' \subset U$. Now let

$$\gamma^V = \beta_1^V - (\beta_1^V \circ \psi), \qquad \gamma^W = \beta_1^W - (\beta_1^W \circ \psi).$$

Then $\gamma^V + \gamma^W = (\beta_1^V + \beta_1^W) - (\beta_1^V + \beta_1^W) \circ \psi = \gamma - \gamma \circ \psi$. Furthermore $\gamma \circ \psi$ is zero because γ is zero on U and the image of ψ is contained in U. So $\gamma^V + \gamma^W = \gamma$. Also γ^V and γ^W are zero on U' by construction, since ψ agrees with the identity on U'. □

8.2. Local homotopy decomposition

Proof of lemma 8.2. Call an open subset P of $Z \times [0, 1]$ *good* if the mapping cycle $\beta_{|P}$ from P to Y can be written as the sum of a mapping cycle from P to V and a mapping cycle from P to W. The goal is to show that every $z \in Z$ has an open neighborhood U such that $U \times [0, 1]$ is good.
The proof is based on two observations.

- Every element of $Z \times [0, 1]$ admits a good open neighborhood.
- If U is open in Z and A, B are open subsets of $[0, 1]$ which are also intervals, and if $U \times A$ and $U \times B$ are both good, then $U \times (A \cup B)$ is good.

To prove the first observation, fix $(z, t) \in Z \times [0, 1]$ and choose an open neighborhood Q of that in $Z \times [0, 1]$ such that $\beta_{|Q}$ can be written as a formal linear combination, with coefficients in $\mathbb{Z}$, of continuous maps from Q to Y. Such a Q exists by the definition of *mapping cycle*. Making Q smaller if necessary, we can arrange that each of the (finitely many) continuous maps which appear in that formal linear combination is either a map from Q to V or a map from Q to W. It follows immediately that Q is good.
In proving the second observation, we can easily reduce to a situation where

$A \cap B$ contains an element t_0, where $0 < t_0 < 1$, and $A \cup B$ is the union of $A \cap [0, t_0]$ and $B \cap [t_0, 1]$. Choose a continuous map $\psi\colon B \to B \cap A$ such that $\psi(s) = s$ for all $s \in B \cap [0, t_0]$. Since $P := U \times A$ is good by assumption, we can write

$$\beta_{|P} = \beta^{V,P} + \beta^{W,P}$$

where the summands in the right-hand side are mapping cycles from P to V and from P to W, respectively. Similarly, letting $Q := U \times B$ we can write

$$\beta_{|Q} = \beta^{V,Q} + \beta^{W,Q}\,.$$

Let $\varphi\colon Q \to P \cap Q$ be given by $\varphi(z, t) = (z, \psi(t))$. Define $\beta^{V,P\cup Q}$, a mapping cycle from $P \cup Q$ to V, as follows:

$$\beta^{V,P\cup Q} = \begin{cases} \beta^{V,P} & \text{on} \quad P \cap (U \times [0, t_0[\,) \\ \beta^{V,Q} - (\beta^{V,Q} \circ \varphi) + (\beta^{V,P} \circ \varphi) & \text{on} \quad Q\,. \end{cases}$$

This is well defined because the two formulas agree on the intersection of Q and $U \times [0, t_0[$, where φ agrees with the identity. Similarly, define $\beta^{W,P\cup Q}$, a mapping cycle from $P \cup Q$ to W, as follows:

$$\beta^{W,P\cup Q} = \begin{cases} \beta^{W,P} & \text{on} \quad P \cap (U \times [0, t_0[\,) \\ \beta^{W,Q} - (\beta^{W,Q} \circ \varphi) + (\beta^{W,P} \circ \varphi) & \text{on} \quad Q\,. \end{cases}$$

An easy calculation shows that $\beta^{V,P\cup Q} + \beta^{W,P\cup Q} = \beta_{|P\cup Q}$. Therefore $P \cup Q = U \times (A \cup B)$ is good. The second observation is established.
Now fix $z_0 \in Z$. By the first of the observations, it is possible to choose for each $t \in [0, 1]$ a good open neighborhood Q_t of (z_0, t) in $Z \times [0, 1]$. By a little exercise, there exists an open neighborhood U of z_0 in Z and a small number $\delta = 1/n$ (where n is a positive integer) such that each of the open sets

$$U \times [0, 2\delta[, \quad U \times]1\delta, 3\delta[, \quad U \times]2\delta, 4\delta[, \quad \dots,$$
$$U \times]1 - 3\delta, 1 - 1\delta[, \quad U \times]1 - 2\delta, 1]$$

in $Z \times [0, 1]$ is contained in Q_t for some $t \in [0, 1]$. Therefore these open sets $U \times [0, 2\delta[$, $U \times]1\delta, 3\delta[$ etc. are also good. By the second of the two observations, applied $(n - 2)$ times, their union, which is $U \times [0, 1]$, is also good. □

8.3. Relationship with fiber bundles

The proof of the homotopy decomposition theorem as given above has many surprising similarities with proofs in section 3 related to fiber bundles (theorem 3.4, corollaries 3.7 and 3.8., and improvements in section 3.4). I cannot resist the temptation to explain these similarities now, after the proof.

Let E and B be topological spaces and let $p\colon E \to B$ be a fiber bundle. We need to be a little more precise by requiring that $p\colon E \to B$ be a fiber bundle *with fiber* F, for a fixed topological space F. This is supposed to mean that every fiber of p is homeomorphic to F in some way. (We learned in section 2 that every fiber bundle over a path connected space is a fiber bundle with fiber F, for some F.) With this situation we can associate two presheaves $\mathcal{T}$ and $\mathcal{H}_F$ on B.

- For an open set U in B, let $\mathcal{H}_F(U)$ be the group of homeomorphisms $h\colon U \times F \to U \times F$ respecting the projection to U.
- For open U in B let $\mathcal{T}(U)$ be the set of trivializations of the fiber bundle $E_{|U} \to U$, that is, the set of all homeomorphisms $p^{-1} \to U \times F$ respecting the projections to U.
- An inclusion of open sets $U_0 \hookrightarrow U_1$ in B induces maps

$$\mathcal{H}_F(U_1) \to \mathcal{H}_F(U_0), \qquad \mathcal{T}(U_1) \to sT(U_0)$$

  by restriction of homeomorphisms.

In fact it is clear that $\mathcal{T}$ and $\mathcal{H}_F$ are sheaves. Clearly $\mathcal{H}_F$ is a sheaf of groups, that is, each set $\mathcal{H}_F(U)$ comes with a group structure and the restriction maps $\mathcal{H}_F(U_1) \to \mathcal{H}_F(U_0)$ are group homomorphisms. By contrast $\mathcal{T}$ is not a sheaf of groups in any obvious way. But there is an *action* of the group $\mathcal{H}_F(U)$ on the set $\mathcal{T}(U)$ given by

$$(h, g) \mapsto h \circ g$$

(composition of homeomorphisms, where $h \in \mathcal{H}_F(U)$ and $g \in \mathcal{T}(U)$). This is compatible with restriction maps (reader, make this precise). Moreover:

(1) for any $g \in \mathcal{T}(U)$, the map $\mathcal{H}_F(U) \to \mathcal{T}(U)$ given by $h \mapsto h \circ g$ is a bijection;
(2) every $z \in B$ has an open neighborhood U such that $\mathcal{T}(U) \neq \emptyset$.

(Of course, despite (1), it can happen that $\mathcal{T}(U)$ is empty for some open subsets U of B, for example, $U = B$.) The proof of (1) is easy and by inspection; (2) holds by the definition of *fiber bundle*. There are words and expressions to describe this situation: we can say that $\mathcal{H}_F$ is a sheaf of groups on B and $\mathcal{T}$ is an $\mathcal{H}_F$*-torsor*.

This reasoning shows that a fiber bundle on B with fiber F determines an $\mathcal{H}_F$-torsor on B. It is also true (and useful, and not very hard to prove, though it will not be explained here) that the process can be reversed: every $\mathcal{H}_F$-torsor on B determines a fiber bundle with fiber F on B. So it transpires that section 3 about fiber bundles could alternatively have been written in the language of sheaves (of sets or groups) and torsors. Note that we are often interested in questions like this one: is $\mathcal{T}(B)$ nonempty? This amounts to asking whether the fiber bundle p is a trivial fiber bundle.

Remark 8.3. For the sake of honesty it should be pointed out that $\mathcal{H}_F$ is a sheaf on *all topological spaces* simultaneously, and this would become important if we really wanted to rewrite section 3 in sheaf language. In more detail:

- We can view $\mathcal{H}_F$ as a contravariant functor from topological spaces to groups. Indeed, for a topological space X let $\mathcal{H}_F(X)$ be the group of homeomorphisms from $X \times F$ to $X \times F$ respecting the projection to X. A continuous map $X_0 \to X_1$ induces a map $\mathcal{H}_F(X_1) \to \mathcal{H}_F(X_0)$ which is a group homomorphism.
- If we evaluate this functor only on open subsets of a fixed space X, and on inclusion maps $U_0 \to U_1$ of open subsets of X, then the resulting presheaf on X is in fact a sheaf on X.

There are also words and expressions for this; to keep it short, I will just say that $\mathcal{H}_F$ is a *big sheaf*.

Now try to forget fiber bundles for a while. We return to the homotopy decomposition theorem. Assume that $Y = V \cup W$ as in the homotopy decomposition theorem. Let Z be any topological space and fix α, a mapping cycle from Z to Y. We introduce two presheaves $\mathcal{F}$ and $\mathcal{G}$ on Z.

- For an open set U in Z, let $\mathcal{G}(U)$ be the abelian group of mapping cycles from U to $V \cap W$.
- For open U in Z let $\mathcal{F}(U)$ be the set of mapping cycles β from U to V such that $\alpha_{|U} - \beta$ is a mapping cycle from U to W. To put it differently: an element β of $\mathcal{F}(U)$ is, or amounts to, a sum decomposition
$$\alpha_{|U} = \beta + (\alpha_{|U} - \beta)$$
where the two summands β and $\alpha_{|U} - \beta$ are mapping cycles from U to V and from U to W, respectively.
- An inclusion of open sets $U_0 \hookrightarrow U_1$ in Z induces maps
$$\mathcal{G}(U_1) \to \mathcal{G}(U_0), \qquad \mathcal{F}(U_1) \to sF(U_0)$$
by restriction of mapping cycles.

It is easy to see that $\mathcal{F}$ and $\mathcal{G}$ are *sheaves*, and $\mathcal{G}$ is even a sheaf of abelian groups on Z. By contrast $\mathcal{F}$ is not in an obvious way a sheaf of abelian groups. But there is an *action* of the group $\mathcal{G}(U)$ on the set $\mathcal{F}(U)$ given by
$$(\lambda, \beta) \mapsto \lambda + \beta\,.$$
(In this formula, $\lambda \in \mathcal{G}(U)$ and $\beta \in \mathcal{F}(U)$; then $\lambda + \beta$ can be viewed as a mapping cycle from U to V and it turns out to be an element of $\mathcal{F}(U)$.) Moreover:

(1) for any $\beta \in \mathcal{F}(U)$, the map $\mathcal{G}(U) \to \mathcal{F}(U)$ given by $\lambda \mapsto \lambda + \beta$ is a bijection;
(2) every $z \in Z$ has an open neighborhood U such that $\mathcal{F}(U) \neq \emptyset$.

(Of course it is quite possible, despite (1), that $\mathcal{F}(U)$ is empty for some open subsets U of Z, for example, $U = Z$.) The proof of (1) is easy and by inspection; the proof of (2) was given in a special case earlier, but it can be repeated. Choose a neighborhood U of z such that $\alpha_{|U}$ can be represented by a formal linear combination, with integer coefficients, of continuous maps from U to Y. Making U smaller if necessary, we can assume that each of the (finitely many) continuous maps which appear in that formal linear combination is either a map from U to V or a map from U to W. Then it is clear that $\alpha_{|U}$ can be written as a sum of two mapping cycles, one from U to V and the other from U to W. So $\mathcal{F}(U)$ is nonempty.

So we see that $\mathcal{G}$ is a sheaf of abelian groups on Z and $\mathcal{F}$ is a $\mathcal{G}$-torsor. Again we are interested in questions like this one: is $\mathcal{F}(Z)$ nonempty? This is equivalent to asking whether our fixed mapping cycle α from Z to Y can be written as a sum of two mapping cycles, one from Z to V and one from Z to W. And again, for the sake of honesty, it should be noted that $\mathcal{G}$ is a big sheaf of abelian groups. (If we wanted to rewrite the proof of the homotopy decomposition theorem in sheaf and torsor language, that would have to be used.)