# Topology: Functor Calculus

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CHAPTER 1

# Prerequisites and previews

## 1.1. Homotopy limits and homotopy colimits

*Related reading material*:
**A.K. Bousfield and D.M. Kan:** *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972. v+348 pp
**J. Hollender and R.M. Vogt:** *Modules of topological spaces, applications to homotopy limits and $E_\infty$ structures*, Arch.Math. 59 (1992), 115-129.
**D. Dugger:** *A primer on homotopy limits*, available on the web, under construction.

Let $\mathcal{D}$ be a small category. Let $E$ be a functor from $\mathcal{D}$ to the category of spaces. In category theory we learn that $E$ has a limit and a colimit. Namely, $\lim E$ is the equalizer of the arrows

$$\prod_d E(d) \rightrightarrows \prod_{f:c\to d} E(d)$$

One of the arrows takes $(x_d)$ to $(y_f)$ where $y_f = f_*(x_c)$ for $f:c \to d$, and the other takes $(x_d)$ to $(z_f)$ where $z_f = x_d$ for $f:c \to d$. Similarly colim $E$ is the coequalizer of

$$\coprod_{f:c\to d} E(c) \rightrightarrows \coprod_d E(d)$$

where one of the arrows takes $x$ in the summand $E(c)$ with label $f:c \to d$ to $x$ in the summand $E(c)$, and the other takes it to $f_*(x)$ in the summand $E(d)$.
We can view $E$ as an object in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$, category of functors from $\mathcal{D}$ to $\mathscr{T}\mathrm{op}$. (The morphisms in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ are the natural transformations.)

Definition 1.1.1. A morphism $E \to F$ in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ will be called a *weak equivalence* if it specializes to a weak equivalence[1] of spaces $E(c) \to F(c)$ for every object $c$ in $\mathcal{D}$.

Remark 1.1.2. We may write $E \xrightarrow{\simeq} F$ to indicate that a morphism is a weak equivalence, although this can give rise to misunderstandings.
The misunderstanding that we want to avoid most of all is as follows. We can say that objects $E$ and $F$ in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ are *naturally homotopy equivalent* if there exist morphisms $u: E \to F$ and $v: F \to E$ and natural homotopies from $uv$ to $\mathrm{id}_F$ and from $vu$ to $\mathrm{id}_E$. It is true that in such a case, $u$ and $v$ are weak equivalences. But the converse tends to fail; in other words if we just know that there is a morphism $u: E \to F$ which is a weak equivalence, then we may not be able to find a morphism $v: F \to E$ and natural homotopies from $uv$ to id and from $vu$ to id.

[1] A map of spaces $Y \to Z$ is a *weak equivalence* if the induced map $[X,Y] \to [X,Z]$ is a bijection for every CW-space $X$.

Now the following problem arises. Let $g\colon E \to F$ be a morphism in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ which is a weak equivalence. Then we get induced maps $\lim E \to \lim F$ and $\operatorname{colim} E \to \operatorname{colim} F$. Unfortunately these are not always weak equivalences of spaces. (This is not a problem related to pathological topologies. It can also happen in cases where all values of $E$ and $F$ are CW-spaces.)

EXAMPLE 1.1.3. Let $\mathcal{D}$ be the poset of proper subsets of $\{0,1\}$. An object of $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ is a diagram of spaces having the shape $X \leftarrow Y \to Z$. The commutative diagram

$$\begin{tikzcd} * \arrow[d] & S^0 \arrow[l] \arrow[r, "\text{incl.}"] \arrow[d] & D^1 \arrow[d] \\ * & S^0 \arrow[l] \arrow[r] & * \end{tikzcd}$$

describes a morphism in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ (vertical arrows). That morphism is a weak equivalence in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$. But the induced map

$$S^1 \simeq \operatorname{colim}(\text{row } 1) \longrightarrow \operatorname{colim}(\text{row } 2) \simeq *$$

is not a weak equivalence of spaces.

EXAMPLE 1.1.4. Let $\mathcal{D}$ be the poset of nonempty subsets of $\{0,1\}$. An object of $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ is a diagram of spaces having the shape $X \to Y \leftarrow Z$. The commutative diagram

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & {[0,1]} \arrow[d] & \{1\} \arrow[l] \arrow[d] \\ {[0,1]} \arrow[r] & {[0,1]} & \{1\} \arrow[l] \end{tikzcd}$$

describes a morphism in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ (vertical arrows). That morphism is a weak equivalence in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$. But the induced map

$$\varnothing = \lim(\text{row } 1) \longrightarrow \lim(\text{row } 2) \simeq *$$

is not a weak equivalence of spaces.

Now we have to invent something to fix this problem. This leads to the concepts holim and hocolim. In the case of hocolim, we proceed as follows. It turns out that for some objects in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$, so-called *cofibrant* objects, the colim has good homotopy invariance properties. For a random object $E$ of $\mathscr{T}\mathrm{op}^{\mathcal{D}}$, we choose a weak equivalence $E' \to E$ where $E'$ is cofibrant, and we set $\operatorname{hocolim} E := \operatorname{colim} E'$. (Extra efforts can be made to ensure that the construction of $E'$ from $E$ is functorial, but I am not planning to emphasize this. See remark 1.1.17 below.) The following definition is rather improvised.

DEFINITION 1.1.5. An object $C$ of $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ is *cofibrant* if it has the following property: for every weak equivalence $p\colon E \to F$ in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ and every morphism $g\colon C \to F$, there exist a morphism $\bar{g}\colon C \to E$ and a (natural) homotopy $h$ from $g$ to $p\bar{g}$.

$$\begin{tikzcd} C \arrow[r, dotted, "\bar{g}"] \arrow[d, "j_0"'] & E \arrow[dd, "p"', "\simeq"] \\ C \times [0,1] \arrow[dr, dotted, "h"] & \\ C \arrow[u, "j_1"] \arrow[r, "g"'] & F \end{tikzcd}$$

EXAMPLE 1.1.6. Suppose that $C$ in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ has the form *representable times CW-space*, in other words $E(c) = \mathrm{mor}(b,c) \times X$ where $b$ is a fixed object in $\mathcal{D}$ and $X$ is a CW-space. Then $C$ is cofibrant. Indeed, natural transformations from $C$ to $F$ correspond to maps from $X$ to $F(b)$, and natural transformations from $C$ to $E$ correspond to maps from $X$ to $E(b)$. Therefore, given a map $g: X \to F(b)$ we have to find a map $\bar{g}: X \to E(b)$ and a homotopy from $p_b\bar{g}: X \to F(b)$ to $g: X \to F(b)$. But this can be solved since $p_b: E(b) \to F(b)$ is a weak equivalence by assumption.

THEOREM 1.1.7. *For every $E$ in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ there exists a weak equivalence $E' \to E$ where $E'$ is cofibrant.*

Before we come to the proof, which is lengthy, let us make some related observations and definitions in the direction of uniqueness.

DEFINITION 1.1.8. Let $E$ and $F$ be objects of $\mathscr{T}\mathrm{op}^{\mathcal{D}}$. We can define a *mapping space* $\mathrm{map}(E,F)$ as (the geometric realization) of a simplicial set, so that a $k$-simplex is a morphism from $E \times \Delta^k$ to $F$. It is always a Kan simplicial set (exercise).

PROPOSITION 1.1.9. *If $E$ is cofibrant and $g: F_1 \to F_2$ is a weak equivalence in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$, then the induced map $g_*: \mathrm{map}(E,F_1) \to \mathrm{map}(E,F_2)$ is a homotopy equivalence (of Kan simplicial sets).*

PROOF. Given $k \geq 1$, let $P \subset \Delta[k]$ be the simplicial subset generated by $d_0\iota$ where $\iota$ is the unique nondegenerate simplex in degree $k$, and let $Q \subset \Delta^k$ be the simplicial subset generated by $d_j\iota$ where $j = 1, 2, \ldots, k$. Note that the pair $(P, P \cap Q)$ is isomorphic to $(\Delta[k-1], \partial\Delta[k-1])$. Since $\mathrm{map}(E,F_1)$ and $\mathrm{map}(E,F_2)$ are Kan simplicial sets, it suffices to show the following (for all $k \geq 1$). Given simplicial maps $u$ and $v$ as in the commutative diagram

$$\begin{tikzcd} P \cap Q \arrow[r, "u"] \arrow[d, "\text{incl.}"] & \mathrm{map}(E,F_1) \arrow[d, "g_*"'] \\ Q \arrow[r, "v"] & \mathrm{map}(E,F_2) \end{tikzcd}$$

there exist simplicial maps $\bar{u}$ and $\bar{v}$ extending $u$ and $v$, respectively, as in the commutative diagram

$$\begin{tikzcd} P \arrow[r, dotted, "\bar{u}"] \arrow[d, "\text{incl.}"] & \mathrm{map}(E,F_1) \arrow[d, "g_*"'] \\ \Delta[k] \arrow[r, dotted, "\bar{v}"] & \mathrm{map}(E,F_2) \end{tikzcd}$$

Translated by adjunction, this means (that we have to show): every commutative diagram in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ of the form

$$\begin{tikzcd} E \times |P \cap Q| \arrow[r] \arrow[d] & F_1 \arrow[d, "g"] \\ E \times |Q| \arrow[r] & F_2 \end{tikzcd}$$

admits an extension

$$\begin{tikzcd} E \times |P| \arrow[r, dotted] \arrow[d] & F_1 \arrow[d, "g"] \\ E \times \Delta^k \arrow[r, dotted] & F_2 \end{tikzcd}$$

Now let $Y$ in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ be defined in such a way that $Y(c)$ for $c$ in $\mathcal{D}$ is the space of solutions of

$$\begin{tikzcd} E(c) \times |P| \arrow[r, dotted] \arrow[d] & F_1(c) \arrow[d, "g"] \\ E(c) \times \Delta^k \arrow[r, dotted] & F_2(c) \end{tikzcd}$$

(where the horizontal arrows are *prescribed* on $E(c) \times |P \cap Q|$ and $E(c) \times |Q|$, respectively). Because $g: F_1(c) \to F_2(c)$ is a weak equivalence, $Y(c)$ is always contractible. In other words $Y$ is weakly equivalent to the terminal object of $\mathscr{T}\mathrm{op}^{\mathcal{D}}$. (The terminal object is $*_{\mathcal{D}}$, the functor on $\mathcal{D}$ which takes every $c$ in $\mathcal{D}$ to a point.) Since $E$ is cofibrant, it follows that there exists a morphism $E \to Y$ (because there certainly exists a morphism $E \to *_{\mathcal{D}}$). This solves our problem. □

Proposition 1.1.10. *If $q: E \to F$ is a weak equivalence in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ where both $E$ and $F$ are cofibrant, then $q$ admits a natural homotopy inverse $F \to E$* (as in remark 1.1.2). □

Next, suppose that $u: E' \to E$ and $v: E'' \to E$ are two morphisms in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$. Suppose that both are cofibrant replacements, i.e., both $u$ and $v$ are weak equivalences and both $E'$ and $E''$ are cofibrant. We have a map

$$\mathrm{map}(E', E'') \to \mathrm{map}(E', E)$$

induced by $v$ which is a homotopy equivalence of Kan simplicial sets. Therefore its homotopy fiber over the 0-simplex $u$ is a contractible (Kan) simplicial set. In particular we can choose a 0-simplex in it. This determines forgetfully a map $f$ from $E'$ to $E''$. In the same way, we obtain a map $g$ from $E''$ to $E'$. It is an exercise to show that these two are reciprocal homotopy inverses, i.e., $fg$ is naturally homotopic to the identity of $E''$ and $gf$ is naturally homotopic to the identity of $E'$. More can be said ....

Now we need to think about the proof of theorem 1.1.7. It is a good idea to begin with a definition.

Definition 1.1.11. An object $E$ of $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ is a free CW-object if it comes with a filtration by subobjects

$$E^{-1} \subset E^0 \subset E^1 \subset E^2 \subset E^3 \subset \dots$$

such that the following conditions are satisfied.

- $E^{-1}(c) = \varnothing$ for all $c$ in $\mathcal{D}$.
- For every $c$ in $\mathcal{D}$, the space $E(c)$ is the (monotone) union of the subspaces $E^i(c)$, with the colimit topology.
- For every $k \geq 0$ there is a family of objects

$$(x_\alpha)_{\alpha \in \Lambda_k}$$

in $\mathcal{D}$ and there is a pushout square

$$\begin{tikzcd} S^{k-1} \times \coprod_{\alpha \in \Lambda_k} \mathrm{mor}_{\mathcal{D}}(x_\alpha, -) \arrow[r] \arrow[d, "\text{incl.}"'] & E^{k-1} \arrow[d, "\text{incl.}"] \\ D^k \times \coprod_{\alpha \in \Lambda_k} \mathrm{mor}_{\mathcal{D}}(x_\alpha, -) \arrow[r] & E^k \end{tikzcd}$$

Remark 1.1.12. If $E$ is a free CW-object, then for every $c$ in $\mathcal{D}$ the filtration of $E(c)$ by subspaces $E^k(c)$ makes $E(c)$ into a CW-space. For every morphism $f: c \to d$ in $\mathcal{D}$, the induced map $E(c) \to E(d)$ is cellular.

PROPOSITION 1.1.13. *A free CW-object in* $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ *is cofibrant.*

PROOF. Exercise. Use induction over skeletons. Use example 1.1.6 for the induction step. The induction beginning is trivial. (The induction has an end which should not be neglected.) $\square$

PROOF OF THEOREM 1.1.7. Given $F$ in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$, we shall construct a weak equivalence $u: E \to F$ where $E$ is a free CW-object. This can be done (again) by induction over skeletons. In step number $k$, we assume that $E^{k-1}$ has already been constructed and that $u^{k-1}$ (restriction of $u$ to $E^{k-1}$) has already been constructed. *Important*: we also assume that, for every $c$ in $\mathcal{D}$, the specialization

$$u^{k-1}: E^{k-1}(c) \longrightarrow F(c)$$

is a $(k-1)$-connected map of spaces. Let's do step number 0 separately because it is a little special and because it is the induction beginning. In this case there is no doubt that $E^{k-1} = E^{-1}$ and $u^{-1}$ have already been constructed, since $E^{-1}(c) = \varnothing$ for all $c$. For every $c$ we choose a set $S_c$ and a map $\lambda_c: S_c \to F(c)$ which is 0-connected. We set

$$E^0 := \coprod_c S_c \times \mathrm{mor}(c, -)$$

and we define $u^0: E^0 \to F$ in such a way that $u^0((x, \mathrm{id}_c)) = \lambda(x) \in F(c)$ for $x \in S_c$. (We can because we remember the Yoneda lemma.) Now let's look at step $k$, where $k > 0$. So we may assume that $u^{k-1}: E^{k-1} \longrightarrow F$ has already been constructed, and if we specialize to any $c$ in $\mathcal{D}$, we have a $(k-1)$-connected map. If this is also $k$-connected (for every $c$), then we don't have to do anything in step number $k$, and we just set $E^k = E^{k-1}$, $u^k = u^{k-1}$. But more likely we are not so lucky. Then for some $c$ in $\mathcal{D}$, at least one of the following takes place:

(i) there is a map $f: S^{k-1} \to E^{k-1}(c)$ which does not extend to a map from $D^k$ to $E^{k-1}(c)$, although its composition with $u^{k-1}$ extends to a map $g$ from $D^k$ to $F(c)$);

(ii) there is a map $g: S^k \to F(c)$ which is not homotopic to a map of the form

$$S^k \to E^{k-1}(c) \xrightarrow{u^{k-1}} F(c).$$

In case (i) we can fix the problem by attaching a free $k$-cell to $E^{k-1}$. In other words we form the pushout of

$$E^{k-1} \longleftarrow S^{k-1} \times \mathrm{mor}(c, -) \longrightarrow D^k \times \mathrm{mor}(c, -)$$

(where we use $f$ to define the left-hand arrow). Then we use $g$ to make a morphism (in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$) from this pushout to $F$. This repairs the problem (for this $f$). In case (ii) we may assume without loss of generality that we have a base point $*$ in $S^k$ and that $g(*) = u^{k-1}(z)$ for some $z \in E^{k-1}(c)$ (because we completed step 0 successfully, long ago). Then we can again fix the problem by attaching a free $k$-cell to $E^{k-1}$. Namely, we form the pushout of

$$E^{k-1} \longleftarrow S^{k-1} \times \mathrm{mor}(c, -) \longrightarrow D^k \times \mathrm{mor}(c, -)$$

where we use the map from $S^{k-1}$ to $E^{k-1}(c)$ with constant value $z$ to define the left-hand arrow. Then we use $g$ to make a morphism (in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$) from this pushout to $F$. In this way, we can fix all problems of type (i) or (ii) by attaching free $k$-cells to $E^{k-1}$. When we have finished, we have constructed $E^k$ and $u^k$, and (little exercise) by construction, the specialization $u^k: E^k(c) \longrightarrow F(c)$ is a $k$-connected map of spaces, for every object $c$ in $\mathcal{D}$. $\square$

DEFINITION 1.1.14. For an object $E$ in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ we can now define $\mathrm{hocolim}\, E$ as follows. We choose a weak equivalence $u: E' \to E$ where $E'$ is cofibrant. We put

$$\mathrm{hocolim}\, E := \mathrm{colim}\, E'.$$

This is admittedly a little ambiguous. But suppose that we have another weak equivalence

$$v\colon E'' \to E$$

where $E''$ is also cofibrant. Then the induced map $v_*\colon \mathrm{map}(E', E'') \to \mathrm{map}(E', E)$ is a homotopy equivalence (of Kan simplicial sets), by proposition 1.1.9. Therefore we can choose a 0-simplex $f$ in $\mathrm{map}(E', E'')$ and a 1-simplex $h$ in $\mathrm{map}(E', E)$ such that $d_0 h = vf$ and $d_1 h = u$. (Briefly, $h$ is a homotopy from $vf$ to $u$.) In particular we have $f\colon E' \to E''$ which induces a map $\operatorname{colim} E' \to \operatorname{colim} E''$. In the same way we obtain $g\colon E'' \to E''$ which gives a map $\operatorname{colim} E'' \to \operatorname{colim} E'$. It is an exercise to show that this gives reciprocal homotopy equivalences from $\operatorname{colim} E'$ to $\operatorname{colim} E''$ and vice versa. (Use proposition 1.1.10.)

Definition 1.1.15. We come to a definition of $\operatorname{holim} F$ for an object $F$ in $\mathscr{T}\mathrm{op}^{D}$. Let's not try to proceed by analogy with definition 1.1.14. Instead, we turn to $*_{\mathcal{D}}$, the constant functor from $\mathcal{D}$ to spaces which takes every object of $\mathcal{D}$ to a point $*$. This is the terminal object of $\mathscr{T}\mathrm{op}^{\mathcal{D}}$. Note that $\lim F$ (the ordinary limit) is the *space of natural transformations* from $*_{\mathcal{D}}$ to $F$. Therefore we choose a weak equivalence $E \to *_{\mathcal{D}}$ where $E$ is cofibrant, and we put

$$\operatorname{holim} F := \mathrm{map}(E, F).$$

(There is a canonical choice for $E$, and we will come to that in a moment.)

Remark 1.1.16. If $F_1 \to F_2$ is a weak equivalence in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$, then the induced map

$$\operatorname{holim} F_1 \to \operatorname{holim} F_2$$

is a homotopy equivalence (of Kan simplicial sets). To have such an induced map, we should stick to the same choice of $E$, cofibrant replacement for $*_{\mathcal{D}}$, in the definition of the holim. Then the claim follows from proposition 1.1.9.

Remark 1.1.17. Something should be said and can be said about canonical choices of cofibrant replacements (i.e., choice of a weak equivalence $E' \to E$ where $E$ in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ is arbitrary but $E'$ is cofibrant). The following is very sketchy and probably hard to decode.
There is a forgetful functor $U\colon \mathscr{T}\mathrm{op}^{\mathcal{D}} \to \mathscr{T}\mathrm{op}^{\mathrm{Ob}(\mathcal{D})}$. This has a left adjoint $L$. The left adjoint takes a family of spaces $(X_c)_{c \in \mathrm{Ob}(\mathcal{D})}$, indexed by the objects of $\mathcal{D}$, to the functor

$$\coprod_c X_c \times \mathrm{mor}_{\mathcal{D}}(c, -).$$

Then the functor $LU\colon \mathscr{T}\mathrm{op}^{\mathcal{D}} \to \mathscr{T}\mathrm{op}^{\mathcal{D}}$ comes with the additional structure of a comonad. Given $E$ in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ and $c$ in $\mathcal{D}$, we can use the comonad structure to make a simplicial object in $\mathscr{T}\mathrm{op}^{\mathcal{D}}$ which in degree $k$ has $(LU)^{k+1}(E)$. It also has an *augmentation* $(LU)^1(E) \to E$. Let $E^\rho$ be the functor sending $c$ in $\mathcal{D}$ to the geometric realization of the simplicial space which in degree $k$ has

$$\big((LU)^{k+1}(E)\big)(c).$$

Then there is a map $E^\rho \to E$ induced by the augmentation. Under mild conditions on $E$ this is a weak equivalence and $E^\rho$ is cofibrant. (Why the letter $\rho$ ... it is supposed to remind you of the word *resolution*.) For example, if $E(c)$ is homotopy equivalent to a CW-space for every object $c$ in $\mathcal{D}$, then $E^\rho$ is cofibrant. For more general $E$, one can first replace $E$ by $E_1$ where $E_1(c)$ is the geometric realization of the singular simplicial set of $E(c)$. Then there is a canonical weak equivalence $E_1 \to E$. The composition $(E_1)^\rho \to E_1 \to E$ is a cofibrant replacement.

## 1.2. First examples and first concepts

*Related reading material*:
**T.G. Goodwillie:** *Calculus I. The first derivative of pseudoisotopy theory*, K-Theory 4 (1990), 1–27.
**T.G. Goodwillie:** *Calculus II. Analytic functors*, K-Theory 5 (1991/92), 295–332.
**M. Weiss: Embeddings from the point of view of immersion theory. I**, Geom.Topol. 3 (1999), 67–101.
**P. Boavida de Brito and M. Weiss**: *Manifold calculus and homotopy sheaves*, Homology Homotopy Appl. 15 (2013), 361–383.

In functor calculus we typically work with functors $F$ from a category $\mathcal{C}$ to the category of spaces. The category $\mathcal{C}$ should satisfy some conditions or be equipped with additional structure so that it makes sense to say that either $F$ is continuous or that it is homotopy invariant. In some of the important cases it makes (some) sense to speak of morphisms in $\mathcal{C}$ which represent a small change. There are two slightly different viewpoints which can lead to calculus-like constructions.

(1) The Newton-Leibniz viewpoint: we ask how $F(X)$ changes under a small change in $X$.
(2) The Taylor viewpoint: we try to define the concept of $n$-polynomial functor (from $\mathcal{C}$ to spaces) and, given $F$ as before, search for a "universal" (i.e. optimized) natural transformation from $F$ to an $n$-polynomial functor. This would be called the $n$-th Taylor approximation of $F$. We try this for $n = 0, 1, 2, 3, ...$

This works surprisingly well (for some choices of $\mathcal{C}$) and functor calculus has many analogies with the ordinary calculus of (differentiable) functions. But there are also some interesting deviations. In ordinary calculus, for example when we study a smooth function $f\colon \mathbb{R} \to \mathbb{R}$, we can produce Taylor expansions at any point $a \in \mathbb{R}$. In some types of functor calculus, we have that kind of freedom, too, but in others we do not (as far as I know). It depends on the choice of $\mathcal{C}$.
On the next few pages we will make a start on (2), but there is a plan to come back to (1).

* * *

*Choice number one*: $\mathcal{C}$ can be the category of spaces. In the category of spaces we have a concept of *weak homotopy equivalence* (abbreviated to *weak equivalence*). A functor $F$ from $\mathcal{C}$ to spaces (i.e. from $\mathcal{C}$ to $\mathcal{C}$) is considered *homotopy invariant* if it takes weak equivalences to weak equivalences. *Vague definition:* a "small change" in $\mathcal{C}$ is a morphism $X \to Y$ which is highly connected. But more specifically we like to make small changes of type $X \hookrightarrow S^m \vee_z X$ (inclusion of a space $X$ into a wedge $S^m \vee X$, where we use $z \in X$ as the base point). Here $m$ should be *large* because this morphism is $(m-1)$-connected.
Question: which functors $F$ from $\mathcal{C}$ to spaces should be called $n$*-polynomial* ? Here we are obviously talking about a definition. Goodwillie's answer is a little surprising, already in the case $n = 1$. For example the identity functor is *not* considered to be 1-polynomial.

Definition 1.2.1. A commutative square of spaces

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is *homotopy cocartesian* (synonymously, a *weak homotopy pushout square*) if the composition

$$\mathrm{hocolim}\,(C \leftarrow A \rightarrow B) \xrightarrow{\text{standard comparison map}} \mathrm{colim}\,(C \leftarrow A \rightarrow B) \xrightarrow{\text{determ. by square}} D$$

is a weak equivalence. The square is *homotopy cartesian* (synonymously, a *weak homotopy pullback square*) if the composition

$$A \xrightarrow{\text{determ. by square}} \lim(B \to D \leftarrow C) \xrightarrow{\text{standard comparison map}} \operatorname{holim}(B \to D \leftarrow C)$$

is a weak equivalence.

A functor $\mathcal{C}$ from spaces to spaces is 1*-polynomial* if it is homotopy invariant and takes homotopy cocartesian squares to homotopy cartesian squares. In other words, for every homotopy cocartesian square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in $\mathcal{C}$, the resulting square

$$\begin{array}{ccc} F(A) & \longrightarrow & F(B) \\ \downarrow & & \downarrow \\ F(C) & \longrightarrow & F(D) \end{array}$$

is a homotopy cartesian square.

This has the consequence that examples of 1-polynomial functors are not easy to make. Here is one. Dold and Thom introduced a functor $SP$ (infinite symmetric product) from based spaces to based spaces and showed, roughly, that for a based *path connected* space $Y$ the homotopy groups of $SP(Y)$ are (naturally isomorphic to) the reduced homology groups of $Y$. If $Y$ is a based CW-space, the definition of $SP(Y)$ can be given as

$$\operatorname*{colim}_{k\to\infty} \underbrace{Y\times Y\times\cdots\times Y}_{k}\big/\Sigma_k$$

where we use the base point of $Y$ to make "inclusion" maps

$$\underbrace{Y\times Y\times\cdots\times Y}_{k}\big/\Sigma_k \quad\longrightarrow\quad \underbrace{Y\times Y\times\cdots\times Y}_{k+1}\big/\Sigma_{k+1}\,.$$

For general $Y$, it might be wiser to apply this formula to a functorial CW-replacement for $Y$ (like the geometric realization of the singular simplicial set of $Y$). On that understanding, the functor $Y \mapsto SP(Y)$ (from based path connected spaces to spaces) can be called 1-polynomial. Here a condition $Y$ *is based and path connected* crept in, and if we object to that we can still make examples like $\Omega^m\circ(SP)\circ(\Sigma_u)^n$ where $\Sigma_u$ is the unreduced suspension ($m\ge 0$, $n\ge 2$). Note that $\Sigma_u$ preserves homotopy cocartesian squares, while $\Omega$ takes homotopy cartesian squares (of based spaces) to homotopy cartesian squares.

An example in a similar spirit is the functor $X\mapsto \Omega^\infty\Sigma^\infty(X_+)$. If $Y$ is a based CW-space, we can define $\Omega^\infty\Sigma^\infty(Y)$ as

$$\operatorname*{colim}_{k\to\infty}\ \Omega^k\Sigma^k(Y) = \operatorname*{colim}_{k\to\infty}\ \operatorname{map}_*(\underbrace{S^1\wedge S^1\wedge\cdots\wedge S^1}_{k}, \underbrace{S^1\wedge S^1\wedge\cdots\wedge S^1}_{k}\wedge Y).$$

For general $Y$, it may be wiser (again) to apply this formula to a functorial CW-replacement for $Y$. Showing that $X\mapsto\Omega^\infty\Sigma^\infty(X_+)$ is 1-polynomial reduces to a theorem about the so-called *stable* homotopy groups of $X$ (the homotopy groups of $\Omega^\infty\Sigma^\infty(X_+)$) which is a consequence of the Blakers-Massey theorem (a weak form of excision property for ordinary homotopy groups).

For the definition of $n$-polynomial functor we need commutative diagrams in the shape of an $(n+1)$-cube. More precisely, let $\mathcal{P}[n]$ be the poset of all subsets of $[n]=\{0,1,\dots,n\}$, ordered by inclusion. An $(n+1)$-cube of spaces is a functor from $\mathcal{P}[n]$ to spaces.

Definition 1.2.2. A functor $Y$ from $\mathcal{P}[n]$ to spaces, a.k.a. an $(n+1)$-cube of spaces, is called

- *homotopy cocartesian* if the composition

$$\underset{\substack{S\in\mathcal{P}[n]\\ S\neq[n]}}{\operatorname{hocolim}}\, Y(S) \xrightarrow{\text{standard comparison map}} \underset{\substack{S\in\mathcal{P}[n]\\ S\neq[n]}}{\operatorname{colim}}\, Y(S) \xrightarrow{\text{determ. by cube}} Y([n])$$

  is a weak homotopy equivalence;
- *strongly homotopy cocartesian* if each of its 2-dimensional faces is homotopy cocartesian (see remark 1.2.3 below);
- *homotopy cartesian* if the composition

$$Y(\varnothing) \xrightarrow{\text{determ. by cube}} \lim_{\substack{S\in\mathcal{P}[n]\\ S\neq\varnothing}} Y(S) \xrightarrow{\text{standard comparison map}} \underset{\substack{S\in\mathcal{P}[n]\\ S\neq\varnothing}}{\operatorname{holim}}\, Y(S)$$

  is a weak equivalence.

A homotopy invariant functor $F$ from spaces is $n$*-polynomial* if, for every $Y$ from $\mathcal{P}[n]$ to spaces which is strongly homotopy cocartesian, the cube $F\circ Y$ is homotopy cartesian.

(Notation: it is often convenient to write $\lim_{c \text{ in } \mathcal{D}} E(d)$ and the like for $\lim E$ (and the like), when $E$ is a functor from a small category $\mathcal{D}$ to spaces. In the definition above, we have used this for $\mathcal{D}=\mathcal{P}[n]\smallsetminus\{[n]\}$ and $\mathcal{D}=\mathcal{P}[n]\smallsetminus\{\varnothing\}$.)

Remark 1.2.3. The *two-dimensional faces* of an $(n+1)$-cube $Y$ are defined/obtained as follows: choose $S,T\subset[n]$ such that $|S\cup T|=|S|+1=|T|+1=|S\cap T|+2$ and restrict the functor $Y$ to $\{S\cap T,S,T,S\cup T\}\subset\mathcal{P}[n]$.
Suppose that we have a collection of maps $e_j\colon X\to V(j)$ where $j=0,1,\dots,n$. Suppose that $X$ and the $V(j)$ are CW-spaces and each $e_j$ is the inclusion of a CW-subspace. Then we can make an $(n+1)$-cube $V$ of spaces by defining $V(S)$ for $S\in\mathcal{P}[n]$ as the "union" of the $V(j)$ for $j\in S$ along the common subspace $X$. Such a cube $V$ is strongly homotopy cocartesian. Up to weak equivalence, every strongly homotopy cocartesian $(n+1)$-cube has this form. In more detail, if $Y$ is any strongly homotopy cocartesian $(n+1)$-cube, we can make an $(n+1)$-cube $V$ following this recipe and a natural transformation $V\to Y$ which is a weak equivalence.

Remark 1.2.4. Let $Y$ be a functor from $\mathcal{P}[n]$ to spaces. The space

$$\underset{\substack{S\in\mathcal{P}[n]\\ S\neq[n]}}{\operatorname{hocolim}}\, Y(S)$$

has an explicit description as a certain quotient of the disjoint union of the spaces $\Delta([n]\smallsetminus S)\times Y(S)$, where $S$ runs through the proper subsets of $[n]$. Here $\Delta(T)$ denotes the geometric simplex with vertex set $T$. ... The space

$$\underset{\substack{S\in\mathcal{P}[n]\\ S\neq\varnothing}}{\operatorname{holim}}\, Y(S)$$

has an explicit description as the space of natural transformations from $S\to\Delta(S)$ (where $S\subset[n]$, $S\neq\varnothing$) to the functor $S\mapsto Y(S)$ (where $S\neq\varnothing$).

Remark 1.2.5. There is another condition that we often impose on a homotopy invariant $F$ from spaces to spaces (before we ask whether it is $n$-polynomial). If it is satisfied, I like to say $F$ *is good* but Goodwillie says $F$ *satisfies the limit condition.* It means the following. Let $X$ be a CW-space. Then we have a canonical map

$$\underset{\text{cpct CW-subspaces } X_\alpha}{\operatorname{hocolim}} F(X_\alpha) \longrightarrow F(X)\,.$$

This is required to be a weak homotopy equivalence. (The condition can also be expressed in terms of homotopy groups.) This condition ensures that $F$ can be recovered (well enough) from its restriction to the category of *compact* CW-spaces and cellular maps between these.

Example of a homotopy invariant functor in this setting which is $n$-polynomial: if $F$ is 1-polynomial, then the functor taking a space $Y$ to

$$F(\underbrace{Y\times Y\times\cdots\times Y\times Y}_{n})$$

is $n$-polynomial. Reason: the functor $Y\mapsto Y\times Y\times\cdots\times Y\times Y$ takes strongly homotopy cocartesian $(n+1)$-cubes to homotopy cocartesian $(n+1)$-cubes and $F$, being 1-polynomial, takes homotopy cocartesian $n$-cubes to homotopy cartesian $n$-cubes. (*Exercise*: verify the last sentence.)

*Choice number one point one.* $\mathcal{C}$ can be the category of spaces over a fixed space $B$. (An object is a map from some other space $X$ to $B$ and a morphism from $u\colon X_0\to B$ to $v\colon X_1\to B$ is a map $f\colon X_0\to X_1$ such that $vf=u$.) A morphism in $\mathcal{C}$ is regarded as a weak equivalence if the underlying map of spaces is a weak equivalence. A commutative diagram in $\mathcal{C}$ in the shape of an $(n+1)$-cube is *strongly homotopy cocartesian* (respectively, homotopy cocartesian, strongly homotopy cartesian, homotopy cartesian) if the underlying diagram in the category of spaces has that property. A functor $F$ (from $\mathcal{C}$ to spaces) is $n$-polynomial if it is homotopy invariant (takes weak equivalences in $\mathcal{C}$ to weak equivalences of spaces) and takes *strongly homotopy cocartesian* $(n{+}1)$*-cubes* in $\mathcal{C}$ to *homotopy cartesian* $(n+1)$*-cubes* in the category of spaces.
A homotopy invariant functor $F$ from the category of spaces to the category of spaces determines, by specialization, a homotopy invariant functor $F_B$ from the category of spaces over $B$ to the category of spaces. If we search for the best $n$-polynomial approximations to $F_B$, for $n=1,2,3,\dots$, then this is like constructing the Taylor expansion of $F$ *at* the object $B$. (It is allowed to take $B=*$; then $F_B$ is much the same as $F$.)

* * *

*Choice number two.* $\mathcal{C}$ can be the opposite of the poset $\mathcal{O}(M)$ of open subsets of a smooth manifold $M$ (without boundary, for simplicity). So we are looking at contravariant functors $F$ from $\mathcal{O}(M)$ to spaces. Such a functor is *isotopy invariant* if, for every inclusion $U_0\to U_1$ of open subsets of $M$ which happens to be an isotopy equivalence (in the category of smooth manifolds and smooth embeddings), the induced map $F(U_1)\to F(U_0)$ is a homotopy equivalence. (Exercise: define isotopy equivalence.) As a rule one wants to impose another condition, *goodness.* This says that, whenever $U$ in $\mathcal{O}(M)$ is a monotone union of open subsets $U_0\subset U_1\subset U_2\subset\cdots$, then the canonical map from $F(U)$ to $\operatorname{holim}_j F(U_j)$ is a weak equivalence. (It has the consequence that the behavior of $F$ on "complicated" open sets in $M$ is sufficiently determined for homotopy theoretic purposes by the behavior of $F$ on "uncomplicated" open subsets of $M$. Exercise: define *uncomplicated.*) Popular example of such an $F$: the functor $V\mapsto \operatorname{emb}(V,N)$ where $N$ is a fixed smooth manifold and $\operatorname{emb}(-,-)$ is for spaces of smooth embeddings. (We do not assume that $N$ is an open subset of $M$, and often $\dim(N)$ will be greater than $\dim(M)$.) — A contravariant functor $F$ from $\mathcal{O}(M)$ to spaces is $n$*-polynomial* if it is isotopy invariant and good and satisfies the following: for every choice of open $V\subset M$ and pairwise disjoint closed subsets $A_0,A_1,\dots,A_n$ of $V$, the cubical diagram (shape of an $(n+1)$-cube)

$$S \mapsto F\big(V \smallsetminus \bigcup_{j\in S} A_j\big)$$

is homotopy cartesian. (Here $S$ runs through the subsets of $\{0,1,\dots,n\}$.) This makes some sense (if we compare with "choice number one") because the cubical diagram $S \mapsto V \smallsetminus \bigcup_{j\in S} A_j$ is a beautiful example of a strongly cocartesian $(n+1)$-cube *in the category of spaces*. In other words this definition of $n$-polynomial (setting of choice number two) has a great deal of similarity with the definition of $n$-polynomial functor in the setting of choice number one. But there are also substantial differences, mainly because in the setting of choice number two we work with a contravariant functor (from "some" spaces and maps, i.e., open subsets of $M$ and their inclusion maps, to spaces) whereas in the setting of choice number one we have a covariant functor.

A famous example of a 1-polynomial functor in this setting is as follows. Recall or accept that a smooth immersion $f\colon M \to N$ (where $M$ and $N$ are smooth manifolds without boundary) is a smooth map such that the differential $T_xM \to T_{f(x)}N$ is an *injective* linear map, for all $x \in M$. Now fix $M$ and $N$, and in such a way that $\dim(N) > \dim(M)$ (or $\dim(N) = \dim(M)$, but in this case we require that no connected component of $M$ be compact). For $U \in \mathcal{O}(M)$ let $F(U)$ be the space $\mathrm{imm}(U,N)$ of smooth immersions from $U$ to $N$. It is a difficult theorem, a.k.a. the Smale-Hirsch $h$-principle for immersions, that $F$ is polynomial of degree 1. We ought to come back to this.

*Choice number two point one.* (Arone and Turchin call this the *context free* variant of choice number two.) Let $\mathrm{man}_d$ be the following category. An object is a smooth manifold of dimension $d$ (empty boundary). A morphism is a smooth (codimension zero) embedding. Instead of homotopy, we have the notion of smooth isotopy in $\mathrm{man}_d$. Let $\mathcal{C}$ be the opposite category of $\mathrm{man}_d$. A functor $F$ from $\mathcal{C} = \mathrm{man}_d^{\mathrm{op}}$ to spaces is *isotopy invariant* if, whenever $g\colon M_0 \to M_1$ is an *isotopy equivalence* in $\mathrm{man}_d$, then $F(g)\colon F(M_1) \to F(M_0)$ is a weak equivalence. (Exercise: define isotopy equivalence.) It is *good* if, whenever $U$ in $\mathrm{man}_d$ is a monotone union of open subsets $U_0 \subset U_1 \subset U_2 \subset \cdots$, then the canonical map from $F(U)$ to $\mathrm{holim}_j F(U_j)$ is a weak equivalence. Popular example of such an $F$: the functor $V \mapsto \mathrm{emb}(V,N)$ where $N$ is a fixed smooth manifold. — A contravariant functor $F$ from $\mathrm{man}_d$ to spaces is *$n$-polynomial* if it is isotopy invariant and good and satisfies the following: for every choice of $V$ in $\mathrm{man}_d$ and pairwise disjoint closed subsets $A_0, A_1, \dots, A_n$ of $V$, the $(n+1)$-cube

$$\mathcal{P}[n] \ni S \quad \mapsto \quad F\big(V \smallsetminus \bigcup_{j\in S} A_j\big)$$

is homotopy cartesian.

* * *

*Choice number three.* Let $\mathcal{C} = \mathcal{J}$ be the following category. Objects are finite dimensional real vector spaces $V, W, \dots$ with inner product (positive definite symmetric bilinear form). A morphism is a linear map respecting the inner product. (Such a linear map is automatically injective, but it does not have to be surjective.) The set of morphisms in $\mathcal{J}$ from $V$ to $W$ should be viewed as a space (Stiefel manifold). In other words, $\mathcal{J}$ is *enriched* over spaces. Therefore we can speak of continuous functors $F$ from $\mathcal{J}$ to spaces. *Examples*: $F(V) = \mathrm{O}(V)$ and $F(V) = \mathrm{Top}(V)$. Here $\mathrm{O}(V)$ is the orthogonal group of $V$ and $\mathrm{Top}(V)$ is the homeomorphism group (both viewed as topological groups). Slightly more radical: $F(V) = B\mathrm{O}(V)$ and $F(V) = B\mathrm{Top}(V)$ (where the $B$ is for *classifying spaces*). We have a concept of small change in $\mathcal{J}$, although it is not very profound: an inclusion of type $V \hookrightarrow V \oplus \mathbb{R}$ is a small change in $\mathcal{J}$. — A continuous functor $F$ from $\mathcal{J}$ to spaces is *$n$-polynomial* if for every $V$ in $\mathcal{J}$, the canonical map

$$F(V) \longrightarrow \underset{0 \neq U \subset \mathbb{R}^{n+1}}{\mathrm{holim}} F(V \oplus U)$$

is a weak equivalence. Here $U$ runs through the linear subspaces of $\mathbb{R}^{n+1}$; we view these $U$ as elements of a poset. *Warning*: this is nevertheless a special type of homotopy limit. It is enriched. It "knows" that the linear subspaces $U$ of $\mathbb{R}^{n+1}$ can vary continuously and it "knows" that $F$ is a continuous functor. Apart from that, the condition reminds us of homotopy cartesian $(n+1)$-cubes of spaces; but instead of a diagram of spaces indexed by the subsets of $\{0,1,\dots,n\}$, we have a diagram of spaces $F(V\oplus U)$ indexed by the linear subspaces $U$ of $\mathbb{R}^{n+1}$. (To formulate the condition we assigned a special role to $U=0$, but it is not really excluded.)

Here are a few more words on the mysterious enriched homotopy limit. Fix $n$ and let $\mathcal{Q}(n+1)$ be the poset of linear subspaces of $\mathbb{R}^{n+1}$. This should be thought of as a topological poset, so it has an underlying space $\mathrm{Ob}(\mathcal{Q}(n+1))$: the disjoint union $\coprod_{k=0}^{n+1}\mathrm{Gr}(k,n+1)$ of the Grassmann manifolds $\mathrm{Gr}(k,n+1)$. We want to allow functors $E$ from $\mathcal{Q}(n+1)$ to spaces which are equipped with the following additional structure (which expresses a form of continuity).

(i) For each $k=0,1,\dots,n+1$ we have a space $E(k)$ and a (continuous) map
$$E(k)\to\mathrm{Gr}(k,n+1).$$
(The fiber $E(k)_V$ for $V\in\mathrm{Gr}(k,n+1)$ can be thought of as $E(V)$, the value of $E$ at the object $V$.)

(ii) For every $k$ and $\ell$ such that $0\le k\le\ell\le n+1$, the evaluation map, from a certain subset of $\mathrm{Gr}(k,n+1)\times\mathrm{Gr}(\ell,n+1)\times E(k)$ to $E(\ell)$, is continuous. That subset consists of triples $(U,V,x)$ where $U\subset V$ and $x\in E(k)_U\subset E(k)$. The evaluation map would take that triple to a point in $E(\ell)_V\subset E(\ell)$.

In such a case, let us say that $E$ is *admissible*. There is also a notion of *admissible map* (or admissible natural transformation) between such admissible functors. (Details omitted.) Such a map $f\colon E\to E'$ is a *weak equivalence* if it specializes to a weak equivalence $E(k)\to E'(k)$ of spaces for $k=0,1,\dots,n+1$.

If for an admissible $E$ the maps $E(k)\to\mathrm{Gr}(k,n+1)$ are Serre fibrations, then we say that the admissible $E$ is *fibrant*. (Example: the functor $U\mapsto F(V\oplus U)$ which appears in the definition of $n$*-polynomial* is admissible and fibrant.) In the situation where $E$ is (admissible and) fibrant, an explicit and illuminating but nevertheless somewhat heavy definition of the enriched $\mathrm{holim}\,E$ can be given. It is the space of all "rules" which to every string of linear subspaces
$$V_k\supset V_{k-1}\supset V_{k-2}\supset\cdots\supset V_0$$
of $\mathbb{R}^{n+1}$ associate a map $u(V_k,\dots,V_0)\colon\Delta^k\to E(V_k)$. Conditions:

- the map $u(V_k,\dots,V_0)$ depends continuously on the string $(V_k,\dots,V_0)$;
- whenever $f\colon[k]\to[\ell]$ is a monotone map, we must have
$$u(V_\ell,V_{\ell-1},\dots,V_0)\circ f_*=E(V_{f(k)}\to V_\ell)\circ u(V_{f(k)},V_{f(k-1)},\dots,V_{f(0)})$$
(both sides are maps from $\Delta^k$ to $E(V_\ell)$).

There is also a non-explicit but more formal definition which I only sketch. Let $F$ be an admissible functor from $\mathcal{Q}(n+1)$ to spaces. We say that $F$ is *cofibrant* if, for every diagram of admissible functors (from $\mathcal{Q}(n+1)$ to spaces) and (admissible) natural transformations
$$\begin{tikzcd} & E_0 \arrow[d, "\eta"] \\ F \arrow[r] & E_1 \end{tikzcd}$$
where $E_0$ and $E_1$ are fibrant and $\eta$ is a weak equivalence, there exists an admissible natural transformation $F\to E_0$ making the diagram commutative up to an admissible (natural) homotopy.

— Now, given a *fibrant* admissible $E$, we choose a cofibrant replacement $F$ of the terminal functor $*_{\mathcal{Q}(n+1)}$ (from $\mathcal{Q}(n+1)$ to spaces), i.e., a weak equivalence $F \to *_{\mathcal{Q}(n+1)}$ where $F$ is admissible and cofibrant. Then we say

$$\operatorname{holim} E := \operatorname{map}(F, E)$$

where $\operatorname{map}(F, E)$ is defined as a simplicial set. (And what are we going to do if $E$ is not fibrant? We choose a fibrant replacement first, i.e., an admissible map $E \to E'$ which is a weak equivalence and where $E'$ is fibrant.)

* * *

*Some observations.* In the setting of choice number two and choice number two point one, it is not very obvious what we mean by a small change (although it is clear what we mean by a polynomial functor). It is not obvious how we might construct Taylor expansions of a functor $F$ at different locations. It seems to me that we only try one Taylor expansion, and it is done "at" the object $\varnothing$ (of $\mathcal{O}(M)$, or of $\mathrm{man}_d$).

In the setting of choice number three, it will become clear (I hope) that we only try one Taylor expansion of a functor $F$, and it is done "at" the idealized object $\mathbb{R}^\infty$. To explain this a little better: we tend to take the space

$$F(\mathbb{R}^\infty) := \operatorname*{hocolim}_{n\to\infty} F(\mathbb{R}^n)$$

for granted and we try to find expressions for $F(V)$ in terms of $F(\mathbb{R}^\infty)$ and some *derivatives of $F$ at infinity*, where $V$ is an honest object of $\mathcal{J}$.

* * *

*Some exercises.*

EXERCISE 1.2.6. Let $X$ be a CW-space which is a union of CW-subspaces $X_0, X_1, \dots, X_n$. Then the cube of spaces defined by

$$S \mapsto \bigcap_{i\in[n]\smallsetminus S} X_i$$

for $S \subset [n] = \{0, 1, \dots, n\}$ is homotopy cocartesian. (Convention: $\bigcap_{i\in\varnothing} X_i = X$.)

EXERCISE 1.2.7. Let $X$ be a CW-space with CW-subspaces $X_0, X_1, \dots, X_n$. Suppose that $\bigcap_{i\in[n]} X_i = \varnothing$. Then the cube of spaces defined by

$$S \mapsto \bigcup_{i\in S} X_i$$

for $S \subset [n] = \{0, 1, \dots, n\}$ is homotopy cocartesian.

EXERCISE 1.2.8. In the terminology of *Choice number three*: the terminal (admissible) functor from $\mathcal{Q}(n+1)$ to spaces is fibrant.

EXERCISE 1.2.9. In the terminology of *Choice number three*: Every admissible functor from $\mathcal{Q}(n+1)$ to spaces has a cofibrant replacement (by a CW-functor). Define CW-functors so that this is true.

CHAPTER 2

# Precursors to functor calculus

This chapter wants to give some "historical" examples of first and second Taylor approximations. This is not the same as giving examples of 1-polynomial and 2-polynomial functors; instead we want to see examples of natural transformations $E \to F$ where $E$ is a functor that we love and want to understand better, $F$ is 1-polynomial or 2-polynomial, and the natural transformation $E \to F$ can make claims to be optimal (under the condition that $F$ is 1-polynomial or 2-polynomial).

## 2.1. Functor calculus in abelian categories

An idea of Eilenberg and MacLane. Not very important here; it is not homotopy theory.

## 2.2. Early notions of functor calculus in the theory of smooth embeddings

Fix smooth manifolds $M$ and $N$, both without boundary (for simplicity). Compactness is not required at this point. We take $\mathcal{C} = \mathcal{O}(M)^{\mathrm{op}}$ where $\mathcal{O}(M)$ is the poset (short for partially ordered set) of open subsets of $M$. As already pointed out in section ..., we can define a functor from $\mathcal{O}(M)^{\mathrm{op}}$ to spaces by

$$V \mapsto \mathrm{emb}(V, N).$$

Here $\mathrm{emb}(V, N)$ could be defined as the space of smooth embeddings from $V$, open subset of $M$, to $N$, with the (compact-open) $C^\infty$ topology. Details available on request. As an alternative, it would be alright to define $\mathrm{emb}(V, N)$ as a (fibrant, a.k.a. Kan) simplicial set so that a $k$-simplex is a smooth embedding

$$\Delta^k \to M \longrightarrow \Delta^k \to N$$

which is at the same time a map *over* $\Delta^k$. (Here too, there are some details to be supplied; it needs to be said what a smooth embedding is in this context.) It was already pointed out that this functor is *isotopy invariant*, and this is also easy to verify. It is slightly more difficult to show that it is *good*. We may come back to that.

But here we need to mention another functor: the functor

$$V \mapsto \mathrm{imm}(V, N)$$

where $\mathrm{imm}(V, N)$ denotes the space of smooth immersions from $V$, open subset of $M$, to $N$. (An immersion from $V$ to $N$ is a smooth map $f: V \to N$ such that the differential $T_xM \to T_{f(x)}N$ is an *injective* linear map, for every $x \in M$.) Again it is easy to show that this functor is isotopy invariant, and slightly more difficult to show that it is good. For the following discussion it is important to keep in mind that $\mathrm{emb}(V, N) \subset \mathrm{imm}(V, N)$. (We have a natural transformation here.)

THEOREM 2.2.1. *Suppose that* $\dim(M) < \dim(N)$, *or suppose that* $\dim(M) = \dim(N)$ *and no connected component of* $M$ *is compact. Then the functor* $V \mapsto \mathrm{imm}(V, N)$ *on* $\mathcal{O}(M)^{\mathrm{op}}$ *is* 1-*polynomial.*

This could be called the *main theorem of immersion theory.* But it is traditionally not stated like that. So let me give the traditional formulation. A *formal immersion* from $V$ to $N$ is a pair $(f,\psi)$ where $f\colon V\to N$ is a continuous map and $\psi$ is a vector bundle monomorphism from $TM$ to $f^*TN$. (Note that $TM$ and $f^*TN$ are two vector bundles on the same space $M$. By monomorphism I mean: vector bundle map which is injective, therefore fiberwise *linearly* injective. Beware: there is no requirement that $\psi$ be the derivative of $f$. This would not even make sense because $f$ was never assumed to be differentiable.) Let

$$\mathrm{fimm}(V,N)$$

be the space of all formal immersion from $V$ to $N$, in other words the space of all such pairs $(f,\psi)$. There is an obvious comparison map

$$\mathrm{imm}(V,N)\longrightarrow \mathrm{fimm}(V,N)$$

given by $f\mapsto (f,df)$ for $f\in \mathrm{imm}(V,N)$. In other words, if we have a smooth immersion $f\colon V\to N$, then we can make a formal immersion by finding the derivative $df$ of $f$ and viewing that as a vector bundle map from $TM$ to $f^*TN$. It will be injective precisely because we assumed that $f$ is an immersion.

THEOREM 2.2.2. *Suppose that* $\dim(M)<\dim(N)$*, or suppose that* $\dim(M)=\dim(N)$ *and no connected component of* $M$ *is compact. Then the standard map* $\mathrm{imm}(M,N)\to \mathrm{fimm}(M,N)$ *is a weak homotopy equivalence.*

We will see that theorem 2.2.1 is an easy corollary of theorem 2.2.2, and vice versa. (The proof of theorem 2.2.2 is quite difficult. Much too long for this course.) Before we come to that, let me give an illustration of theorem 2.2.2.

EXAMPLE 2.2.3. Let us look at smooth embeddings and smooth immersions from $S^{n-1}$ to $\mathbb{R}^n$, and more specifically let us make a comparison between $\pi_0\mathrm{emb}(S^{n-1},\mathbb{R}^n)$ and $\pi_0\mathrm{imm}(S^{n-1},\mathbb{R}^n)$.
If we assume only the rudiments of the Schoenflies theorem, then we may say that for a smooth embedding $f\colon S^{n-1}\to\mathbb{R}^n$ the complement of $f(S^{n-1})$ in $\mathbb{R}^n$ has two connected components: the outside component and the inside component. As a result, the submanifold $f(S^{n-1}$ of $\mathbb{R}^n$ has an *outward* unit normal vector field. This leads to a Gauss map associated with $f$: to $x\in S^{n-1}$ we associate the ouward unit normal to $f(S^{n-1}$ at $f(x)$, an element of $S^{n-1}$. So the Gauss map of $f$ is a continuous map $S^{n-1}\to S^{n-1}$. Now it is easy to give examples of $f$ such that the Gauss map of $f$ has degree 1, or degree $-1$. In this way we see that $\mathrm{emb}(S^{n-1},\mathbb{R}^n)$ has at least two path components (which we can distinguish by *degree of Gauss map*).
If we have a smooth immersion $f\colon S^{n-1}\to\mathbb{R}^n$ instead of a smooth embedding, then it looks as if much of this should still work. It is possible to define two unit normal vector fields (to $f$) but it is not easy to decide which is outward and which is inward. So let us use theorem 2.2.2 instead. Therefore we want to understand $\pi_0\mathrm{fimm}(S^{n-1},\mathbb{R}^n)$. The forgetful map

$$\mathrm{fimm}(S^{n-1},\mathbb{R}^n)\to \mathrm{map}(S^{n-1},\mathbb{R}^n)$$

given be $(f,\psi)\mapsto f$ is easily seen to be a fibration and its target, $\mathrm{map}(S^{n-1},\mathbb{R}^n)$, is easily seen to be contractible. Therefore $\mathrm{fimm}(S^{n-1},\mathbb{R}^n)$ is homotopy equivalent to the fiber of that map over the base point, which is the space of vector bundle monomorphisms $\psi\colon TS^{n-1}\to S^{n-1}\times\mathbb{R}^3$. (Here $S^{n-1}\times\mathbb{R}^3$ is sloppy notation for the trivial vector bundle on $S^{n-1}$ with fiber $\mathbb{R}^3$. Generally, it is sloppy but convenient to confuse a vector bundle with its total space.) It is an exercise to show that the space of vector bundle monomorphisms $\psi\colon TS^{n-1}\to S^{n-1}\times\mathbb{R}^3$ is homotopy equivalent to the space of orientation-preserving vector bundle isomorphisms from $S^{n-1}\times\mathbb{R}^3$ to $S^{n-1}\times\mathbb{R}^3$. (Use the standard inclusion of vector bundles $TS^{n-1}\to S^{n-1}\times\mathbb{R}^n$.) Therefore we obtain

$$\mathrm{fimm}(S^{n-1},\mathbb{R}^n)\simeq \mathrm{map}(S^{n-1},\mathrm{GL}^+(n,\mathbb{R}))\simeq \mathrm{map}(S^{n-1},S\mathrm{O}(n))$$

where $\mathrm{GL}^+(n,\mathbb{R})$ denotes the subgroup of $\mathrm{GL}(n,\mathbb{R})$ consisting of the $(n\times n)$-matrices with positive determinant. In particular, taking $n=2$ we obtain

$$\pi_0\mathrm{fimm}(S^1,\mathbb{R}^2)\cong\mathbb{Z}$$

(infinitely many path components, not very surprising). Taking $n=3$ we obtain

$$\pi_0\mathrm{fimm}(S^2,\mathbb{R}^3)\qquad\text{has only one element}$$

because $\pi_2\mathrm{SO}(3)\cong 0$. (There is a theorem in the theory of Lie groups saying that $\pi_2$ of a connected Lie group is always $0$.) This is very surprising. (*End of example.*)

Next, we wanted to see how theorem 2.2.1 follows from theorem 2.2.2. The first thing to note is that if the conditions on $M$ in theorem 2.2.2 are satisfied, then they are also satisfied for every open $V\subset M$. Therefore

$$\mathrm{imm}(V,N)\xrightarrow{\simeq}\mathrm{fimm}(V,N)$$

for every $V\in\mathcal{O}(M)$ (since we are assuming theorem 2.2.2). Consequently it is enough to show that the functor $V\mapsto\mathrm{fimm}(V,N)$ is 1-polynomial. This is supposed to include the properties *isotopy invariant* and *good*. Now *isotopy invariant* is obvious; in fact it should be clear that $V\mapsto\mathrm{fimm}(V,N)$ is homotopy invariant, which is stronger. *Goodness* can be reduced to the following. Suppose that $U$ is open in $M$ and that $U$ is a monotone union of *compact subsets*

$$K_0\subset K_1\subset K_2\subset K_3\subset\dots$$

where each $K_j$ is a codimension 0 smooth compact submanifold of $M$, and $K_j$ is contained in $\mathrm{int}(K_{j+1})$. Then we need to show that the standard map from $\mathrm{fimm}(U,N)$ to

$$\operatorname*{holim}_j\ \mathrm{fimm}(K_{j+1},N)$$

is a weak homotopy equivalence. (See exercise ...) Here one might object that $\mathrm{fimm}(K_j,N)$ was never defined. But it is clear that it *can be* defined and it is clear how. Now we observe that the standard map from $\mathrm{fimm}(U,N)$ to the genuine (inverse) limit $\lim_j\mathrm{fimm}(K_j,N)$ is a homeomorphism (or isomorphism, if we rely on the simplicial set interpretation). Furthermore the restriction maps $\mathrm{fimm}(K_{j+1},N)\to\mathrm{fimm}(K_j,N)$ are all fibrations (respectively, Kan fibrations). Therefore we can say in this case that the comparison map from $\lim_j\mathrm{fimm}(K_j,N)$ to $\mathrm{holim}_j\mathrm{fimm}(K_j,N)$ is a weak homotopy equivalence. (See exercise ... )
Finally let us turn to the key property: for open subsets $U,V$ of $M$ we need to show that the commutative square of restriction maps

$$\begin{tikzcd}
\mathrm{fimm}(U\cup V,N) \arrow[r] \arrow[d] & \mathrm{fimm}(U,N) \arrow[d] \\
\mathrm{fimm}(V,N) \arrow[r] & \mathrm{fimm}(U\cap V,N)
\end{tikzcd}$$

is homotopy cartesian. By goodness, we can reduce to the situation where the closures $\bar U$ of $U$ and $\bar V$ of $V$ in $M$ are smooth compact codimension zero submanifolds of $M$, and their boundaries intersect transversely (in general position). Now we can look at

$$\begin{tikzcd}
\mathrm{fimm}(\bar U\cup\bar V,N) \arrow[r] \arrow[d] & \mathrm{fimm}(\bar U,N) \arrow[d] \\
\mathrm{fimm}(\bar V,N) \arrow[r] & \mathrm{fimm}(\bar U\cap\bar V,N)
\end{tikzcd}$$

instead. This square is a strict pullback square (strictly cartesian) and all the arrows in it are fibrations. Therefore it is a homotopy cartesian square. (See exercise ... )

We also want to know that theorem 2.2.2 follows from theorem 2.2.1. (After much confusion, I decided that it is not obvious, after all.) Here is a reduction to a "formal" statement. (To which we may return later, as usual. It is not a very hard statement.) The formal statement is as follows. Suppose that $E$ and $F$ are 1-polynomial functors on $\mathcal{O}(M)^{\mathrm{op}}$. Let $\tau\colon E \to F$ be a natural transformation.

> *If $\tau_V\colon E(V) \to F(V)$ is a weak homotopy equivalence for every $V$ in $\mathcal{O}(M)$ which is abstractly diffeomorphic to $\mathbb{R}^m$, then $\tau_V\colon E(V) \to F(V)$ is a weak homotopy equivalence for all $V \in \mathcal{O}(M)$.*

Let us use this, taking $E(V) = \mathrm{imm}(V,N)$ and $F(V) = \mathrm{fimm}(V,N)$. We are *assuming* that $E$ is 1-polynomial and we *know* already that $F$ is 1-polynomial. Therefore it is enough to show that the standard comparison map $\mathrm{imm}(V,N) \to \mathrm{fimm}(V,N)$ is a weak homotopy equivalence if $V$ is (abstractly) diffeomorphic to $\mathbb{R}^m$. In other words we have to show that the standard map

$$\mathrm{imm}(\mathbb{R}^m,N) \longrightarrow \mathrm{fimm}(\mathbb{R}^m,N)$$

is a weak homotopy equivalence. We can view this map as a map over $N$:

$$\begin{tikzcd} \mathrm{imm}(\mathbb{R}^m,N) \arrow[rr] \arrow[dr] & & \mathrm{fimm}(\mathbb{R}^m,N) \arrow[dl] \\ & N & \end{tikzcd}$$

The reference maps to $N$ are fibrations. Therefore it suffices to show that, for fixed $y \in N$, the induced map of fibers over $y$ is a weak homotopy equivalence:

$$\mathrm{imm}(\mathbb{R}^m,N)_y \longrightarrow \mathrm{fimm}(\mathbb{R}^m,N)_y$$

Here $\mathrm{imm}(\mathbb{R}^m,N)_y$ is the space of smooth immersions $\mathbb{R}^m \to N$ taking 0 to $y$. While $\mathrm{fimm}(\mathbb{R}^m,N)_y$ has a slightly complicated description, it is easy to see that it is homotopy equivalent (by a suitable restriction) to $\mathrm{injhom}_{\mathbb{R}}(\mathbb{R}^m,T_yN)$, where injhom for injective linear maps. Briefly, it remains to show that the map

$$\mathrm{imm}(\mathbb{R}^m,N)_y \longrightarrow \mathrm{injhom}_{\mathbb{R}}(\mathbb{R}^m,T_yN)$$

defined by $f \mapsto df(0)$ is a weak homotopy equivalence. For this, choose an open neighborhood $W$ of $y$ in $N$ such that $W$ is diffeomorphic to $\mathbb{R}^n$. First we verify that the inclusion

$$\mathrm{imm}(\mathbb{R}^m,W)_y \to \mathrm{imm}(\mathbb{R}^m,N)_y$$

is a homotopy equivalence. (Exercise; use a smooth isotopy $(e_t\colon\mathbb{R}^m \to \mathbb{R}^m)_{t\in[0,\infty)}$ such that $e_0$ is the identity, and the image of $e_t$ for $t>0$ is contained in the disk of radius $t^{-1}$.) Secondly we verify that the standard map

$$\mathrm{imm}(\mathbb{R}^m,\mathbb{R}^n)_0 \to \mathrm{injhom}_{\mathbb{R}}(\mathbb{R}^m,\mathbb{R}^n)$$

is a homotopy equivalence. We can view the target as a subspace of the source. It is a strong deformation retract: the deforming homotopy is given by

$$f \mapsto (x \mapsto tf(t^{-1}x))_{t\in[1,\infty)}$$

and $x \mapsto df(0)(x)$ for $t=\infty$.

THEOREM 2.2.4. *The inclusions* $\mathrm{emb}(V,N) \to \mathrm{imm}(V,N)$ *define a natural transformation of functors on $\mathcal{O}(M)^{\mathrm{op}}$. If the conditions of theorem 2.2.2 are satisfied, then this qualifies as a first Taylor approximation for the functor $V \mapsto \mathrm{emb}(V,N)$.*

We do not really have the vocabulary (yet) to make or prove such a claim. But let me reduce it to a formal statement once again. The formal statement is as follows. Suppose that $u: E \to F$ is a natural transformation of isotopy invariant and good functors from $\mathcal{O}(M)^{\mathrm{op}}$ to spaces. Suppose that $F$ is 1-polynomial.

> *Then $u: E \to F$ qualifies as the first Taylor approximation (for the functor $E$) if and only if, for every $V \in \mathcal{O}(M)$ which is abstractly diffeomorphic to $\mathbb{R}^m$, the map*
>
> $$u_V: E(V) \to F(V)$$
>
> *is a weak equivalence.*

Consequently we only have to show that the maps $\mathrm{emb}(V,N) \to \mathrm{imm}(V,N)$ are weak equivalences for the very special $V \in \mathcal{O}(M)$ which are abstractly diffeomorphic to $\mathbb{R}^m$. In the diagram

$$\mathrm{emb}(V,N) \to \mathrm{imm}(V,N) \to \mathrm{fimm}(V,N)$$

the second map is a weak homotopy equivalence and the composition of the two maps is also a weak homotopy equivalence by much the same argument. Therefore the first arrow is a weak homotopy equivalence.

Here is an alternative formulation which may in fact be better. No conditions on $M$ and $N$ are needed. The proof is much the same.

THEOREM 2.2.5. *The inclusions* $\mathrm{emb}(V,N) \to \mathrm{fimm}(V,N)$ *define a natural transformation of functors on $\mathcal{O}(M)^{\mathrm{op}}$. This qualifies as a first Taylor approximation for the functor $V \mapsto \mathrm{emb}(V,N)$.*

The functor $V \mapsto \mathrm{emb}(V,N)$ on $\mathcal{O}(M)^{\mathrm{op}}$ also has a second Taylor approximation which was (nearly) well known (though not under that name) before functor calculus emerged. This is due to A Haefliger (Commentarii Math Helv. 36 (1962), 47-82), except for a small modification. The new idea here is that we pay attention to maps $g: M \times M \to N \times N$ which are equivariant for the (permutation) action of the group $\mathbb{Z}/2$ on $M \times M$ and $N \times N$. The space of all such maps would be denoted

$$\mathrm{map}^{\mathbb{Z}/2}(M \times M, N \times N).$$

But there are notions sharper than equivariance. A map $g: M \times M \to N \times N$ is *isovariant* (for the permutation actions of $\mathbb{Z}/2$) if it is equivariant and

$$g^{-1}(\mathrm{diag}_N) = \mathrm{diag}_M$$

where $\mathrm{diag}_N \subset N \times N$ and $\mathrm{diag}_M \subset M \times M$ are the diagonals. *Example*: if $f: M \to N$ is a continuous map, then $f \times f: M \times M \to N \times N$ is certainly equivariant, but it will be isovariant if and only if $f$ is injective. We need an even stronger notion, *strongly isovariant*. This is only a meaningful condition for smooth maps. A *smooth* map $g: M \times M \to N \times N$ is *strongly isovariant* if it is isovariant and, for every $x \in M$, the differential $dg$ at $(x,x)$ induces a linear *injection*

$$T_{(x,x)}(M \times M)/T_{(x,x)}\mathrm{diag}_M \longrightarrow T_{(x,x)}(N \times N)/T_{(x,x)}\mathrm{diag}_N\,.$$

The space of these (smooth) strongly isovariant maps is denoted

$$\mathrm{ivmap}^{\mathbb{Z}/2}(M \times M, N \times N).$$

There is a commutative square

$$\begin{tikzcd}
\mathrm{emb}(M,N) \arrow[r] \arrow[d] & \mathrm{map}(M,N) \arrow[d] \\
\mathrm{ivmap}^{\mathbb{Z}/2}(M \times M, N \times N) \arrow[r] & \mathrm{map}^{\mathbb{Z}/2}(M \times M, N \times N)
\end{tikzcd}$$

where the vertical maps are both given by the same formula, $f \mapsto f \times f$ (and the horizontal ones are forgetful, i.e., they forget conditions). (It does not matter much whether we define $\mathrm{map}(M,N)$ and $\mathrm{map}^{\mathbb{Z}/2}(M \times M, N \times N)$ using smooth maps or just continuous maps; the homotopy types do not change if we impose smoothness.) Haefliger showed that the resulting map from $\mathrm{emb}(M,N)$ to

$$\mathrm{holim}\left(\begin{tikzcd} & \mathrm{map}(M,N) \arrow[d] \\ \mathrm{ivmap}^{\mathbb{Z}/2}(M \times M, N \times N) \arrow[r] & \mathrm{map}^{\mathbb{Z}/2}(M \times M, N \times N) \end{tikzcd}\right)$$

is 1-connected if $m+1 < 2n/3$, where $n = \dim(N)$ and $m = \dim(M)$. This was later improved by Dax: he showed that the same map is is $(2n-3-3m)$-connected. For comparison, the forgetful map from $\mathrm{emb}(M,N)$ to $\mathrm{imm}(M,N)$ is $(n-2m-1)$-connected under the mild conditions of theorem 2.2.2.

The proofs of these statements by Haefliger and Dax rely on transversality arguments. (They are not very easy; Dax' paper is quite long.) We skip these, with many regrets. For us it is important to verify two formal statements:

(i) the functor

$$V \;\mapsto\; F(V) := \mathrm{holim}\left(\begin{tikzcd} & \mathrm{map}(V,N) \arrow[d] \\ \mathrm{ivmap}^{\mathbb{Z}/2}(V \times V, N \times N) \arrow[r] & \mathrm{map}^{\mathbb{Z}/2}(V \times V, N \times N) \end{tikzcd}\right)$$

on $\mathcal{O}(M)^{\mathrm{op}}$ is 2-polynomial;

(ii) if $V$ happens to be diffeomorphic to a disjoint union of $k$ copies of $\mathbb{R}^m$ where $k \le 2$, then Haefliger's comparison map $\mathrm{emb}(V,N) \to F(V)$ is a (weak) homotopy equivalence.

In particular, (ii) is the condition which ensures that $\mathrm{emb}(V,N) \to F(V)$ is a second Taylor approximation.

Sketch-proving (i): it is enough to show that each of the three functors

$$V \mapsto \mathrm{map}(V,N),\ \mathrm{map}^{\mathbb{Z}/2}(V \times V, N \times N),\ \mathrm{ivmap}^{\mathbb{Z}/2}(V \times V, N \times N)$$

on $\mathcal{O}(M)^{\mathrm{op}}$ is 2-polynomial. (Why?) *First case*: we know already that $V \mapsto \mathrm{map}(V,N)$ is 1-polynomial, therefore 2-polynomial. *Second case*: $V \mapsto \mathrm{map}^{\mathbb{Z}/2}(V \times V, N \times N)$. So we choose $V$ and 3 pairwise disjoint closed subsets $A_0, A_1, A_2$ of $V$. Write $A_S$ for $\bigcup_{t \in S} A_t$ whenever $S \subset \{0,1,2\}$. Form the 3-cube of spaces with $\mathbb{Z}/2$-actions (and maps respecting the actions)

$$S \mapsto (V \smallsetminus A_S) \times \big(V \smallsetminus A_S\big)$$

where $S \subset \{0,1,2\}$. (This is contravariant in $S$; not a big problem.) This is a standard 3-cube associated with the covering of $V \times V$ by open subsets $(V - A_t) \times (V - A_t)$ for $t = 0,1,2$. Therefore we believe that this is a homotopy cocartesian 3-cube, because we have seen this phenomenon before. But that is not enough: we need to know that it is a homotopy cocartesian 3-cube *of spaces with action of* $\mathbb{Z}/2$. (It is: justification using iterated mapping cylinders and partitions of unity. The task is to show that the hocolim of the terms for nonempty $S$ maps by a $\mathbb{Z}/2$-homotopy equivalence to $V \times V$, which is the term corresponding to $S = \varnothing$.) Therefore the covariant 3-cube

$$S \mapsto \mathrm{map}^{\mathbb{Z}/2}\big((V \smallsetminus A_S) \times (V \smallsetminus A_S),\ N \times N\big)$$

is homotopy cartesian. *Third case*: a lazy argument. If we inspect the second case carefully, we may discover that we have an explicit formula for a homotopy inverse for

$$\mathrm{map}^{\mathbb{Z}/2}(V \times V, N \times N) \longrightarrow \operatorname*{holim}_{\varnothing \neq S \subset \{0,1,2\}} \mathrm{map}^{\mathbb{Z}/2}((V \smallsetminus A_S) \times (V \smallsetminus A_S),\ N \times N)$$

and explicit formulas for the requisite homotopies. Now "restrict" these to get an explicit formula for a homotopy inverse for

$$\mathrm{ivmap}^{\mathbb{Z}/2}(V \times V, N \times N) \longrightarrow \operatorname*{holim}_{\varnothing \neq S \subset \{0,1,2\}} \mathrm{ivmap}^{\mathbb{Z}/2}((V \smallsetminus A_S) \times (V \smallsetminus A_S),\ N \times N)$$

and explicit formulas for the requisite homotopies.
Sketch-proving (ii): we can just try it out, assuming for example that $V$ is diffeomorphic to $\mathbb{R}^m \sqcup \mathbb{R}^m$ (the most interesting case). Let $A := \{0\} \sqcup \{0\} \subset V$, a set with 2 elements. Let's put $\mathrm{emb}(V, N)$ back into the diagram, and separate $V \times V$ respectively $A \times A$ into two on-diagonal and two off-diagonal copies of $\mathbb{R}^m \times \mathbb{R}^m$ respectively $*$, denoted $V \times^{\mathrm{on}} V$ and $V \times^{\mathrm{off}} V$ and $A \times^{\mathrm{on}} A$ and $A \times^{\mathrm{off}} A$. Then we get a square of the form

$$\begin{tikzcd}
\mathrm{emb}(V,N) \arrow[r] \arrow[d] & \begin{array}{rl} & \mathrm{map}(V,N) \\ \simeq & \mathrm{map}(A,N) \\ \simeq & N \times N \end{array} \arrow[d, "(x,y)\mapsto(x,y,x,y)"] \\
\begin{array}{rl} & \mathrm{ivmap}^{\mathbb{Z}/2}(V \times V, N \times N) \\ \simeq & \mathrm{ivmap}^{\mathbb{Z}/2}(V \times^{\mathrm{on}} V, N \times N) \\ & ((N \times N) \smallsetminus \mathrm{diag}_N) \\ \simeq & \mathrm{fimm}(V,N) \times ((N \times N) \smallsetminus \mathrm{diag}_N) \end{array} \arrow[r] & \begin{array}{rl} & \mathrm{map}^{\mathbb{Z}/2}(V \times V, N \times N) \\ \simeq & \mathrm{map}^{\mathbb{Z}/2}(A \times A, N \times N) \\ \simeq & \mathrm{map}^{\mathbb{Z}/2}(A \times^{\mathrm{on}} A, N \times N) \\ & \times\ \mathrm{map}^{\mathbb{Z}/2}(A \times^{\mathrm{off}} A, N \times N) \\ \simeq & (N \times N) \times (N \times N) \end{array}
\end{tikzcd}$$

Now we can verify that it is homotopy cartesian. Observation:

$$\mathrm{holim}\left(\begin{tikzcd} & C \arrow[d, "z\mapsto(z,z)"] \\ A \times B \arrow[r, "(a,b)\mapsto(f(a),g(b))"] & C \times C \end{tikzcd}\right) \cong \mathrm{holim}\ (A \xrightarrow{f} C \xleftarrow{g} B).$$

Use this with $C = N \times N$, $A = \mathrm{fimm}(V, N)$, $B = (N \times N) \smallsetminus \mathrm{diag}_N$.

REMARK 2.2.6. There is a forgetful map from

$$\mathrm{holim}\left(\begin{tikzcd} & \mathrm{map}(M,N) \arrow[d] \\ \mathrm{ivmap}^{\mathbb{Z}/2}(M \times M, N \times N) \arrow[r] & \mathrm{map}^{\mathbb{Z}/2}(M \times M, N \times N) \end{tikzcd}\right)$$

i.e., from Haefliger's second order approximation to $\mathrm{emb}(M, N)$, to the older first order approximation,

$$\mathrm{fimm}(M, N).$$

Indeed a strongly isovariant (smooth) map $g: M \times M \to N \times N$ determines $(f, \psi) \in \mathrm{fimm}(M, N)$ where $f: M \to N$ and $\psi: TM \to f^*TN$ are given by

$$g(x, x) = (f(x), f(x))$$

and $\psi_x$ equal to $dg$ at $(x, x)$, but viewed as a linear injection from $T_xM \cong T_{(x,x)}(M \times M)/T_{(x,x)}\mathrm{diag}_M$ to $T_{f(x)}N \cong T_{g(x,x)}(N \times N)/T_{g(x,x)}\mathrm{diag}_N$.

EXAMPLE 2.2.7. If $N = \mathbb{R}^n$ then $\mathrm{map}(M, N)$ and $\mathrm{map}^{\mathbb{Z}/2}(M \times M, N \times N)$ are both contractible. Therefore the Haefliger-Dax theory gives a $(2n - 3 - 3m)$-connected map from $\mathrm{emb}(M, N)$ to $\mathrm{ivmap}^{\mathbb{Z}/2}(M \times M, N \times N)$.

EXERCISE 2.2.8. Let $j: S^2 \to S^2 \times \mathbb{R}$ be given by $j(z) = (z, 0)$. Show that

$$j^*: \mathrm{imm}(S^2 \times \mathbb{R}, \mathbb{R}^3) \to \mathrm{imm}(S^2, \mathbb{R}^)$$

is not a homotopy equivalence.

EXERCISE 2.2.9. There is a standard mistake in immersion theory which goes roughly like this. Let $f: M \to \mathbb{R}^n$ be a smooth immersion with normal vector bundle $V \to M$. Let $j: M \to V$ be the zero section. *Then* $j^*: \mathrm{imm}(V, \mathbb{R}^n) \to \mathrm{imm}(M, \mathbb{R}^n)$ *is a (weak) homotopy equivalence.* The previous exercise is already a counterexample. But what is the easiest counterexample? And is it possible to make a corrected statement along the same lines?

## 2.3. An early instance of orthogonal calculus

There is a well known map $\mathbb{R}P^{n-1} \to \mathrm{O}(n)$ defined by $H \mapsto \rho_H$, where $H$ denotes a linear codimension 1 subspace of $\mathbb{R}^n$ (element of $\mathbb{R}P^{n-1}$) and $\rho_H$ is the reflection at $H$ (fixing $H$ pointwise). James invented a *stable left inverse* for this.

THEOREM 2.3.1. *There is a map* $f_n: \mathrm{O}(n) \to \Omega^\infty\Sigma^\infty(\mathbb{R}P^{n-1}_+)$ *so that the composition*

$$\mathbb{R}P^{n-1} \longrightarrow \mathrm{O}(n) \xrightarrow{f_n} \Omega^\infty\Sigma^\infty(\mathbb{R}P^{n-1}_+)$$

*is an instance of the standard inclusion* $Y \to \Omega^\infty\Sigma^\infty Y_+$ *, meaningful for any space* $Y$.

We may review the construction of $f_n$ later. For now the formal aspects are important. As usual. For a start, $f_n$ is natural in $n$, so there are commutative diagrams

$$\begin{tikzcd}
\mathrm{O}(n) \arrow[r] \arrow[d, "f_n"] & \mathrm{O}(n+1) \arrow[d, "f_{n+1}"] \\
\Omega^\infty\Sigma^\infty(\mathbb{R}P^{n-1}_+) \arrow[r] & \Omega^\infty\Sigma^\infty(\mathbb{R}P^{n}_+)
\end{tikzcd}$$

But it is better than that.

THEOREM 2.3.2. *For each* $n$ *there is a commutative diagram whose rows are (well known) homotopy fiber sequences:*

$$\begin{tikzcd}
\mathrm{O}(n) \arrow[r] \arrow[d, "f_n"] & \mathrm{O}(n+1) \arrow[r] \arrow[d, "f_{n+1}"] & S^n \arrow[d, "\text{stab.}"] \\
\Omega^\infty\Sigma^\infty(\mathbb{R}P^{n-1}_+) \arrow[r] & \Omega^\infty\Sigma^\infty(\mathbb{R}P^{n}_+) \arrow[r] & \Omega^\infty\Sigma^\infty S^n
\end{tikzcd}$$

This is an indication that the maps $f_n$ are good approximations in a relative sense. Namely, the right-hand vertical arrow is known to be $(2n-1)$-connected.

For a famous application it is convenient to modify the last diagram slightly to get

$$\begin{tikzcd} \mathrm{O}(n) \arrow[r] \arrow[d, "\tilde f_n"] & \mathrm{O}(n+1) \arrow[r] \arrow[d, "\tilde f_{n+1}"] & S^n \arrow[d, "\text{stab.}"] \\ \Omega^\infty\Sigma^\infty(\mathbb{R}P^{n-1}) \arrow[r] & \Omega^\infty\Sigma^\infty(\mathbb{R}P^n) \arrow[r] & \Omega^\infty\Sigma^\infty S^n \end{tikzcd}$$

where the rows are still homotopy fiber sequences. (If we can agree on a best choice of base point in $\mathbb{R}P^{n-1}$, then we have a canonical map of based spaces $\mathbb{R}P^{n-1}_+ \to \mathbb{R}P^n$ by taking that $+$ point to the new base point.)

Now let us extend the rows in the last diagram to the left, following Barratt-Puppe. In the upper row we get

$$\Omega S^n \to \mathrm{O}(n) \to \mathrm{O}(n+1) \to \cdots .$$

The map $\Omega S^n \to \mathrm{O}(n)$ is also known as $\Omega$ of the classifying map $\tau_n$ for the tangent bundle $TS^n$. In the lower row we get

$$\Omega^1\Omega^\infty\Sigma^\infty S^n \to \Omega^\infty\Sigma^\infty(\mathbb{R}P^{n-1}) \to \Omega^\infty\Sigma^\infty(\mathbb{R}P^n)$$

which we may also write in the form

$$\Omega^\infty\Sigma^\infty S^{n-1} \to \Omega^\infty\Sigma^\infty(\mathbb{R}P^{n-1}) \to \Omega^\infty\Sigma^\infty(\mathbb{R}P^n).$$

The left-hand arrow is the attaching map $\alpha\colon S^{n-1} \to \mathbb{R}P^{n-1}$ for the $n$-cell of $\mathbb{R}P^n$ in the standard CW-structure (decorated with $\Omega^\infty\Sigma^\infty$). Therefore we can summarize:

Corollary 2.3.3. *The composition $\tilde f_n \circ \Omega\tau_n$ is in agreement with*

$$\Omega S^n \longrightarrow \Omega(\Omega^\infty\Sigma^\infty S^n) \simeq \Omega^\infty\Sigma^\infty S^{n-1} \xrightarrow{\Omega^\infty\Sigma^\infty\alpha} \Omega^\infty\Sigma^\infty\mathbb{R}P^{n-1}.$$

This was used by Adams in around 1961 when he solved the "vector fields on spheres" problem. The question was as follows, in honest homotopy theory terms. Let

$$\tau_n\colon S^n \longrightarrow B\mathrm{O}(n)$$

be the classifying map for the tangent vector bundle of $S^n$. We ask for a lift up to homotopy

$$(A) \qquad \begin{tikzcd} & B\mathrm{O}(n-k) \arrow[d] \\ S^n \arrow[ur, dotted] \arrow[r, "\tau_n"] & B\mathrm{O}(n) \end{tikzcd}$$

with $k = k_n$ as large as possible. (The existence of such a lift is equivalent to the existence of $k$ everywhere nonzero and everywhere linearly independent smooth vector fields on $S^n$.) The Hurwitz-Radon theorem gives a lower bound for $k_n$ (based on some form of linear algebra). Adams made the following observation, based on the work of James: solving $(A)$ is equivalent to solving

$$(B) \qquad \begin{tikzcd} & \mathrm{O}(n-k) \arrow[d] \\ S^{n-1} \arrow[ur, dotted] \arrow[r, "\Omega\tau_n\circ u"] & \mathrm{O}(n) \end{tikzcd}$$

(where $u: S^{n-1} \to \Omega S^n$ is the usual stabilaztion) and by the above, a solution of $(B)$ implies a solution of

$$(C) \qquad \begin{tikzcd} & \Omega^\infty\Sigma^\infty\mathbb{R}P^{n-k-1} \arrow[d] \\ S^{n-1} \arrow[r, "\alpha"] \arrow[ur, dotted] & \Omega^\infty\Sigma^\infty\mathbb{R}P^{n-1} \end{tikzcd}$$

In fact, if $k$ is not too large, solving $(C)$ is equivalent to solving $(B)$. (There is a comparison map from $(B)$ to $(C)$ and the induced map of vertical homotopy fibers is $(2n-2k-2)$-connected, which is good enough if $n-1 < 2n-2k-2$; equivalently, good enough if $2k < n-1$.) Using $KO$-homology and the newly invented Adams operations, Adams was able to produce upper bounds on $k$ in $(C)$. They happened to agree with the lower bounds on $k$ in $(B)$ or $(A)$ found by Hurwitz-Radon.

Now I must try to explain how the James maps $f_n: \mathrm{O}(n) \to \Omega^\infty\Sigma^\infty\mathbb{R}P^{n-1}$ are defined. Here is the necessary vocabulary.

(1) Retractive space over $X$: a space $E$ with maps $p: E \to X$ and $\zeta: X \to E$ such that $p\zeta = \mathrm{id}_X$.
(2) Fiberwise suspension of a retractive space $(E, p, \zeta)$ over $X$: the retractive space over $X$ obtained from $S^1 \times E$ by introducing relations $(t, y) \sim (*, \zeta p(y))$ if either $t = * \in S^1$ or $y = \zeta p(y) \in E$. Provisional notation: $S^1 \wedge_X E$ (many drawbacks).
(3) Fibered spectrum $\mathbf{E}$ over a space $X$: a sequence of retractive spaces $(E_n, p_n, \zeta_n)$ over $X$ (where $n = 0, 1, 2, \dots$) together with maps
$$\sigma_n: S^1 \wedge_X E_n \longrightarrow E_{n+1}$$
over $X$. In addition: each $p_n$ is a (Serre) fibration and each $\zeta_n$ is a cofibration. We call this a fibered $\Omega$-spectrum over $X$ if for each $x \in X$, the ordinary spectrum $\mathbf{E}/X$ formed by the spaces $E_{n,x} := p_n^{-1}(x)$ is an $\Omega$-spectrum (i.e., the maps $E_{n,x} \to \Omega E_{n+1,x}$ adjoint to the structure maps $S^1 \wedge E_{n,x} \to E_{n+1,x}$ are weak homotopy equivalences).
(4) *Homology groups* of $X$ with coefficients in a fibered spectrum $(E_n, p_n, \zeta_n)_{n\ge 0}$ over $X$: the homotopy groups of the ordinary spectrum $\mathbf{E}/X$. Notation: $H_n(X; \mathbf{E}) := \pi_n(\mathbf{E}/X)$. If $\mathbf{E}$ is trivially fibered over $X$ with constant fiber $\mathbf{F}$, then $H_n(X; \mathbf{E})$ is what we would normally denote $H_n(X; \mathbf{F})$.
(5) *Cohomology groups* of $X$ with coefficients in a fibered spectrum $\mathbf{E} = (E_n, p_n, \zeta_n)_{n\ge 0}$ over $X$: if $X$ is homotopy equivalent to a compact CW-space, these are the homotopy groups of the ordinary spectrum $\Gamma(\mathbf{E})$ of sections, $(\Gamma(p_n: E_n \to X), \dots)_{n\ge 0}$. If $X$ is not homotopy equivalent to a compact CW-space, this definition still makes sense but we should insist on a fibered $\Omega$-spectrum. In that case $\Gamma(\mathbf{E})$ is also an $\Omega$-spectrum. Notation: $H^n(X; \mathbf{E}) := \pi_{-n}\Gamma(\mathbf{E})$. Again, if $\mathbf{E}$ is trivially fibered over $X$ with constant fiber $X$, then $H^n(X; \mathbf{E})$ is what we would normally denote by $H^n(X; \mathbf{F})$.
(6) If $X$ is locally compact and $\mathbf{E}$ is a fibered spectrum over $X$, then we can also define $\Gamma_c(\mathbf{E})$ and cohomology with compact supports $\pi_{-n}(\Gamma_c(\mathbf{E}))$, in terms of sections with compact support.
(7) Let $\mathbf{E} = (E_n, p_n, \zeta_n)_{n\ge 0}$ be a fibered spectrum over $X$ and let $V \to X$ be a vector bundle on $X$ (of fiber dimension $m$). We make a new fibered spectrum $(L_n, q_n, \xi_n)$ over $X$ such that
$$q_n^{-1}(x) := S^{V_x} \wedge p_n^{-1}(x)$$
for $x \in X$ (where $S^{V_x}$ is the one-point compactification of $V_x$ , with $\infty$ as base point). (Some extra conditions should be satisfied by $\mathbf{E}$ if we want to be sure that this is really a fibered spectrum on $X$.) Notation for this: $\mathbf{E}[V]$.

(8) Let $M$ be a smooth closed manifold with tangent bundle $TM$. Let $\mathbf{E}$ be any fibered spectrum over $M$. Then there is a canonical homotopy equivalence

$$\mathbf{E}/M \longrightarrow \Gamma(\mathbf{E}[TM])\,.$$

The name of this game is *Poincaré duality* (by scanning). Passing to homotopy groups, we obtain isomorphisms

$$H_k(M;\mathbf{E}) \cong H^{-k}(M;\mathbf{E}[TM]).$$

More generally, if $M$ is smooth (without boundary) but not necessarily compact, we still have a canonical homotopy equivalence $\mathbf{E}/M \longrightarrow \Gamma_c(\mathbf{E}[TM])$.

How does it work? Assuming $M$ closed for simplicity, choose a Riemannian metric on $M$ and choose $\varepsilon > 0$ small enough so that the exponential map $\exp_x\colon T_xM \to M$ smoothly embeds the disk of radius $\varepsilon$ about $0 \in T_xM$ in $M$. Let $C(x) \subset M$ be the closure of the complement of that little disk. Let $\mathbf{E}^{(x)}$ be the restriction of $\mathbf{E}$ to $C(x)$. For each $x \in M$ we obtain an ordinary spectrum

$$\frac{\mathbf{E}/M}{\mathbf{E}^{(x)}/C(x)},$$

the (homotopy) cofiber of the inclusion of ordinary spectra $\mathbf{E}^{(x)}/C(x) \hookrightarrow \mathbf{E}/M$. These (homotopy) cofibers make up a fibered spectrum $\mathbf{E}^{\mathrm{scan}}$ over $M$ (as we vary $x \in M$) and we have an obvious map

$$\mathbf{E}/M \longrightarrow \Gamma(\mathbf{E}^{\mathrm{scan}})$$

determined by the quotient maps from $\mathbf{E}/M$ to $(\mathbf{E}/M)/(\mathbf{E}^{(x)}/C_x)$ for all $x \in M$. It only remains to show that $\mathbf{E}^{\mathrm{scan}}$ is weakly equivalent, as a fibered spectrum over $M$, to $\mathbf{E}[TM]$. (Exercise.)

(9) Warning: Poincaré duality by scanning as described in the previous item does not agree 100% with the generally accepted standard form of Poincaré duality (at this level of generality), named after Atiyah and Milnor (and Poincaré as usual). But the difference is small and easy to pin down.

Now for the James map. We fix a real vector space $V$ with inner product (positive definite) because we want to see a James map in the form $\mathrm{O}(V) \to \Omega^\infty\Sigma^\infty(\mathbb{R}P(V)_+)$. — Write $M = \mathbb{R}P(V)$. We begin by making a sphere bundle $E \to M$ so that the fiber at/over $H \in M$ (codimension 1 linear subspace of $V$) is $\mathrm{O}(V)/\mathrm{O}(H)$. This has a preferred zero section, so that we can apply $\Sigma^\infty$ fiberwise:

$$\Sigma^\infty_M E$$

is a fibered spectrum over $M$. We obtain a map

$$f_V\colon \mathrm{O}(V) \to \Omega^\infty\Gamma(\Sigma^\infty_M E)$$

by associating to $g \in \mathrm{O}(V)$ the section of $E$ whose value at $H \in M$ is the coset $g \cdot \mathrm{O}(H)$, element of $\mathrm{O}(V)/\mathrm{O}(H)$. But $E \to M$ is the same as the fiberwise one-point compactification of the tangent bundle, $TM \to M$. Therefore

$$\Omega^\infty\Gamma(\Sigma^\infty_M E) \simeq \Omega^\infty\Sigma^\infty M_+$$

by the above description of Poincaré duality. Remembering $M = \mathbb{R}P(V)$, we can write

$$f_V\colon \mathrm{O}(V) \longrightarrow \Omega^\infty\Sigma^\infty(\mathbb{R}P(V)_+).$$

(*End of story.* Perhaps this is not exactly how James constructed this map. Perhaps it is how he should have constructed it.)

Connection with orthogonal calculus: the James maps make up a continuous natural transformation

$$f_V\colon \mathrm{O}(V) \to \Omega^\infty\Sigma^\infty(\mathbb{R}P(V)_+)$$

between functors on $\mathcal{J}$. The target functor is 1-polynomial. See exercise 2.3.5. Surprisingly perhaps, this approximation of the functor $V \mapsto \mathrm{O}(V)$ by a 1-polynomial functor is not best possible, i.e., it is not the first Taylor approximation. Instead we get the first Taylor approximation of that functor by setting up a commutative square

$$\begin{tikzcd}
\mathrm{O}(V) \arrow[r, "f_V"] \arrow[d] & \Omega^\infty\Sigma^\infty(\mathbb{R}P(V)_+) \arrow[d] \\
\mathrm{O}(V\oplus\mathbb{R}^\infty) \arrow[r, "f_{V\oplus\mathbb{R}^\infty}"] & \Omega^\infty\Sigma^\infty(\mathbb{R}P(V\oplus\mathbb{R}^\infty)_+)
\end{tikzcd}$$

and extracting a map from $\mathrm{O}(V)$ to the homotopy pullback of the other three terms:

$$(\dagger) \qquad \mathrm{O}(V) \longrightarrow \operatorname{holim}\left(\begin{tikzcd}
 & \Omega^\infty\Sigma^\infty(\mathbb{R}P(V)_+) \arrow[d] \\
\mathrm{O}(V\oplus\mathbb{R}^\infty) \arrow[r, "f_{V\oplus\mathbb{R}^\infty}"] & \Omega^\infty\Sigma^\infty(\mathbb{R}P(V\oplus\mathbb{R}^\infty)_+)
\end{tikzcd}\right)$$

Exercise 2.3.4. The map $(\dagger)$ is $(\dim(V)-2)$-connected.

Exercise 2.3.5. An attempt to explain in not-very-technical terms why the functor

$$V \mapsto \Omega^\infty\Sigma^\infty(\mathbb{R}P(V)_+)$$

from $\mathcal{J}$ to spaces is 1-polynomial. More precisely this has been arranged so as to look non-technical, but the truth is that many technical points are left to you, gentle reader.

(1) $\mathcal{Q}(n)$ is the topological poset of linear subspaces of $\mathbb{R}^n$. Details as in section 1.2. Let $F$ be a continuous functor from $\mathcal{Q}(n)$ to spaces. Such an $F$ can be called *homotopy cartesian* (in this exercise) if the standard comparison map

$$F(0) \longrightarrow \operatorname*{holim}_{\substack{V\in\mathcal{Q}(n)\\ V\neq 0}} F(V)$$

is a weak equivalence, and *homotopy cocartesian* if the standard comparison map

$$\operatorname*{hocolim}_{\substack{V\in\mathcal{Q}(n)\\ V\neq \mathbb{R}^n}} F(V) \longrightarrow F(\mathbb{R}^n)$$

is a weak equivalence. (The definition of holim and hocolim may require fibrant/cofibrant replacements.) Similar notions for a continuous functor $F$ from $\mathcal{Q}(n)$ to spectra.

(2) Let $F$ be a continuous functor from $\mathcal{Q}(n)$ to spectra. Suppose that it is homotopy cartesian. *Then $\Omega^\infty F$ is also homotopy cartesian.*

(3) Let $F$ be a continuous functor from $\mathcal{Q}(n)$ to spaces. Suppose that it is homotopy cocartesian. *Then $\Sigma^\infty F_+$ is also homotopy cocartesian.*

(4) Suppose that $F$ from $\mathcal{Q}(n)$ to spectra is of finite type, i.e., each value $F(V)$ is weakly equivalent to $\Sigma^{\infty+k}X$ for some $k\in\mathbb{Z}$ and compact based CW-space $X$. Define $F^*$ by $F^*(V) := $ functional dual of $F(V)$ = mapping spectrum $\mathrm{map}(F(V),\mathbf{S})$, where $\mathbf{S}$ is the sphere spectrum. Then $F^*$ is also of finite type. (Strictly speaking $F^*$ is from $\mathcal{Q}(n)^{\mathrm{op}}$ to spaces, but $\mathcal{Q}(n)^{\mathrm{op}}$ is isomorphic to $\mathcal{Q}(n)$ as a topological poset.) *Show that $F$ is homotopy cartesian iff $F^*$ is homotopy cocartesian.*

(5) Take $n=2$. Show that $F$ defined by $F(V)=\Sigma^\infty S(U\oplus V)_+$ for $V\in\mathcal{Q}(2)$ and fixed $U$ in $\mathcal{J}$ is homotopy cocartesian. Here $S(U\oplus V)$ is the unit sphere of $U\oplus V$.
(6) Show that $F$ as in previous item is also homotopy cartesian.
(7) Deduce that $F$ defined by $F(V)=\Sigma^\infty\mathbb{R}P(U\oplus V)_+$ is homotopy cartesian.
(8) Deduce that $V\mapsto\Omega^\infty\Sigma^\infty(\mathbb{R}P(V)_+)$ is 1-polynomial as a functor from $\mathcal{J}$ to spaces.

### 2.4. Early notions of homotopy functor calculus

This section is about first and second order polynomial approximations to the identity functor from spaces to spaces; more precisely, to the identity functor from based path-connected spaces to based path-connected spaces.

For a based path-connected space $X$, let $F_1(X):=QX:=\Omega^\infty\Sigma^\infty X$. The functor $F_1$ on based spaces is 1-polynomial. (This is a consequence of three well-known facts: (i), the functor $\Sigma^\infty$ on based spaces respects homotopy cocartesian squares; (ii), in the category of spectra a commutative square is homotopy cocartesian if and only if it is homotopy cartesian; (iii), the functor $\Omega^\infty$ from spectra to based spaces respects homotopy cartesian squares.) The standard map $X\to F_1(X)$ is the best approximation of id by a 1-polynomial functor. We don't have ready criteria for proving that now, but one strong indication is as follows:

if $X$ is $n$-connected, then $X\to F_1(X)$ is $(2n+1)$-connected.

This is (again) a consequence of the Blakers-Massey theorem. More precisely $\pi_kF_1(X)$ for $k\ge 1$ is the colimit of one of the rows in the commutative diagram

$$(\ddagger)\qquad \begin{tikzcd} \pi_kX \arrow[r] \arrow[d, "="] & \pi_k\Omega\Sigma X \arrow[r] \arrow[d, "\cong"] & \pi_k\Omega^2\Sigma^2X \arrow[r] \arrow[d, "\cong"] & \pi_k\Omega^3\Sigma^3X \arrow[r] \arrow[d, "\cong"] & \pi_k\Omega^4\Sigma^4X \arrow[r] \arrow[d, "\cong"] & \cdots \\ \pi_kX \arrow[r] & \pi_{k+1}\Sigma X \arrow[r] & \pi_{k+2}\Sigma^2X \arrow[r] & \pi_{k+3}\Sigma^3X \arrow[r] & \pi_{k+4}\Sigma^4X \arrow[r] & \cdots \end{tikzcd}$$

and the Blakers-Massey theorem implies that if $X$ is $n$-connected and $k\le 2n+1$, then the first arrow in the lower row is onto and all the other arrows in the lower row are isomorphisms.

This little excursion gives us an indication that the calculus of the identity functor has something to do with "unstable homotopy groups versus stable homotopy groups", and questions on how fast homotopy groups stabilize when we suspend a space (many times), and so on. From this point of view it is appropriate to recall the James construction $JX$ for a path-connected based space $X$.

Roughly $JX$ is the free (associative) monoid generated by $X$, with only one relation specifying that the base point of $X$ becomes the neutral element of the monoid. (This works best if $X$ is a CW-space. It is similar to the Dold-Thom construction. Just don't divide by the action of $\Sigma_k$.) There are canonical comparison maps

$$X\hookrightarrow JX\longrightarrow\Omega\Sigma X\,.$$

James was able to prove that the second of these is a weak homotopy equivalence. (More details for example in my Topo 3 lecture notes.) The space $JX$ has a filtration by subspaces,

$$*=J^{(0)}\subset J^{(1)}(X)\subset J^{(2)}(X)\subset J^{(3)}(X)\subset\cdots$$

such that the quotient $J^{(k)}X/J^{(k-1)}X$ is

$$\underbrace{X\wedge X\wedge\cdots\wedge X}_{k}\,.$$

It is well known and not very hard to show that this filtration turns into a splitting if we suspend $JX$, in other words,

$$\Sigma(JX)\simeq\Sigma\Big(\bigvee_{i=1}^{\infty}X^{\wedge i}\Big).$$

We can use this to set up the EHP sequence, traditionally a (medium to-) long exact sequence relating the homotopy groups of $X$ to those of $\Sigma X$ and those of $\Sigma(X^{\wedge 2})$. For simplicity, assume that $X$ is a based connected CW-space. Then we have the homotopy cocartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ JX \arrow[r] & JX/X \end{tikzcd}$$

Suppose that $X$ is $(n-1)$-connected. Then $X \to *$ is $n$-connected and $X \to JX$ is $(2n-1)$-connected by inspection. By Blakers-Massey (homotopy excision theorem), the induced map of horizontal homotopy fibers is $(n+(2n-1)-1)$-connected. In other words the resulting map

$$(*) \qquad\qquad X \to \text{hofiber}[\, JX \to JX/X\,]$$

is $(3n-2)$-connected. But because of the splitting just above, we also have a canonical map from $\Sigma(JX/X)$ to $\Sigma(X^{\wedge 2})$, and its adjoint from $JX/X$ to $\Omega\Sigma(X^{\wedge 2})$; the latter is $(3n)$-connected. Therefore it is safe to replace $JX/X$ by $\Omega\Sigma(X^{\wedge 2})$ in $(*)$. We get a comparison map

$$X \longrightarrow \text{hofiber}[\, JX \to \Omega\Sigma(X^{\wedge 2})\,]$$

which is still $(3n-2)$-connected. This leads to the (exact) EHP-sequence relating the homotopy groups of $X$ and $\Sigma X$ and $\Sigma(X^{\wedge 2})$ (in the range of dimensions from 0 to $3n-2$ approximately, if $X$ is $(n-1)$-connected).

EXAMPLE 2.4.1. Take $X = S^n$. The EHP-sequence has the form

$$\cdots \to \pi_k(S^n) \xrightarrow{E} \pi_{k+1}(S^{n+1}) \xrightarrow{H} \pi_{k+1}(S^{2n+1}) \xrightarrow{P} \pi_{k-1}(S^n) \to \cdots$$

where $\pi_{k+1}(S^{n+1}) \cong \pi_k(JS^n)$ and $\pi_{k+1}(S^{2n+1}) \cong \pi_k(\Omega\Sigma S^{2n})$. It begins with the term $\pi_{3n-2}(S^n)$ on the extreme left. Rumour has it that $E$ is for Einhängung, H is for Hopf invariant and $P$ is for products as in Whitehead product. The notation (and maybe the concept) probably goes back to George Whitehead.

Analogous to the James model $JX$ for $\Omega\Sigma X$ is the *configuration space* model for $F_1(X) = QX = \Omega^\infty\Sigma^\infty X$. For $m,k>0$ let $W(\mathbb{R}^m;k;X)$ be the following space: a point is a subset $S \subset \mathbb{R}^m$ with exactly $k$ elements (a.k.a. *unordered configuration* in $\mathbb{R}^m$) together with a map $f\colon S \to X$. Fixing $m$ for the moment, we introduce relations in

$$\coprod_{k\ge 1} W(\mathbb{R}^m;k;X)$$

by saying that $(S,f) \in W(\mathbb{R}^m;k;X)$ is $\sim (T,g) \in W(\mathbb{R}^m;\ell;X)$ if $S \subset T$ (as finite subsets of $\mathbb{R}^m$) and $g|_S = f$ and $g(z) = * \in X$ if $z \in T \smallsetminus S$. Write

$$W(\mathbb{R}^m;X)$$

for the quotient space. Finally we note that the standard inclusions $\mathbb{R}^m \to \mathbb{R}^{m+1}$ induce inclusions $W(\mathbb{R}^m;X) \to W(\mathbb{R}^{m+1};X)$. Taking the colimit, we get $W(\mathbb{R}^\infty;X)$. The configuration space experts have established that

$$W(\mathbb{R}^m;X) \simeq \Omega^m\Sigma^m X$$

and consequently $W(\mathbb{R}^\infty;X) \simeq \Omega^\infty\Sigma^\infty X$. (It is not difficult to guess/describe the underlying map from $W(\mathbb{R}^m;X)$ to $\Omega^m\Sigma^m X$. Maybe we will see it in the lectures.) The case $m=1$ is almost exactly the same as the James statement, $JX \simeq \Omega\Sigma X$. But now we are more interested in $W(\mathbb{R}^\infty;X) \simeq \Omega^\infty\Sigma^\infty X = QX$.

The space $W(\mathbb{R}^\infty;X)$ has a filtration by subspaces $W(\mathbb{R}^\infty;k;X)$ where $W(\mathbb{R}^\infty;k;X)$ is, predictably, the colimit of

$$W(\mathbb{R}^1;k;X)\to W(\mathbb{R}^2;k;X)\to W(\mathbb{R}^3;k;X)\to\cdots.$$

It is not hard to identify the homotopy cofiber of the inclusion

$$W(\mathbb{R}^\infty;k-1;X)\longrightarrow W(\mathbb{R}^\infty;k;X)$$

with the reduced homotopy orbit construction

$$(X^{\wedge k})_{h_r\Sigma_k}.$$

(I am writing $h_r$ for the reduced homotopy orbit construction; same as ordinary homotopy orbit construction divided out by $(*)_{h\Sigma_k}$.) This is quite remarkable; we learn that $QX$ has a filtration such that the successive filtration quotients are these reduced homotopy orbit consructions,

$$(X^{\wedge k})_{h_r\Sigma_k}$$

for $k=1,2,3,\dots$ .. Moreover Vic Snaith proved that this filtration splits stably (naturally in $X$):

$$\Sigma^\infty(QX)\simeq\Sigma^\infty\big((\bigvee_{k=1}^{\infty}(X^{\wedge k})_{h_r\Sigma_k}\big).$$

(Note how this is analogous to the splitting of $JX$ after one suspension which we saw above.) This leads us as before to a map from $X$ to

$$F_2(X):=\text{hofiber}[\,QX\longrightarrow Q(X\wedge X)_{h_r\Sigma_2})\,]$$

where the unlabeled arrow[1] is obtained by composing

$$QX\longrightarrow\Omega^\infty\Sigma^\infty(QX)\simeq\Omega^\infty\Sigma^\infty\big(\bigvee_{k=1}^{\infty}(X^{\wedge k})_{h_r\Sigma_k}\big)\longrightarrow\Omega^\infty\Sigma^\infty\big((X^{\wedge 2})_{h_r\Sigma_2}\big).$$

Again if $X$ is $(n-1)$-connected, then this map (from $X$ to $F_2(X)$) is $(3n-2)$-connected. The functor $F_2$ on based spaces is 2-polynomial and the map $X\to F_2(X)$ which we have constructed is the best approximation of id by a 2-polynomial functor. ... Now we also have a forgetful map

$$F_2(X)\to F_1(X)$$

of based spaces whose homotopy fiber is $\Omega Q((X\wedge X)_{h_r\Sigma_2})$. That's an example of a 2-homogeneous functor.

[1]This is known as a James-Hopf map, which suggests that it was known before Snaith established his splitting.

CHAPTER 3

# Newton-Leibniz point of view

## 3.1. Newton-Leibniz in homotopy functor calculus

*Related reading material*:
**T.G. Goodwillie:** *Calculus III: Taylor Series*, Geometry and Topology 7 (2003), 645–711.
**J.W. Milnor:** *On the construction $FK$*, chapter in the book *Algebraic Topology — a student's guide* edited by J.F. Adams, London Math. Soc. Lecture Note Series no. 4, Cambridge University Press 1972.
**G. Arone and M. Kankaanrinta**, *A functorial model for iterated Snaith splitting with applications to calculus of functors*, in: Stable and unstable homotopy (Toronto, ON, 1996), 1—30, Fields Inst. Commun., 19, Amer. Math. Soc.

Easiest case: a homotopy invariant functor $F$ from spaces to based spaces. We want to make lots of spectra measuring rates of change of $F$. For *based* spaces $X_1,\dots,X_k$ let

$$\mathrm{cro}(F;X_1,\dots,X_k) := \mathrm{tohofiber}\left[\, R \mapsto F(\bigvee_{t\notin R} X_t)\,\right]$$

where $R \subset \{1,2,\dots,k\}$. Pronunciation: cross-effect. Beware: this is meant to be a covariant cube and so we are using the collapse maps of wedges (not inclusions of smaller wedges into bigger wedges). And tohofiber means: total homotopy fiber. (For $k>0$, it is the homotopy fiber of the standard map from initial term in the cube to the holim of the other terms. It measures how much the cube fails to be homotopy cartesian.) It would be nice if we could say: there are "structure maps"

$$\mathrm{cro}(F;X_1,\dots,X_k) \longrightarrow \Omega\,\mathrm{cro}(F;X_1,\dots,X_{i-1},S^1\wedge X_i,X_{i+1},\dots X_k).$$

Then we could say: for fixed $k>0$ the spaces

$$\mathrm{cro}(F;S^{n_1},S^{n_2},\dots,S^{n_k})$$

(where $n_1,\dots,n_k$ are arbitrary non-negative integers) form a $k$-fold spectrum (with $k$ looping coordinates/directions). Instead we have something less explicit.

Lemma 3.1.1. *If one of the $X_i$ is $\simeq *$, then* $\mathrm{cro}(F;X_1,\dots,X_k)\simeq *$.

Proof. Use the following principle: for a $k$-cube $\mathcal{Y}$ of based spaces, $\mathrm{tohofiber}(\mathcal{Y})$ is weakly equivalent to $\mathrm{tohofiber}(\mathcal{X})$ where $\mathcal{X}$ is the $(k-1)$-cube defined by

$$\mathcal{X}(S) := \mathrm{hofiber}[\,\mathcal{Y}(S)\to\mathcal{Y}(S\cup k)\,]$$

for $S\subset\{1,\dots,k-1\}$. Without loss, we can assume $i=k$. Then for $\mathcal{Y}(S)=F(\bigvee_{t\notin S}X_t)$ we get $\mathcal{X}(S)\simeq *$ for all $S\subset\{1,2,\dots,k-1\}$. □

There is a commutative square

$$\begin{tikzcd}
\mathrm{cro}(F;X_1,\dots,X_k) \arrow[r] \arrow[d] & \mathrm{cro}(F;X_1,\dots,X_{i-1},S^{1,+}\wedge X_i,X_{i+1},\dots X_k) \arrow[d] \\
\mathrm{cro}(F;X_1,\dots,X_{i-1},S^{1,-}\wedge X_i,X_{i+1},\dots X_k) \arrow[r] & \mathrm{cro}(F;X_1,\dots,X_{i-1},S^{1}\wedge X_i,X_{i+1},\dots X_k)
\end{tikzcd}$$

where $S^{1,+}$ and $S^{1,-}$ are closed half-circles. By the lemma, the off-diagonal terms are weakly contractible. So we obtain a weak equivalence from $\Omega\,\mathrm{cro}(F;X_1,\dots,X_{i-1},S^1\wedge X_i,X_{i+1},\dots X_k)$ to

$$\mathrm{holim}\left(\begin{tikzcd}
 & \mathrm{cro}(F;X_1,\dots,X_{i-1},S^{1,+}\wedge X_i,X_{i+1},\dots X_k) \arrow[d] \\
\mathrm{cro}(F;X_1,\dots,X_{i-1},S^{1,-}\wedge X_i,X_{i+1},\dots X_k) \arrow[r] & \mathrm{cro}(F;X_1,\dots,X_{i-1},S^{1}\wedge X_i,X_{i+1},\dots X_k)
\end{tikzcd}\right)$$

and another map (not claimed to be a weak equivalence) from $\mathrm{cro}(F;X_1,\dots,X_k)$ to the same holim .

$$\begin{tikzcd}
\mathrm{cro}(F;X_1,\dots,X_k) \arrow[d] & \\
\mathrm{holim}\left(\cdots\right) & \Omega\,\mathrm{cro}(F;X_1,\dots,X_{i-1},S^1\wedge X_i,X_{i+1},\dots X_k) \arrow[l, "\simeq"']
\end{tikzcd}$$

Together these are a substitute for the structure map(s) that we could not produce.

Remark 3.1.2. How can we make an honest spectrum out of this? I propose the following.

(1) Make a poset $\mathcal{E}$ with elements $(n,f)$ where $n$ is a non-negative integer and $f$ is a map from $\{1,2,\dots,n\}$ to $\{-1,0,1\}$. We decree that $(m,e)\le(n,f)$ if and only if $m\le n$ and $e$ agrees with the restriction of $f$ to $\{1,\dots,m\}$ as a subset of $\{1,\dots,n\}$. Note that $\mathcal{E}$ has a minimal element $(0,f)$ where $f$ is the unique map from $\varnothing$ to $\{-1,0,1\}$.

(2) We introduce a functor from $\mathcal{E}$ to based spaces as follows: the value of the functor on $(n,f)\in\mathcal{E}$ is denoted $S^{n,f}$, and it is the smash product $Z_1\wedge Z_2\wedge\dots Z_n$ where $Z_i$ is one of $S^{1,-}$, $S^1$, $S^{1,+}$ depending on $f(i)\in\{-1,0,+1\}$.

(3) We define an enhanced variant $\mathrm{cro}^{\sim}(F;X_1,\dots,X_k)$ of $\mathrm{cro}(F;X_1,\dots,X_k)$. This is the space of natural transformations

$$S^{n_1+n_2+\cdots+n_k,f_1\sqcup f_2\sqcup\cdots\sqcup f_k} \Rightarrow \mathrm{cro}(F;S^{n_1,f_1}\wedge X_1,\dots,S^{n_k,f_k}\wedge X_k)$$

between functors from $\mathcal{E}^k$ to spaces.

(4) There is a forgetful map

$$\mathrm{cro}^{\sim}(F;X_1,\dots,X_k)\longrightarrow\mathrm{cro}(F;X_1,\dots,X_k)$$

because a natural transformation (as in the previous item) can be evaluated on the minimal element of $\mathcal{E}^k$. I believe that this forgetful map is a weak equivalence. Exercise.

(5) I believe that there are now canonical structure maps

$$\mathrm{cro}^{\sim}(F;X_1,\dots,X_k)\longrightarrow\Omega\,\mathrm{cro}^{\sim}(F;X_1,\dots,X_{i-1},S^1\wedge X_i,X_{i+1},\dots X_k)$$

for each $i\in\{1,2,\dots,k\}$.

(*End of remark.*)

Specializing to $X_1,\dots,X_k=S^{n_1},\dots,S^{n_k}$ we obtain an honest $k$-fold spectrum (using the remark) with an action of $\Sigma_k$ permuting the looping directions.

DEFINITION 3.1.3. These spectra are the higher derivative spectra of $F$ at $Y = *$.
Here is a mild generalization. Let $Y$ be a space and select points $y_1, \dots, y_k \in Y$. We can make a spectrum (with $k$ looping coordinates) out of the spaces

$$\begin{array}{rl} & \mathrm{cro}_Y(F; Y \vee_{x_1} S^{n_1}, Y \vee_{x_2} S^{n_2}, \dots, Y \vee_{x_k} S^{n_k}) \\ := & \mathrm{tohofiber}\Big[ R \mapsto F\big(\mathrm{colim}\,(\coprod_{t\notin R} Y \vee_{y_t} S^{n_t} \longleftarrow \coprod_{t\notin R} Y \xrightarrow{\text{fold}} Y)\big)\Big] \end{array}$$

(plus rectification as in remark 3.1.2). Goodwillie's notation for these spectra appears to be

$$\partial^{(k)}_{y_1,\dots,y_k} F(Y).$$

If $Y = *$, we have only one choice for $y_1, \dots, y_k$ and we can write $\partial^k F(*)$. In the general case, there is still something like a $\Sigma_k$-action but it is slightly complicated. A permutation $\sigma \in \Sigma_k$ induces an isomorphism of $k$-fold spectra

$$\partial^{(k)}_{y_1,\dots,y_k} F(Y) \longrightarrow \partial^{(k)}_{y_{\sigma(1)},\dots,y_{\sigma(k)}} F(Y)$$

but in addition this permutes the $k$ looping directions.
One more generalization: if $F$ is a functor from unbased spaces to *unbased* spaces, we can still make sense of

$$\mathrm{cro}_Y(F; Y \vee_{x_1} S^{n_1}, Y \vee_{x_2} S^{n_2}, \dots, Y \vee_{x_k} S^{n_k})$$

and consequently of $\partial^{(k)}_{y_1,\dots,y_k} F(Y)$, but we must choose a base point in $F(Y)$.

Now the usual questions arise: firstly, what is this good for, and secondly, can we see some computations. In section 3.2 we may get an idea of what it is good for. Therefore let's think about computations now. My plan is that we take $F = \mathrm{id}$. This is a functor from unbased spaces to unbased spaces, but we have a preferred choice of base point in $F(*)$ and so we can make sense of the $k$-th derivative spectrum $\partial^k F(*)$. To "compute" it, we should try first to understand

$$\mathrm{cro}(F; S^{n_1}, S^{n_2}, \dots, S^{n_k}) = \mathrm{tohofiber}\Big[ R \mapsto \bigvee_{t\notin R} S^{n_t} \Big].$$

At this point I can only quote from Goodwillie's calculus III paper. We can view

$$\Omega\ \mathrm{cro}(F; S^{n_1}, S^{n_2}, \dots, S^{n_k})$$

as the total homotopy fiber of the $k$-cube $\mathcal{Y}$, where

$$\mathcal{Y}(R) := \Omega\Sigma\big( \bigvee_{t\notin R} S^{n_t - 1} \big)$$

for $R \subset \{1, 2, \dots, k\}$. By the Hilton-Milnor theorem (see remark 3.1.4), each term $\mathcal{Y}(R)$ has a description as a (restricted) product,

$$\mathcal{Y}(R) \xleftarrow{\ \simeq\ } \mathcal{X}(R) := \prod_{u\geq 1}{}' \Omega\Sigma\ \big( X_1^{\wedge m_{1,u}} \wedge X_2^{\wedge m_{2,u}} \wedge \cdots \wedge X_k^{\wedge m_{k,u}} \big).$$

The numbers $m_{t,u}$ are described in remark 3.1.4, and they depend on $k$ (plus certain choices), but the meaning of $X_t$ depends on $R$ (though the notation does not indicate it). If $t \notin R$ it is $S^{n_t - 1}$, but if $t \in R$ it is a point. (Without loss of generality, $n_t > 0$ for all $t$.) This means that the factor corresponding to an index $u$ is a point if some $m_{t,u}$ with $t \in R$ is $> 0$.
Although the Hilton-Milnor formula for $\Omega\Sigma(X_1 \vee \cdots \vee X_k)$ (where the $X_j$ could be arbitrary based connected CW-spaces, though we were interested mainly in spheres and single points) is very dependent on certain choices, it is natural in the variables $X_1, \dots, X_k$ once the choices have been made. Consequently the maps in the cube $\mathcal{X}$ are seen to be projection maps. In other words if we have $S \subset R$ then the induced map $\mathcal{X}(S) \to \mathcal{X}(R)$ is the map which drops the coordinates corresponding

to $u$ where $m_{t,u} > 0$ for some $t$ in $R \smallsetminus S$. It follows instantly that the total homotopy fiber of the cube $\mathcal{X} \simeq \mathcal{Y}$ is

$$\prod_{\substack{u\geq 1\\ \forall t:\ m_{t,u}>0}}{}' \, \Omega\Sigma\left(S^{(n_1-1)m_{1,u}} \wedge S^{(n_2-1)m_{2,u}} \wedge \cdots \wedge S^{(n_k-1)m_{k,u}}\right).$$

which simplifies to

$$\prod_{\substack{u\geq 1\\ \forall t:\ m_{t,u}>0}}{}' \, \Omega\, S^{(n_1-1)m_{1,u}+(n_2-1)m_{2,u}+\cdots+(n_k-1)m_{k,u}+1}\,.$$

The naturality properties of the Hilton-Milnor formula imply that the $k$-fold spectrum made out of the spaces

$$\Omega \ \mathrm{cro}(F; S^{n_1}, S^{n_2}, \ldots, S^{n_k})$$

is a (restricted) product of spectra, with one factor for each $u$ (in the above product formula) such that $m_{t,u} > 0$ for all $t \in \{1, 2, \ldots, k\}$. That factor is a $k$-fold spectrum whose term in degree $(n_1, n_2, \ldots, n_k)$ is precisely

$$\Omega\, S^{(n_1-1)m_{1,u}+(n_2-1)m_{2,u}+\cdots+(n_k-1)m_{k,u}+1}.$$

The $r$-th homotopy group of that factor is the colimit of the groups

$$\begin{array}{rl} & \pi_{r+n_1+n_2+\cdots+n_k}\Omega\, S^{(n_1-1)m_{1,u}+(n_2-1)m_{2,u}+\cdots+(n_k-1)m_{k,u}+1} \\ \cong & \pi_{r+n_1+n_2+\cdots+n_k-1}\, S^{(n_1-1)m_{1,u}+(n_2-1)m_{2,u}+\cdots+(n_k-1)m_{k,u}+1} \end{array}$$

(as $n_1, n_2, \ldots, n_k$ tend to $\infty$). This is easily seen to be zero if at least one of the $m_{t,u}$ is $> 1$. Therefore only the factors corresponding to $u$ such that $m_{t,u} = 1$ for all $t$ are of interest. These factors are clearly of the form

$$\Omega\, S^{n_1+n_2+\cdots+n_k-k+1}$$

in degree $(n_1, n_2, \ldots, n_k)$. It is easy to guess what the structure maps are and the naturality properties of Hilton-Milnor confirm that they are the standard maps

$$\begin{tikzcd} S^1 \wedge \Omega\, S^{n_1+n_2+\cdots+n_k-k+1} \arrow[d] & \\ \Omega(S^1 \wedge S^{n_1+n_2+\cdots+n_k-k+1}) \arrow[r] & \Omega\, S^{n_1+n_2+\cdots+(n_j+1)+n_{j+1}+\cdots+n_k-k+1} \end{tikzcd}$$

so that this factor is weakly equivalent to $\Omega^k\mathbf{S}$ where $\mathbf{S}$ denotes a ($k$-fold) sphere spectrum. With more insight into the combinatorics of the Hilton-Milnor theorem one can show that the number of $u$ such that $m_{t,u} = 1$ for $t = 1, 2, \ldots, k$ is $(k-1)!$. (Theorem 4 in Milnor's paper has a counting formula which takes care of this.) Therefore we have "shown" that $\Omega\partial^{(k)}F(*)$ is a wedge sum of $(k-1)!$ copies of $\Omega^k\mathbf{S}$. Equivalently, we may write

$$(\dagger) \qquad\qquad \partial^{(k)}F(*) \simeq \bigvee_{j=1}^{(k-1)!} \Omega^{k-1}\mathbf{S}\,.$$

This does not reveal the action of $\Sigma_k$. It is a difficult action (for $k \geq 2$). Goodwillie writes the following about the induced action of $\Sigma_k$ on the integral homology group

$$H_{1-k}(\partial^{(k)}F(*))$$

which by (†) is a free abelian group of rank $(k-1)!$. Let $L(k)$ be the free Lie algebra[1] (over $\mathbb{Z}$) on generators $x_1, \dots, x_k$. This has an action of $\Sigma_k$ given by permutation of generators. Let $M_k \subset L(k)$ be the (abelian) subgroup which is additively generated by the iterated brackets

$$[[[[\cdots[x_{\sigma(k)}, x_{\sigma(k-1)}], x_{\sigma(k-2)}], x_{\sigma(k-3)}], \dots], x_{\sigma(1)}]$$

where $\sigma \in \Sigma_k$. This is clearly invariant under the $\Sigma_k$-action on $L(k)$. But it turns out that $M_k$ is freely generated (as an abelian group) by the iterated brackets

$$[[[[\cdots[x_{\sigma(k)}, x_{\sigma(k-1)}], x_{\sigma(k-2)}], x_{\sigma(k-3)}], \dots], x_{\sigma(1)}]$$

where $\sigma(k) = k$, in other words $\sigma \in \Sigma_{k-1}$. (This is a theorem due to Marshall Hall; see the references in Milnor's FK article.) Therefore $M_k$ is free abelian of rank $(k-1)!$, and it is in fact isomorphic to $H_{1-k}(\partial^{(k)} F(*))$ as a $\Sigma_k$-module. (Notice that this is just enough to determine the action of $\Sigma_k$ on the spectrum $\partial^{(k)} F(*)$, as an object of the *homotopy category* of $k$-fold spectra. But notice also that this is not good enough.)
More information on $\partial^{(k)} F(*)$ with the action of $\Sigma_k$ can be found in the paper by Arone and Kankaanrinta cited at the beginning of this section.

REMARK 3.1.4. Hilton-Milnor theorem: an attempt to give a summary. Let $A$ and $B$ be based simplicial sets. Let $Z$ be the free group construction, a functor from *based* simplicial sets to simplicial groups. (Milnor writes $F$ instead of $Z$.) In the paper mentioned at the start of this section, Milnor shows or observes (depending on circumstances) the following.

(1) $|ZA| \simeq \Omega\Sigma|A|$ where $\Sigma = S^1 \wedge$. This is very similar and very closely related to James' theorem, $JX \simeq \Omega\Sigma X$ for a based CW-space $X$. There are advantages and disadvantages. Showing that $|ZA| \simeq \Omega\Sigma|A|$ is harder than showing $JX \simeq \Omega\Sigma X$. But $ZA$ has inverses (which $JX$ has not) and as a result we can speak of commutators in $ZA$. These are very important in the Hilton-Milnor theorem.
(2) $Z(A \vee B)$ can also be described as the coproduct of $ZA$ and $ZB$ in the category of simplicial groups. (Milnor writes $*$ for this coproduct.)
(3) Let $L$ be the kernel of the homomorphism $Z(A \vee B) \to Z(A)$ induced by $A \vee B \to A$. The short exact sequence $L \to Z(A \vee B) \to Z(A)$ has an (obvious) splitting, so that $Z(A \vee B)$ is the semi-direct product of $L$ and $Z(A)$ (as a simplicial group); and we may write
$$Z(A \vee B) \cong L \times Z(A)$$
in the category of based simplicial sets (not in the category of simplicial groups).
(4) There are simplicial maps $B \wedge A^{\wedge i} \longrightarrow L$ for $i = 0, 1, 2, \dots$ obtained by forming iterated commutators (Lie bracket notation) such as
$$b, [b, a_1], [[b, a_1], a_2], \dots, [[[\cdots[[b, a_1], a_2], a_3], \dots, a_i], \dots$$
where $b \in B_k$ and $a_1, a_2, \dots, a_i \in A_k$ for some $k$ (the same $k$).[2] These induce a homomorphism of simplicial abelian groups from the coproduct
$$*_{i \ge 0} Z(B \wedge A^{\wedge i}) \cong Z\Big(\bigvee_{i \ge 0} (B \wedge A^{\wedge i})\Big)$$
to $L$. *Theorem*: this homomorphism is a homotopy equivalence *if $A$ is connected*. In that situation we may write, using (3) as well:
$$Z(A \vee B) \simeq ZA \times Z\Big(\bigvee_{i \ge 0} B \wedge A^{\wedge i}\Big).$$

[1] The condition $[y, y] = 0$ for all $y$ is in force.
[2] $A^{\wedge 0}$ means $S^0$, therefore $A^{\wedge 0} \wedge B \cong B$.

Notice how this makes us just a little wiser, going from left to right. It may look as if we have traded a wedge of two things for a wedge of infinitely many things. That would be disappointing, but we have isolated one factor $ZA$ (in the right-hand side) which we need not touch again. And if $A$ and $B$ are both connected, then the new wedge summands $B \wedge A^{\wedge i}$ for $i \geq 1$ which we have introduced are more highly connected than $A$ and $B$. To repeat the trick, we can write $C = \bigvee_{i\geq 1}(B \wedge A^{\wedge i})$ and then

$$Z\Big(\bigvee_{i\geq 0} B \wedge A^{\wedge i}\Big) = Z(B \vee C) \simeq ZB \times Z\Big(\bigvee_{j\geq 0} C \wedge B^{\wedge j}\Big) \cong ZB \times Z\Big(\bigvee_{i\geq 1, j\geq 0} B \wedge A^{\wedge i} \wedge B^{\wedge j}\Big)$$

using a certain distributive law.

(5) This leads to the following (semi-)algorithm. Given a set $S_u$ with more than one element and a map $f_u \colon S_u \to \{1, 2, 3, \dots\}$ which has finite point preimages. The symbol $u$ is an integer and a step counter.
Choose $b_u \in S_u$ such that $f_u(b_u)$ is as small as possible. (Make a note of $b_u$.) Make a new set $S_{u+1}$ with elements $(s, j)$ where $s \in S_u \smallsetminus \{b_u\}$ and $j \in \{0, 1, 2, \dots\}$. Make a new map $f_{u+1} \colon S_{u+1} \to \{1, 2, \dots\}$ by $f_{u+1}((s, j)) = f_u(s) + j f_u(b_u)$.

Suppose we begin with a finite set $S_1 = \{1, 2, \dots, k\}$ and $f_1 \equiv 1$. Form $(S_2, f_2)$, $(S_3, f_3)$, $(S_4, f_4)$ etc. following the recipe given. We can associate an iterated Lie-bracket expression or Lie word $w(s)$ in the letters $x_1, \dots, x_k$ to each $s \in S_u$ for all $u \geq 1$. The total number of letters $x_j$ in $w(s)$, counted with multiplicities, will be $f_u(s)$. Namely, to $i \in S_1$ we simply associate the word $x_i$ in one letter, no brackets. Suppose that the words $w(s)$ for $s \in S_u$ have already been determined. Let $p \in S_{u+1}$; by construction we have $p = (s, j)$ for some $s \in S_u \smallsetminus \{b_u\}$ and $j \geq 0$, and then we say that $w(p)$ is the expression

$$[[[[\dots[w(s), \underbrace{w(b_u)], w(b_u)], w(b_u)], \dots], w(b_u)}_{j}].$$

In particular we obtain the so-called *basic words* $w(b_1), w(b_2), w(b_3), \dots$ .

(6) Suppose now that $A(1), \dots, A(k)$ are based and *connected* simplicial sets, where $k \geq 2$. For each basic word $w(b_u)$ as in item (5) we obtain a map from a certain smash product

$$A(1)^{\wedge m_{1,u}} \wedge A(2)^{\wedge m_{2,u}} \wedge \cdots \wedge A(k)^{\wedge m_{k,u}}$$

to $Z(A(1) \vee \cdots \vee A(k))$ by interpreting the Lie bracket as a commutator and by interpreting each instance of the letter $x_j$ in $w(b_u)$ as an invitation to substitute an element from $A(j)$ (in some degree). Of course, $m_{j,u}$ is the number of times that the letter $x_j$ appears in the word $w(b_u)$.

(7) Using the ordinary multiplication in $Z(A(1) \vee \cdots \vee A(u))$ we can extend these maps to a map from the restricted product (colimit of the finite products)

$$\prod_{u\geq 1}{}' Z\Big(A(1)^{\wedge m_{1,u}} \wedge A(2)^{\wedge m_{2,u}} \wedge \cdots \wedge A(k)^{\wedge m_{k,u}}\Big)$$

to $Z(A(1) \vee \cdots \vee A(k))$. The theorem of Hilton-Milnor states that this is a weak equivalence. I hope we can agree that the essence of this is item (4), but some observations related to high connectedness must be added. Namely, $Z(A(1)^{\wedge m_{1,u}} \wedge A(2)^{\wedge m_{2,u}} \wedge \cdots \wedge A(k)^{\wedge m_{k,u}})$ is $((\sum_j m_{j,u}) - 1)$-connected.

## 3.2. The tower in homotopy functor calculus

To begin with let $F$ be a homotopy invariant functor from spaces to spaces. (There is no need for base points here.) The goal is to construct, for every $k \geq 0$, a homotopy invariant functor $P_k F$

which is $k$-polynomial and a natural transformation

$$F \Rightarrow P_k F$$

which can claim to be universal among natural transformations from $F$ to $k$-polynomial functors. The description of $P_k F$ is easy, but it takes a little more work to show that $P_k F$ is in fact $k$-polynomial. We start with an easier construction which Goodwillie denotes $T_k F$.

Remember that the *join* $X * U$ of two spaces $X$ and $U$ is the homotopy pushout of $X \leftarrow X \times U \rightarrow U$ (where the two arrows are the projections). Here we are mainly interested in the cases where $U$ is a finite set with the discrete topology, while $X$ is arbitrary. In such a case $X * U$ can be imagined and should be imagined as a union of cones on $X$, one for every $u \in U$, along their common boundary. (In particular if $U = \{-1, +1\}$, then $X * U$ is the unreduced suspension of $X$.)

Define $T_k F(X)$ to be the holim of the *punctured* $(k+1)$-cube of spaces $U \mapsto F(X * U)$, where $U$ runs through the *nonempty* subsets of $\{1, \ldots, k+1\}$. There is a canonical map

$$t_k = t_{k,F} \colon F(X) \longrightarrow T_k F(X).$$

Namely, the inclusions $X \rightarrow X * U$ induce (compatible) maps from $F(X)$ to $F(X * U)$, so a map from $F(X)$ to the limit of the spaces $F(X * U)$ for $\varnothing \neq U \subset \{1, 2, \ldots, k\}$, and we can pass from there to the holim. A weak motivation for the construction $T_k F$ comes from the following easy lemma.

LEMMA 3.2.1. *If $F$ is $k$-polynomial, then $t_k \colon F(X) \rightarrow T_k F(X)$ is a weak equivalence for every $X$.*

PROOF. The $(k+1)$-cube $U \mapsto X * U$, now with arbitrary $U \subset \{1, 2, \ldots, k+1\}$, is strongly homotopy cocartesian. Therefore the $(k+1)$-cube $U \mapsto F(X * U)$ is homotopy cartesian by our assumption on $F$. But this means exactly that $t_k \colon F(X) \cong F(X * \varnothing) \rightarrow T_k F(X)$ is a weak equivalence. ☐

DEFINITION 3.2.2. $P_k F(X)$ is the homotopy colimit (telescope) of the diagram

$$F(X) \xrightarrow{t_k} T_k F(X) \xrightarrow{t_{k,T_k F}} T_k(T_k F)(X) \xrightarrow{t_{k,T_k(T_k F)}} \cdots$$

Consequently there is a canonical inclusion $p_k = p_{k,F} \colon F(X) \rightarrow P_k F(X)$.

COROLLARY 3.2.3. *If $F$ is $k$-polynomial, then $p_k \colon F(X) \rightarrow P_k F(X)$ is a weak equivalence for every space $X$.*

Goodwillie shows that $P_k F$ is always $k$-polynomial. He writes that there are in some sense two proofs: an older proof where $F$ needs to satisfy a few conditions (which tend to be satisfied in most cases of interest) and a newer proof which is unconditional. (The newer proof was simplified by C Rezk.)

First let's talk about conditions on $F$ which make it easy to show that $P_k F$ is $k$-polynomial. Goodwillie has a condition which he calls $E_k(c, \alpha)$, where $c$ and $\alpha$ are integers.

DEFINITION 3.2.4. $F$ satisfies $E_k(c, \alpha)$ if the following holds. If $\mathcal{X}$ is a strongly homotopy cocartesian $(k+1)$-cube, indexed by the subsets of $\{1, 2, \ldots, k+1\}$, and if for all $s \in \{1, 2, \ldots, k\}$ the map $\mathcal{X}(\varnothing) \rightarrow \mathcal{X}(\{s\})$ in the cube is $a_s$-connected where $a_s \geq \alpha$, then the cube $F(\mathcal{X})$ is homotopy $(-c + \sum a_s)$-cartesian (i.e, the standard comparison map from $F(\mathcal{X}(\varnothing))$ to

$$\operatorname*{holim}_{\substack{U \subset \{1,2,\ldots,k+1\} \\ U \neq \varnothing}} F(\mathcal{X}(U))$$

is $(-c + \sum a_s)$-connected).

This condition becomes stronger if $c$ and/or $\alpha$ decrease. Goodwilie shows:

LEMMA 3.2.5. *If $F$ satisfies $E_k(c,\alpha)$, then $T_kF$ satisfies $E_k(c-1,\alpha-1)$.*

(The proof is not very difficult. The idea is to look at the functors $X \mapsto F(X*U)$ for fixed nonempty $U \subset \{1,2,\dots,k+1\}$ separately. They satisfy $E_n(c-1-(k+1),\alpha-1)$.)

COROLLARY 3.2.6. *If $F$ satisfies $E_k(c,\alpha)$ for some $c$ and $\alpha$, then $P_kF$ satisfies $E_k(-\infty,-\infty)$. In other words, $P_kF$ is $k$-polynomial.*

For the unconditional proof that $P_kF$ is $k$-polynomial, Goodwillie uses the following observation:

LEMMA 3.2.7. *Let $\mathcal{X}$ be any strongly homotopy cocartesian $(k+1)$-cube of spaces. Then the map of cubes*

$$F(\mathcal{X}) \xrightarrow{(t_kF)(\mathcal{X})} (T_kF)(\mathcal{X})$$

*factors through some homotopy cartesian cube.*

PROOF. (Following Rezk) For $U \subset \{1,\dots,k+1\}$ define a $(k+1)$-cube $\mathcal{X}_U$ of spaces by

$$\mathcal{X}_U(T) := \operatorname{hocolim}\Big(\mathcal{X}(T) \leftarrow \coprod_{s\in U}\mathcal{X}(T) \rightarrow \coprod_{s\in U}\mathcal{X}(T\cup\{s\})\Big)$$

where $T \subset \{1,2,\dots,k+1\}$, and $\coprod$ is the coproduct in the category of spaces. We have $\mathcal{X}_\varnothing(T) \cong \mathcal{X}(T)$, and there is an evident map $\beta\colon \mathcal{X}_U(T) \to \mathcal{X}(T)*U$, which is natural in both $T$ and $U$. The map $(t_kF)(\mathcal{X})$ of $k$-cubes factors as follows:

$$F(\mathcal{X}(T)) \to \operatorname*{holim}_U\, F(\mathcal{X}_U(T)) \xrightarrow{\beta_*} \operatorname*{holim}_U\, F(\mathcal{X}(T)*U) \cong (T_kF)(\mathcal{X}(T))$$

where $U$ runs through the nonempty subsets of $\{1,2,\dots,k+1\}$. Now suppose that $\mathcal{X}$ is strongly homotopy cocartesian. Then there are natural weak equivalences $\mathcal{X}_U(T) \to \mathcal{X}(T\cup U)$ induced by the inclusions $T\cup\{s\} \hookrightarrow T\cup U$ for $s \in U$, and $T \hookrightarrow T\cup U$. The maps $\mathcal{X}(T\cup U) \to \mathcal{X}(T\cup\{s\}\cup U)$ are isomorphisms for $s\in U$ (especially when $s\notin T$), and thus if $U \neq \varnothing$ the cube $T \mapsto F(\mathcal{X}_U(T))$ is homotopy cartesian. Therefore

$$T \mapsto \operatorname*{holim}_U\, F(\mathcal{X}_U(T))$$

is a homotopy limit, over nonempty $U \subset \{1,2,\dots,k+1\}$, of homotopy cartesian cubes, and thus is homotopy cartesian. □

Another thing we need to discuss: is the canonical map $F(X) \to P_kF(X)$ a good approximation? We would like it to be highly connected if $k$ is large. There is no point in trying to prove something of that kind without imposing conditions on $F$ (and sometimes on $X$ as well).

PROPOSITION 3.2.8. *If $F$ satisfies $E_k(c,\alpha)$, then for every $X$ which is $(a-1)$-connected where $a \geq \alpha$, the map $F(X) \to P_kF(X)$ is $(-c+(k+1)a)$-connected.*

The proof is probably mechanical, therefore omitted. The conditions $E_k(c,\alpha)$ are introduced and investigated in Goodwillie's calculus I and II papers.

EXAMPLE 3.2.9. The functor $F = \mathrm{id}$ satisfies condition $E_k(k,-1)$ for all $k \geq 1$. (Big theorem in Goodwillie's calulus II.) Therefore, if $X$ is $(m-1)$-connected, the map $F(X) \to P_kF(X)$ is $(-k+(k+1)m)$-connected. Notice that this is not very useful if $m < 2$. But if $m = 2$, in other words if we know that $X$ is 1-connected, then we can already say that $F(X) \to P_kF(X)$ is $(k+2)$-connected. If $X$ is 5-connected, then $F(X) \to P_kF(X)$ is $(5k+6)$-connected.

DEFINITION 3.2.10. $F$ is $\rho$-analytic if there is an integer $q$ such that $F$ satisfies $E_k(k\rho-q,\rho+1)$ for all $k \geq 1$.

For example the identity functor is 1-analytic. (Take $q = 0$. It satisfies $E_k(k, -1)$ which is slightly better than $E_k(k, 2)$.)
Goodwillie writes (somewhere) that most reasonable homotopy functors (in homotopy functor calculus) are $\rho$-analytic for some $\rho$. This is probably true. The concept of *analytic functor* can be exported to other forms of functor calculus (manifold calculus, orthogonal calculus) but it turns out to be hard or impossible to verify for some important functors.

And now the tower: there are nearly-obvious projection maps $P_kF(X) \to P_{k-1}F(X)$ because there are nearly-obvious projection maps $T_kF(X) \to T_{k-1}F(X)$. The latter are projection maps from the homotopy limit of a bigger diagram to the homotopy limit of a smaller diagram. (We are using $\{1, 2, \ldots, k-1\} \hookrightarrow \{1, 2, \ldots, k\}$.) Therefore we obtain a commutative diagram

$$\begin{tikzcd}
& \vdots \arrow[d] \\
& P_3F(X) \arrow[d] \\
& P_2F(X) \arrow[d] \\
& P_1F(X) \arrow[d] \\
F(X) \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & P_0F(X)
\end{tikzcd}$$

*Variant of the above.* Suppose that $F$ is a homotopy invariant functor from the category of *spaces over* $Y$ to the category of spaces. In that case, the join construction $X \mapsto X * U$ (for a finite set $U$) should be replaced by the homotopy pushout of

$$X \leftarrow X \times U \to Y \times U\,.$$

This leads to a definition of $P_kF$ such that $P_kF$ is again a functor from spaces over $Y$ to spaces. Condition $E_k(c, \alpha)$ remains meaningful as it stands. Proposition 3.2.8 remains correct but the condition on $X$ needs restating: we want the reference map $X \to Y$ to be $a$-connected. (If $Y$ is a point, then the reference map $X \to *$ is $a$-connected if and only if $X$ is $(a-1)$-connected.)

### 3.3. Homogeneous layers in homotopy functor calculus

Here we assume, to begin with, that $F$ is a homotopy invariant functor from the category of based spaces to the category of based spaces.[3] (If $F$ was "originally" a functor from unbased spaces to unbased spaces, then we may restrict it to the category of based spaces and we may choose a point $z \in F(*)$. This makes a functor from based spaces to based spaces. It can happen that we cannot choose a point in $F(*)$ because $F(*)$ is empty, but then $F(X)$ must be empty for all $X$ and the case is not very interesting.)
The definition of $P_kF$ goes through and $P_kF$ is again a functor from based spaces to based spaces. What can be said about the homotopy fiber $D_kF(X)$ of $P_kF(X) \to P_{k-1}F(X)$ (over the base point) ?

[3] And I suspect we want based spaces with a nondegenerate base point, to ensure that *wedge sum* is a weakly homotopy invariant operation. We are going to make heavy use of wedge sums.

Definition 3.3.1. A homotopy functor $E$ (e.g. from based spaces to based spaces) is *$k$-homogeneous* if it is $k$-polynomial and $P_{k-1}E$ is homotopy terminal, i.e., all values $P_{k-1}E(X)$ are weakly contractible.

Proposition 3.3.2. *$D_kF$ is $k$-homogeneous.*

Proof. (Sketch.) The first thing one should verify is that $P_{k-1}$ preserves homotopy fiber sequences (of functors from based spaces to based spaces).[4] This is not too hard, because it boils down to showing that $T_{k-1}$ preserves homotopy fiber sequences. More generally, $T_{k-1}$ and $P_{k-1}$ both preserve homotopy cartesian $r$-cubes (in the category of homotopy functors from based spaces to based spaces) for any $r \geq 1$.
Similarly, $T_{k-1}$ and therefore $P_{k-1}$ respect *sequential homotopy colimits* (homotopy colimits taken over the poset of natural numbers) up to weak equivalence.
Now we verify: $(*)$ the map $P_{k-1}F \to P_{k-1}(P_jF)$ induced by $p_j\colon F \to P_jF$ is a weak equivalence if $j \geq k-1$. For that it suffices to show that the map $P_{k-1}F \to P_{k-1}(T_jF)$ induced by $t_j\colon F \to T_jF$ is a weak equivalence. For a fixed $X$, based space, this map has the form

$$P_{k-1}\big[F(X) \to \operatorname*{holim}_U \; F(X * U)\big]$$

where $U$ runs over the nonempty subsets of $\{1, 2, \ldots, j\}$. Since $P_{k-1}$ respects homotopy cartesian $j$-cubes (and here we view $X \mapsto F(X * U)$ as a functor of the variable $X$, for fixed $U$), this last map can also be written in the form

$$(P_{k-1}F)(X) \longrightarrow \operatorname*{holim}_U \; (P_{k-1}F)(X * U).$$

(Maybe we should make a distinction between $(P_{k-1}F)(X * U)$ and $(P_{k-1}F_U)(X)$ where $F_U$ is the functor taking $X$ to $F(X * U)$, but it is not very necessary or helpful.) Therefore it suffices to show that $P_{k-1}F$ is $j$-polynomial. But this is easy (since $j \geq k-1$), so $(*)$ is established.
Combining statement $(*)$ for $j = k-1$ and $j = k$ we deduce (by two-out-of-three for weak equivalences) that the map $P_{k-1}(P_kF) \to P_{k-1}(P_{k-1}F)$ induced by the forgetful $P_kF \to P_{k-1}F$ is a weak equivalence. Therefore $P_{k-1}D_k$ is weakly contractible (since $P_{k-1}$ preserves homotopy fiber sequences). □

Remark 3.3.3. There is a very complete and satisfying classification of $k$-homogeneous functors as in definition 3.3.1. This is based on the following correspondence: $k$-homogeneous functors correspond to symmetric $k$-linear functors (in $k$ variables, which are all based spaces). The classification of symmetric $k$-linear functors is approachable because it reduces to the case $k = 1$ (which is the case of 1-homogeneous functors).
This correspondence has a pretty analogue in ordinary calculus, or perhaps more correctly, in multilinear algebra. Suppose that $f$ is a $k$-homogeneous polynomial function from $\mathbb{R}^m$ to $\mathbb{R}^n$. With such an $f$ we can associate a symmetric $k$-linear function

$$\mathrm{cro}_k f\colon (\mathbb{R}^m)^k \longrightarrow \mathbb{R}^n$$

as follows: $(\mathrm{cro}_k f)(v(1), v(2), \ldots, v(k))$ is the alternating sum

$$\sum_{S \subset \{1,2,\ldots,k\}} (-1)^{|S|} f\big(\sum_{t \notin S} v(t)\big)\,.$$

[4] I always preach that *homotopy fiber sequence* is unclean language; the correct expression is *homotopy pullback square in which one off-diagonal term is weakly contractible.*

We can recover $f$ from $\mathrm{cro}_k f$ using the formula

$$f(v) = \frac{(\mathrm{cro}_k f)(\overbrace{v, v, v, \dots, v}^{k})}{k!}\,.$$

These elementary algebraic operations have counterparts in homotopy theory. In particular the outer (alternating) sum in the definition of $\mathrm{cro}_k f(v(1), \dots, v(k))$ corresponds to the total homotopy fiber $\mathrm{cro}_k F(X_1, \dots, X_k)$ (in section 3.1 this was denoted $\mathrm{cro}(F; X_1, \dots, X_k)$) of a certain $k$-cube. The $k$-cube is obtained by evaluating a $k$-homogeneous functor $F$ (corresponding to the function $f$) on wedge sums

$$\bigvee_{t \in \{1,2,\dots,k\} \smallsetminus S} X_t$$

(so that these wedge sums correspond to the inner sums in the definition of $\mathrm{cro}_k f$.) One operation which is not easy to translate from algebra to homotopy theory is division by $k!$. On the homotopy theory side this corresponds vaguely to a homotopy orbit construction, for the action of the symmetric group $\Sigma_k$ on $(\mathrm{cro}_k F)(X, \dots, X)$. Unfortunately this is not an ordinary homotopy orbit construction for spaces, but one for infinite loop spaces or spectra. So we need to understand why $\mathrm{cro}_k F$ takes values in infinite loop spaces (or spectra). But to some extent we did that already in section 3.1.

Now the following definition is obvious:

Definition 3.3.4. For a $k$-homogeneous functor $E$ from based spaces to based spaces, $\mathrm{cro}_k E$ is the symmetric $k$-linear functor given by

$$\mathrm{cro}_k E(X_1, \dots, X_k) = \mathrm{tohofiber}\left( S \mapsto \bigvee_{\substack{t \in \{1,2,\dots,k\} \\ t \notin S}} X_t \right)_{S \subset \{1,2,\dots,k\}}$$

We need to explain and justify the words *symmetric* and *$k$-linear* in this definition.

Definition 3.3.5. A functor $C$ of $k$ variables (which are based spaces) taking values in based spaces is symmetric $k$-linear if (i) there are natural homeomorphisms

$$e_\sigma : C(X_1, \dots, X_k) \xrightarrow{\cong} C(X_{\sigma(1)}, \dots, X_{\sigma(k)})$$

such that $e_\tau e_\sigma = e_{\tau\sigma}$, and (ii) for fixed choice of $j \in \{1, 2, \dots, k\}$ and based spaces $X_i$ where $i \in \{1, 2, \dots, k\} \smallsetminus \{j\}$, the functor $Y \mapsto C(X_1, X_2, \dots, X_{i-1}, Y, X_{i+1}, \dots, X_k)$ is 1-homogeneous (a.k.a. linear).

It is clear that $\mathrm{cro}_k E$ in definition 3.3.4 has property/structure (i) in definition 3.3.5. Property (ii) is less obvious. Without loss of generality, we can take $j = 1$. Let

$$\begin{tikzcd} Y_{11} \arrow[r, "f_1"] \arrow[d, "g_1"] & Y_{12} \arrow[d, "g_2"] \\ Y_{21} \arrow[r, "f_2"] & Y_{22} \end{tikzcd}$$

be a homotopy cocartesian square. We use this to make four $k$-dimensional cubes of the form

$$\left( S \mapsto \bigvee_{\substack{t\in\{1,2,\dots,k\}\\ t\notin S}} X_t \right)_{S\subset\{1,2,\dots,k\}}$$

where $X_2, X_3, \dots, X_k$ are selected in advance whereas $X_1$ can mean any one of $Y_{ab}$ for $a, b \in \{1,2\}$. These cubes $\mathcal{X}_{ab}$ are related by maps induced by $f_1, f_2, g_1, g_2$; so that we get a $(k+2)$-cube in all. That $(k+2)$-cube is *strongly homotopy cocartesian* by inspection. Therefore, if we apply $F$, we obtain a homotopy *cartesian* $(k+2)$-cube. From that we must deduce that the commutative square, a.k.a. 2-cube, formed by the total homotopy fibers of the sub-cubes $F(\mathcal{X}_{ab})$ (of dimension $k$) is again homotopy *cartesian*. There is a general statement of that kind; a little like Fubini. (I think we can obtain this by $k$-fold application of Prop 1.18 in Goodwillie's Calc II. Each single application will reduce the cube dimension by one. See also: his inductive definition of total homotopy fiber of a cube, just before Def. 1.1 in Calc II.)
We must also show: $\mathrm{cro}_k E(X_1, X_2, \dots, X_k)$ is weakly contractible if one of the $X_i$ is weakly contractible. Without loss of generality, $X_1$ is weakly contractible. Then we can write the $k$-cube

$$\mathcal{X} := \left( S \mapsto \bigvee_{\substack{t\in\{1,2,\dots,k\}\\ t\notin S}} X_t \right)_{S\subset\{1,2,\dots,k\}}$$

as a map between two $(k-1)$-cubes (one for the $S$ which do contain 1, and another for the $S$ which do not contain 1). That map between $(k-1)$-cubes will be a levelwise equivalence, and this remains true when we apply $F$; therefore the total homotopy fiber of $F(\mathcal{X})$ is weakly contractible (here we can again use the aforementioned Prop. 1.18).

EXAMPLE 3.3.6. Let $k = 2$ and let $E$ be the functor

$$X \;\mapsto\; \Omega^\infty\Sigma^\infty((X \wedge X)_{h_*\mathbb{Z}/2})$$

where the subscript $h_*\mathbb{Z}/2$ is for a reduced homotopy orbit construction (a.k.a. Borel construction).[5] The functor $E$ is 2-homogeneous; I am not planning to explain this in detail. (Proving it is a good exercise.) Then $\mathrm{cro}_2 E(X_1, X_2)$ is the total homotopy fiber of

$$\begin{tikzcd}
\Omega^\infty\Sigma^\infty(((X_1\wedge X_1)\vee(X_1\wedge X_2)\vee(X_2\wedge X_1)\vee(X_2\wedge X_2))_{h_*\mathbb{Z}/2}) \arrow[r] \arrow[d] & \Omega^\infty\Sigma^\infty((X_1\wedge X_1)_{h_*\mathbb{Z}/2}) \arrow[d] \\
\Omega^\infty\Sigma^\infty((X_2\wedge X_2)_{h_*\mathbb{Z}/2}) \arrow[r] & *
\end{tikzcd}$$

which simplifies to $\Omega^\infty$ of total homotopy fiber of

$$\begin{tikzcd}
\Sigma^\infty(((X_1\wedge X_1)\vee(X_1\wedge X_2)\vee(X_2\wedge X_1)\vee(X_2\wedge X_2))_{h_*\mathbb{Z}/2}) \arrow[r] \arrow[d] & \Sigma^\infty((X_1\wedge X_1)_{h_*\mathbb{Z}/2}) \arrow[d] \\
\Sigma^\infty((X_2\wedge X_2)_{h_*\mathbb{Z}/2}) \arrow[r] & *
\end{tikzcd}$$

[5] Suppose that $Y$ is a based space with an action of a (discrete) group $G$ fixing the base point. Let $U$ be your favorite contractible CW-space with a cellular action of $G$ permuting the cells freely. Then $Y_{h_*G}$ is the quotient of orbit spaces $(Y \times U)_G)/(* \times U)_G$ where $G$ acts diagonally on $Y \times G$.

which simplifies to $\Omega^\infty\Sigma^\infty(((X_1 \wedge X_2) \vee (X_2 \wedge X_1))_{h_*\mathbb{Z}/2})$, which simplifies (up to a natural weak equivalence respecting symmetries) to $\Omega^\infty\Sigma^\infty(((X_1 \wedge X_2) \vee (X_2 \wedge X_1))_{\mathbb{Z}/2})$, which boils down to nothing but

$$\Omega^\infty\Sigma^\infty(X_1 \wedge X_2).$$

Note that, as a functor of two variables $X_1$ and $X_2$, this is still *symmetric*, as it should be. And certainly 2-linear. This formula for $\mathrm{cro}_2 E$ may come as a surprise[6] since $\mathrm{cro}_2 E$ looks like one of the most basic examples of a 2-linear functor, whereas $E$ itself did not look like one of the most basic example of a 2-homogeneous functor. (But it is.)

Theorem 3.3.7. Classification of symmetric $k$-linear functors: *Every symmetric $k$-linear functor $C$ as in definition 3.3.5 has the form (up to a zigzag of natural weak equivalences preserving the symmetries)*

$$C(X_1, X_2, \ldots, X_k) = \Omega^\infty\big(X_1 \wedge X_2 \wedge \cdots \wedge X_k \wedge \mathbf{V}\big)$$

*where $\mathbf{V}$ is a spectrum with action of the group $\Sigma_k$.*

NB: in this formulation, the symmetry isomorphism corresponding to $\sigma \in \Sigma_k$ is supposed to be the one induced by the isomorphism of spectra

$$X_1 \wedge X_2 \wedge \cdots \wedge X_k \wedge \mathbf{V} \longrightarrow X_{\sigma(1)} \wedge X_{\sigma(2)} \wedge \cdots \wedge X_{\sigma(k)} \wedge \mathbf{V}$$

taking $(x_1, \ldots, x_k, v)$ to $(x_{\sigma(1)}, \ldots, x_{\sigma(k)}, \sigma v)$ (in slightly symbolic notation). The action of $\Sigma_k$ on $\mathbf{V}$ does not influence the functor as such, but it determines the additional structure described by the adjective *symmetric*.

Let us take the view that this theorem is not very surprising.(I don't want to give a proof, except for some indications.) A symmetric $k$-linear functor $C$ gives rise to a $k$-fold spectrum defined (roughly) by

$$(n_1, n_2, \ldots, n_k) \mapsto C(S^{n_1}, S^{n_2}, \ldots, S^{n_k}).$$

The symmetry structure of $C$ can be used to define an action of $\Sigma_k$ on that spectrum. We saw ideas like that in section 3.1.

Next topic: if we have a symmetric $k$-linear functor $C$ as in definition 3.3.5, we want to associate to it a $k$-homogeneous functor $E$. Ideally this construction should be inverse to $\mathrm{cro}_k$. An obvious attempt is to try

$$X \;\mapsto\; C(\underbrace{X, X, \ldots, X}_{k}).$$

But example 3.3.6 shows us that this is not inverse to $\mathrm{cro}_k$. Namely, if we take $k = 2$ and begin with $E$ where

$$E(X) = \Omega^\infty\Sigma^\infty((X \wedge X)_{h_*\mathbb{Z}/2})$$

then $C$ has the form $C(X_1, X_2) = \Omega^\infty\Sigma^\infty(X_1 \wedge X_2)$, as explained in the example. If we evaluate $C$ on $(X, X)$ we get

$$\Omega^\infty\Sigma^\infty(X \wedge X)$$

which is not nearly the $E(X)$ that we started with, although it is somewhat related.
Also, it emerges that in our attempt $X \mapsto C(X, X, \ldots, X)$ we lost the symmetry; more specifically there is a symmetry action of $\Sigma_k$ on $C(X, X, \ldots, X)$ (because $C$ has a symmetry structure by assumption) and we did not use it.

[6] It did for me.

Therefore our second attempt could be

$$X \;\mapsto\; \Big(C(\underbrace{X, X, \dots, X}_{k})\Big)_{h_*\Sigma_k}$$

and this is better, but still not good enough. In our example, where $C(X_1, X_2) = \Omega^\infty\Sigma^\infty(X_1 \wedge X_2)$, we would get

$$X \;\mapsto\; \Big(\Omega^\infty\Sigma^\infty(X \wedge X)\Big)_{h_*\Sigma_2}$$

which is not always an infinite loop space, and more importantly, not in good agreement with

$$\Omega^\infty\Sigma^\infty((X \wedge X)_{h_*\mathbb{Z}/2}) \;\cong\; \Omega^\infty((\Sigma^\infty(X \wedge X))_{h\mathbb{Z}/2})$$

So the plan should be as follows:

- (i) show that the classification of $k$-homogeneous functors from based spaces to spaces is "the same" as the classification of $k$-homogeneous functors from spaces to spectra;
- (ii) show that the classification of symmetric $k$-linear functors from $k$-tuples of based spaces to based spaces is "the same" as the classification of symmetric $k$-linear functors from $k$-tuples of based spaces to spectra;
- (iii) note that $\mathrm{cro}_k$ transforms a $k$-homogeneous functor from spaces to spectra into a symmetric $k$-linear functor from $k$-tuples of based spaces to spectra;
- (iv) note that $X \mapsto C(X, X, \dots, X)_{h\Sigma_k}$ transforms a symmetric $k$-linear functor from $k$-tuples of based spaces to spectra into a $k$-homogeneous functor from spaces to spectra;
- (v) show that procedure (iv) is the inverse of (iii), up to (zigzags of) weak equivalences etc. preserving symmetries where appropriate.

Goodwillie follows this outline in Calculus III.
More on the relationship between Taylor tower and Newton-Leibniz derivatives.

## 3.4. Newton-Leibniz point of view in orthogonal calculus

Let $E$ be a continuous functor from $\mathcal{J}$ to *based* spaces (for simplicity). As a first attempt in defining rates of change, we might introduce

$$E^{(1)}(V) := \mathrm{hofiber}[\,E(V) \longrightarrow E(\mathbb{R} \oplus V)\,].$$

Then $E^{(1)}$ is again a continuous functor from $\mathcal{J}$ to based spaces. But we will discover that it has much more structure. First let us look at the spaces $E^{(1)}(\mathbb{R}^n)$ where $n = 0, 1, 2, \dots$.

LEMMA 3.4.1. *Suppose that in a commutative square of based spaces*

$$\begin{array}{ccc} W & \xrightarrow{u} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{v} & Z \end{array}$$

*a based map $e\colon Y \to X$ can be found and based homotopies $ef \simeq u$ and $ge \simeq v$. Then the map* $\mathrm{hofiber}(f) \to \mathrm{hofiber}(g)$ *determined by the commutative square admits a nullhomotopy; more precisely, choices of homotopies $ef \simeq u$ and $ge \simeq v$ determine such a nullhomotopy.*

Proof: exercise. — We can apply this lemma in the following case:

$$\begin{array}{ccc} E(\mathbb{R}^n) & \longrightarrow & E(\mathbb{R}_b \oplus \mathbb{R}^n) \\ \downarrow & & \downarrow \\ E(\mathbb{R}_a \oplus \mathbb{R}^n) & \longrightarrow & E(\mathbb{R}_a \oplus \mathbb{R}_b \oplus \mathbb{R}^n) \end{array}$$

where $\mathbb{R}_a$ and $\mathbb{R}_b$ are two distinct copies of $\mathbb{R}$, and all maps are induced by obvious inclusions of vector spaces. Here we can make a map from lower left-hand term to upper right-hand term by identifying $\mathbb{R}_a$ with $\mathbb{R}_b$ in the most obvious way. This makes the upper triangle commutative. The lower one is commutative up to a homotopy obtained from a path in the orthogonal group $\mathrm{O}(\mathbb{R}_a \oplus \mathbb{R}_b)$ given by

$$t \mapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

where $t \in [0, \pi/2]$. So we get a nullhomotopy for the map $E^{(1)}(\mathbb{R}^n) \to E^{(1)}(\mathbb{R}_b \oplus \mathbb{R}^n)$ induced by the inclusion $\mathbb{R}^n \to \mathbb{R}_b \oplus \mathbb{R}^n$ (from now on simply $\mathbb{R}^n \to \mathbb{R} \oplus \mathbb{R}^n$). And similarly, we get another nullhomotopy by identifying $\mathbb{R}_a$ with $\mathbb{R}_b$ in the less obvious way ($z \mapsto -z$) and using the path

$$t \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

in $\mathrm{O}(\mathbb{R}_a \oplus \mathbb{R}_b)$, where $t \in [0, \pi/2]$. Together the nullhomotopies give us a map

$$S^1 \wedge E^{(1)}(\mathbb{R}^n) \to E^{(1)}(\mathbb{R} \oplus \mathbb{R}^n).$$

We begin to see that the spaces $E^{(1)}(\mathbb{R}^n)$, where $n = 0, 1, 2, \ldots$, form a spectrum. But this construction, although inspired, was rather pedestrian. Let us try to make a cleaner statement.

Proposition 3.4.2. *The functor $E^{(1)}$ has the additional structure of a "coordinate free" spectrum; i.e., it comes with a binatural transformation*

$$S^V \wedge E^{(1)}(W) \longrightarrow E^{(1)}(V \oplus W)$$

*(where $S^V := V \cup \infty$) satisfying associativity and unit conditions.*

But this is so clean that we need some preparation. We introduce a variant $\mathcal{J}_1$ of $\mathcal{J}$. It has the same objects as $\mathcal{J}$. The space of morphisms from $V$ to $W$ in $\mathcal{J}_1$ is the Thom space of a certain vector bundle $\gamma_{V,W}$ on $\mathrm{mor}_{\mathcal{J}}(V, W)$. The fiber of $\gamma_{V,W}$ over $f \in \mathrm{mor}_{\mathcal{J}}(V, W)$ is $\mathrm{im}(f)^\perp$ (the orthogonal complement of $f(V)$ in $W$). In other words: a morphism from $V$ to $W$ in $\mathcal{J}_1$ is typically a pair $(f, w)$ where $f \in \mathrm{mor}_{\mathcal{J}}(V, W)$ and $w \in W$ is perpendicular to $f(V)$, but untypically it can also be $\infty$ (the base point of the Thom space). Composition in $\mathcal{J}_1$ is as follows: the composition of

$$U \xrightarrow{(e,v)} V \xrightarrow{(f,w)} W$$

is $(fe, w + f(v))$. But if at least one of the morphisms to be composed is $*$ (base point of the Thom space), then the composition is also $*$. Note that $\mathcal{J}_1$ is "enriched" in based spaces, and composition has the form of a map

$$\mathrm{mor}_{J_1}(V, W) \wedge \mathrm{mor}_{J_1}(U, V) \longrightarrow \mathrm{mor}_{J_1}(U, W).$$

Also important: $\mathcal{J}_1$ contains $\mathcal{J}$; a morphism $f: V \to W$ in $\mathcal{J}$ can be viewed as a morphism $(f, 0)$ from $V$ to $W$ in $\mathcal{J}_1$.

A continuous functor $F$ from $\mathcal{J}_1$ to based spaces (always taking the base point in $\mathrm{mor}_{\mathcal{J}_1}(V, W)$ to the zero map from $F(V)$ to $F(W)$) is the same thing as a continuous functor from $\mathcal{J}$ to based spaces (of the same name $F$) together with a binatural map

$$S^V \wedge F(W) \longrightarrow F(V \oplus W)$$

satisfying associativity and unit conditions. (Every $v \in V \subset S^V$ determines a morphism $(j, v)$ in $\mathcal{J}_1$ from $W$ to $V \oplus W$, where $j: W \to V \oplus W$ is the standard inclusion. We can define the binatural map by $(v, x) \mapsto (j, v)_*(x)$.)

So if we really want to prove proposition 3.4.2, our task is as follows. We must extend the functor $E^{(1)}$, currently defined only on $\mathcal{J}$, to a functor on $\mathcal{J}_1$.
This leads us somewhat mechanically to a few definitions and a little theorem. Let $\mathcal{E}_0$ be the category of continuous functors from $\mathcal{J}$ to based spaces. Let $\mathcal{E}_1$ be the category of continuous functors from $\mathcal{J}_1$ to based spaces (taking base points of morphism spaces in $\mathcal{J}_1$ to zero maps). There is an obvious *restriction* functor from $\mathcal{E}_1 \to \mathcal{E}_0$.

THEOREM 3.4.3. *The restriction functor $\mathcal{E}_1 \to \mathcal{E}_0$ has a right adjoint.*

PROOF. Suppose to begin that it has a right adjoint $Q$. For $f$ in $\mathcal{E}_0$ we must have

$$Q(F)(V) \cong \mathrm{mor}_{\mathcal{E}_1}(\mathrm{mor}_{\mathcal{J}_1}(V,-), Q(F))$$

by the Yoneda lemma. The right-hand expression is the space of natural transformations from the representable functor

$$\mathrm{mor}_{\mathcal{J}_1}(V,-)$$

to $Q(F)$. — By adjunction, the right-hand side is also

$$\cong \mathrm{mor}_{\mathcal{E}_0}(\mathrm{mor}_{\mathcal{J}_1}(V,-), F).$$

But this means that we have a definition of $Q(F)$ on objects $V$ of $\mathcal{J}$. Namely: $Q(F)(V)$ is the space of natural transformations from

$$\mathrm{mor}_{\mathcal{J}_1}(V,-)$$

(representable functor from $\mathcal{J}_1$ to based spaces, but here restricted to $\mathcal{J}$) to $F$. It is mechanical to extend this definition to morphisms. Meanwhile the definition of $Q(F)$ is rather "big"; we may ask *does the space of these natural transformations ... really exist.* To dispel doubts, we can make the following observation:

$$\mathrm{mor}_{\mathcal{J}_1}(V,W) \cong \mathrm{cone}[\, \mathrm{mor}_{\mathcal{J}}(\mathbb{R}\oplus V, W) \to \mathrm{mor}_{\mathcal{J}}(V,W)\,].$$

Namely, the Thom space of $\gamma_{V,W}$ (left-hand side) is the mapping cone of the projection from unit sphere bundle of $\gamma_{V,W}$, also known as $\mathrm{mor}_{\mathcal{J}}(\mathbb{R}\oplus V, W)$, to the base space $\mathrm{mor}_{\mathcal{J}}(V,W)$ of $\gamma_{V,W}$. (The "projection" is by restriction of morphisms from $\mathbb{R}\oplus V$ to $V$.) Here $W$ should be seen as a variable, also known as "$-$". It follows that

$$\begin{aligned}\mathrm{mor}_{\mathcal{E}_0}(\mathrm{mor}_{\mathcal{J}_1}(V,-), F) &\cong \mathrm{hofiber}[\mathrm{mor}_{\mathcal{E}_0}(\mathrm{mor}_{\mathcal{J}}(V,-), F) \to \mathrm{mor}_{\mathcal{E}_0}(\mathrm{mor}_{\mathcal{J}}(\mathbb{R}\oplus V,-), F)\,]\\ &\cong \mathrm{hofiber}[\, F(V) \to F(\mathbb{R}\oplus V)\,]\end{aligned}$$

(the last step uses the Yoneda lemma again). So the space of these natural transformations is really quite easy to grasp. □

PROOF OF PROPOSITION 3.4.2. The last sentences of the last proof tell us that the functor $E^{(1)}$, also known as $V \mapsto \mathrm{hofiber}[\, E(V) \to E(\mathbb{R}\oplus V)\,]$ has an extension to $\mathcal{J}_1$ which is

$$Q(E) = \mathrm{mor}_{\mathcal{E}_0}(\mathrm{mor}_{\mathcal{J}_1}(V,-), E);$$

here $Q$ is the right adjoint in theorem 3.4.5. □

REMARK 3.4.4. The spaces $E^{(1)}(V)$ come with a preferred action of $\mathrm{O}(1)$, simply because $\mathrm{O}(1)$ acts in the usual way on the $\mathbb{R}$ which appears in the definition of $E^{(1)}(V)$:

$$\mathrm{hofiber}[\, E(V) \to E(\mathbb{R}\oplus V)\,].$$

It is therefore tempting to conclude that $E^{(1)}$ is a coordinate free spectrum with action of $\mathrm{O}(1)$, but the matter is slightly more complicated. The structure maps

$$S^V \wedge E^{(1)}(W) \longrightarrow E^{(1)}(V \oplus W)$$

are indeed $\mathrm{O}(1)$-maps, but we must use the (nontrivial) action of $\mathrm{O}(1)$ on $S^V \cong S^{\mathrm{hom}(\mathbb{R},V)}$ determined by the standard action of $\mathrm{O}(1)$ on $\mathbb{R}$. And then the diagonal action on the $\wedge$ product.

Let $F$ be a continuous functor from $\mathcal{J}_1$ to based spaces (taking base points of morphism spaces in $\mathcal{J}_1$ to zero maps). Then $V \mapsto \Omega^V F(V)$ is a continuous functor from $\mathcal{J}$ to based spaces (where $\Omega^V(-) = \mathrm{map}_*(S^V, -)$). Therefore, given $E$, continuous functor from $\mathcal{J}$ to spectra, we can make a sequence of (coordinate free) spectra as follows:

- make $E^{(1)}$, coordinate free spectrum;
- make a new functor $V \mapsto \Omega^V E^{(1)}(V)$ from $\mathcal{J}$ to spaces;
- repeat procedure with $V \mapsto \Omega^V E^{(1)}(V)$ in place of $E$ and obtain another coordinate free spectrum;
- .............

Again this is a little pedestrian and it does not show us all the symmetries that we need to see. So let us return to the point of view of theorem 3.4.5.
We introduce a variant $\mathcal{J}_k$ of $\mathcal{J}$. It has the same objects as $\mathcal{J}$. The space of morphisms from $V$ to $W$ in $\mathcal{J}_k$ is the Thom space of $k \cdot \gamma_{V,W}$ (Whitney sum of $k$ copies of $\gamma_{V,W}$. Composition in $\mathcal{J}_k$ is as follows: the composition of

$$U \xrightarrow{(e,v)} V \xrightarrow{(f,w)} W$$

is $(fe, w+f_*(v))$. But if at least one of the morphisms to be composed is $*$ (base point of the Thom space), then the composition is also $*$. Note that $\mathcal{J}_k$ is "enriched" in based spaces, and composition has the form of a map

$$\mathrm{mor}_{J_k}(V,W) \wedge \mathrm{mor}_{J_k}(U,V) \longrightarrow \mathrm{mor}_{J_k}(U,W).$$

Also important: $\mathcal{J}_k$ contains $\mathcal{J}$; a morphism $f\colon V \to W$ in $\mathcal{J}$ can be viewed as a morphism $(f,0)$ from $V$ to $W$ in $\mathcal{J}_1$.
A continuous functor $F$ from $\mathcal{J}_k$ to based spaces (always taking the base point in $\mathrm{mor}_{\mathcal{J}_k}(V,W)$ to the zero map from $F(V)$ to $F(W)$) is the same thing as a continuous functor from $\mathcal{J}$ to based spaces (of the same name $F$) together with a binatural map

$$S^{\mathbb{R}^k \otimes V} \wedge F(W) \longrightarrow F(V \oplus W)$$

satisfying associativity and unit conditions.
Let $\mathcal{E}_k$ be the category of continuous functors from $\mathcal{J}_k$ to based spaces (taking base points of morphism spaces in $\mathcal{J}_k$ to zero maps). There is an obvious *restriction* functor from $\mathcal{E}_k \to \mathcal{E}_0$.

THEOREM 3.4.5. *The restriction functor $\mathcal{E}_k \to \mathcal{E}_0$ has a right adjoint.*

PROOF. This proof follows the lines of the previous one. At the end (where the previous proof says *To dispel doubts*) it gets more complicated. *To dispel doubts*, we can make the following observation:

$$\mathrm{mor}_{\mathcal{J}_k}(V,W) \cong \mathrm{cone}\Big[ \operatorname*{hocolim}_{\substack{U \subset \mathbb{R}^k \\ 0 \neq U}} \mathrm{mor}_{\mathcal{J}}(U \oplus V, W) \to \mathrm{mor}_{\mathcal{J}}(V,W)\Big].$$

But this is a longer story (and some clarifications are needed because we are dealing with an enriched homotopy colimit). The first step is to reduce to the claim

$$S(k \cdot \gamma_{V,W}) \cong \operatorname*{hocolim}_{\substack{U \subset \mathbb{R}^k \\ 0 \neq U}} \mathrm{mor}_{\mathcal{J}}(U \oplus V, W)$$

(left-hand side: total space of the sphere bundle of $k \cdot \gamma_{V,W}$). This reduction is easy. The second step is to reduce to the case $V = 0$. This is also easy because both sides are total spaces of bundles over $\mathrm{mor}_{\mathcal{J}}(V,W)$, and the fibers over some fixed $f \in \mathrm{mor}_{\mathcal{J}}(V,W)$ are precisely

$$S(k \cdot \gamma_{0,W_f}) \text{ resp. } \underset{\substack{U \subset \mathbb{R}^k \\ 0 \neq U}}{\mathrm{hocolim}}\ \mathrm{mor}_{\mathcal{J}}(U, W_f)$$

where $W_f = \mathrm{im}(f)^\perp \subset W$. — In the case $V = 0$ we have to show

$$S(\hom_{\mathbb{R}}(\mathbb{R}^k, W)) \cong \underset{\substack{U \subset \mathbb{R}^k \\ 0 \neq U}}{\mathrm{hocolim}}\ \mathrm{mor}_{\mathcal{J}}(U, W).$$

This is a very interesting exercise combining linear algebra with enriched homotopy colimits. (Who would have guessed it.) The vector space $\hom_{\mathbb{R}}(\mathbb{R}^k, W)$ is already equipped with an inner product (since we identify it with a direct sum of $k$ copies of $W$), but this is identical with the Hilbert-Schmidt inner product

$$\langle f, g \rangle = \text{trace of } f^\diamond g$$

where $f^\diamond : W \to \mathbb{R}^k$ is the adjoint[7] (in linear operator language) of $f : \mathbb{R}^k \to W$. (This uses the inner product on $W$ and the standard inner product on $\mathbb{R}^k$.) For every $g \in S(\hom_{\mathbb{R}}(\mathbb{R}^k, W))$ we have a symmetric operator $g^\diamond g : \mathbb{R}^k \to \mathbb{R}^k$ and a splitting into pairwise orthogonal eigenspaces,

$$\mathbb{R}^k = \bigoplus_{\lambda \geq 0} V(\lambda)$$

so that $g^\diamond g \equiv \lambda I$ on $V(\lambda)$. (Only finitely many of the $V(\lambda)$ are nonzero.) Let $\lambda_0 > \lambda_1 > \cdots > \lambda_r$ be the list of *nonzero* eigenvalues. There is quite a lot of information here, and tidying this up we obtain the following (all depending on $g$, even though the notation may not show it):

(i) an ascending chain of nonzero linear subspaces of $\mathbb{R}^k$,

$$U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_r$$

where $U_j = V(\lambda_0) \oplus V(\lambda_1) \oplus \cdots \oplus V(\lambda_j)$;

(ii) an (injective) linear isometry $U_r \to W$ which agrees with $\lambda_j^{-1/2} g$ on the summand $V(\lambda_j)$;

(iii) a string of positive real numbers $(c_0, c_1, \ldots, c_r)$ such that $\sum_j c_j = 1$; namely

$$c_j = \frac{\lambda_j - \lambda_{j+1}}{\lambda_0}$$

for $j < r$ and $c_r = \lambda_r / \lambda_0$. The string $(c_0, c_1, \ldots, c_r)$ describes a point in the interior of $\Delta^r$.

The data (i),(ii),(iii) taken together describe exactly a point in

$$\underset{0 \neq U \subset \mathbb{R}^k}{\mathrm{hocolim}}\ \mathrm{mor}_{\mathcal{J}}(U, W).$$

Moreover, it is not hard to recover $g$ from the data (i),(ii),(iii). The only small challenge is to recover the eigenvalues $\lambda_0, \ldots, \lambda_k$ from the string $(c_0, c_1, \ldots, c_k)$. But clearly $\lambda_j / \lambda_0 = c_j + c_{j+1} + \cdots + c_r$ for $j = 0, 1, \ldots, k$. And we know that the sum of the numbers $\dim(V(\lambda_j)) \cdot \lambda_j$ is $1$, since we assumed $g \in S(\hom_{\mathbb{R}}(\mathbb{R}^k, W))$. Therefore

$$\lambda_0^{-1} = \sum_{j=0}^{r} \dim(V(\lambda_j)) \cdot (\lambda_j / \lambda^0) = \sum_{j=0}^{r} \dim(V(\lambda_j)) \cdot (c_j + c_{j+1} + \cdots + c_r) = \sum_{j=0}^{r} \dim(U_j) c_j$$

so that $\lambda_j = \dfrac{c_j + c_{j+1} + \cdots + c_r}{\sum_{i=0}^{r} \dim(U_i) c_i}$. ☐

[7] I did not want to use a star since we use the star in so many other places.