Topology 3

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CHAPTER 1

Homotopy groups

1.1. Homotopy groups: definition and first deductions

DEFINITION 1.1.1. Let X be a space with base point $\star$ and let n be a non-negative integer. Write $\pi_n(X,\star)$ for the set $[S^n, X]_*$ (based homotopy classes of based maps from S^n to X). It is clear that π_n is a covariant functor from $\mathscr{H}\mathrm{o}\mathscr{T}\mathrm{op}_\star$ (the homotopy category of based spaces) to sets.

The case $n = 1$ has already been looked at in detail and we saw that $\pi_1(X,\star)$ is a group in a natural way.
The case $n = 0$ is also useful. Namely, $\pi_0(X,\star)$ is just the set of path components of X. Indeed, a based map $f: S^0 \to X$ must send the base point -1 of S^0 to the base point of X. So the only interesting feature it has is the value $f(1) \in X$. And if we pass to homotopy classes, only the path component of $f(1)$ remains.
There is no point in trying to put a natural group structure on $\pi_0(X,\star)$. We must accept that it is in most cases just a set with a distinguished element. The distinguished element is the path component of the base point. (There are exceptions: if X has the structure of a topological group, then $\pi_0(X)$ also has the structure of a group in an obvious way, and that can be useful.)

DEFINITION 1.1.2. *For $n \geq 2$, the set $\pi_n(X,\star)$ has the structure of an abelian group in a natural way. In other words we can equip $\pi_n(X,\star)$ with a structure of abelian group in such a way that, for every based map $f: X \to Y$, the induced map of sets*

$$\pi_n(X,\star) \to \pi_n(Y,\star)$$

becomes a homomorphism of abelian groups. The neutral element of $\pi_n(X,\star)$ is represented by the unique constant based map from S^n to X.

For the proof, we note first that

$$\pi_n(X,\star) \times \pi_n(X,\star) = [S^n, X]_\star \times [S^n, X]_\star \cong [S^n \vee S^n, X]_\star$$

(where $\cong$ is used for an obvious bijection). Therefore it is reasonable to try to construct a multiplication map

$$\mu: \pi_n(X,\star) \times \pi_n(X,\star) \to \pi_n(X,\star)$$

by writing this in the form $\mu: [S^n \vee S^n, X]_\star \longrightarrow [S^n, X]_\star$ and defining it as pre-composition with some fixed element $\kappa \in [S^n, S^n \vee S^n]_*$.

Elementary description of κ. Think of S^n as the quotient space of $[0,1]^n$ obtained by collapsing the subspace consisting of all points which have some coordinate equal to 0 or 1. Think of $S^n \vee S^n$ as the quotient space of $[0,2] \times [0,1]^{n-1}$ obtained by collapsing all points which have some coordinate equal to 0 or 1, or first coordinate 2. Then κ can be defined by $\kappa(x_1, x_2, \dots, x_n) := (2x_1, x_2, \dots, x_n)$, where $x_1, x_2, \dots, x_n \in [0,1]$. It is easy to verify the following directly: the compositions

$$S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{\mathrm{id} \vee \kappa} S^n \vee (S^n \vee S^n)$$

and

$$S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{\kappa \vee \mathrm{id}} (S^n \vee S^n) \vee S^n$$

are based homotopic. This implies that our formula for the multiplication μ on $[S^n, X]_\star$ is *associative*. Next, it is easy to verify the following directly: the composition

$$S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{\text{permute summands}} S^n \vee S^n$$

is based homotopic to κ. (Here we need $n > 1$.) This implies that our formula for the multiplication μ on $[S^n, X]_\star$ is *commutative*. Furthermore, it is easy to verify directly that the constant based map $S^n \to X$ is a two-sided neutral element for the multiplication μ. (In cubical coordinates for S^n, multiplication with the constant map has the effect of replacing a based map

$$f\colon \frac{[0,1]^n}{\sim} \quad \longrightarrow \quad X$$

by the map g where $g(x_1, \dots, x_n) = f(2x_1, x_2, \dots, x_n)$ when $2x_1 \le 1$ and $g(x_1, \dots, x_n) = \star \in X$ when $2x_1 \ge 1$. So the task is to show that f is based homotopic to g, and that is easy.) Next, it is easy to verify directly that an element $[f] \in [S^n, X]_\star$ has an inverse given by $[f \circ \eta]$ where $\eta\colon S^n \to S^n$ is given in cubical coordinates by

$$(x_1, x_2, \dots, x_n) \mapsto (1 - x_1, x_2, \dots, x_n).$$

(In cubical coordinates for S^n, the product of $[f]$ and $[f \circ \eta]$ is given by g where $g(x_1, \dots, x_n) = f(2x_1, x_2, \dots, x_n)$ when $2x_1 \le 1$ and $g(x_1, \dots, x_n) = f(2 - 2x_1, x_2, \dots, x_n)$ when $2x_1 \ge 1$.)

Although the homotopy groups π_n have a great deal of theoretical importance, they are very hard to compute in general, especially for large n. Recently I read in an article about homotopy theory: *not a single compact simply connected CW-space X is known for which we have a formula describing $\pi_n(X)$ for all $n > 0$*, except for the totally uninteresting case where X is contractible (so that $\pi_n(X)$ is the trivial group for all $n > 0$). In particular nobody has a really convincing formula for $\pi_n(S^2)$, for all $n \ge 1$ (although there are some deep results which describe these abelian groups in algebraic/combinatorial terms ... but not in such a way that we can easily read off how many elements they have). But there are many partial results, especially about $\pi_n(S^m)$. For example, we know that $\pi_n(S^m)$ is always a finitely generated abelian group ($m, n > 1$). It is known that $\pi_n(S^m)$ is the trivial group if $n < m$ and that $\pi_n(S^m) \cong \mathbb{Z}$ if $n = m$; see theorem 1.2.1 below. It is known that $\pi_n(S^m)$ is infinite if and only if m is even and $n = m$ or $n = 2m - 1$. An example of that is $\pi_3(S^2) \cong \mathbb{Z}$. Recall that $\pi_3(S^2)$ is not trivial according to example 2.5.3 in the Topo 1 lecture notes. This was conditional at the time on S^3 not being contractible, but later in Topo 1 we showed that S^3 is not contractible.

1.2. Homotopy groups of spheres: the easy cases

THEOREM 1.2.1. *For $0 < n < m$, the group $\pi_n(S^m)$ is trivial. For all $n > 0$, the group $\pi_n(S^n)$ is isomorphic to $\mathbb{Z}$, with $[\mathrm{id}]$ as the generator.*

PROOF. The proof is fiddly, but it is an important result. The case $n < m$ is an easy consequence of cellular approximation. By remark 11.5.2 in the lecture notes for WS 2014-2015, any based map from S^n to S^m is based homotopic to a cellular map. But a cellular map from S^n to S^m must be constant. (Use the CW structure on S^m which has one 0-cell and one m-cell.)
For the case $m = n$, it suffices to show that $\pi_n(S^n)$ is generated by the element $[\mathrm{id}]$. Indeed, this gives us an upper bound on the size of $\pi_n(S^n)$. A lower bound comes from the map $\pi_n(S^n) \to H_n(S^n)$ which takes the homotopy class of a map f to the class of the mapping cycle f. It is an exercise to show that this is a homomorphism.[1] Since $[\mathrm{id}] \in \pi_n(S^n)$ maps to a generator of $H_n(S^n)$, this homomomorphism $\pi_n(S^n) \to H_n(S^n)$ is onto.
With that in mind, the most important tool is Sard's theorem. (We used this earlier in connection with approximation of maps by cellular maps). This states that for a smooth map $f: U \to \mathbb{R}^m$ where U is open in $\mathbb{R}^n$, the set of critical values of f is a set of Lebesgue measure zero (in $\mathbb{R}^m$). An element $y \in \mathbb{R}^m$ is a *critical value* of f if there exists $x \in U$ such that $f(x) = y$ and the derivative $f'(x)$, viewed as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, is not surjective. We can also assume $n > 1$ since $\pi_1(S^1, \star)$ is well understood. We need a few observations.

(i) Any based map $S^n \to S^n$ can be written in the form of a map
$$f: \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\},$$
and after a homotopy we can assume that f is smooth in a neighborhood U of the compact set $f^{-1}(D^n)$.

(ii) In the situation of (i), if $f^{-1}(0)$ contains exactly one element $x \in \mathbb{R}^n$ and the derivative $f'(x)$ is an invertible linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, then f is based homotopic either to the identity map or to the map
$$\eta: (x_1, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$$
from $\mathbb{R}^n \cup \{\infty\}$ to itself.

(iii) The inclusion of the wedge $S^n \vee S^n$ into the product $S^n \times S^n$ induces an isomorphism from $\pi_n(S^n \vee S^n)$ to $\pi_n(S^n \times S^n) \cong \pi_n(S^n) \times \pi_n(S^n)$.

(iv) Let $\alpha: S^n \to S^n \vee S^n$ be any based map. Let $\varphi: S^n \vee S^n \to S^n$ be the fold map (which is the identity on the first summand S^n and also on the second summand S^n). Let $q_i: S^n \vee S^n \to S^n \vee S^n$ be the map which is the identity on summand i and takes the other summand to the base point (for $i = 1, 2$). Then we have
$$[\alpha] = [q_1\alpha] + [q_2\alpha] \in \pi_n(S^n \vee S^n)$$
and consequently
$$[\varphi\alpha] = [\varphi q_1 \alpha] + [\varphi q_2 \alpha] \in \pi_n(S^n),$$
writing $+$ for the multiplication in $\pi_n(S^n \vee S^n)$ and $\pi_n(S^n)$, respectively.

Observation (iii) is a good exercise in cellular approximation; $n > 1$ is important. Observation (iv) follows from observation (iii). Namely, (iii) shows that α is homotopic to a based map obtained by composing $\kappa: S^n \to S^n \vee S^n$ with a map $S^n \vee S^n \to S^n \vee S^n$ which agrees with $q_1\alpha$ on the first wedge summand S^n and with $q_2\alpha$ on the second.
As for (ii), it is easy to reduce to the situation where $x = 0 \in \mathbb{R}^n$. Then $f^{-1}(0) = \{0\}$ and $f'(0)$ is an invertible linear map. The next idea is to show that f is based homotopic to

[1] Hint: you need to say what $\kappa: S^n \vee S^n \to S^n$ does in homology.

the map $g: \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$ where g is the linear map $f'(0)$ (except for $g(\infty) = \infty$). A based homotopy is given by

$$(h_t: \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\})$$

where $h_t(v) = t^{-1}f(tv)$ for $v \in \mathbb{R}^n$ and t runs from 1 to 0. To be more precise, h_1 is of course f and h_0 is of course not really defined by our formula for h_t , but if you (re)define $h_0 = g$ then it ought to make a good homotopy, by definition of differentiability. The next idea is to note that the space of linear isomorphisms from $\mathbb{R}^n \to \mathbb{R}^n$, also known as $\mathrm{GL}_n(\mathbb{R})$, is a space with exactly two path components. One of these path components contains the identity matrix and the other one contains the diagonal matrix with -1 in row one, column one and $+1$ in the other diagonal positions. Therefore our (linear) map

$$g: \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$$

is based homotopic (by a homotopy through invertible linear maps) to either the identity map or to the map η from $\mathbb{R}^n \cup \{\infty\}$ to itself. This proves observation (ii).
Now let's turn to the proof of this theorem, properly speaking. We start with f as in (i). We want to show that $[f] \in \pi_n(S^n)$ is in the subgroup generated by $[\mathrm{id}]$. By Sard, we know that f has a regular value arbitrarily close to 0 and it is easy to reduce to the case where 0 itself is regular value (by composing with a translation of $\mathbb{R}^n$). The preimage $f^{-1}(0)$ is compact and discrete with the subspace topology (since $f'(x)$ is invertible for any $x \in f^{-1}(0)$ use; the inverse function theorem). Therefore $f^{-1}(0)$ is a finite set. Assume that it has k distinct elements

$$x^{(1)}, \dots, x^{(k)}.$$

We want to argue by induction on k. The case $k = 1$ has already been settled in observation (ii). We can therefore assume $k > 1$.
Choose a small open ball B_ε of radius ε about the origin $0 \in \mathbb{R}^n$ such that $f^{-1}(B_\varepsilon)$ is a *disjoint* union of k open sets $U_1, \dots, U_k$ (so that $x^{(i)} \in U_i$) in such a way that f restricts to a diffeomorphism from U_i to B_ε. (This is possible by the inverse function theorem.) Choose a map

$$e: \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$$

which maps B_ε diffeomorphically to all of $\mathbb{R}^n$ and maps the complement of B_ε to ∞ and has $e'(0)$ equal to the identity (matrix). Then we know that $e \simeq \mathrm{id}$ and so $ef \simeq f$. But ef can also be written as a composition

$$S^n \xrightarrow{\ \gamma\ } S^n \vee S^n \xrightarrow{\ \varphi\ } S^n$$

where $S^n = \mathbb{R}^n \cup \{\infty\}$, the first map takes U_1 to the first wedge summand S^n by ef and takes $\bigcup_{i>1} U_i$ to the second wedge summand by ef, and takes all remaining points to the base point ∞ of the wedge. Then by (iv) we have

$$[f] = [ef] = [\varphi\gamma] = [\varphi q_1 \gamma] + [\varphi q_2 \gamma]$$

where $\varphi q_1 \gamma$ and $\varphi q_2 \gamma$ are maps as in (i) for which $0 \in \mathbb{R}^n \cup \{\infty\}$ is a regular value with fewer than k preimage points. By inductive assumption, $[\varphi q_1 \gamma]$ and $[\varphi q_2 \gamma]$ are in the subgroup of $\pi_n(S^n)$ generated by $[\mathrm{id}]$ and therefore $[f]$ is also in that subgroup. □

1.3. Change of base point

This looks like a tedious topic, but it is not without pitfalls (and that makes it more interesting). The following seems to be a useful organizing principle (reminiscent of statements from abstract homotopy theory).

(†) *Let Y be a space and let $A \subset Y$ be a closed subset. Suppose that the inclusion $A \hookrightarrow Y$ is a cofibration (has the HEP, homotopy extension property), and suppose also that it is a homotopy equivalence. Then A is a retract of Y. In other words there exists a continuous $r\colon Y \to A$ such that $r|_A = \mathrm{id}$.*

PROPOSITION 1.3.1. *Let X be a space, $x_0, x_1 \in X$ and $n \geq 2$. If x_0, x_1 are in the same path component of X, then $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic as abelian groups. More precisely, any path γ in X from x_0 to x_1 determines a group isomorphism ι_γ from $\pi_n(X, x_0)$ to $\pi_n(X, x_1)$. The isomorphism ι_γ depends only on the homotopy class of γ with start- and endpoints fixed.*

PROOF. The definition of $\iota_\gamma\colon \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1)$ is as follows. Suppose that $\gamma\colon [0,1] \to X$ has $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Let $\alpha\colon S^n \to X$ be a map such that $\alpha(\star) = x_0$ where $\star \in S^n$ is the base point. Choose a homotopy

$$\big(h_t\colon S^n \to X\big)_{t\in[0,1]}$$

such that $h_0 = \alpha$ and $h_t(\star) = \gamma(t)$. This is possible because the inclusion $\star \to S^n$ is a cofibration. Let $\iota_\gamma[\alpha] \in \pi_n(X, x_1)$ be the based homotopy class of h_1 (a map from S^n to X taking $\star$ to x_1).
We need to show that ι_γ is well defined. Suppose that $\alpha'\colon S^n \to X$ is another map such that $\alpha'(\star) = x_0$ and $[\alpha] = [\alpha'] \in \pi_n(X, x_0)$. Suppose that

$$\big(h'_t\colon S^n \to X\big)_{t\in[0,1]}$$

is a homotopy such that $h'_0 = \alpha'$ and $h'_t(\star) = \gamma(t)$. We need to show that

$$[h_1] = [h'_1] \in \pi_n(X, x_1).$$

Choose a based homotopy $(g_t)_{t\in[0,1]}$ from α to α'. Our problem is solved if we can construct a homotopy

$$\big(H_t\colon S^n \times [0,1] \to X\big)_{t\in[0,1]}$$

in such a way that $H_0(x,s) = g_s(x)$ for all $x \in S^n$, $s \in [0,1]$ and $H_t(x,0) = h_t(x)$, $H_t(x,1) = h'_t(x)$ for all $x \in S^n$ and $t \in [0,1]$, and $H_t(\star, s) = \gamma(t)$ for all $s, t \in [0,1]$. Then H_1 is the required homotopy showing that $[h_1] = [h'_1] \in \pi_n(X, x_1)$.
To show that such a homotopy (H_t) exists we view it as a map to X which is to be defined on

$$Y := S^n \times [0,1] \times [0,1] = \{(x,s,t) \mid x \in S^n,\ s,t \in [0,1]\}.$$

Let A be the closed subset of Y where that map is already defined: the set of (x,s,t) such that either $x = \star$ or $t = 0$ or $s = 0$ or $s = 1$. We want to apply (†), so we need to verify that the inclusion $A \to Y$ is a cofibration and a homotopy equivalence. (Details omitted.) Then we get a retraction $r\colon Y \to A$ and therefore we can define the required map $Y \to X$ by precomposing that map $A \to X$ which we already have with r.
Next we ask whether ι_γ is a homomorphism. In fact this is true by inspection. In slightly more detail, if we have $\alpha, \beta\colon S^n \to X$ such that $\alpha(\star) = x_0 = \beta(\star)$ and homotopies

$$\big(h^\alpha_t\colon S^n \to X\big)_{t\in[0,1]}, \qquad \big(h^\beta_t\colon S^n \to X\big)_{t\in[0,1]}$$

as above, satisfying $h^\alpha_0 = \alpha$ and $h^\beta_0 = \beta$ and $h^\alpha_t(\star) = \gamma(t) = h^\beta_t(\star)$, then the homotopy

$$\Big((h^\alpha_t \vee h^\beta_t) \circ \kappa\Big)_{t\in[0,1]}$$

demonstrates that $\iota_\gamma(\alpha+\beta)=\iota_\gamma(\alpha)+\iota_\gamma(\beta)$, where we use "+" for the group operation in π_n. (Recall that κ is a based map from S^n to $S^n\vee S^n$ which we have used to define the group structure in π_n.)
Next we need to show that ι_γ is bijective. From the definition of ι_γ, it is clear that an inverse is given by $\iota_{\bar\gamma}$ where $\bar\gamma(t)=\gamma(1-t)$ as usual.
Next we need to show that ι_γ depends only on the homotopy class (start- and endpoints fixed) of γ. So let $\Gamma\colon[0,1]\times[0,1]\to X$ be a map such that $\Gamma(s,0)=x_0$ for all s and $\Gamma(s,1)=x_1$ for all s. Let $\alpha\colon S^n\to X$ be a map such that $\alpha(\star)=x_0$. We need to show that

$$\iota_{\Gamma_0}=\iota_{\Gamma_1}$$

where $\Gamma_0(t):=\Gamma(0,t)$ and $\Gamma_1(t):=\Gamma(1,t)$. Since the inclusion of $\star\times[0,1]$ in $S^n\times[0,1]$ is a cofibration, we can construct a homotopy

$$\big(H_t\colon S^n\times[0,1]\to X\big)_{t\in[0,1]}$$

in such a way that $H_0(x,s)=\alpha(x)$ for all $x\in S^n$ and $H_t(\star,s)=\Gamma(s,t)$ for all $s,t\in[0,1]$. Then H_1 is the required homotopy showing that ι_{Γ_0} and ι_{Γ_1} take the same value on the class $[\alpha]$. □

Remark 1.3.2. Suppose that $\beta,\gamma\colon[0,1]\to X$ are paths such that $\beta(1)=\gamma(0)$. Then the concatenated path $\gamma\circ\beta$ is defined. (It is parameterized by the interval $[0,2]$; you can re-parameterize if you wish.) We have

$$\iota_{\gamma\circ\beta}=\iota_\gamma\circ\iota_\beta$$

where both sides of the equation describe isomorphisms from $\pi_n(X,\beta(0))$ to $\pi_n(X,\gamma(1))$. This should be clear from the construction.

Corollary 1.3.3. *For a space X with base point x_0 and $n\geq 2$, the abelian group $\pi_n(X,x_0)$ is a* module *over the fundamental group $\pi_1(X,x_0)$; that is to say, the group $\pi_1(X,x_0)$ acts on $\pi_n(X,x_0)$ by group automorphisms.*[2]

Proof. A formula for the action is $[\gamma]\cdot[\alpha]=\iota_\gamma[\alpha]$, where $[\gamma]\in\pi_1(X,x_0)$ and $[\alpha]\in\pi_n(X,x_0)$. Note that since $\gamma(0)=\gamma(1)=x_0$, the isomorphism ι_γ is an automorphism of $\pi_n(X,x_0)$. □

In many cases this action of π_1 on π_n also has another neat description using universal covering spaces. To set this up we start with a proposition about higher homotopy groups of covering spaces.
Let $q\colon E\to X$ be a covering space, alias fiber bundle with discrete fibers. Suppose also E and X are based spaces, with base points $\star_E$ and $\star_X=q(\star_E)$, so that q is a based map.

Proposition 1.3.4. *Then $q_*\colon\pi_n(E,\star_E)\to\pi_n(X,\star_X)$ is an isomorphism for all $n\geq 2$.*

Proof. This is a consequence of the lifting lemma (see lecture notes on the fundamental group and covering spaces). According to that, for any based map $f\colon S^n\to X$, there exists a unique based map $g\colon S^n\to E$ such that $f=qg$ (assuming $n\geq 2$ to ensure that $\pi_1(S^n,\star)$ is trivial). This argument applies also with $S^n\times[0,1]$ instead of S^n, so that q induces a bijection $[S^n,E]_*\to[S^n,X]_*$. □

[2]For a group G, a *G-module* is understood to be an abelian group A with a homomorphism from G to the group of automorphisms of the abelian group A. This terminology is not completely absurd because the group G determines a group ring $\mathbb{Z}[G]$ whose elements are finite formal linear combinations $\Sigma_{g\in G}n_g\cdot g$ where the coefficients n_g are integers. It is easy to see that a G-module A is the same thing as a module over the ring $\mathbb{Z}[G]$.

Now suppose that X is path connected and locally path connected, with base point $\star$, and that it has a universal covering space

$$q\colon \tilde{X} \longrightarrow X\,.$$

In other words, the action of $\pi_1(X,\star)$ on the set $q^{-1}(\star)$ (given by path lifting) is free and transitive. We can make this q unique up to unique isomorphism (of covering spaces of X) by specifying a base point $\star_1 \in q^{-1}(\star)$ for $\tilde{X}$. (That is to say, if two universal coverings of X are given, both with a base point in the fiber over $\star \in X$, then there exists a unique based homeomorphism between them which respects the maps to X.) Now we make a few observations.

- Proposition 1.3.4 is applicable to this covering space q (set $E := \tilde{X}$).
- Since $\tilde{X}$ is path connected and $\pi_1(\tilde{X},\star_1)$ is trivial, proposition 1.3.1 tells us that $\pi_n(\tilde{X},y)$ is totally independent of the choice of base point y, and we can therefore write $\pi_n(\tilde{X})$. *Little exercise*: the forgetful map from $\pi_n(\tilde{X},y)$ to $[S^n,X]$ is a bijection ... where $[S^n,X]$ is the set of unbased homotopy classes of maps from S^n to X.
- The translation action of $\pi_1(X,\star)$ on $\tilde{X}$ therefore induces an action of $\pi_1(X,\star)$ on $\pi_n(\tilde{X})$.

(This translation action on $\tilde{X}$ is a confusing theme. Let $G = \pi_1(X,\star)$. We know already that an automorphism of the covering space q is determined by the induced permutation of the set $q^{-1}(\star)$. This permutation is a G-map and as such it can be any G-map we like. We constructed q in such a way that $q^{-1}(\star)$ is a free G-orbit $G\cdot\star_1$. What are the automorphisms of $G\cdot\star_1$ as a G-set? They are given by multiplication with elements of G on the *right*; i.e., for fixed $h \in G$ the map $\rho_h\colon G\cdot\star_1 \to G\cdot\star_1$ given by $g\star_1 \mapsto gh\star_1$ is a G-map. Indeed $\rho_h(fg\star_1) = fgh\star_1 = f\rho_h(g\star_1)$ for $f,g \in G$. Unfortunately $h \mapsto \rho_h$ is not a homomorphism, but an antihomomorphism:

$$\rho_{h_1h_2} = \rho_{h_2}\rho_{h_1}\ .$$

Therefore the translation action mentioned above is best defined as follows: an element $h \in G$ determines a G-set automorphism $\rho_{h^{-1}} = (\rho_h)^{-1}$ ... which extends uniquely to an automorphism of the covering space q.)
Showing that the two descriptions of the action of $\pi_1(X,\star)$ on $\pi_n(X,\star)$ agree: let $h \in \pi_1(X,\star)$ be represented by a path $\gamma\colon[0,1]\to X$ from $\star$ to $\star$. Let

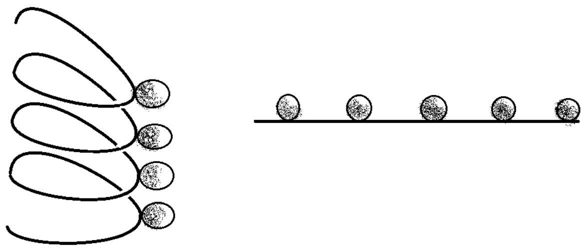
$$\beta\colon[0,1]\to\tilde{X}$$

be a path in $\tilde{X}$ which covers γ, begins at $h^{-1}\star_1$ and so ends at $\star_1$.

$$\begin{tikzcd}
\pi_n(X,\star) \arrow[rr, dotted] & & \pi_n(X,\star) \\
\pi_n(\tilde{X},\star_1) \arrow[u, "q_*", "\cong"'] \arrow[r, "(\text{action of } h^{-1} \text{ on } \tilde{X})_*"] & \pi_n(\tilde{X},h^{-1}\cdot\star_1) \arrow[r, "\iota_\beta"] & \pi_n(\tilde{X},\star_1) \arrow[u, "q_*", "\cong"']
\end{tikzcd}$$

In the lower row, if we identify $\pi_n(\tilde{X},\star_1)$ and $\pi_n(\tilde{X},h^{-1}\cdot\star_1)$ forgetfully with $[S^n,\tilde{X}]$, then the left-hand arrow is the interesting one; the other one, labeled ι_β , is the identity! To make the diagram commutative, the dotted arrow has to be ι_γ .

EXAMPLE 1.3.5. Let's look at $\pi_2(X,\star)$ where X is $S^2\vee S^1$ with the standard base point. The following picture gives two ways of drawing $\tilde{X}$:



From the picture or otherwise, we get that

$$\tilde{X} \simeq \bigvee_{k\in\mathbb{Z}} S^2 \ ,$$

a wedge of spheres S^2 indexed by the integers. In this description the action of $\ell \in \mathbb{Z} \cong \pi_1(X,\star)$ takes the summand S^2 with label k to the summand S^2 with label $k-\ell$ in the obvious way. An argument like observation (iii) in the proof of theorem 1.2.1 then shows that

$$\pi_2(X,\star) \cong \pi_2(\tilde{X},\star_1) \cong \bigoplus_{k\in\mathbb{Z}} \mathbb{Z}\,.$$

The action of $\ell \in \mathbb{Z} \cong \pi_1(X,\star)$ takes the summand $\mathbb{Z} \subset \pi_2(X,\star)$ with label k to the summand $\mathbb{Z}$ with label $k-\ell$ in the obvious way. As an abelian group, $\pi_2(X,\star)$ is obviously not finitely generated. But as a module over the group ring $\mathbb{Z}[\pi_1(X,\star)] = \mathbb{Z}[\mathbb{Z}]$ it is free on one generator, and therefore certainly finitely generated.
This raises the question: if X is a compact CW-space with base point $\star$, and $n \ge 2$, is $\pi_n(X,\star)$ always finitely generated as a module over $\mathbb{Z}[\pi_1(X,\star)]$? See exercises.

1.4. Cup product in cohomology and homotopy groups

Let X be a based path connected space and $f\colon S^n \to X$ a based map, where $n \ge 1$. We form $Y = \mathrm{cone}(f)$, the mapping cone of f. Often by taking a hard look at Y, we can show that $[f]$ is not the trivial element of $\pi_n(X,\star)$. This is based on the following observation.

Lemma 1.4.1. *Let $u,v\colon A \to B$ be any maps. If u is homotopic to v, then* $\mathrm{cone}(u)$ *is homotopy equivalent to* $\mathrm{cone}(v)$.

Proof. Exercise. (As an exercise in WS2014-15, this did not find many friends, but the formulation was more complicated at the time. I hope that it will find more friends this time.) But we can make a stronger statement. There exists a homotopy commutative diagram of the shape

$$\begin{tikzcd} B \arrow[r, "\text{incl.}"] \arrow[d, "="] & \mathrm{cone}(u) \arrow[r] \arrow[d, "\simeq"] & \Sigma A = \mathrm{cone}(u)/B \arrow[d, "="] \\ B \arrow[r, "\text{incl.}"] & \mathrm{cone}(v) \arrow[r] & \Sigma A = \mathrm{cone}(v)/B \end{tikzcd}$$

where the horizontal maps are the usual ones. $\square$

Now let's return to the based map $f: S^n \to X$ and $Y = \mathrm{cone}(f)$ and the quotient map from Y to $Y/X = S^{n+1}$.

COROLLARY 1.4.2. *If f is nullhomotopic, then there exists a graded ring homomorphism $H^*(Y) \to H^*(S^{n+1})$ such that the composition*

$$H^*(S^{n+1}) \xrightarrow{\text{induced by quot. map}} H^*(Y) \longrightarrow H^*(S^{n+1})$$

is the identity.

PROOF. If f is nullhomotopic, then we can assume (by the lemma) that it is the map which sends every point of S^n to the base point of X. Then Y is $X \vee S^{n+1}$. The inclusion $S^{n+1} \to Y$ of the wedge summand induces a homomorphism in cohomology which has the stated property. □

EXAMPLE 1.4.3. Let $f: S^3 \to S^2$ be the Hopf map. (Write $S^2 = \mathbb{C}P^1 = S^3/\sim$ where $S^3 \subset \mathbb{C}^2$; the equivalence relation is $(z_1, z_2) \sim (uz_1, uz_2)$ for $u \in S^1 \subset \mathbb{C}$ and $z_1, z_2 \in \mathbb{C}$ with $|z_1|^2 + |z_2|^2 = 1$. Let f be the quotient map.) Here $X = S^2$ and Y can be identified with $\mathbb{C}P^2$. (To put it differently: $\mathbb{C}P^2$ has a well-known CW structure with one 0-cell, one 2-cell and one 4-cell; the attaching map for the 4-cell happens to be the Hopf map $S^3 \to S^2$.) The cohomology ring $H^*(Y) = H^*(\mathbb{C}P^2)$ is well known: it is the graded ring $\mathbb{Z}[x]/(x^3)$ where x lives in degree 2. It follows that a graded ring homomorphism from $H^*(\mathbb{C}P^2)$ to $H^*(S^4)$ can never be surjective (because it must take x to 0). Therefore f is not nullhomotopic. (We have already seen other proofs of this fact.)
More generally, let $f: S^{2n-1} \to \mathbb{C}P^{n-1}$ be the usual quotient map (where S^{2n-1} is viewed as the unit sphere in $\mathbb{C}^n$). Then $X = \mathbb{C}P^{n-1}$ and Y can be identified with $\mathbb{C}P^n$. The cohomology ring $H^*(Y) = H^*(\mathbb{C}P^n)$ is well known:[3] it is the graded ring $\mathbb{Z}[x]/(x^{n+1})$ where x lives in degree 2. It follows that a graded ring homomorphism from $H^*(\mathbb{C}P^n)$ to $H^*(S^{2n})$ can never be surjective. Therefore f is not nullhomotopic.

DEFINITION 1.4.4. The *Hopf* invariant of a based map $f: S^{4n-1} \to S^{2n}$, where $n \geq 1$, is defined as follows. Form $Y = \mathrm{cone}(f)$, a CW-space with three cells: a 0-cell, a $2n$-cell and a $4n$-cell. (The 0-cell and the $2n$-cell together make up S^{2n}.) The cohomology $H^*(Y)$ as a graded group is then given by

$$H^r(Y) = \begin{cases} \mathbb{Z} & \text{if } r = 0, 2n, 4n \\ 0 & \text{otherwise.} \end{cases}$$

Let $x_{2n} \in H^{2n}(Y)$ and $x_{4n} \in H^{4n}(Y)$ be the preferred generators of these infinite cyclic groups. We have

$$(x_{2n})^2 = a \cdot x_{4n}$$

for some $a \in \mathbb{Z}$, inevitably. This integer a determines the ring structure in $H^*(Y)$. It is the Hopf invariant of f. (By corollary 1.4.2, if the Hopf invariant of f is $\neq 0$, then f is not nullhomotopic.)

EXAMPLE 1.4.5. The Hopf invariant of the Hopf map $S^3 \to S^2$ is 1, as we have seen. There are similar maps $S^7 \to S^4$ (constructed using the Hamilton Quaternions instead of $\mathbb{C}$) and $S^{15} \to S^8$ (constructed using the Cayley Octonions). These, too, have Hopf invariant 1. It is a theorem (J.F. Adams 1961) that there is no map $S^{4n-1} \to S^{2n}$ of odd Hopf invariant except in the cases $n = 1, 2, 4$. The original proof by Adams was very

[3] Although well known, this is not easy. We came very close to it in WS 2014/15 with problems 3,4,5 on exercise sheet 11.

difficult, but an easier proof using K-theory (a *generalized* form of cohomology) became available a few years later. — But there are maps $S^{4n-1} \to S^{2n}$ of Hopf invariant 2 for any $n \geq 1$. We shall return to this in a little while.

Example 1.4.6. To see more applications of corollary 1.4.2 it is a good idea to work backwards, i.e., to begin with Y. So take $Y = S^m \times S^n$ where $m, n \geq 1$. This has a standard CW-structure with 4 cells: a 0-cell, an m-cell, an n-cell and an $(m+n)$-cell. We allow $m = n$. The graded cohomology *ring* $H^*(Y)$ can be described as $\mathbb{Z}[x,y]/(x^2,y^2)$ where x is in degree m and y is in degree n. (This notation indicates that xy is in degree $m+n$, not zero, and $H^{m+n}(Y)$ is the infinite cyclic group generated by xy. There is also an understanding that $xy = (-1)^{mn}yx$.) In any case we see that any graded ring homomorphism $H^*(Y) \to H^*(S^{m+n})$ must take xy to zero because it will take x and y to zero. So there cannot be a surjective ring homomorphism from $H^*(Y)$ to $H^*(S^{m+n})$. Therefore, if we take $X = S^m \vee S^n$ to be the $(m+n-1)$-skeleton of Y, then the attaching map for the unique $(m+n)$-cell of Y is a map $w\colon S^{m+n-1} \to S^m \vee S^n$ and it is not nullhomotopic. This is called the *Whitehead* map (in honor of JHC Whitehead again). For an explicit description of w it is best to think of S^{m+n-1} as the boundary of $D^m \times D^n$:

$$S^{m+n-1} \cong \{(y,z) \in D^m \times D^n \mid \|y\| = 1 \text{ or } \|z\| = 1\}.$$

The right-hand expression can be written as $K \cup L$ where $K = D^m \times S^{n-1}$ and $L = S^{m-1} \times D^n$, so that $K \cap L = S^{m-1} \times S^{n-1}$. In these coordinates, w is the map which takes $(y,z) \in K$ to the class of $y \in D^m/S^{m-1} \cong S^m \subset S^m \vee S^n$ and which takes $(y,z) \in L$ to the class of $z \in D^n/S^{n-1} \cong S^n \subset S^m \vee S^n$. Note that this takes $K \cap L$ to the base point. We want to think of w as a based map, so it is probably best to choose the base point of S^{m+n-1} as (y,z) in the above coordinates, where $y = (-1,0,0,\dots) \in D^m$ and $z = (-1,0,0,\dots) \in D^n$.

Definition 1.4.7. Let X be a based space and $a \in \pi_m(X,\star)$, $b \in \pi_n(X,\star)$, where $m,n \geq 2$. The *Whitehead product* $\lceil a,b \rceil$ of a and b is the element of $\pi_{m+n-1}(X,\star)$ obtained as follows. Choose representatives $\alpha\colon S^m \to X$ and $\beta\colon S^n \to X$ for a and b and let $\lceil a,b \rceil$ be the based homotopy class of the composition of $\alpha \vee \beta$ with the Whitehead map w:

$$S^{m+n-1} \xrightarrow{\ w\ } S^m \vee S^n \xrightarrow{\ \alpha\vee\beta\ } X$$

(Official notation for the Whitehead product of a and b is $[a,b]$, but since we use the square brackets in so many ways for homotopy classes and sets of homotopy classes, I prefer to write $\lceil a,b \rceil$ instead.)

Example 1.4.8. Let $\iota = [\mathrm{id}] \in \pi_{2m}(S^{2m},\star)$, where $m \geq 1$. Then the Whitehead product $\lceil \iota,\iota \rceil \in \pi_{4m-1}(S^{2m},\star)$ is $\neq 0$. In fact it is an element of Hopf invariant 2. — To see this let $X = S^{2m} \times S^{2m}$ and $A = S^{2m} \vee S^{2m}$ and let Y be the pushout of

$$X \xleftarrow{\text{incl.}} A \xrightarrow{\ \varphi\ } S^{2m}$$

where φ is the fold map. In other words Y is obtained from X by gluing together the two cells of dimension $2m$ in X using the fold map. The ring $H^*(X)$ is isomorphic to $\mathbb{Z}[s,t]/(s^2,t^2)$ where s and t are in degree $2m$. We view X and Y as CW-spaces with 4 and 3 cells, respectively. The quotient map $X \to Y$ is cellular. Comparing cellular chain complexes, it is therefore easy to see that the graded ring homomorphism $H^*(Y) \to H^*(X)$ determined by the quotient map $X \to Y$ is injective and its image is the graded subring of $H^*(X)$ generated by $u = s+t$ and $v = st$. Since $u^2 = s^2 + 2st + t^2 = 2st = 2v$ in $H^*(X)$, we have $H^*(Y) \cong \mathbb{Z}[u,v]/(u^2-2v, uv, v^2)$, where u is in degree $2m$ and v is in degree $4m$. This proves that the attaching map $S^{4m-1} \to S^{2m} = Y^{2m}$ for the $4m$-dimensional cell of Y has Hopf invariant 2. But that attaching map can also be written as the attaching map

$$w: S^{4m-1} \to S^{2m} \vee S^{2m} = X^{2m}$$

for the $4m$-dimensional cell of X, followed by the fold map

$$\varphi: S^{2m} \vee S^{2m} \longrightarrow S^{2m} .$$

Its homotopy class is therefore $[\iota,\iota]$ by the definition of the Whitehead product in terms of the Whitehead map w.

1.5. Homotopy groups of pairs

A *pair of spaces* (X,A) means a space X with a distinguished subspace A. If a base point in A has been selected, then we speak of a pair of based spaces. The base point in A also serves as base point in X.

An important example of a pair of spaces is (D^n, S^{n-1}). The preferred base point for me is probably $(-1,0,0,\dots)$.

A map of pairs from (X,A) to (Y,B) is a map $f: X \to Y$ such that $f(A)$ is contained in B. We write $f:(X,A) \to (Y,B)$ in this situation. The map is *based* if f of the base point is the base point (assuming that we are talking about pairs with base point).

Two maps $f:(X,A) \to (Y,B)$ and $g:(X,A) \to (Y,B)$ are *homotopic* as maps of pairs if there exists a map of pairs $h:(X \times [0,1], A \times [0,1]) \to (Y,B)$ such that $h_0 = f$ and $h_1 = g$, where $h_t(x) := h(x,t)$ for $t \in [0,1]$. Such an h is called a *homotopy* from f to g. If f and g are based maps, and each h_t is also a based map, then we call h a based homotopy (between based maps of pairs). Homotopy (based homotopy) in this sense is an equivalence relation on the set of (based) maps from (X,A) to (Y,B).

DEFINITION 1.5.1. For $n > 0$, the n-th homotopy set $\pi_n(X,A,\star)$ of the pair (X,A) with base point $\star \in A$ is the set of based homotopy classes of based maps from the pair (D^n, S^{n-1}) to (X,A). For $n = 0$ we define $\pi_0(X,A,\star)$ to be the quotient of $\pi_0(X,\star)$ by the image of $\pi_0(A,\star)$.

It is already routine to verify that $\pi_n(X,A,\star)$ is an abelian group for $n \geq 3$, and still a group for $n = 2$. To define the group structure we use a map of pairs

$$\bar{\kappa}: (D^n, S^{n-1}) \longrightarrow (D^n \vee D^n, S^{n-1} \vee S^{n-1})$$

which extends the map $\kappa: S^{n-1} \to S^{n-1} \vee S^{n-1}$ which we used previously to define the group structure in $\pi_{n-1}(A,\star)$. In more detail: let $I = [0,1]$ and $\partial I = \{0,1\}$ and use a homeomorphism of your choice to identify the pair (D^n, S^{n-1}) with the pair $(I^n/K, \partial I^n/K)$, where

> ∂I^n consists of all points in I^n which have at least one of their n coordinates in ∂I;
> $K \subset \partial I^n$ consists of all points in I^n which have at least one of the first $(n-1)$ coordinates in ∂I, or the n-th coordinate equal to 0. (In the case $n = 2$ this looks like $\sqcup$, the union of three edges of the square $\square = \partial I^2$.)

Then define $\bar{\kappa}$ by

$$(x_1,\dots,x_n) \mapsto \begin{cases} (2x_1, x_2, \dots, x_n) & \text{if } 2x_1 \leq 1 \\ (2x_1 - 1, x_2, \dots, x_n) & \text{if } 2x_1 \geq 1. \end{cases}$$

This description makes the proof of the following statement mechanical.

Proposition 1.5.2. *There is a forgetful map* $\partial\colon\pi_n(X,A,\star)\to\pi_{n-1}(A,\star)$ *which is a homomorphism of groups for* $n\ge 2$.

Even more obvious: a based map of pairs $f\colon(X,A)\to(Y,B)$ induces a map

$$\pi_n(X,A,\star)\to\pi_n(Y,B,\star)$$

which is a homomorphism of groups for $n\ge 2$. In particular the inclusion of $(X,\star)$ in (X,A) induces a map from $\pi_n(X,\star,\star)=\pi_n(X,\star)$ to $\pi_n(X,A,\star)$ which is a homomorphism for $n\ge 2$.

For $n\le 1$ we can only say (in general) that $\pi_n(X,A,\star)$ is a set with a distinguished base point: the class of the constant map with value $\star$ from $(D^n,\partial D^n)$ to X.

Theorem 1.5.3. *For a based pair of spaces* (X,A), *the sequence*

$$\cdots\longrightarrow\pi_n(X,\star)\longrightarrow\pi_n(X,A,\star)\xrightarrow{\partial}\pi_{n-1}(A,\star)\longrightarrow\pi_{n-1}(X,\star)\longrightarrow\cdots$$

is exact.

While the proof is not very exciting, the interpretation of exactness for low values of n is interesting. We agree that a sequence of based sets and based maps $\cdots\to\cdot\to\cdot\to\cdots$ is *exact* if for each map in the sequence, the preimage of the base point is equal to the image of the previous map. If the based sets happen to be abelian groups (with the zero element as base point) then this definition agrees with our standard concept of exactness. *Example*: the exactness theorem above implies that the image of $\pi_2(X,\star)$ in $\pi_2(X,A,\star)$ is a normal subgroup of $\pi_2(X,A,\star)$. *Example*: the sequence in the theorem is supposed to end with the term $\pi_0(X,A,\star)$. Consequently it is claimed that the map from $\pi_0(X,\star)$ to $\pi_0(X,A,\star)$ is onto and the preimage of the base element under that map is the image of $\pi_0(A,\star)$ in $\pi_0(X,\star)$ ... this is obviously correct by the very definition of $\pi_0(X,A,\star)$.

Proof. Let's prove exactness at $\pi_n(X,A,\star)$, assuming $n\ge 1$. It is clear that the composition of the two arrows (with that target/source) is zero. Suppose that

$$\alpha\colon(D^n,S^{n-1})\longrightarrow(X,A)$$

is a based map of pairs representing an element $[\alpha]\in\pi_n(X,A,\star)$. If $\partial[\alpha]=0\in\pi_{n-1}(A,\star)$, then we know that the restriction of α to S^{n-1} is based nullhomotopic as a based map from S^{n-1} to A. By the homotopy extension property for the inclusion $S^{n-1}\to D^n$, once we choose such a homotopy $h=(h_t)_{t\in[0,1]}$ we can also extend it to a homotopy $(\bar h_t)$ from α to another map $\beta\colon D^n\longrightarrow X$. Each $\bar h_t$ is automatically a based map of pairs from (D^n,S^{n-1}) to (X,A), since $\bar h_t$ agrees with h_t on A. Therefore α is based homotopic as a map of pairs to $\beta=\bar h_1$. But since $\beta(A)=\star$ we can say that $[\beta]$ is in the image of the map from $\pi_n(X,\star)$ to $\pi_n(X,A,\star)$.

Next, let's prove exactness at $\pi_n(X,\star)$, assuming $n\ge 1$. The composition of the two maps is the zero map because there is a commutative square

$$\begin{tikzcd}
\pi_n(A,\star) \arrow[r] \arrow[d] & \pi_n(X,\star) \arrow[d] \\
\pi_n(A,A,\star) \arrow[r] & \pi_n(X,A,\star)
\end{tikzcd}$$

where the term $\pi_n(A,A,\star)$ is trivial (a based set with only one element). If α is a based map from $S^n\cong I^n/\partial I^n$ to X such that the class of α in $\pi_n(X,A,\star)$ is zero, then α is based nullhomotopic as a map of pairs from $(I^n/K,\partial I^n/K)$ to (X,A). Let h be such

a homotopy. So h is a map from I^{n+1} to X which takes $2n$ of the $2n+2$ faces of that cube to the base point. The exceptional faces are $I^n \times 0$, where h agrees with α, and $I^{n-1} \times 1 \times I$ which is mapped to A. They intersect in $I^{n-1} \times 1 \times 0$ which is mapped to $\star \in A \subset X$. Parametrizing this somewhat differently (details left to you, gentle reader) we see that h is a homotopy from the restriction of h to one of the two exceptional faces to the restriction of h to the other exceptional face. One of these restricted maps, viewed as a based map from $I^n/\partial I^n$ to X, is just α. The other restricted map can be viewed as a based map from $I^n/\partial I^n$ to A.
Exactness at $\pi_n(A,\star)$ is straightforward and left to the reader. □

1.6. Homotopy groups and fibrations

THEOREM 1.6.1. *Let $p: E \to B$ be a fibration* (w.r.t. the class of compact spaces) *which is also a based map of based spaces (base points $\star_s \in E$ and $\star_t \in B$). Suppose that B is path connected. Let F be the fiber of p over $\star_t$, $F = p^{-1}(\star_t)$. Then the map*

$$\pi_n(E,F,\star_s) \to \pi_n(B,\star_t)$$

induced by p is a bijection for all $n \geq 0$.

COROLLARY 1.6.2. *In the circumstances of theorem 1.6.1 there is a long exact sequence of homotopy groups/sets*

$$\cdots \longrightarrow \pi_n(F,\star_s) \longrightarrow \pi_n(E,\star_s) \longrightarrow \pi_n(B,\star_t) \longrightarrow \pi_{n-1}(F,\star_s) \longrightarrow \pi_{n-1}(E,\star_s) \longrightarrow \cdots$$

ending in $\cdots \longrightarrow \pi_0(E,\star_s) \longrightarrow \pi_0(B,\star_t)$.

PROOF. In the long exact sequence of the pair (E,F), replace $\pi_n(E,F,\star_s)$ by $\pi_n(B,\star_t)$ using the bijection of theorem 1.6.1. □

PROOF OF THEOREM 1.6.1. Case $n=0$: here the claim is that the inclusion of F in E induces a surjection $\pi_0(F) \to \pi_0(E)$. Proof of this: given $x \in E$, choose a path ω from $p(x)$ to $\star_t$ in B. Use the path lifting property of p to lift this to a path $\tilde{\omega}$ in E from x itself to some point $y \in E$. The $p(y) = \star_t$, so $y \in F$.
Case $n > 0$, surjectivity. Represent an element of $\pi_n(B,\star_t)$ by a map $g: D^n \to B$ such that $g(z) = \star_t$ for all $z \in S^{n-1}$. View this as a homotopy $(h_t: S^{n-1} \to B)_{t\in[0,1]}$ where

$$h_t(z) = g(z + (1-t)b)$$

for $z \in S^{n-1}$; here $b \in D^n$ is the base point $(-1,0,0,\dots,0)$. The homotopy begins with h_0 which is the constant map with value $\star_t$ and ends with h_1 which is again the constant map with value $\star_t$. By the HLP for the map p, there exists a homotopy $(\bar{h}_t: S^{n-1} \to E)_{t\in[0,1]}$ such that $p\bar{h}_t = h_t$ for all t and $\bar{h}_0$ is the constant map with value $\star_s$. The homotopy $(\bar{h}_t)$ can also be viewed as a single map

$$\bar{g}: (D^n, S^{n-1}) \to (E,F)$$

determined by $\bar{g}(z + (1-t)b) = \bar{h}_t(z)$ for $z \in S^{n-1}$. Then $p \circ \bar{g} = g$.
Case $n > 0$, injectivity. Represent an element of $\pi_n(E,F,\star_s)$ by a based map

$$f: (D^n, S^{n-1}) \to (E,F).$$

Suppose that $p \circ f$ is nullhomotopic as a based map from D^n/S^{n-1} to B. Let

$$(k_t: D^n/S^{n-1} \longrightarrow B)_{t\in[0,1]}$$

be a based nullhomotopy, so that $k_0 = p \circ f$ and k_1 is constant with value $\star_t$. We can also write this in the form $(k_t\colon D^n \longrightarrow B)_{t\in[0,1]}$. Use the HLP for p to find a homotopy

$$(\bar{k}_t\colon D^n \to E)_{t\in[0,1]}$$

such that $\bar{k}_0 = f$ and $p \circ \bar{k}_t = k_t$ for all t. Each $\bar{k}_t$ is then a based map of pairs from (D^n, S^{n-1}) to (E,F), and $\bar{k}_1$ is a based map from (D^n, S^{n-1}) to (F,F). Therefore $(\bar{k}_t)$ is not exactly a nullhomotopy for $f = \bar{k}_0$, but it is nevertheless the kind of homotopy that we require because we believe that any based map from (D^n, S^{n-1}) to (F,F) is nullhomotopic as such. □

1.7. The Serre construction for turning a map into a fibration

Example 1.7.1. *Serre construction*: every map $f\colon X \to Y$ has a factorization

$$\begin{tikzcd} & X^\sharp \arrow[d, "f^\sharp"] \\ X \arrow[ur, "j"] \arrow[ur, phantom, "\simeq", swap] \arrow[r, "f"'] & Y \end{tikzcd}$$

where $f^\sharp$ is a fibration, j is "often" a cofibration and, more importantly, j is always a homotopy equivalence. Definition of $X^\sharp$: it is the space of all pairs (x,ω) where $x \in X$ and $\omega\colon I \to Y$ is a path such that $\omega(0) = f(x)$. It is a subspace of $X \times \mathrm{map}(I,Y)$. Here $I = [0,1]$ and the set $\mathrm{map}(I,Y)$ becomes a topological space with the compact-open topology. Definition of $f^\sharp$: let $f^\sharp(x,\omega) = \omega(1) \in Y$. Definition of j: let $j(x) := (x,\omega)$ where ω is the constant path in Y at $f(x)$.

Showing that $f^\sharp\colon X^\sharp \to Y$ is a fibration. Let's first solve the path lifting problem. For a path $\gamma\colon I \to Y$ and a choice of element $(x,\omega) \in X^\sharp$ such that $f^\sharp(x,\omega) = \gamma(0)$, define $\bar{\gamma}\colon I \to X^\sharp$ as follows. The first coordinate of $\bar{\gamma}(t)$ is x. The second coordinate is the path in Y obtained by concatenating ω with $\gamma|_{[0,t]}$ and reparameterizing, $s \mapsto (1+t)s$. This works because $\omega(1) = f^\sharp(x,\omega) = \gamma(0)$. Now $\bar{\gamma}$ satisfies $f^\sharp \circ \bar{\gamma} = \gamma$ and $\bar{\gamma}(0) = (x,\omega)$. So $\bar{\gamma}$ is a solution for this particular path lifting problem. — Since this solution depends very nicely (continuously) on the problem, we can use it to solve the homotopy lifting problem in general. Let P be any space, let

$$(g_t\colon P \to Y)_{t\in[0,1]}$$

be a homotopy and let $G\colon P \to X^\sharp$ be a map such that $f^\sharp G = g_0$. Then for each $p \in P$ we obtain a path $\gamma = \gamma_p$ in Y by $\gamma_p(t) = g_t(p)$, and an element $G(p)$ in $X^\sharp$ such that $f^\sharp(G(p)) = \gamma_p(0)$. This path lifting problem has a solution $\bar{\gamma}_p$, constructed exactly as above. Then the map $(p,t) \mapsto \bar{\gamma}_p(t)$ from $P \times I$ to $X^\sharp$ is continuous, and it solves the homotopy lifting problem consisting of (g_t) and G.

Showing that j is a homotopy equivalence. We start with the observation that $j(X) \subset X^\sharp$ consists of all $(x,\omega) \in X^\sharp$ where the path ω is constant. There is a projection $X^\sharp \to X$ given by $(x,\omega) \mapsto x$. Restricting this to $j(X)$, we see that j is a homeomorphism onto its image, $j(X)$, and that $j(X)$ is closed in $X^\sharp$. That's a good start. Now, for $t \in I$, let $\mu_t\colon I \to I$ be the map $s \mapsto st$. A homotopy $(h_t\colon X^\sharp \to X^\sharp)_{t\in[0,1]}$ is defined by $h_t(x,\omega) := (x, \omega\circ\mu_t)$. This homotopy is a deformation retraction to $j(X)$; that is, $h_0 = \mathrm{id}$, the image of h_1 is $j(X)$, and h_t restricted to $j(X)$ is the identity, for all t.

Showing that $j\colon X \to X^\sharp$ is a cofibration when Y is a CW-space. (Not very important and not very reliable.) I believe that every closed subset in a CW-space is the intersection of a countable decreasing sequence of open subsets. Suggested proof: induction on skeleta. Apply this to the diagonal ΔY as a subspace of the CW-space $Y \times_{CW} Y$. It follows (a variant of the Tietze-Urysohn extension lemma, applicable since $Y \times_{CW} Y$ is a normal space) that there exists a continuous function $q: Y \times_{CW} Y \to I$ such that $q^{-1}(0) = \Delta Y$. Define a function $\bar{q}: X^\sharp \to I$ by $\bar{q}(x,\omega) := \max\{q(\omega(0), \omega(t)) \mid t \in I\}$. I hope that this is continuous. The preimage of 0 under $\bar{q}$ is exactly the closed subset $j(X)$ of $X^\sharp$. Make a map r from $X^\sharp \times I$ to itself by

$$r((x,\omega),t) := ((x,\gamma),s)$$

where

- if $t \geq \bar{q}(x,\omega)$, then γ is constant with value $\omega(0)$, and $s := t - \bar{q}(x,\omega)$;
- if $t = (1-u)\cdot\bar{q}(x,\omega)$ where $u \in I$, then γ is $\omega \circ \mu_u$ where μ_u is multiplication by u, and s is 0.

Now I hope that $rr = r$ and that the image of r is exactly the union of $X^\sharp \times \{0\}$ and $j(X) \times I$. By a well-known criterion, this would prove that j is a cofibration.

For $z \in Y$, the fiber of $f^\sharp$ over z is called the *homotopy fiber of f over z*; if Y has a preferred base point and z is that base point, then simply the *homotopy fiber of f*. Notation: $\mathrm{hofiber}_z(f: X \to Y)$ or $\mathrm{hofiber}(f: X \to Y)$.
If $f: X \to Y$ is the inclusion of a subspace (and the subspace X has a base point $\star$), then there is a canonical bijection (isomorphism of groups for $k \geq 1$)

$$\pi_{m-1}(\mathrm{hofiber}_\star(f)) \longrightarrow \pi_m(Y, X, \star) = \pi_m(Y, X).$$

This can be seen by writing the pair (D^m, S^{m-1}), which we use in the definition of m-th homotopy group of pairs, as $(\mathrm{cone}(S^{m-1}), S^{m-1})$, where cone(...) is the *reduced* cone. A map of pairs from $(\mathrm{cone}(S^{m-1}), S^{m-1})$ to (Y, X) is "the same" (by a form of adjunction) as a map from S^{m-1} to $\mathrm{hofiber}_\star(X \hookrightarrow Y)$.
More generally, if $f: X \to Y$ is any map of based spaces, then the standard projection $\mathrm{cyl}(f) \to Y$ is a based map and induces a map

$$\mathrm{hofiber}_\star(X \hookrightarrow \mathrm{cyl}(f)) \quad \longrightarrow \quad \mathrm{hofiber}_\star(f: X \to Y).$$

(Here we ought to choose the base point in $\mathrm{cyl}(f)$ so that the inclusion $X \to \mathrm{cyl}(f)$ is a based map. This has the slight disadvantage that the inclusion $Y \to \mathrm{cyl}(f)$ is not a based map, but that is not an issue here.) It is an exercise to show that this map is a weak equivalence. Therefore we can still say that $\pi_m(\mathrm{cyl}(f), X)$ is isomorphic to $\pi_{m-1}(\mathrm{hofiber}_\star(f))$.

1.8. The Barratt-Puppe sequence

Let $f: X_1 \to X_0$ be a based map of based spaces. We form the homotopy fiber

$$\mathrm{hofiber}(f) = \{\, (x,\omega) \mid x \in X_1 \ ,\ \omega: [0,1] \to X_0 \ ,\ \omega(0) = f(x) \ ,\ \omega(1) = * \,\}$$

(over the base point of X_0). It is a based space. There is a forgetful map

$$\mathrm{hofiber}(f) \to X_1$$

which takes (x,ω) to x. It is a based map. Now let us iterate this procedure. So we redefine $f_1 := f$ and we define recursively

$$X_{k+1} := \mathrm{hofiber}(f_k: X_k \to X_{k-1})$$

(for $k \geq 1$) and define recursively $f_{k+1} \colon X_{k+1} \to X_k$ as the forgetful map. In this way we get a sequence of based spaces and based maps

$$\cdots \xrightarrow{f_4} X_3 \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

This is the Barratt-Puppe sequence. (In Wikipedia it is called the Puppe sequence, which may seem a little unfair, but I must admit that it is easier to find it in D Puppe's papers than in M Barratt's. An additional source of confusion and injustice is that it has an analogue where homotopy cofibers alias mapping cones replace the homotopy fibers. This may also be called the Barratt-Puppe sequence.)

Things that one should know about the Barratt-Puppe sequence:

(i) There are homotopy equivalences $X_{k+3} \to \Omega X_k$ for $k \geq 0$, where ΩX (for an arbitrary base space X) can be defined as the homotopy fiber of the inclusion $* \to X$. In other words, ΩX is the space of maps from $[0,1]$ to X which take 0 and 1 to the base point.

(ii) For any fixed $n \geq 0$, the sequence

$$\cdots \longrightarrow \pi_n(X_3) \longrightarrow \pi_n(X_2) \longrightarrow \pi_n(X_1) \xrightarrow{f_1} \pi_n(X_0)$$

is an exact sequence (of based sets for $n = 0$; of groups for $n = 1$; of abelian groups for $n \geq 2$). In fact it "agrees" with the appropriate portion of the long exact sequence of homotopy groups of the pair $(\mathrm{cyl}(f_1), X_1)$.

Details: *under construction.*

CHAPTER 2

JHC Whitehead theorem

2.1. Homotopy groups and homotopy equivalences

THEOREM 2.1.1. (J.H.C. Whitehead) *Let $f: X \to Y$ be a map between nonempty CW-spaces such that, for all $x_0 \in X$ and all $n \geq 0$, the map*

$$f_*: \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

is an isomorphism (bijection for $n = 0$)*. Then f is a homotopy equivalence.*

As a preparation for the proof we make a few observations.

It is not a serious restriction to assume that f is cellular. In any case we know that f is homotopic to a cellular map (call it g for now), and if $f_*: \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for all $x_0 \in X$ and $n \geq 0$, then $g_*: \pi_n(X, x_0) \to \pi_n(Y, g(x_0))$ will also be an isomorphism for all x_0 and $n \geq 0$. (Here we need to remind ourselves how higher homotopy groups depend on base points. A homotopy from f to g determines a path γ from $f(x_0)$ to $g(x_0)$ and that path determines an isomorphism (bijection) ι_γ from $\pi_n(Y, f(x_0))$ to $\pi_n(Y, g(x_0))$. It is easy to see from the definition of ι_γ that $g_*: \pi_n(X, x_0) \to \pi_n(Y, g(x_0))$ is the composition of f_* from $\pi_n(X, x_0)$ to $\pi_n(Y, f(x_0))$ with ι_γ.)

Next, if we assume that f is cellular then we can easily reduce to the case where it is the inclusion of a CW-subspace. For that reduction step we replace Y by the mapping cylinder $\text{cyl}(f)$. This is defined as

$$\frac{Y \sqcup [0,1] \times X}{\sim}$$

where $\sim$ means that we identify $(1, x)$ with $f(x) \in Y$, for all $x \in X$. Note that $\text{cyl}(f)$ contains a copy of $X \cong \{0\} \times X$, and more obviously a copy of Y, and we have $\text{cyl}(f)/X = \text{cone}(f)$. There is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\text{inclusion}"] \arrow[d, equal] & \text{cyl}(f) \arrow[d, "\simeq"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Moreover $\text{cyl}(f)$ has a preferred CW structure. (The k-skeleton of that is the union of Y^k and the image of the k-skeleton of $[0,1] \times X$.) With that preferred CW-structure, the inclusion of $X \cong \{0\} \times X$ in $\text{cyl}(f)$ is the inclusion of a CW-subspace. If f has the property that

$$f_*: \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

is a bijection for all x_0 and $n \geq 0$, then it follows easily that the inclusion $X \to \text{cyl}(f)$ has the analogous property. (Use the commutative square just above.)

Next, suppose that $f: X \to Y$ is the inclusion of a CW-subspace and that

$$f_*: \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

is a bijection for all x_0 and $n \geq 0$. Then we can use the exact sequence of theorem 1.5.3 to deduce that $\pi_n(Y, X, x_0)$ is trivial for all x_0 and $n \geq 0$.

LEMMA 2.1.2. *Let (Y, X) be a pair of spaces and let $n \geq 0$ be an integer such that $\pi_n(Y, X, x_0)$ is trivial for all $x_0 \in X$. Then for every map*

$$g: (D^n, S^{n-1}) \to (Y, X)$$

there exists a homotopy $(h_t: D^n \to Y)_{t \in [0,1]}$ such that $h_0 = g$ and $h_1(D^n)$ is contained in X, and h is stationary on S^{n-1} (meaning $h_t(z) = g(z)$ for all $z \in S^{n-1}$ and all t).

PROOF. In the important case $n = 0$, the claim is that every point in Y can be connected by a path in Y to a point in the subspace X. Once we select a base point $x_0 \in X$, this is equivalent to saying that the map from $\pi_0(X, x_0)$ to $\pi_0(Y, x_0)$ induced by the inclusion $X \to Y$ is onto. And that is equivalent to saying that $\pi_0(Y, X, x_0)$ is trivial, by our definition of $\pi_0(Y, X, x_0)$. — Now assume $n > 0$. For a map g as in the statement, let $x_0 = g(\star)$ where $\star \in S^{n-1} \subset D^n$ is the base point. Since $\pi_n(Y, X, x_0)$ is trivial and g represents an element in it, we know that there exists a map of pairs

$$G: (D^n \times [0,1], S^n \times [0,1]) \longrightarrow (Y, X)$$

such that $G(z, 0) = g(z)$ and $G(z, 1) = x_0$ for all $z \in D^n$. Let L be the union of $S^n \times [0,1]$ and $D^n \times \{1\}$ in $D^n \times [0,1]$. Then $G(L)$ is contained in the subspace X of Y. Now it is easy to construct a homotopy

$$(k_t: D^n \to D^n \times [0,1])_{t \in [0,1]}$$

such that $k_0(z) = (z, 0)$ and $k_1(z) \in L$ for all $z \in D$, and k_t is stationary on S^{n-1}, so that $k_t(z) = k_0(z)$ for all $z \in S^{n-1}$ and $t \in [0,1]$. Define $h_t := G \circ k_t$ for $t \in [0,1]$. This gives the required homotopy. □

PROOF OF THEOREM 2.1.1. In view of the above observations it suffices to show the following. Suppose that Y is a CW-space with a CW-subspace X. Suppose that for every $n \geq 0$ and every map of pairs

$$g: (D^n, S^{n-1}) \to (Y, X)$$

there exists a homotopy $(h_t: D^n \to Y)_{t \in [0,1]}$ such that $h_0 = g$ and $h_1(D^n)$ is contained in X, and the homotopy is stationary on S^{n-1}. *Then the inclusion $X \to Y$ is a homotopy equivalence.*

Indeed we are going to construct a homotopy $(F_t: Y \to Y)_{t \in [0,\infty]}$ such that $F_0 = \mathrm{id}_Y$ and (F_t) is stationary on X and $F_\infty(Y) \subset X$. This is clearly enough.[1] It turns out to be convenient for induction purposes to parameterize the homotopy by a compact interval of the form $[0, \infty]$; think of this as the one-point compactification of $\{x \in \mathbb{R} \mid x \geq 0\}$. (We have used this idea before to establish the HEP for inclusions of CW-subspaces; lecture notes WS 2014-2015.)

The idea is to construct (F_t) in steps corresponding to conditions $t \in [k, k+1]$ where k runs through the non-negative integers. This will be done in such a way that F_{k+1} takes the k-skeleton Y^k of Y to X and $F_t(y) = F_{k+1}(y)$ whenever $y \in Y^k$ and $t \geq k+1$. In

[1] Such a homotopy is called a strong deformation retraction of Y onto X. We can say that X is a strong deformation retract of Y.

words, the homotopy (F_t) is stationary on the k-skeleton of Y for $t \geq k+1$.
Suppose then that F_t has already been constructed for $t \in [0,k]$ where k is a positive integer, and that $F_k(Y^{k-1}) \subset X$. Let

$$\varphi\colon (D^k, S^{k-1}) \to (Y^k, Y^{k-1})$$

be a characteristic map for a k-cell E of Y. Then $F_k \circ \varphi$ is a map of pairs from (D^k, S^{k-1}) to (Y, X). By our assumption on the pair (Y, X) there exists a homotopy

$$(h_t\colon D^k \to Y)_{t\in[0,1]}$$

which is stationary on S^{k-1} and has $h_0 = F_k \circ \varphi$ and $h_1(D^k) \subset X$. We want to define F_t for $t \in [k, k+1]$ in such a way that $F_t \circ \varphi = h_{t-k}$. This seems to define F_t only on $Y^{k-1} \cup E$. (Note that F_t for $t \in [k, k+1]$ is already defined on Y^{k-1} because it is supposed to agree there with F_k.) But we can proceed in the same way to define F_t for $t \in [k, k+1]$ on all other k-cells of Y, so that it is defined on all of Y^k. By the definition of a CW-space, specialized here to Y, there is no problem with that in regard to continuity. Then we use the homotopy extension theorem for the inclusion $Y^k \to Y$ to extend to a homotopy, parameterized by a time interval $[k, k+1]$, of maps from Y to Y, beginning with $F_k\colon Y \to Y$ which is already given.
This induction process has a trivial beginning, $F_0 = \mathrm{id}_Y$, but it has a slightly nontrivial end. We decree $F_\infty(y) = F_{k+1}(y)$ if $y \in Y^k$. Since every $y \in Y$ is contained in Y^k for some k, this takes care of all $y \in Y$. By construction of F_t for $t < \infty$, this definition of $F_\infty(y)$ is unambiguous. By the definition of a CW-space, there is no problem whatsoever in regard to continuity. □

CHAPTER 3

Comparing homotopy groups and homology groups

3.1. Homotopy and homology

We have already seen the *Hurewicz homomorphism.* It is a map

$$\pi_n(X,\star) \longrightarrow H_n(X)$$

defined by $[\alpha] \mapsto \alpha_*(1)$. Here $\alpha\colon S^n \to X$ is a based map and $1 \in H_n(S^n) \cong \mathbb{Z}$ is informal notation for the standard generator (also known as the standard fundamental class of the sphere as an oriented manifold). I am assuming $n \geq 1$. The Hurewicz homomorphism is indeed a homomorphism of groups. In the case $n = 1$, when X is path connected, it is surjective and its kernel is the commutator subgroup (the smallest normal subgroup of $\pi_n(X,\star)$ with a commutative quotient). This section is about similar statements for higher n under strong conditions on X.

THEOREM 3.1.1. (Hurewicz.) *Let X be a based CW-space such that $\pi_k(X,\star)$ is trivial for $k = 0,1,2,\ldots,n-1$, where $n \geq 2$. Then the Hurewicz homomorphism $\pi_n(X,\star) \to H_n(X)$ is an isomorphism.*

The theorem is an easy consequence of what we already know about homology of CW-spaces, modulo the following lemma.

LEMMA 3.1.2. *If X is a based connected CW-space such that $\pi_k(X,\star)$ is trivial for $k = 1,2,\ldots,n-1$, where $n \geq 1$, then X is based homotopy equivalent to a CW-space Y such that Y^{n-1} is a single point, which is the base point.*

We postpone the proof of lemma 3.1.2.

PROOF OF THEOREM MODULO LEMMA. We can assume that X in the theorem has X^{n-1} equal to a single point, the base point. Choose characteristic maps for all cells of X. We know already that the homomorphism

$$\pi_n(X^n,\star) \to \pi_n(X,\star)$$

is onto. Moreover X^n is a wedge of (possibly many) n-spheres, say $X^n = \bigvee_{\lambda\in\Lambda} S^n$ where $n \geq 2$. We know (see remark 3.1.3 below) that the inclusion of these wedge summands in X^n induces an isomorphism from

$$\bigoplus_\lambda \mathbb{Z} \ \cong \ \bigoplus_\lambda \pi_n(S^n,\star)$$

to $\pi_n(X^n)$. Now let $\psi\colon S^n \to X^n$ be an attaching map for an $(n+1)$-cell of X. Then

$$[\psi] \in [S^n, X^n] \cong [S^n, X^n]_* = \pi_n(X^n,\star) \ \cong \ \bigoplus_\lambda \mathbb{Z}$$

goes to zero in

$$[S^n, X^{n+1}] = [S^n, X] = [S^n, X]_* = \pi_n(X,\star)$$

because ψ extends to a map from D^{n+1} to X^n (the characteristic map for that n-cell). Therefore the element of $\bigoplus_\lambda \mathbb{Z}$ determined by $[\psi]$ is in the kernel of the surjective map from $\bigoplus_\lambda \mathbb{Z} \cong \pi_n(X^n,\star)$ to $\pi_n(X,\star)$. This reasoning, carried out for all $(n+1)$-cells of X, gives us a *lower bound* on that kernel (a subgroup contained in the kernel). We do not need more because of the following commutative diagram:

$$\begin{tikzcd} \bigoplus_\lambda \mathbb{Z} \cong \pi_n(X^n,\star) \arrow[r] \arrow[d, "\text{Hurew.}"', "\cong"] & \pi_n(X^{n+1},\star) \cong \pi_n(X,\star) \arrow[d, "\text{Hurew.}"] \\ \bigoplus_\lambda \mathbb{Z} \cong H_n(X^n) \arrow[r] & H_n(X^{n+1}) \cong H_n(X) \end{tikzcd}$$

We know the kernel of the lower horizontal arrow and we therefore get an *upper bound* on the kernel of the upper horizontal arrow (a subgroup containing that kernel) using the commutativity of the diagram. But we already had a lower bound for it, and the upper bound agrees with the lower bound. Therefore, since the horizontal arrows are both surjective, the right-hand vertical arrow must be an isomorphism. $\square$

REMARK 3.1.3. For a wedge of spheres $\bigvee_{\lambda\in\Lambda} S^n$, where $n \ge 2$ is fixed, the inclusions of the wedge summands induce an isomorphism

$$\bigoplus_{\lambda\in\Lambda} \pi_n(S^n,\star) \to \pi_n(\bigvee_{\lambda\in\Lambda} S^n,\star)\,.$$

This was already mentioned in example 1.8 (lecture notes weeks 1 and 2). There are two steps to the proof. Firstly, it is clear that every element of $\pi_n(\bigvee_{\lambda\in\Lambda} S^n,\star)$ comes from $\pi_n(\bigvee_{\lambda\in\Lambda'} S^n,\star)$ for some finite subset $\Lambda' \in \Lambda$. Moreover $\pi_n(\bigvee_{\lambda\in\Lambda'} S^n,\star)$ injects into $\pi_n(\bigvee_{\lambda\in\Lambda} S^n,\star)$ as a direct summand (since $\bigvee_{\lambda\in\Lambda'} S^n$ is a *retract* of $\bigvee_{\lambda\in\Lambda} S^n$). Secondly, as we saw in lecture notes weeks 1 and 2, the inclusion

$$\bigvee_{\lambda\in\Lambda'} S^n \longrightarrow \prod_{\lambda\in\Lambda'} S^n$$

induces an isomorphism

$$\pi_n(\bigvee_{\lambda\in\Lambda'} S^n,\star) \;\longrightarrow\; \pi_n(\prod_{\lambda\in\Lambda'} S^n,\star) \cong \prod_{\lambda\in\Lambda'} \pi_n(S^n,\star) = \bigoplus_{\lambda\in\Lambda'} \pi_n(S^n,\star).$$

COROLLARY 3.1.4. *Let X be a path connected CW-space X with base point such that $\pi_1(X,\star)$ is trivial and the homology groups $H_n(X)$ are trivial for $n \ge 1$. Then X is contractible.*

PROOF. Suppose for a contradiction that X is not contractible. Then by the JHC Whitehead theorem 2.1.1, there is $n \ge 1$ such that $\pi_n(X,\star)$ is nontrivial. Find the minimal such n. It is > 1 by assumption. Then $\pi_n(X,\star) \cong H_n(X)$ for this n, by the Hurewicz theorem 3.1.1. So $H_n(X)$ is also nontrivial. Contradiction. $\square$

3.2. Homotopy of pairs and homology

THEOREM 3.2.1. (Hurewicz.) *Let (Y,X) be a pair of based connected CW-spaces such that the fundamental group $\pi_1(X,\star)$ is trivial and $\pi_k(Y,X,\star)$ is trivial for $k = 0,1,2,\dots,n-1$, where $n \ge 2$. Then the composition*

$$\pi_n(Y,X,\star) \xrightarrow{\text{ind. by quot. map}} \pi_n(Y/X,\star) \xrightarrow{\text{Hurewicz homom.}} H_n(Y/X)$$

is an isomorphism.

For the proof we need a lemma similar to lemma 3.1.2 but slightly more general. (And once again we postpone the proof.)

LEMMA 3.2.2. *Let (Z,X) be a pair of based connected CW-spaces such that $\pi_k(Z,X,\star)$ is trivial for $k=0,1,2,\dots,n-1$, where $n\ge 1$. Then there exists a CW-space Y containing X as a CW-subspace, and a map $Y\to Z$ which is the identity on X, such that*

- *the map $Y\to Z$ is a homotopy equivalence*
- *$Y^{n-1}=X^{n-1}$ (i.e., all cells in $Y\smallsetminus X$ have dimension $\ge n$).*

REMARK 3.2.3. Let (Y,X) be a pair of based spaces where X is path connected, and let $x_0,x_1\in X$. Then for $n\ge 2$ the groups $\pi_n(Y,X,x_0)$ and $\pi_n(Y,X,x_1)$ are isomorphic; in fact a choice of path in X from x_0 to x_1 determines an isomorphism ι_γ between the two. This can also be used to define an action (by group automorphisms) of $\pi_1(X,x_0)$ on $\pi_n(Y,X,x_0)$. The forgetful map from $\pi_n(Y,X,x_0)$ to the set of homotopy classes of unbased maps from (D^n,S^{n-1}) to (Y,X) is always surjective; two elements of $\pi_n(Y,X,x_0)$ determine the same unbased homotopy class if and only if they are in the same orbit of the action of $\pi_1(X,x_0)$. (The proof of these statements is an exercise.)

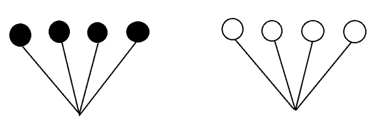
We need one more lemma. Assuming lemma 3.2.2, we can restate the assumptions of the Hurewicz theorem 3.2.1 as saying that we have a pair of CW-spaces (Y,X) where X is based, connected and has trivial fundamental group, and $Y^{n-1}=X^{n-1}$, where $n\ge 2$ is fixed. Choose characteristic maps

$$\varphi_j\colon (D^n,S^{n-1})\longrightarrow (Y^n,Y^{n-1})=(Y^n,X^{n-1})$$

for the n-cells of Y not contained in X. These maps need not take base point to base point, but for each j we can choose a path γ_j in X^n from φ_j of the base point (of S^{n-1}) to $\star$, the base point of X^n. Let J be the disjoint union of the base points in each copy of S^{n-1}. Together the φ_j and the γ_j define a based map of pairs

$$f\colon \big(\mathrm{cone}(J\hookrightarrow \textstyle\coprod_j D^n),\mathrm{cone}(J\hookrightarrow \coprod_j S^{n-1})\big)\longrightarrow (Y^n,X^n).$$

(We are using unreduced mapping cones. An unreduced mapping cone has its own standard base point.) Here is a budding artist's impression of the two mapping cones.



LEMMA 3.2.4. *The map $\pi_n(\mathrm{cone}(J\hookrightarrow\coprod_j D^n),\mathrm{cone}(J\hookrightarrow\coprod_j S^{n-1}))\longrightarrow \pi_n(Y^n,X^n,\star)$ induced by f is surjective.*

PROOF. By remark 3.2.3, when we represent elements of $\pi_n(Y^n,X^n,\star)$ by maps of pairs there is no need to pay attention to base points. We begin with

$$g\colon (D^n,S^{n-1})\to (Y^n,X^n)$$

representing an element of $\pi_n(Y^n,X^n,\star)$. The goal is to show that g is unbased homotopic, as a map of pairs, to a map in the form of a composition

$$(D^n,S^{n-1}) \dashrightarrow \big(\mathrm{cone}(J\hookrightarrow \textstyle\coprod_j D^n),\mathrm{cone}(J\hookrightarrow \coprod_j S^{n-1})\big)\xrightarrow{\ f\ }(Y^n,X^n)$$

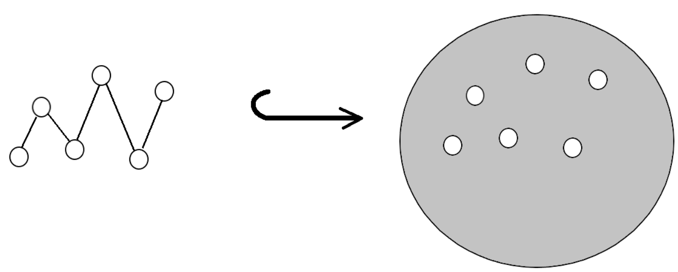
where f was defined above. (It is not important whether the broken arrow is a based map or not — it is certainly homotopic to a based map since $\mathrm{cone}(J \hookrightarrow \coprod S^{n-1})$ is path connected.) — By smooth approximation and Sard's theorem, we can assume that the sets $S_j := g^{-1}(\varphi_j(0))$ are finite and that for each $z \in S_j$ there is a small compact disk K_z inside D^n, centered at z, such that $\varphi_j^{-1}g$ maps a neighborhood of K_z smoothly and diffeomorphically to a neighborhood of 0 in $D^n \smallsetminus S^{n-1}$. The disks K_z for $z \in \bigcup_j S_j$ are pairwise disjoint and we can also suppose that their images under the first projection $p_1 : D^n \to \mathbb{R}$ are pairwise disjoint. (If not, pre-compose g with a suitable perturbation, alias diffeomorphism $D^n \to D^n$ which is the identity in a neighborhood of the boundary.) This gives us a way to number the $z \in \bigcup S_j$ consecutively by comparing their first coordinates; so we write $z(1), z(2), \dots, z(r)$. Now draw a straight line segment $L(1)$ of minimal length from $\partial K_{z(1)}$ to $\partial K_{z(2)}$; next a straight line segment $L(2)$ of minimal length from $\partial K_{z(2)}$ to $\partial K_{z(3)}$, etc. Let Q be the union of the little disks $K_{z(1)}$, $K_{z(2)}$, $K_{z(3)}$, ..., $K_{z(r)}$ and the segments $L_{z(1)}$, $L_{z(2)}$, $L_{z(3)}$, ... $L_{z(r-1)}$. *Claim*:

$$\partial Q := \partial K_{z(1)} \cup L(1) \cup \partial K_{z(2)} \cup L(2) \cup \cdots \cup L(q-1) \cup K_{z(q)}$$

is a strong deformation retract of

$$D^n \smallsetminus \mathrm{int}(Q) \;=\; D^n \smallsetminus \mathrm{int}\left(K_{z(1)} \cup K_{z(2)} \cup \cdots \cup K_{z(q)}\right).$$

Meaning: there exists a homotopy $(h_s : D^n \smallsetminus \mathrm{int}(Q) \to D^n \smallsetminus \mathrm{int}(Q))_{s \in [0,1]}$ which is stationary on ∂Q and such that $h_0 = \mathrm{id}$ whereas $h_1(D^n \smallsetminus \mathrm{int}(Q)) \subset \partial Q$. The proof of the claim is left to the gentle reader, but the budding artist is back trying to help us visualize the inclusion of ∂Q into $D^n \smallsetminus \mathrm{int}(Q)$:



Let $h_t^e : D^n \to D^n$ be defined like h_t on $D^n \smallsetminus Q$ and like the identity on Q. The homotopy $(gh_t^e)_{t \in [0,1]}$ can be viewed as an unbased homotopy of maps

$$(D^n, \partial D^n) \longrightarrow (Y^n, U)$$

where U is Y^n minus all the points $\varphi_j(0)$ (the *center points* of the n-cells of Y not contained in X). The inclusion of (Y^n, X^n) in (Y^n, U) is a homotopy equivalence of pairs, so we can in fact replace (Y^n, X^n) by (Y^n, U) without loss of essential information. The homotopy $(gh_t^e)_{t \in [0,1]}$ begins with $gh_0^e = g$ and ends with gh_1^e. But gh_1^e is the composition of h_1^e viewed as a map from (D^n, S^{n-1}) to $(Q, \partial Q)$ with

$$g|_Q : (Q, \partial Q) \longrightarrow (Y^n, U).$$

Therefore it only remains to show that $g|_Q\colon(Q,\partial Q)\to(Y^n,U)$ is homotopic to a composition

$$(Q,\partial Q) \dashrightarrow \big(\mathrm{cone}(J\hookrightarrow \textstyle\coprod_j D^n),\mathrm{cone}(J\hookrightarrow \coprod_j S^{n-1})\big) \xrightarrow{f} (Y^n,U).$$

Here it is convenient to replace S^{n-1} by $D^n\smallsetminus 0$. So now we are hoping to show that $g|_Q\colon(Q,\partial Q)\to(Y^n,U)$ is homotopic to a composition

$$(Q,\partial Q) \dashrightarrow \big(\mathrm{cone}(J\hookrightarrow \textstyle\coprod_j D^n),\mathrm{cone}(J\hookrightarrow \coprod_j (D^n\smallsetminus 0))\big) \xrightarrow{f} (Y^n,U).$$

But this is easy. Define $\bar g$ on $K_z\subset Q$ so that it agrees with $\varphi_j^{-1}g$, where $g(z)=\varphi_j(0)$. Then $\bar g(\partial K_z)$ is contained in

$$(\coprod_j D^n\smallsetminus 0)\subset \mathrm{cone}(J\hookrightarrow \coprod_j (D^n\smallsetminus 0)).$$

Now try to extend the definition of $\bar g$ to the segments $L(i)$. They have to be mapped to $\mathrm{cone}(J\hookrightarrow\coprod_j D^n\smallsetminus 0)$. This extension problem has a solution because $\mathrm{cone}(J\hookrightarrow\coprod_j D^n\smallsetminus 0)$ is path connected. In this way $\bar g$ can be defined on all of Q. On the little disks K_z we have agreement of $f\bar g$ with g. Therefore it suffices to show that $f\bar g$ restricted to a segment $L(i)$ is homotopic to g restricted to the segment $L(i)$, by a homotopy (of maps to U) which is stationary on the boundary points of the segment. This is clear since the fundamental group of $U \simeq X^n$ is trivial. □

Proof of theorem 3.2.1 modulo lemma 3.2.2. We can assume that $Y^{n-1}=X^{n-1}$. By analogy with the proof of theorem 3.1.1 we start with a commutative diagram

$$\begin{array}{ccc} \pi_n(Y^n,X^n) & \longrightarrow & \pi_n(Y,X) \\ \downarrow & & \downarrow \\ \bigoplus_j \mathbb{Z}\cong H_n(Y^n/X^n) & \longrightarrow & H_n(Y/X) \end{array}$$

where j runs through a set of labels for the n-cells of $Y\smallsetminus X$. (Base points have been suppressed.) The horizontal arrows are induced by the inclusion $Y^n\to Y$, and are known to be surjective (the upper horizontal arrow by cellular approximation). By lemma 3.2.4, there is a surjective homomorphism

$$f_*\colon \pi_n(\mathrm{cone}(J\hookrightarrow \textstyle\coprod_j D^n),\mathrm{cone}(J\hookrightarrow \coprod_j S^{n-1})) \longrightarrow \pi_n(Y^n,X^n).$$

If $n\ge 3$, source and target of this homomorphism are abelian and the source group is isomorphic to $\bigoplus_j\mathbb{Z}$, from the long exact sequence of homotopy groups of the pair $(\mathrm{cone}(J\hookrightarrow\coprod_j D^n),\mathrm{cone}(J\hookrightarrow\coprod_j S^{n-1}))$. If $n=2$, the target group is abelian, from the long exact sequence of homotopy groups of the pair (Y^n,X^n). The source group is a free group with generators corresponding to the labels j; this follows again from the long exact sequence of homotopy groups of the pair $(\mathrm{cone}(J\hookrightarrow\coprod_j D^n),\mathrm{cone}(J\hookrightarrow\coprod_j S^{n-1}))$. Therefore, in all cases, we obtain a surjection from $\bigoplus_j\mathbb{Z}$ to the abelian group $\pi_n(Y^n,X^n)$. Using this, it follows that the left-hand vertical arrow in the little square above is an isomorphism (of abelian groups). Therefore, as in the proof of theorem 3.1.1, it suffices to show that the kernel of the upper horizontal arrow "contains" the kernel of the lower horizontal arrow. (Quotation marks apply because the two kernels are subgroups of two different abelian groups, which are however related by a preferred isomorphism.) This is easy to establish by looking at the element of $\pi_n(Y^n,X^n)$ defined by the attaching map

$\alpha: S^n \to Y^n$ for an $(n+1)$-cell of Y not in X. That element is in the kernel of the upper horizontal arrow. □

3.3. Trading cells

DEFINITION 3.3.1. A map of spaces $f: X \to Y$ is 0-*connected* if it induces a surjection of path components, $f_*: \pi_0(X) \to \pi_0(Y)$. (The sets $\pi_0(X)$ and $\pi_0(Y)$ do not really depend on base points, so none has been specified.)
A map of spaces $f: X \to Y$ is n-*connected*, where n is a positive integer, if it is 0-connected and for every $x_0 \in X$ the map

$$f_*: \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$$

is bijective for $k = 0, 1, 2, \ldots, n-1$ and surjective for $k = n$.
A map of spaces $f: X \to Y$ which is n-connected for all $n \geq 0$ is also called a *weak equivalence*.

EXAMPLE 3.3.2. (i) Let X be a CW-space. We have seen that the inclusion $X^n \to X$ is n-connected.
(ii) Let (Y, X) be a pair of nonempty spaces. The long exact sequence of homotopy groups (homotopy sets) of the pair (Y, X) implies that the inclusion $X \to Y$ is n-connected if and only if $\pi_k(Y, X, x_0)$ is trivial (has just one element) for $k = 0, 1, \ldots, n$.
(iii) If $f: X \to Y$ is n-connected, where $n > 0$, then it is also $(n-1)$-connected.
(iv) If $f: X \to Y$ is a map of CW-spaces which is a weak equivalence, then it is a homotopy equivalence according to JHC Whitehead's theorem.

PROOF OF LEMMA 3.1.2. Given a CW-space X with the stated properties, we construct a CW-space Y such that $Y^{n-1} = \star$ and a map $g: Y \to X$ which is a weak equivalence (and therefore a homotopy equivalence). The plan is to construct Y^k and a map $g^k: Y^k \to X$ simultaneously, by induction on k, so that g^k is k-connected and g^k agrees with $g^{(k+1)}$ on $Y^k \subset Y^{k+1}$. The induction begins with $Y^{n-1} = \star$ and $g^{n-1}: Y^{n-1} \to X$ equal to the inclusion of the base point. By our assumptions on X, this is indeed an $(n-1)$-connected map. Now assume that $g^k: Y^k \to X$ has already been constructed and is k-connected, where $k \geq n-1$ is fixed. We distinguish two cases.
(i) If $n = 1$ and $k = n - 1 = 0$, then $Y^0 = \star$. The map from $\pi_0(Y^0, \star)$ to $\pi_0(X, \star)$ determined by g^0 is a bijection, but the map from $\pi_1(Y^0, \star)$ to $\pi_1(X, \star)$ determined by g^0 need not be surjective. Choose based maps $\gamma_i: S^1 \to X$ such that the classes $[\gamma_i] \in \pi_1(X, \star)$ form a generating set for that group. Define Y^1 to be the wedge $\bigvee_i S^1$ of as many circles and define g^1 so that it agrees with γ_i on the circle (wedge summand) with label i. Then $g^1: Y^1 \to X$ is 1-connected.
(ii) Otherwise start by observing that the map $\pi_k(Y^k, \star) \to \pi_k(X, \star)$ determined by g^k is a surjective homomorphism of groups. Choose based maps $\alpha_i: S^k \to Y$ such that the classes $[\alpha_i]$ generate the kernel of that homomorphism, and for each i choose a map $\beta_i: D^{k+1} \to X$ which extends $g^k \circ \alpha_i$. Choose based maps $\gamma_j: S^{k+1} \to X$ such that the classes $[\gamma_j]$ generate the group $\pi_{k+1}(X, \star)$. Define Y^{k+1} to be

$$\left(Y^k \cup_{\vee \alpha_i} \bigvee_i D^{k+1} \right) \vee \bigvee_j S^{k+1}.$$

In words: Y^{k+1} is the $(k+1)$-dimensional CW-space obtained from the k-dimensional CW-space Y^k by first using the maps α_i as attaching maps for so many $(k+1)$-cells, and then taking the wedge sum with so many spheres S^{k+1}. Define g^{k+1} so that the composition

$$\bigvee_i D^{k+1} \longrightarrow Y^{k+1} \xrightarrow{g^{k+1}} X$$

agrees with $\bigvee_i \beta_i$ and so that the composition

$$\bigvee_j S^{k+1} \longrightarrow Y^{k+1} \xrightarrow{g^{k+1}} X$$

agrees with $\bigvee_i \gamma_i$. — Finally let $Y = \bigcup Y^k$ and define g on Y so that it agrees with g^k on Y^k. □

Proof of lemma 3.2.2. This is very similar to the proof of lemma 3.1.2. Given a CW-pair (Z,X) with the stated properties, we construct a CW-space Y containing X as a CW-subspace such that $Y^{n-1} = X^{n-1}$ and a map $g: Y \to Z$ which is a weak equivalence (and therefore a homotopy equivalence). The plan is to construct Y^k and a map $g^k: Y^k \to Z$ simultaneously, by induction on k, so that g^k is k-connected and g^k agrees with $g^{(k+1)}$ on $Y^k \subset Y^{k+1}$. The induction begins with $Y^{n-1} = X^{n-1}$ and $g^{n-1}: Y^{n-1} \to Z$ equal to the inclusion of X^{n-1} in Z. By our assumptions on the pair (Z,X), this is indeed an $(n-1)$-connected map. Now assume that $g^k: Y^k \to Z$ has already been constructed and is k-connected, where $k \geq n-1$ is fixed. We distinguish two cases.
(i) If $n = 1$ and $k = n-1 = 0$, then $Y^0 = X^0$. Choose based maps $\gamma_j: S^1 \to Z$ such that the classes $[\gamma_j]$ generate the group $\pi_1(Z,\star)$. Let

$$Y^1 := X^1 \vee \bigvee_j S^1.$$

Define $g^1: Y^1 \to Z$ so that it agrees with the inclusion $X^1 \to Z$ on X^1 and with $\bigvee_j \gamma_j$ on $\bigvee_j S^1$.
(ii) Otherwise start by observing that the map $\pi_k(Y^k,\star) \to \pi_k(Z,\star)$ determined by g^k is a surjective homomorphism of groups. Choose based maps $\alpha_i: S^k \to X$ such that the classes $[\alpha_i]$ generate the kernel of that homomorphism, and for each i choose a map $\beta_i: D^{k+1} \to X$ which extends $g^k \circ \alpha_i$. Choose based maps $\gamma_j: S^{k+1} \to X$ such that the classes $[\gamma_j]$ generate the group $\pi_{k+1}(X,\star)$. Define Y^{k+1} to be ... (continue as in the proof of lemma 3.1.2). □

3.4. Homotopy equivalences and homology

Theorem 3.4.1. (G. Whitehead) *Let $f: X \to Y$ be a map between path connected CW-spaces which induces an isomorphism $H_k(X) \to H_k(Y)$ for all k. Suppose also that $\pi_1(X,x_0)$ and $\pi_1(Y,y_0)$ are trivial (for some or all $x_0 \in X$, $y_0 \in Y$). Then f is a homotopy equivalence.*

Proof. Without loss of generality, f is the inclusion $X \hookrightarrow Y$ of a CW-subspace. The long exact sequence of homology groups implies that $\tilde{H}_k(Y/X)$ is zero for all k. From our assumptions we also get that $\pi_1(Y,X,\star)$ is trivial for any choice of base point $\star \in X$. The second Hurewicz theorem 3.2.1 then implies that $\pi_k(Y,X,\star)$ is trivial for every choice of $\star \in X$ and $k \geq 2$. (If not, choose minimal $k \geq 2$ for which $\pi_k(Y,X,\star)$ is nontrivial; note that this $\pi_k(Y,X,\star)$ is isomorphic to $H_k(Y/X)$ which is zero, contradiction.) Since $\pi_k(Y,X,\star)$ is trivial for all $k \geq 1$, it follows that the inclusion $X \to Y$ is a weak equivalence and therefore a homotopy equivalence by JHC Whitehead's theorem 2.1.1. □

3.5. Related thoughts

REMARK 3.5.1. Under the assumptions of the second Hurewicz theorem 3.2.1, the space Y/X is path connected and has trivial fundamental group. This follows from lemma 3.2.2. By that lemma we can pretend that $Y^{n-1} = X^{n-1}$, in which case Y/X has no cells in dimension $< n$ other than the base point. (And n is at least 2.) Therefore by the *first* Hurewicz theorem 3.1.1, the Hurewicz homomorphism $\pi_n(Y/X, \star) \to H_n(Y/X)$ is an isomorphism. Therefore the *second* Hurewicz theorem is equivalent to the statement that

$$\pi_n(Y, X, \star) \to \pi_n(Y/X, \star)$$

(induced by the quotient map ...) is an isomorphism of groups, under such and such assumptions.
One may ask whether this homomorphism $\pi_n(Y, X, \star) \to \pi_n(Y/X, \star)$ is an isomorphism in more general circumstances. That is what the Blakers-Massey theorem is about. We will probably get to know it later.

REMARK 3.5.2. The G. Whitehead theorem 3.4.1 becomes false if the condition that $\pi_1(X, \star)$ be trivial is dropped. Specifically, there exist connected based CW-spaces X with nontrivial $\pi_1(X, \star)$ such that the unique map from X to a point induces isomorphisms in homology (that is, X has the homology of a point, $H_0(X) \cong \mathbb{Z}$ and $H_k(X) = 0$ for $k > 0$). Note that this map from X to a point is not a homotopy equivalence because it does not induce an isomorphism of fundamental groups. — See the exercises for more (counter)examples.

REMARK 3.5.3. There are more complicated variants of the second Hurewicz theorem 3.2.1 in which X is allowed to have a nontrivial fundamental group. I recommend to work around them as follows.
(i) *Case* $n \geq 3$. Let (Y, X) be a pair of based connected CW-spaces with the property that $\pi_k(Y, X, \star)$ is trivial for $k = 0, 1, 2, \dots, n-1$, where $n \geq 3$. Then the inclusion-induced homomorphism from $\pi_1(X, \star)$ to $\pi_1(Y, \star)$ is an isomorphism. We can pass to the pair of universal covers

$$(\tilde{Y}, \tilde{X}).$$

We have $\pi_n(Y, X, \star) \cong \pi_n(\tilde{Y}, \tilde{X}, \star)$ for very general reasons (as in Prop. 1.7, lecture notes for weeks 1 and 2). But the pair $(\tilde{Y}, \tilde{X})$ satisfies the assumptions of theorem 3.2.1 and so we get

$$\pi_n(Y, X, \star) \cong \pi_n(\tilde{Y}, \tilde{X}, \star) \cong H_n(\tilde{Y}/\tilde{X}) \ .$$

This is quite satisfactory in my opinion. But if you still wish to make a connection with $H_n(Y/X)$, then you are asking how $H_n(\tilde{Y}/\tilde{X})$ is related to $H_n(Y/X)$. This can be answered by comparing the corresponding cellular chain complexes. The result should be that

$$H_n(Y/X) \cong \mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X,\star)]} H_n(\tilde{Y}/\tilde{X}) \, .$$

Exercise: make sense of that and prove it.
(ii) *Case* $n = 2$. Let (Y, X) be a pair of based connected CW-spaces with the property that $\pi_1(Y, X, \star)$ is trivial. Then the inclusion-induced homomorphism $\pi_1(X, \star) \to \pi_1(Y, \star)$ is onto. Let $\tilde{Y} \to Y$ be the universal covering of Y and let

$$\tilde{Y}|_X \longrightarrow X$$

be the *connected* covering space of X obtained by restricting that to X. Then we have the following commutative diagram with exact rows:

$$\begin{tikzcd} \pi_2(\tilde{Y}) \arrow[r] \arrow[d, "\cong"] & \pi_2(\tilde{Y}, \tilde{Y}|_X) \arrow[r, "\text{onto}"] \arrow[d] & \pi_1(\tilde{Y}|_X) \arrow[d, "\text{understood}"] \\ H_2(\tilde{Y}) \arrow[r] & H_2(\tilde{Y}/(\tilde{Y}|_X)) \arrow[r, "\text{onto}"] & H_1(\tilde{Y}|_X) \end{tikzcd}$$

It follows that the vertical arrow in the middle is an *abelianization* like the vertical arrow on the right; i.e., it is onto and the kernel is the smallest normal subgroup of the source with an abelian quotient. This is again satisfactory in my opinion! But if you still wish to make a connection with $H_2(Y/X)$, then you are asking how $H_2(\tilde{Y}/(\tilde{Y}|_X))$ is related to $H_2(Y/X)$. This can be answered by comparing the corresponding cellular chain complexes. The result should be that

$$H_2(Y/X) \cong \mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(Y,\star)]} H_2(\tilde{Y}/\tilde{Y}|_X).$$

REMARK 3.5.4. The ideas in the proofs of lemma 3.1.2 and lemma 3.2.2 can also be used to prove the following.

(i) For any space Z there exists a CW-space Y and a map $g: Y \to Z$ which is a weak equivalence.
(ii) For any space Z and map f from a CW-space X to Z, there exists a CW-space Y containing X as a CW-subspace and a map $g: Y \to Z$ which is a weak equivalence and satisfies $g|_X = f$.

Obviously (ii) implies (i); so here is a sketch proof of (ii). We proceed by induction. Suppose that Y^k has already been constructed and contains X^k; also $g^k: Y^k \to Z$ has been constructed and is k-connected, and agrees with f on X^k. (The induction beginning is easy; so assume $k \geq 0$.) Then

$$(g^k \cup f): Y^k \cup X \longrightarrow Z$$

is also k-connected. For every choice of $\star \in Y^0$ and element in the kernel of

$$(g^k \cup f)_*: \pi_k(Y^k \cup X, \star) \to \pi_k(Z, g^k(\star)),$$

represent the element by a based cellular map $\alpha_i: S^k \to Y^k$ and choose an extension $\beta_i: D^{k+1} \to Z$ of $g^k \circ \alpha_i$. For every element of $\pi_{k+1}(Z, g^k(\star))$, represent the element by a based map $\gamma_j: S^{k+1} \to Z$. Define Y^{k+1} to be

$$\left((Y^k \cup X^{k+1}) \cup_{\vee \alpha_i} \bigvee_i D^{k+1}\right) \vee \bigvee_j S^{k+1}.$$

Define g^{k+1} so that the composition

$$\textstyle\bigvee_i D^{k+1} \longrightarrow Y^{k+1} \xrightarrow{g^{k+1}} Z$$

agrees with $\bigvee_i \beta_i$ and so that the composition

$$\textstyle\bigvee_j S^{k+1} \longrightarrow Y^{k+1} \xrightarrow{g^{k+1}} Z$$

agrees with $\bigvee_i \gamma_i$.

CHAPTER 4

Homotopy pushouts and homotopy pullbacks

4.1. Pullbacks, pushouts, homotopy pullbacks and homotopy pushouts

The *pushout* P of a diagram of spaces

$$Y \xleftarrow{f} X \xrightarrow{g} Z$$

is the quotient space obtained from the disjoint union $Y \sqcup Z$ by introducing the relations

$$Y \ni f(x) \sim g(x) \in Z\,.$$

(This is also known as the *colimit* of the above diagram.) From the definition, there is a commutative square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d] \\ Y \arrow[r] & P \end{tikzcd}$$

and this has a universal property. (It is the initial object in a certain category whose objects are commutative squares which extend the given diagram $Y \leftarrow X \rightarrow Z$.)
The *pullback* P of a diagram of spaces

$$Y \xrightarrow{f} X \xleftarrow{g} Z$$

is the subspace of the product $Y \times Z$ consisting of all (y,z) such that $f(y) = g(z)$. (This is also known as the *limit* of the above diagram.) From the definition, there is a commutative square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

and this has a universal property. (It is the terminal object in a certain category whose objects are commutative squares which extend the given diagram $Y \rightarrow X \leftarrow Z$.)

Definition 4.1.1. The *homotopy pushout* of a diagram of spaces $Y \xleftarrow{f} X \xrightarrow{g} Z$ (also known as the *homotopy colimit* of that diagram) is the quotient space Q obtained from the disjoint union

$$Y \sqcup \big(X \times [-1,1]\big) \sqcup Z$$

by introducing the relations

$$Y \ni f(x) \sim (x,-1) \in X \times [-1,1], \qquad X \times [-1,1] \ni (x,1) \sim g(x) \in Z$$

for all $x \in X$. The homotopy pushout Q comes with a canonical map to the pushout P which takes an element of Q represented by $y \in Y$ or $z \in Z$ to the element of P represented by y, respectively z, and which takes an element of Q represented by $(x,t) \in X \times [-1,1]$ to the element of P represented by $f(x) \in Y$ or equivalently by $g(x) \in Z$.

Associated with the homotopy pushout we have a long exact "Mayer-Vietoris" sequence of homology groups,

$$\cdots \longrightarrow H_n(X) \longrightarrow H_n(Y) \oplus H_n(Z) \longrightarrow H_n(Q) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

This is the long exact homology sequence of the pair $(Q, Y \sqcup Z)$. Indeed the inclusion $Y \sqcup Z \to Q$ is clearly a cofibration. We may therefore identify $H_n(Q, Y \sqcup Z)$ with the reduced n-th homology of the quotient $Q/(X \sqcup Y)$ which is $\cong H_{n-1}(X)$.

Definition 4.1.2. The *homotopy pullback* of a diagram of spaces $Y \xrightarrow{f} X \xleftarrow{g} Z$ (also known as the *homotopy limit* of that diagram) is the subspace Q of the product $Y \times X^{[-1,1]} \times Z$ consisting of all triples (y, γ, z) where the path $\gamma\colon [-1,1] \to X$ satisfies $\gamma(-1) = f(y)$ and $\gamma(1) = g(z)$. There is a canonical map from the pullback P of that diagram to the homotopy pullback Q. This takes a point $(y,z) \in P$ (where $f(y) = g(z)$ to the triple (y, γ, z) where $\gamma\colon [-1,1] \to X$ is the constant path with value $f(y) = g(z)$.

Associated with the homotopy pullback we have a long exact "Mayer-Vietoris" sequence of homotopy groups,

$$\cdots \longrightarrow \pi_n(Q) \longrightarrow \pi_n(Y) \oplus \pi_n(Z) \longrightarrow \pi_n(X) \longrightarrow \pi_{n-1}(Q) \longrightarrow \cdots$$

This does of course require a choice of base point in Q (which determines base points y_0 in Y and z_0 in Z, and almost determines a base point in X, although strictly speaking we get two base points $f(y_0), g(z_0)$ in X connected by a preferred path). Except for some extra features near the right-hand end, it is the long exact homotopy group sequence of the forgetful map $Q \to Y \times Z$ which is clearly a fibration. The fiber over the base point is the space of paths in X connecting $f(y_0)$ to $g(z_0)$. Since we have a preferred connecting path, it is easy to show that this fiber is homotopy equivalent to the space of paths in X from $f(y_0)$ to $f(y_0)$, usually denoted ΩX or $\Omega(X, *)$ with the abbreviation $* = f(y_0)$.

Proposition 4.1.3. *If, in a diagram $Y \leftarrow X \to Z$, one of the arrows is a cofibration, then the canonical map $Q \to P$ from homotopy pushout to pushout is a homotopy equivalence.*

Instructions for a proof: suppose that $X \to Z$ is a cofibration, and take this to mean: *inclusion* of a closed subspace which has the homotopy extension property. Show that the pair (Z, X) is homotopy equivalent, relative to X, to the pair (Z^e, X) where Z^e is now the mapping cylinder of the inclusion $X \to Z$. More precisely, show that the projection $(Z^e, X) \to (Z, X)$ is a homotopy equivalence relative to X. Here it is convenient to (re)define the mapping cylinder as the union of $(X \times [-1,1])$ and $Z \times \{1\}$ in $Z \times [-1,1]$. This implies that the projection from the pushout of $Y \leftarrow X \to Z^e$ (which is Q) to the pushout of $Y \leftarrow X \to Z$ (which is P) is a homotopy equivalence relative to X. So this proves more than we wanted to know. □

Proposition 4.1.4. *If, in a diagram $Y \to X \leftarrow Z$, one of the arrows is a fibration, then the canonical map $P \to Q$ from pullback to homotopy pullback is a homotopy equivalence.*

Instructions for a proof: suppose that $Y \to X$ is a fibration, and call it f. Show that Y is homotopy equivalent, as a space over X, to the space Y_e of pairs

$$(y, \gamma) \in Y \times X^{[-1,1]}$$

which satisfy $\gamma(-1) = f(y)$. *Beware*: we make Y_e into a space over X using the map which takes $(y,\gamma) \in Y_e$ to $\gamma(1)$. More precisely, show that a certain "inclusion" of Y in Y_e is a homotopy equivalence in the category of spaces over X. This implies that the inclusion of the pullback of $Y \to X \leftarrow Z$ (which is P) in the pullback of $Y_e \to X \leftarrow Z$ (which is Q) is a homotopy equivalence over X. Again, this proves more than we wanted to know. ☐

PROPOSITION 4.1.5. *If, in a commutative diagram of spaces*

$$\begin{tikzcd} Y_0 \arrow[d] & X_0 \arrow[l, "f_0"'] \arrow[r, "g_0"] \arrow[d] & Z_0 \arrow[d] \\ Y_1 & X_1 \arrow[l, "f_1"'] \arrow[r, "g_1"] & Z_1 \end{tikzcd}$$

the vertical arrows are homotopy equivalences, then the induced map $Q_0 \to Q_1$ of homotopy pushouts of the rows is a homotopy equivalence.

PROPOSITION 4.1.6. *If, in a commutative diagram of spaces*

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & X_0 \arrow[d] & Z_0 \arrow[l] \arrow[d] \\ Y_1 \arrow[r] & X_1 & Z_1 \arrow[l] \end{tikzcd}$$

the vertical arrows are homotopy equivalences, then the induced map $Q_0 \to Q_1$ of homotopy pullbacks of the rows is a homotopy equivalence.

Remark. From the long exact sequences mentioned earlier, we can see immediately that under the assumptions of proposition 4.1.5 the map $Q_0 \to Q_1$ induces an isomorphism in H_n for all n, and under the assumptions of proposition 4.1.6 the map $Q_0 \to Q_1$ induces an isomorphism in π_n for all n (for any choice of base point in Q_0). But we may object that this is not good enough.

Outline of a proof. Concentrating on the proof of proposition 4.1.5, let us introduce a category $\mathscr{C}$ whose objects are diagrams spaces and maps of the form $B \leftarrow A \to C$. A morphism in $\mathscr{C}$ from $B_1 \leftarrow A_1 \to C_1$ to $B_2 \leftarrow A_2 \to C_2$ is a natural transformation, i.e., a commutative diagram

$$\begin{tikzcd} B_1 \arrow[d, "q"] & A_1 \arrow[l] \arrow[r] \arrow[d, "p"] & C_1 \arrow[d, "r"] \\ B_2 & A_2 \arrow[l] \arrow[r] & C_2 \end{tikzcd}$$

To avoid clutter, we may also use notation like

$$(B_1 \leftarrow A_1 \to C_1) \xrightarrow{(q,p,r)} (B_2 \leftarrow A_2 \to C_2).$$

for morphisms. We can also talk about homotopies between morphisms in $\mathscr{C}$. (A homotopy between morphisms from $B_1 \leftarrow A_1 \to C_1$ to $B_2 \leftarrow A_2 \to C_2$ is the same thing as a morphism in $\mathscr{C}$ from $B_1 \times [0,1] \leftarrow A_1 \times [0,1] \to C_1 \times [0,1]$ to $B_2 \leftarrow A_2 \to C_2$.) Now let us try the following lifting problem

$$\begin{tikzcd} & (B_1 \leftarrow A_1 \rightarrow C_1) \arrow[d, "{(q,p,r)}"] \\ (B_0 \leftarrow A_0 \rightarrow C_0) \arrow[r] \arrow[ur, dotted, "{(v,u,w)}"] & (B_2 \leftarrow A_2 \rightarrow C_2) \end{tikzcd}$$

under the condition that the maps $q: B_1 \to B_2$, $p: A_1 \to A_2$, $r: C_1 \to C_2$ are (individually) homotopy equivalences. More precisely, we want to see the dotted arrow (v,u,w) and a (natural) homotopy connecting (qv, pu, rw) to the horizontal morphism. In general this may be hard, but if we assume that the object $(B_0 \leftarrow A_0 \to C_0)$ is *cofibrant* (which means simply that the maps $B_0 \leftarrow A_0$ and $A_0 \to C_0$ are cofibrations) then this lifting problem has a solution. (Exercise.) We can draw the following conclusion. Let

$$(B_1 \leftarrow A_1 \rightarrow C_1) \xrightarrow{(q,p,r)} (B_2 \leftarrow A_2 \rightarrow C_2)$$

be a morphism in $\mathscr{C}$ where both source and target are cofibrant objects, and suppose that the maps q, p, r are (individually) homotopy equivalences. Then the morphism admits a *right* homotopy inverse, i.e., there exists a morphism

$$(B_2 \leftarrow A_2 \rightarrow C_2) \xrightarrow{(v,u,w)} (B_1 \leftarrow A_1 \rightarrow C_1)$$

such that (qv, pu, rw) is homotopic to the identity of $B_1 \leftarrow A_1 \to C_1$. But now it follows that v, u, w are (individually) homotopy equivalences, and so we may reason that the morphism (v,u,w) also has a right homotopy inverse. Then it follows in a formal manner that (q,p,r) and (v,u,w) are reciprocal homotopy equivalences. *And* it follows that the map from the (ordinary) pushout of $B_1 \leftarrow A_1 \to C_1$ to the pushout of $B_2 \leftarrow A_2 \to C_2$ induced by (q,p,r) is a homotopy equivalence. This proves proposition 4.1.5 because we can view the homotopy pushouts there as pushouts. More precisely we can convert the diagram there to a diagram

$$\begin{tikzcd} \mathrm{cyl}(Y_0 \leftarrow X_0) \arrow[d] & X_0 \arrow[l] \arrow[r] \arrow[d] & \mathrm{cyl}(X_0 \rightarrow Z_0) \arrow[d] \\ \mathrm{cyl}(Y_1 \leftarrow X_1) & X_1 \arrow[l] \arrow[r] & \mathrm{cyl}(X_1 \rightarrow Z_1) \end{tikzcd}$$

which can be viewed as a morphism in $\mathscr{C}$ between cofibrant objects. The ordinary pushouts of the rows are now the homotopy pushouts that we are after. □

4.2. Homotopy pushout squares and homotopy pullback squares

We start with a commutative diagram of spaces

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

By the universal properties of pushouts and pullbacks, this determines two maps

$$\mathrm{colim}(B \leftarrow A \rightarrow C) \longrightarrow D\,, \qquad A \longrightarrow \lim(B \rightarrow D \leftarrow C)$$

(from the pushout of $B \leftarrow A \rightarrow C$ to D and from A to the pullback of $B \rightarrow D \leftarrow C$). If the first of these map is a homeomorphism, we say that the square is a *pushout square*, or that it is cocartesian. If the second map is a homeomorphism, we say that the square is a *pullback square*, or that it is *cartesian*.
We may compose with the canonical maps from homotopy pushout to pushput and from pullback to homotopy pullback:

$$\operatorname{hocolim}(B \leftarrow A \rightarrow C) \rightarrow \operatorname{colim}(B \leftarrow A \rightarrow C), \qquad \lim(B \rightarrow D \leftarrow C) \rightarrow \operatorname{holim}(B \rightarrow D \leftarrow C).$$

DEFINITION 4.2.1. If the composite map from $\operatorname{hocolim}(B \leftarrow A \rightarrow C)$ to D is a homotopy equivalence, then we say that the square is a *homotopy pushout square*, or that it is *homotopy cocartesian*. If the composite map from A to $\operatorname{holim}(B \rightarrow D \leftarrow C)$ is a weak homotopy equivalence, we say that the square is a *homotopy pullback square*, or that it is *homotopy cartesian*.

EXAMPLE 4.2.2. If the square is a pushout square and at least one of the two arrows $A \rightarrow C$, $A \rightarrow B$ is a cofibration, then it is a homotopy pushout square. This follows from proposition 4.1.3. Similarly, if the square is a pullback square and at least one of the arrows $B \rightarrow D$, $C \rightarrow D$ is a fibration, then it is a homotopy pullback square. This follows from proposition 4.1.4.

Two more remarks which are nearly obvious in the light of propositions 4.1.5 and 4.1.6. Suppose that we have two commutative squares of spaces

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d] & C_0 \arrow[d] \\ B_0 \arrow[r] & D_0 \end{tikzcd} , \quad \begin{tikzcd} A_1 \arrow[r] \arrow[d] & C_1 \arrow[d] \\ B_1 \arrow[r] & D_1 \end{tikzcd}$$

and a natural transformation between them, consisting of maps $f_a: A_0 \rightarrow A_1$, $f_b: B_0 \rightarrow B_1$, $f_c: C_0 \rightarrow C_1$ and $f_d: D_0 \rightarrow D_1$. If the two squares are homotopy pushout squares and f_a, f_b, f_c are homotopy equivalences, then f_d is a homotopy equivalence. If the two squares are homotopy pullback squares and f_b, f_c, f_d are homotopy equivalences, then f_a is a homotopy equivalence.
Another remark: there are variants of these definitions where the notion of *homotopy equivalence* is replaced by *weak homotopy equivalence*. In that setting it is customary to replace the notion of *fibration* by *Serre fibration*.

The following two lemmas will be useful later.

LEMMA 4.2.3. *Let*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

be a commutative square of spaces and a homotopy pullback square. Let $p: E \rightarrow D$ *be a fibration and let*

$$\begin{tikzcd} A^* \arrow[r] \arrow[d] & C^* \arrow[d] \\ B^* \arrow[r] & D^* \end{tikzcd}$$

be the commutative square obtained from the above (with spaces A, B, C, D*) by taking the pullback with* p *along the reference maps to* D *(for example,* A^* *is the pullback of*

$A \to D \leftarrow E$, and D^* is homeomorphic to E). Then this new square is again a homotopy pullback square.

Proof. If the first square happens to be an ordinary pullback square and the vertical arrows in it are fibrations, then the second square is also an ordinary pullback square and the vertical arrows in it are fibrations. In that situation both squares are homotopy pullback squares. Therefore we try to reduce to that situation.
The first step in the reduction consists in replacing the right-hand vertical arrow by a fibration, using the Serre method. This replaces C by a larger space, call it C_1. We make no changes to D and $p: E \to D$. It is easy to show that the inclusion of C^* in $(C_1)^*$ is a homotopy equivalence (because the inclusion $C \to C_1$ is a homotopy equivalence.) Therefore this modification or reduction does not change the content of the assertion. In the second reduction step, we may assume that the map from C to D is already a fibration and the change that we make is to replace A by the pullback A_1 of $(B \to D \leftarrow C)$. By proposition 4.1.6, the comparison map from A to A_1 is a homotopy equivalence. It follows that the comparison map from A^* to $(A_1)^*$ is again a homotopy equivalence. Therefore this second modification or reduction does not change the content of the assertion either. This completes the reduction process and the proof. □

Lemma 4.2.4. *Let*

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

be a commutative square of spaces; suppose that all spaces involved are homotopy equivalent to CW-spaces. The following conditions are equivalent:

(i) *the square is a homotopy pullback square;*
(ii) *for every $z \in D$, the map*

$$\mathrm{hofiber}_z[\,A \to D\,] \longrightarrow \mathrm{hofiber}_z[\,B \to D\,] \times \mathrm{hofiber}_z[\,C \to D\,]$$

determined by the square is a homotopy equivalence.

Proof. The implication (i) $\Rightarrow$ (ii) is really a special case of lemma 4.2.3. Namely, for E take the space of paths $\gamma: [0,1] \to D$ starting at z, and for $p: E \to D$ take the map given by $p(\gamma) = \gamma(1)$, where $\gamma: [0,1] \to D$ is a typical element of E. Then the new square with spaces A^*, B^*, C^*, D^* takes the form

$$\begin{array}{ccc} \mathrm{hofiber}_z[\,A \to D\,] & \longrightarrow & \mathrm{hofiber}_z[\,C \to D\,] \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{hofiber}_z[\,B \to D\,] & \longrightarrow & \mathrm{hofiber}_z[\,D \to D\,] \end{array}$$

and it is a homotopy pullback square. Since $\mathrm{hofiber}_z[\,D \to D\,]$, also known as E, is contractible, we may conclude that (ii) holds.
For the converse, suppose that (ii) holds. We compare the square with spaces A, B, C, D with a new commutative square

$$\begin{array}{ccc} A_1 & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ B_1 & \longrightarrow & D \end{array}$$

constructed as follows. For the map $B_1 \to D$ we take the Serre construction on $B \to D$. For the map $C_1 \to D$ we take the Serre construction on $C \to D$. For A_1 we take

$\lim(B_1 \to D \leftarrow C_1)$, the ordinary pullback. There are standard inclusion maps $B \to B_1$ and $C \to C_1$. These are maps over D and so induce a map from A to A_1 by the universal property of the pullback. By hypothesis (ii), for any choice of base point $y \in A$ with image $z \in D$, that map from A to A_1 induces a homotopy equivalence of based spaces

$$\mathrm{hofiber}_z[A \to D] \longrightarrow \mathrm{hofiber}_z[A_1 \to D]$$

and consequently an isomorphism of homotopy groups/sets,

$$\pi_k(\mathrm{hofiber}_z[A \to D]) \longrightarrow \pi_k(\mathrm{hofiber}_z[A_1 \to D])$$

for $k \geq 0$. These isomorphisms extend to a map between long exact sequences (the long exact sequence connecting the groups/sets

$$\pi_k(A, y),\ \pi_k(D, z),\ \pi_k(\mathrm{hofiber}_z[A \to D], y),$$

and the long exact sequence connecting the groups/sets

$$\pi_k(A_1, y),\ \pi_k(D, z),\ \pi_k(\mathrm{hofiber}_z[A_1 \to D], y)).$$

The five lemma (with small adjustments in low dimensions) allows us to conclude that the comparison map $A \to A_1$ is a weak homotopy equivalence. Since both A and A_1 are homotopy equivalent to CW-spaces, it follows that $A \to A_1$ is a homotopy equivalence. □

LEMMA 4.2.5. *Let*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \end{tikzcd}$$

be a commutative diagram of spaces and maps. If the small squares in the diagram are homotopy pullback squares, then the outer rectangle (with spaces A, C, X, Z) is also a homotopy pullback square. Similarly, if the small squares are homotopy pushout squares, then the outer rectangle is a homotopy pushout square.

PROOF. First statement: using the unnumbered remark after example 4.2.2, and the Serre construction, we may first reduce to a situation where the map from C to D is a fibration. Then we may reduce to a situation where in addition the right-hand square is a strict pullback square. In that case the map from B to Y is a fibration and using that same remark once more, we may reduce to a situation where the left-hand square is a strict pullback square (and the right-hand square is a pullback square, and the map from C to Z is a fibration). Now the outer rectangle is a strict pullback square in which the arrow from C to Z is a fibration. Therefore it is a homotopy pullback square. — The proof of the statement about homotopy pushout squares is similar. □

4.3. The magic cube

The magic cube is really a theorem stating this. Suppose we have a commutative diagram of spaces and maps

$$\begin{tikzcd} X_0 \arrow[d] & X_1 \arrow[l] \arrow[r] \arrow[d] & X_2 \arrow[d] \\ Y_0 & Y_1 \arrow[l] \arrow[r] & Y_2 \end{tikzcd}$$

where the little squares are homotopy pullback squares, and the horizontal arrows in the left-hand square (or in the right-hand square) are cofibrations. Let X and Y be the pushouts of the rows. (By assumption on the rows, these pushouts X and Y also qualify as homotopy pushouts.) I also want to assume that all spaces involved are CW-spaces or at least homotopy equivalent to CW-spaces. (There is a price to pay for that. We need to know that certain constructions, most importantly taking the homotopy pullback of a diagram $B \leftarrow A \rightarrow C$, produces spaces homotopy equivalent to CW-spaces if A, B and C are homotopy equivalent to CW-spaces. I have probably already mentioned and recommended a paper by J Milnor on this topic: *On spaces having the homotopy type of a CW-complex*, 1959.)

THEOREM 4.3.1. *Under the above conditions, the squares*

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & X \arrow[d] \\ Y_i \arrow[r] & Y \end{tikzcd}$$

are also homotopy pullback squares for $i = 1, 2$ (and consequently for $i = 0$, too).

To put it differently, we have a diagram in the shape of a 3-dimensional cube in which two of the six 2-dimensional faces (the two front faces) are homotopy pullback squares by assumption, and two more faces (top and bottom face) are homotopy pushout squares by assumption, and two more faces (the two back faces) are homotopy pullback squares because that is the theorem.

$$\begin{tikzcd} & & X \arrow[d] & & \\ X_0 \arrow[urr] \arrow[d] & & Y & & X_2 \arrow[ull] \arrow[d] \\ Y_0 \arrow[urr] & & X_1 \arrow[ull] \arrow[urr] \arrow[d] & & Y_2 \arrow[ull] \\ & & Y_1 \arrow[ull] \arrow[urr] & & \end{tikzcd}$$

In the book by Jeffrey Strom, *Modern classical homotopy theory*, AMS 2011, such a cube is called a *Mather* cube. (I was pleased to discover that Strom takes this concept very seriously and somewhat relieved that he does not give a proof of theorem 4.3.1 which makes my proof look foolish. The Strom book is an unusual book which has a lot to offer, but the offer includes an amazing number of typos and other small errors.)

EXAMPLE 4.3.2. Start with a pushout square of CW-spaces

$$\begin{tikzcd} & Y & \\ Y_0 \arrow[ur] & & Y_2 \arrow[ul] \\ & Y_1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where, for simplicity, all arrows are inclusions of CW-subspaces (so that this is also a homotopy pushout square). Let $p\colon X \rightarrow Y$ be any fibration, where X is homotopy equivalent to a CW-space. Let $X_i = p^{-1}(Y_i)$ for $i = 0, 1, 2$. Then the inclusions of the X_i in X make up a homotopy pushout square

$$\begin{tikzcd}
 & X & \\
X_0 \arrow[ur] & & X_2 \arrow[ul] \\
 & X_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

(by lemma 4.3.3 below). The map p and appropriate restrictions now give us a commutative cube

$$\begin{tikzcd}
 & & X \arrow[d] & & \\
X_0 \arrow[urr] \arrow[d] & & Y & & X_2 \arrow[ull] \arrow[d] \\
Y_0 \arrow[urr] & & X_1 \arrow[ull] \arrow[urr] \arrow[d] & & Y_2 \arrow[ull] \\
 & & Y_1 \arrow[ull] \arrow[urr] & &
\end{tikzcd}$$

which satisfies the conditions, and clearly also the conclusion, of the magic cube theorem.

LEMMA 4.3.3. *Let $p: E \to B$ be a fibration. Let $A \subset B$ be a closed subspace such that the inclusion is a cofibration. Then the inclusion $p^{-1}(A) \to E$ is again a cofibration.*

PROOF. Since the inclusion of A in B is a cofibration, we know that the cylinder $Z := (A \times [0,1]) \cup B \times \{1\}$ is a retract of $B \times [0,1]$. It follows that Z is a strong deformation retract[1] of $B \times [0,1]$. Now let

$$(h_s: B \times [0,1] \to B \times [0,1])_{s \in [0,1]}$$

be a strong deformation retraction retracting $B \times [0,1]$ to Z; so $h_0 = \mathrm{id}$, $h_1(B \times [0,1]) = Z$ and $h_s(x,t) = (x,t)$ for all s and all $(x,t) \in Z$. To upgrade this to a better strong deformation retraction we choose a continuous function[2] $\psi: B \to [0,1]$ such that $\psi^{-1}(0) = A$. Then we define

$$(g_s: B \times [0,1] \to B \times [0,1])_{s \in [0,1]}$$

by $g_s(x,t) = h_{s/\psi(t)}(x,t)$ for all $s, t \in [0,1]$ such that $s \le \psi(t)$ and $\psi(t) > 0$. This means that we still need to define $g_s(x,t)$ for $s > \psi(t)$ (and also for $\psi(t) = 0$, any s); we let $g_s(x,t) = h_1(x,t)$ in those cases. (Continuity by inspection.) By the homotopy lifting property of the fibration

$$q: E \times [0,1] \longrightarrow B \times [0,1] \; ; \; q(y,t) = (p(y),t)$$

we can find a homotopy $(G_s: E \times [0,1] \to E \times [0,1])_{s \in [0,1]}$ such that $G_0 = \mathrm{id}$ and $qG_s = g_s q$ for all $s \in [0,1]$. And now of course we define a retraction $E \times [0,1] \to E \times [0,1]$ with image $q^{-1}(Z)$ by $(y,t) \mapsto G_{\psi(t)}(y,t)$. Since $q^{-1}(Z)$ is the mapping cylinder of $p^{-1}(A) \to E$, this completes the proof. $\square$

[1] Maybe this has been pointed out a few times before. Let $r = (r_1, r_2)$ be a retraction with image Z, where $r_1: B \times [0,1] \to B$ and $r_2: B \times [0,1] \to [0,1]$. Then for each $(x,t) \in B \times [0,1]$ we obtain a path

$$[0,1] \ni s \mapsto (r_1(x,st), (1-s)t + sr_2(t))$$

in $B \times [0,1]$. The value for $s = 0$ is always (x,t) and the value for $s = 1$ is $r(x,t)$, therefore in Z. The path is constant if $(x,t) \in Z$. So these paths make up a strong deformation retraction.

[2] Using notation as in the previous footnote let $\psi(x) = \max\{t - r_2(x,t) \mid t \in [0,1]\}$. This formula comes from chapter VI, Satz 2.15 in Tammo tom Dieck, *Topologie*, 2nd Edition, De Gruyter Lehrbuch, Walter de Gruyter 2000. It was kindly pointed out to me by students.

Let X be a simplicial complex with underlying vertex scheme $(V,\mathscr{S})$. (Some people might prefer to say that X is a *geometric simplicial complex* whereas $(V,\mathscr{S})$ is an *abstract simplicial complex* or a *combinatorial simplicial complex*. In any case $\mathscr{S}$ is a set of finite nonempty subsets of V, subject to conditions.) As a set, X is the set of functions $x: V \to [0,1]$ such that the support of x is an element of $\mathscr{S}$ and $\sum_{s\in V} x(s) = 1$. For $S \in \mathscr{S}$ let $\Delta(S) \subset X$ be the simplex spanned by S, which is the set of all $x \in X$ whose support is contained in S. Let $\partial\Delta(S)$ be the union of all $\Delta(T)$ for proper nonempty $T \subset S$.

LEMMA 4.3.4. *Let $p: E \to X$ be a map of spaces, where X is a simplicial complex with vertex set V. Suppose that*

(i) *for every simplex $\Delta(S) \subset X$, the inclusion $p^{-1}(\partial\Delta(S)) \to p^{-1}(\Delta(S))$ is a cofibration;*
(ii) *for simplices $\Delta(T)) \subset \Delta(S) \subset X$, the inclusion $p^{-1}(\Delta(T)) \to p^{-1}(\Delta(S))$ is a homotopy equivalence;*
(iii) *E is the topological colimit of the subspaces $p^{-1}(K)$, where K runs through the compact simplicial subcomplexes K of X.*

If in addition X is contractible, then for any $v \in V$ the inclusion $p^{-1}(v) \to E$ admits a homotopy left inverse $E \to p^{-1}(v)$.

PROOF. (Still very sketchy.) Fix some $v \in V$. Write F for $p^{-1}(v)$. The conditions give us, for all v, w in V, a preferred homotopy class of homotopy equivalences from $p^{-1}(v)$ to $p^{-1}(w)$. Here we also use the fact that X is simply connected. —
Let Y be simplicial subcomplex of X such that $Y \subset \Delta(S) \subset X$ for some simplex $\Delta(S)$ of X. There is a map

$$p^{-1}(Y) \longrightarrow Y \times p^{-1}(\Delta(S))$$

determined by the projection $p^{-1}(Y) \to Y$ and the inclusion $p^{-1}(Y) \to p^{-1}(\Delta(S))$. *Sub-claim*: this map is a homotopy equivalence. Proof of this: induction on the number of simplices (equivalently, number of cells) of Y. Use proposition 4.1.5 for the induction step. The justification for that comes from condition (i) and the necessary information comes from condition (ii). —
Now let X^k be the k-skeleton and let $E^{(k)} = p^{-1}(X^k)$. We proceed by induction on k. At stage k in the induction, we assume that we have already found a map

$$f^k: E^{(k)} \longrightarrow F$$

which is homotopy left inverse to the inclusion $F \to E^{(k)}$. (The induction beginning is as follows. As was already pointed out, we have a preferred homotopy class of maps from $E^{(0)}$, disjoint union of the $p^{-1}(w)$ for $w \in V$, to F. We choose f^0 in that homotopy class. This guarantees that we can also find f^1 extending f^0.) In the induction step, we ask whether the map f^k (now $k \geq 1$) can be extended to $E^{(k+1)}$. The obstruction is a cocycle on X of dimension $k+1$ with values in the abelian group $\pi_k(\mathrm{map}(F,F),\mathrm{id})$. (Indeed, taking $Y = \partial\Delta(S)$ in the sub-claim just above makes it rather clear that we are talking about a homomorphism from the abelian group of $(k+1)$-chains in the simplicial/cellular chain complex of X to $\pi_k(\mathrm{map}(F,F),\mathrm{id})$. Showing that this homomorphism is a cocycle is an interesting exercise! Showing that $\pi_k(\mathrm{map}(F,F),\mathrm{id})$ is abelian for $k = 1$ is also an exercise, but a well-known one. The point is that $\mathrm{map}(F,F)$ has a structure of topological monoid.) If we allow modifications of f^k , but not of f^{k-1}, then it is only the cohomology class of this cocycle which encodes the remaining obstruction. Since X is contractible, that cohomology class is zero. $\square$

Lemma 4.3.5. *The general case of theorem 4.3.1 follows from the less general case where Y is contractible.*

Proof. Start with a general cube

$$\begin{tikzcd}
& X \arrow[d] & \\
X_0 \arrow[ur] \arrow[d] & Y & X_2 \arrow[ul] \arrow[d] \\
Y_0 \arrow[ur] & X_1 \arrow[ul] \arrow[ur] \arrow[d] & Y_2 \arrow[ul] \\
& Y_1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

satisfying the *conditions* of theorem 4.3.1. (We should not assume here that it satisfies the *conclusion* of the theorem.) Choose a point $b \in Y$; note that Y is the *terminal* object in the cube. Let $p: E \to Y$ be the result of applying the Serre construction to the inclusion $\{b\} \hookrightarrow Y$. In other words, E is the space of paths $\gamma: [0,1] \to Y$ satisfying $\gamma(0) = b$, and p is the map taking such a path γ to $\gamma(1) \in Y$. The map p is a fibration. We now define X^*, X_i^* and Y_i^* for $i = 0, 1, 2$ by taking the (ordinary) pullback of $p: E \to Y$ along the reference maps from X, X_i and Y_i respectively to Y. For example X_1^* is the pullback of

$$X_1 \longrightarrow Y \xleftarrow{\ p\ } Y^*$$

where $X_1 \to Y$ is the map from the cube. The result of this operation is that we have a new commutative diagram in the shape of a cube,

$$\begin{tikzcd}
& X^* \arrow[d] & \\
X_0^* \arrow[ur] \arrow[d] & Y^* & X_2^* \arrow[ul] \arrow[d] \\
Y_0^* \arrow[ur] & X_1^* \arrow[ul] \arrow[ur] \arrow[d] & Y_2^* \arrow[ul] \\
& Y_1^* \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

in which Y^* is the same as E (strictly speaking: homeomorphic to E), and as such is contractible. *Sub-claim*: the new cube also satisfies the conditions of theorem 4.3.1. Indeed, it is clear from the construction that top face and bottom face of the new cube are still pushout squares. By lemma 4.3.3, the arrows $X_0^* \leftarrow X_1^*$ and $Y_0^* \leftarrow Y_1^*$ are cofibrations (if the arrows $X_0 \leftarrow X_1$ and $Y_0 \leftarrow Y_1$ in the original cube were cofibrations, as we may assume without loss of generality). By lemma 4.2.3, the square with vertices X_0^*, X_1^*, Y_0^*, Y_1^* (for example) is a homotopy pullback square. Therefore the sub-claim is proved.
Now, if the theorem holds for cubes whose terminal object is contractible (as we want to pretend), then we may conclude that the new cube satisfies the *conclusion* of the theorem. In particular the square with vertices X_0^*, Y_0^*, X^* and Y^* is a homotopy pullback square. Since Y^* is contractible, this implies that the map from X_0^* to $Y_0^* \times X^*$ determined by

that square is a homotopy equivalence. By lemma 4.2.4, it follows that the face

$$\begin{tikzcd} & X \arrow[d] \\ X_0 \arrow[ur] \arrow[d] & Y \\ Y_0 \arrow[ur] & \end{tikzcd}$$

in the original cube is a homotopy pullback square. This completes the reduction. $\square$

Proof of theorem 4.3.1. First step: we enlarge the terms in the top row of the six-term diagram

$$\begin{tikzcd} X_0 \arrow[d] & X_1 \arrow[l] \arrow[r] \arrow[d] & X_2 \arrow[d] \\ Y_0 & Y_1 \arrow[l] \arrow[r] & Y_2 \end{tikzcd}$$

by applying the Serre construction to the vertical arrows, turning them into fibrations:

$$\begin{tikzcd} X_{0,1} \arrow[d] & X_{1,1} \arrow[l] \arrow[r] \arrow[d] & X_{2,1} \arrow[d] \\ Y_{0,1} & Y_{1,1} \arrow[l] \arrow[r] & Y_{2,1} \end{tikzcd}$$

where $Y_{i,1} := Y_i$. (This may spoil the cofibration condition, i.e., it is not clear that any of the arrows in the top row are cofibrations. But we will repair this later.) Second step: approximate the $Y_i = Y_{i,1}$ by simplicial complexes. More precisely, extend the lower row of the previous diagram to a commutative diagram

$$\begin{tikzcd} Y_{0,2} \arrow[d, "\simeq"] & Y_{1,2} \arrow[l] \arrow[r] \arrow[d, "\simeq"] & Y_{2,2} \arrow[d, "\simeq"] \\ Y_{0,1} & Y_{1,1} \arrow[l] \arrow[r] & Y_{2,1} \end{tikzcd}$$

where the spaces $Y_{i,2}$ are simplicial complexes and the maps in the top row are inclusions of simplicial subcomplexes. Third step: form the pullbacks $X_{i,2}$ of

$$Y_{i,2} \to Y_{i,1} \leftarrow X_{i,1}$$

to obtain a commutative diagram

$$\begin{tikzcd} X_{0,2} \arrow[d] & X_{1,2} \arrow[l] \arrow[r] \arrow[d] & X_{2,2} \arrow[d] \\ Y_{0,2} & Y_{1,2} \arrow[l] \arrow[r] & Y_{2,2} \end{tikzcd}$$

Summary of achievements so far: the vertical arrows in the last diagram are fibrations, the spaces in the lower row are simplicial complexes and the maps in the lower row are inclusions of simplicial subcomplexes. Fourth step: we replace the outer terms in the last diagram by suitable mapping cylinders. This gives

$$\begin{tikzcd} X_{0,3} \arrow[d] & X_{1,3} \arrow[l] \arrow[r] \arrow[d] & X_{2,3} \arrow[d] \\ Y_{0,3} & Y_{1,3} \arrow[l] \arrow[r] & Y_{2,3} \end{tikzcd}$$

where $X_{1,3} = X_{1,2}$ and $Y_{1,3} = Y_{1,2}$ and

$$X_{0,3} = \mathrm{cyl}(X_{0,2} \leftarrow X_{1,2}), \qquad X_{2,3} = \mathrm{cyl}(X_{1,2} \to X_{2,2}),$$
$$Y_{0,3} = \mathrm{cyl}(Y_{0,2} \leftarrow Y_{1,2}), \qquad Y_{2,3} = \mathrm{cyl}(Y_{1,2} \to Y_{2,2}).$$

Now the horizontal arrows (in the last diagram) are cofibrations. We would still like to say that the spaces in the lower row are simplicial complexes and that the maps in the lower row are inclusions of simplicial complexes. This calls for some decisions as in: given a simplicial complex Z, how do we extend the obvious simplicial complex structure on $Z \times \{0,1\}$ to a simplicial complex structure on $Z \times [0,1]$. This is left to the reader.
Finally we rename the terms in the last diagram by omitting the second subscript 3, .i.e. we write (again)

$$\begin{tikzcd} X_0 \arrow[d] & X_1 \arrow[l] \arrow[r] \arrow[d] & X_2 \arrow[d] \\ Y_0 & Y_1 \arrow[l] \arrow[r] & Y_2 \end{tikzcd}$$

but we are now allowed to say: without loss of generality, certain additional conditions are satisfied (e.g., the terms in the lower row are simplicial complexes).
After these unexciting preparations we come to the point. By lemma 4.3.5, we may assume that the pushout Y of the lower row in the last diagram is contractible. At the same time Y has a preferred structure of simplicial complex. Let X be the pushout of the upper row. The map $X \to Y$ now satisfies the conditions of lemma 4.3.4 (the notation there was $E \to X$, whereas here it is $X \to Y$). *More details should be given here.* Therefore we may conclude that the inclusion of any fiber F of $X \to Y$ admits a homotopy left inverse, $X \to F$. The resulting composite maps $X_i \to X \to F$ for $i = 0, 1, 2$ lead to product structures, more precisely, to commutative squares

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & F \arrow[d] \\ Y_i \arrow[r] & * \end{tikzcd}$$

which are homotopy pullback squares by inspection. Therefore we may replace X_i by $Y_i \times F$ and our diagram then simplifies to

$$\begin{tikzcd} Y_0 \times F \arrow[d] & Y_1 \times F \arrow[l] \arrow[r] \arrow[d] & Y_2 \times F \arrow[d] \\ Y_0 & Y_1 \arrow[l] \arrow[r] & Y_2 \end{tikzcd}$$

where the vertical arrows are the projections. In this situation it is clear that the pushout (and homotopy pushout) of the top row is $Y \times F$ while the homotopy pushout of the lower row is still Y by definition. It is also clear that the conclusion of the magic cube theorem is satisfied in this case. □

CHAPTER 5

The James construction

5.1. The free monoid generated by a CW-space

Let X be a CW-space with base point $*$; the base point is assumed to be a 0-cell. For simplicity we assume that X is compact and connected. Let

$$J^{(n)}X := \underbrace{X \times X \times X \times \cdots \times X}_{n} \Big/ \sim$$

(quotient of the n-th power of X), where $\sim$ denotes the following relations:

$$(x_1, x_2, x_3, \ldots, x_j, x_{j+1}, \ldots, x_n) \sim (x_1, x_2, x_3, \ldots, x_{j+1}, x_j, \ldots, x_n)$$

if $x_j = *$ (where $j \in \{1, 2, \ldots, n-1\}$ arbitrary). There is an "inclusion"

$$J^{(n)}X \longrightarrow J^{(n+1)}X$$

defined by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, *)$. Let JX be the monotone union of the $J^{(n)}X$ with the colimit topology; a subset of JX shall be open if and ony if its intersection with every $J^{(n)}X$ is open. It is convenient to represent elements of JX by *words* of finite length in letters taken from X, such as $(x_1, x_2, \ldots, x_m)$, where $m \geq 0$.

(i) The product CW structures on the powers of X determine CW structures on the quotients $J^{(n)}X$ for all n and thereby a CW structure on JX.

(ii) JX comes with a structure of topological monoid[1]. If we think of elements of JX as words with letters taken from X, then the multiplication is given by concatenating words:

$$(x_1, x_2, \ldots, x_m) \cdot (y_1, y_2, \ldots, y_n) := (x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n).$$

The multiplication $JX \times JX \to JX$ is continuous and in fact it is a cellular map.

Let E_X be the quotient space obtained from two disjoint copies of $\mathrm{cone}(X) \times JX$ by gluing points (x, w) in the first copy (where $x \in X \subset \mathrm{cone}(X)$ and $w \in JX$) to (x, wx) in the second copy. (The gluing instruction is meaningful because $X \cong J^{(1)}X \subset JX$ and because JX has this monoid structure. By $\mathrm{cone}(X)$ we mean the reduced cone; this is again meaningful since X has a base point.)

LEMMA 5.1.1. *E_X is contractible.*

PROOF. Exercise. The following alternative to E_X could be useful: the quotient space obtained from the disjoint union of $\mathrm{cone}(X) \times JX$ and JX by gluing points (x, w) in $X \times JX \subset \mathrm{cone}(X) \times JX$ to wx in JX. There is a quotient map from E_X to the proposed alternative which is easily seen to be a homotopy equivalence. (It is still a hard exercise.

[1]... has an asociative multiplication with a neutral element. It is not claimed that every element has a left or right inverse.

Advice: do not try to find a contracting homotopy directly. Instead, find an ascending filtration of E_X and investigate the filtration quotients.)

THEOREM 5.1.2. *$JX \simeq \Omega\Sigma X$ where $\Sigma X = S^1 \wedge X$ is the reduced suspension of X.*

PROOF. There is a commutative diagram

$$\begin{tikzcd} \mathrm{cone}(X) \times JX \arrow[d, "\text{proj.}"] & X \times J(X) \arrow[l, "f"'] \arrow[r, "\text{incl.}"] \arrow[d, "\text{proj.}"] & \mathrm{cone}(X) \times JX \arrow[d, "\text{proj.}"] \\ \mathrm{cone}(X) & X \arrow[l, "\text{incl.}"'] \arrow[r, "\text{incl.}"] & \mathrm{cone}(X) \end{tikzcd}$$

where $f(x, w) = (x, wx)$. Taking the pushouts of the rows we obtain a map

$$\begin{tikzcd} E_X \arrow[d] \\ \Sigma X \end{tikzcd}$$

(where ΣX comes in as the union of two copies of $\mathrm{cone}(X)$ along their common boundary X). By the theorem of the cube, this fits into a homotopy pullback square

$$\begin{tikzcd} \mathrm{cone}(X) \times J(X) \arrow[r] \arrow[d] & E_X \arrow[d] \\ \mathrm{cone}(X) \arrow[r] & \Sigma X \end{tikzcd}$$

Therefore the homotopy fiber of $E_X \to \Sigma X$ is identified with (maps by a homotopy equivalence to) the homotopy fiber of the projection $\mathrm{cone}(X) \times JX \to \mathrm{cone}(X)$, which is JX. But we know also that E_X is contractible, so that the homotopy fiber of $E_X \to \Sigma X$ must be homotopy equivalent to the homotopy fiber of $* \hookrightarrow \Sigma X$, which is ΩX. □

In formulating theorem 5.1.2, we could have done better by stating that such-and-such an explicit *map* from JX to ΩX is a homotopy equivalence. Let us try that.[2] First of all we interpret ΩX as the space of all pairs (a, γ) where $a \in \mathbb{R}$, $a \geq 0$, and $\gamma\colon [0, a] \to X$ is a path satisfying $\gamma(0) = * = \gamma(a)$. This has the advantage that we have a strictly associative multiplication on ΩX given by $(a, \gamma) \cdot (b, \tau) := (a + b, \tau \circ \gamma)$ where $(\tau \circ \gamma)(t)$ means $\gamma(t)$ for $t \in [0, a]$ and $\tau(t - a)$ for $t \in [a, a + b]$. (This variant of ΩX is often called the *Moore* loop space of X, after John Moore presumably.) Choose a continuous map $v\colon X \to [0, 1]$ such that $v^{-1}(0)$ is precisely the base point, and write $v_y := v(y)$. There is a based map from X to $\Omega\Sigma X$ given by

$$y \mapsto (v_y, \gamma_y)$$

for $y \in X$, $y \neq *$, where $\gamma_y(v_y t) = (e^{2\pi i t}, y)$ for $t \in [0, 1]$ (if we think of ΣX as a quotient of $S^1 \times X$). This extends uniquely to a continuous homomorphism

$$\alpha\colon JX \to \Omega\Sigma X .$$

Now we ought to show that α is a homotopy equivalence (in addition to being a homomorphism).

[2] Leon H pointed out that my first attempt was not quite correct.

Here is a sketchy argument for that. Let us construct $\bar{E}_X$ like E_X , but using $\Omega\Sigma X$ instead of JX throughout. So $\bar{E}_X$ is the pushout of the upper row in the commutative diagram

$$\begin{tikzcd}
\mathrm{cone}(X) \times \Omega\Sigma X \arrow[d, "\text{proj.}"] & X \times \Omega\Sigma(X) \arrow[l, "g"'] \arrow[r, "\text{incl.}"] \arrow[d, "\text{proj.}"] & \mathrm{cone}(X) \times \Omega\Sigma X \arrow[d, "\text{proj.}"] \\
\mathrm{cone}(X) & X \arrow[l, "\text{incl.}"'] \arrow[r, "\text{incl.}"] & \mathrm{cone}(X)
\end{tikzcd}$$

where $g(x,(a,\tau)) = (x,(1,\gamma_x)\cdot(a,\tau))$. Taking the pushout of the lower row as well we obtain a map from $\bar{E}_X$ to ΣX. In fact we obtain a commutative triangle

$$\begin{tikzcd}
E_X \arrow[rr] \arrow[dr] & & \bar{E}_X \arrow[dl] \\
& \Sigma X &
\end{tikzcd}$$

where the horizontal arrow extends α. More precisely, the restriction of the horizontal arrow to the fibers over the base point (of ΣX) is the map α. Now, if we can prove that $\bar{E}_X$ is *also* contractible (and if we are willing to apply the theorem of the magic cube not only to E_X but also to $\bar{E}_X$), then we may conclude that α is a (weak) homotopy equivalence.

LEMMA 5.1.3. *$\bar{E}_X$ is contractible.*

PROOF. (Sketch.) Again it is convenient to use an economy version of $\bar{E}_X$: the quotient space obtained from the disjoint union of $\mathrm{cone}(X)\times\Omega\Sigma X$ and $\Omega\Sigma X$ by gluing points $(x,(a,\tau))$ in $X\times\Omega\Sigma X \subset \mathrm{cone}(X)\times\Omega\Sigma X$ to $(1,\gamma_x)\cdot(a,\tau)$ in $\Omega\Sigma X$. (And allow me to use the *unreduced* cone of X throughout.) Make a map from the economy version of $\bar{E}_X$ to the space $P\Sigma X$ of (Moore) paths in Σ_X starting at the base point, as follows: a point in $\bar{E}_X$ represented by a pair $(y,(a,\tau))$ in $\mathrm{cone}(X)\times\Omega\Sigma X$ is taken to the path in ΣX obtained by first running through the loop τ in the time interval $[0,a]$, then through the standard path from $*\in\mathrm{cone}(X)$ to $y\in\mathrm{cone}(X)$ in the time interval $[a,a+s]$ where $s\in[0,1]$ is the "radial" distance from y to the base point of the cone. (Check that this is well defined! We used representatives.) We now have a map $\bar{E}_X \to P\Sigma_X$ over ΣX. For each choice of $z\in\Sigma X$, the induced map of fibers over z is a homotopy equivalence by inspection. By writing this map as a map between two homotopy pushouts induced by a map between two diagrams of the shape $\bullet\leftarrow\bullet\rightarrow\bullet$ (details omitted), we can conclude that $\bar{E}_X \to P\Sigma X$ is a homotopy equivalence. □

5.2. The theorem of Hilton-Milnor

The theorem of Hilton-Milnor is a formula for $J(X\vee Y)$ in terms of JX and JY (where X and Y are based connected CW-spaces). The special case in which X and Y are spheres was found by P Hilton and the more general version is due to Milnor. Milnor's formulation expresses $J(X\vee Y)$ as a product of JX and JY and spaces of higher commutators of elements from JX and JY, using the monoid structure in $J(X\vee Y)$. This makes important connections with Lie algebra theory (because of the commutator calculus) and as a result it is quite ambitious. Brayton Gray developed a lighter version which does not emphasize commutators and Lie algebra concepts. It still relies on the James construction, but not as much as the proofs due to Hilton and Milnor. This version due to Gray is the one which I will describe. It would be nice to know that the Milnor formulation can be obtained from the Gray formulation with some extra bookkeeping efforts, but I have not had time to explore this.

LEMMA 5.2.1. *Let X and Y be connected based CW-spaces. Then*

$$\Sigma(X \times Y) \simeq \Sigma((X \wedge Y) \vee X \vee Y)\,.$$

PROOF. Let a, b, c be the inclusions of $X \wedge Y$, X and Y in $(X \wedge Y) \vee X \vee Y$. Let $u: X \times Y \to X \wedge Y$ be the quotient map and let v, w be the projections from $X \times Y$ to X and Y, respectively. Let

$$\lambda: \Sigma(X \times Y) \longrightarrow \Sigma(X \times Y) \vee \Sigma(X \times Y) \vee \Sigma(X \times Y)$$

be the based map induced by a suitable map $S^1 \longrightarrow S^1 \vee S^1 \vee S^1$ which represents $z_1 z_2 z_3 \in \pi_1(S^1 \vee S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ (coproduct of groups), where the z_i are generators for each copy of $\mathbb{Z}$. All these maps can be set up as cellular maps. Then

$$\Sigma(au \vee bv \vee cw) \circ \lambda$$

is a based cellular map from $\Sigma(X \times Y)$ to $\Sigma((X \wedge Y) \vee X \vee Y)$. The corresponding map of (reduced) cellular chain complexes is clearly a chain homotopy equivalence, and indeed an isomorphism. (Note that the reduced cellular chain complex $\tilde{C}(X \times Y)$ is canonically isomorphic to

$$\tilde{C}(X \vee Y) \oplus \tilde{C}(X) \oplus \tilde{C}(Y)$$

and we have now *realized* this isomorphism by a cellular map after suspending $X \times Y$, $X \wedge Y$, X and Y.) Therefore we can apply the Hurewicz theorem. □

COROLLARY 5.2.2. (D. Puppe.) *For a connected based (compact) CW-space X we have*

$$\Sigma(JX) \simeq \Sigma(\bigvee_{k \geq 1} X^{(k)})$$

where $X^{(k)}$ means $\underbrace{X \wedge X \wedge \cdots \wedge X}_{k}$.

PROOF. First let $X_1, \ldots, X_n$ be any based connected CW-spaces. By iterating the decomposition of the lemma we obtain a homotopy equivalence

$$\Sigma(X_1 \times X_2 \times \cdots \times X_n) \xrightarrow{\ \simeq\ } \Sigma\Big(\bigvee_{s_1 < s_2 < \cdots < s_k} X_{s_1} \wedge X_{s_2} \wedge \cdots \wedge X_{s_k}\Big)$$

where the s_i are elements of $\{1, 2, \ldots, n\}$. In particular we extract from this a map

$$\Sigma(X_1 \wedge X_2 \wedge \cdots \wedge X_n) \longrightarrow \Sigma(X_1 \times X_2 \times \cdots \times X_n)$$

(inclusion of a wedge summand followed by a homotopy inverse to the above map). Now we specialize by taking $X_1 = X_2 = \cdots = X_n := X$ and write

$$g_n: \Sigma(\underbrace{X \wedge X \wedge \cdots \wedge X}_{n}) \longrightarrow \Sigma(\underbrace{X \times X \times \cdots \times X}_{n}).$$

Let f_n be the composition

$$\Sigma(\underbrace{X \wedge X \wedge \cdots \wedge X}_{n}) \xrightarrow{g_n} \Sigma(\underbrace{X \times X \times \cdots \times X}_{n}) \longrightarrow \Sigma(J^{(n)}X) \xrightarrow{\subset} \Sigma(JX)$$

where the second arrow is induced by the quotient map. By inspection of the reduced cellular chain complexes, the map

$$\bigvee_{n \geq 1} f_n : \bigvee_{n \geq 1} \Sigma(\underbrace{X \wedge X \wedge \cdots \wedge X}_{n}) \qquad \longrightarrow \qquad \Sigma(JX)$$

induces an isomorphism in homology. Since it is also a map between simply connected CW-spaces, it is a homotopy equivalence. □

Now fix two based connected CW-spaces X and Y. Out of these we make a homotopy pushout square

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \vee Y \end{tikzcd}$$

We have the standard map $X \vee Y \to X$ and use this to view the entire square as a square of spaces over X, and we form the homotopy fibers over Y. Let the result be

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

We unravel: $D = \text{hofiber}[X \vee Y \to X]$, $B \simeq *$, $C \simeq Y \times \Omega X$, $A = \Omega X$. We know that it is a homotopy pushout square and so we obtain

$$\text{hofiber}[X \vee Y \to X] \simeq \text{cone}[A \to B] = \frac{Y \times \Omega X}{* \times \Omega X}.$$

But it is easy to see (remark 5.2.6 below) that

$$\Omega(X \vee Y) \simeq \Omega X \times \Omega\,\text{hofiber}[X \vee Y \to X]$$

so that we can conclude, following Gray:

Proposition 5.2.3. $\Omega(X \vee Y) \simeq \Omega X \times \Omega\left[\frac{Y \times \Omega X}{* \times \Omega X}\right]$.

Continuing with that formula, let us plug in ΣX and ΣY instead of X and Y. We get

$$\Omega(\Sigma(X \vee Y)) \simeq \Omega\Sigma X \times \Omega\left[\frac{\Sigma Y \times \Omega\Sigma X}{* \times \Omega\Sigma X}\right]$$

for which we can also write

$$J(X \vee Y) \simeq JX \times \Omega\left[\frac{\Sigma Y \times JX}{* \times JX}\right].$$

Now another unexpected move due to Gray (where I use the notation $(-)_+$ for the operation of adding a new point by disjoint union and taking that as the base point):

$$\frac{\Sigma Y \times JX}{* \times JX} \cong \Sigma Y \wedge (JX)_+ \cong Y \wedge \Sigma((JX)_+) \simeq Y \wedge (\Sigma JX \vee S^1) \cong \Sigma Y \vee (Y \wedge \Sigma JX).$$

Now we can still use corollary 5.2.2 and we get

$$\Sigma Y \vee (Y \wedge \Sigma JX) \simeq \Sigma Y \vee \Sigma(Y \wedge \bigvee_{k\geq 1} X^{(k)}).$$

Putting all that together we have the Hilton-Milnor-Gray formula:

Theorem 5.2.4. $J(X \vee Y) \simeq JX \times J\Big(Y \vee (Y \wedge X) \vee (Y \wedge X \wedge X) \vee \cdots\Big)$.

This can be used recursively. It may look as if the right-hand side is no better than the left-hand side, but it is better because the wedge summands $Y \wedge X$, $Y \wedge X \wedge X$ etc. are more highly connected than X (unless X is contractible). Therefore we can, by iterating the procedure and taking a (homotopy) colimit, express $J(X \vee Y)$ as a product of factors which have the form

$$J(X^{(p)} \wedge Y^{(q)})$$

for some $p, q \geq 0$. But there are many such factors and it is not easy to keep track of them. (This is where the "advanced" Hilton-Milnor theorem does a better job.)

Twofold application of the theorem gives a more symmetric expression:

$$J(X \vee Y) \;\simeq\; JX \times JY \times J\Big(\bigvee_{p,q \geq 1} X^{(p)} \wedge Y^{(q)}\Big).$$

Example 5.2.5. We might want to understand $\pi_{10}(S^4 \vee S^4)$. Take $X = Y = S^3$ in the Hilton-Milnor-Gray theorem.

$$\begin{aligned} \pi_{10}(S^4 \vee S^4) &\cong \pi_9 J(S^3 \vee S^3) \\ &\cong \pi_9 JS^3 \times \pi_9 JS^3 \times \pi_9 J(S^6 \vee S^9 \vee S^9 \vee \cdots) \\ &\cong \pi_9 JS^3 \times \pi_9 JS^3 \times \pi_9 JS^6 \times \pi_9 JS^9 \times \pi_9 JS^9 \\ &\cong \pi_{10} S^4 \times \pi_{10} S^4 \times \pi_{10} S^7 \times \pi_{10} S^{10} \times \pi_{10} S^{10}\,. \end{aligned}$$

Remark 5.2.6. Let B be a based connected CW-space, A a CW-subspace of B containing the base point. Suppose that $r\colon B \to A$ satisfies $r|_A = \mathrm{id}_A$. Let $F := \mathrm{hofiber}(r)$. Then $\Omega B \simeq \Omega A \times \Omega F$. (Exercise.)

5.3. Whitehead products and Samelson products

Let X be a based CW-space. The Whitehead product

$$\pi_m(X) \times \pi_n(X) \longrightarrow \pi_{m+n-1}(X)$$

was defined in example 1.4.6 and definition 1.4.7.
The Whitehead product is not associative. Instead it satisfies a type of Jacobi identity, like the product in a Lie algebra. At one time, when the Whitehead product was new, the Jacobi identity for it was considered very difficult to establish. But it is easy to understand in a reformulation.

Definition 5.3.1. Suppose first that K is a topological group. The identity element of K will be viewed as the base point. There is a product

$$\pi_m(K) \times \pi_n(K) \longrightarrow \pi_{m+n}(K)$$

defined as follows. For a class in $\pi_m(K)$ represented by a based map $f\colon S^m \to K$ and a class in $\pi_n(K)$ represented by based map $g\colon S^n \to K$, we have a map

$$S^m \times S^n \ni (x, y) \;\mapsto\; f(x)g(y)f(x)^{-1}g(x)^{-1} \in K\,.$$

This takes all elements of the form $(*, y)$ and all elements of the form $(x, *)$ to the base point $1 \in K$. Therefore it amounts to a based map from the smash product $S^m \wedge S^n$, also known as S^{m+n}, to K. The homotopy class of that is well defined in terms of $[f]$ and $[g]$. It is denoted by

$$\langle\, [f], [g]\, \rangle \in \pi_{m+n}(K)$$

and called the *Samelson* product of $[f]$ and $[g]$. This depends very much on the topological group structure of K.

Next, suppose that K is a topological monoid (i.e., it comes with a strictly associative multiplication, with unit). We want to assume that K is *group-like*, which means that the shear map

$$\varphi\colon (x,y) \mapsto (x,xy)$$

from $K\times K$ to $K\times K$ is a based homotopy equivalence. Then we can choose a homotopy inverse ψ for that, and a homotopy from $\psi\varphi$ to id. Specializing ψ to elements of the form $(x,1)$, we learn that there exists a based map $c\colon K\to K$ such that the map $x\mapsto xc(x)$ from K to K is based homotopic to the trivial map $\to K$ taking all elements to the base point. And moreover we have a preferred choice of such a homotopy. Of course we view c as a substitute for a possible nonexistent map $x\mapsto x^{-1}$. Therefore we can define a slightly more general Samelson product as follows. For a class in $\pi_m(K)$ represented by a based map $f\colon S^m\to K$ and a class in $\pi_n(K)$ represented by based map $g\colon S^n\to K$, we have a map

$$S^m\times S^n \ni (x,y) \;\mapsto\; f(x)g(y)c(f(x))c(g(x)) \in K\,.$$

The restriction of this to the wedge $S^m\vee S^n$ is based nullhomotopic in a preferred way. Making use of the homotopy extension property for the inclusion $S^m\vee S^n\to S^m\times S^n$, we obtain a map from $S^m\wedge S^n$, also known as S^{m+n}, to K. The homotopy class of that is well defined in terms of $[f]$ and $[g]$. It is again denoted by

$$\langle\, [f],[g]\,\rangle \in \pi_{m+n}(K)$$

and called the Samelson product of $[f]$ and $[g]$.
[In the case where K is a group-like topological monoid, there is a more elegant description of the Samelson product which avoids substitutes for inverses at the element level. Suppose that (X,Y) is a pair of CW-spaces. Let $v,w\colon X\to K$ be two maps such that $v|_Y = w|_Y$. Then there exists a based map

$$u\colon X/Y \longrightarrow K$$

such that $u\cdot v$ is homotopic to w, by a homotopy which is stationary on Y. (Here we view u as a map from X to Y and we use pointwise multiplication in K to make sense of $u\cdot v$.) Moreover the based homotopy class of u, as a based map from X/Y to K, is uniquely determined by this condition.
Therefore we can re-define the Samelson product as follows. For a class in $\pi_m(K)$ represented by a based map $f\colon S^m\to K$ and a class in $\pi_n(K)$ represented by based map $g\colon S^n\to K$, we have maps $v,w\colon S^m\times S^n\to K$ given by $v(x,y)=g(y)f(x)$ and $w(x,y)=f(x)g(y)$. They agree on $S^m\vee S^n$. Therefore we obtain as above

$$u\colon S^m\wedge S^n \longrightarrow K$$

with the property that $u\cdot v$ is homotopic to w, relative to $S^m\vee S^n$. The homotopy class of u is well defined in terms of $[f]$ and $[g]$. It is denoted by

$$\langle\, [f],[g]\,\rangle \in \pi_{m+n}(K)$$

and called the Samelson product of $[f]$ and $[g]$. — If K happens to be a topological group after all, then the equation $u\cdot v = w$ can be solved uniquely and we obtain $u(x,y)=f(x)g(y)f(x)^{-1}g(y)^{-1}$, so that this third definition of the Samelson product is in agreement with the first one. Similar arguments can be used to show that it is in agreement with the second one.]

THEOREM 5.3.2. *Let X be a based connected CW-space. Let ΩX be the "Moore" loop space, an associative topological monoid. Under the isomorphisms*

$$\pi_k(X) \cong \pi_{k-1}(\Omega X)$$

for $k \geq 1$, the Whitehead product on $\pi_*(X)$ corresponds to the Samelson product on $\pi_*(\Omega X)$, except for a sign. More precisely, the diagram

$$\begin{tikzcd}
\pi_{m+1}(X) \times \pi_{n+1}(X) \arrow[r, "\text{Whitehead prod.}"] \arrow[d, "\cong"] & \pi_{m+n+1}(X) \arrow[d, "\cong"] \\
\pi_m(\Omega X) \times \pi_n(\Omega X) \arrow[r, "\text{Samelson prod.}"] & \pi_{m+n}(\Omega X)
\end{tikzcd}$$

commutes up to a sign $(-1)^m$.

This explains why the Whitehead product satisfies a Jacobi identity: it does because the Samelson product satisfies a Jacobi identity. I am not planning to prove this for the Samelson product[3], but the statement is as follows. If K is a group-like topological monoid and $u \in \pi_p(K)$, $v \in \pi_q(K)$, $w \in \pi_r(K)$, then

$$(-1)^{pr}\langle u, \langle v, w\rangle\rangle + (-1)^{pq}\langle v, \langle w, u\rangle\rangle + (-1)^{qr}\langle w, \langle u, v\rangle\rangle = 0\,.$$

In addition, the Samelson product is graded anticommutative, i.e., it satisfies

$$\langle u, v\rangle = (-1)^{pq+1}\langle v, u\rangle$$

for $u \in \pi_p(K)$ and $v \in \pi_q(K)$.

PROOF OF THEOREM 5.3.2. By naturality it is enough to consider a universal case. For X we take $S^{m+1} \vee S^{n+1}$. There are certain elements a, b in $\pi_{m+1}X$ and $\pi_{n+1}X$ respectively given by the inclusions of the wedge summands:

$$S^{m+1} \to S^{m+1} \vee S^{n+1}, \qquad S^{n+1} \to S^{m+1} \vee S^{n+1}.$$

Let α and β be the corresponding elements of $\pi_m(\Omega X)$ and $\pi_n(\Omega X)$, respectively. We need to understand what happens to the element $(a, b) \in \pi_{m+1}X \times \pi_{n+1}X$ when we chase it through the upper itinerary of the diagram, towards $\pi_{m+n}(\Omega X)$. The goal is to show that we get $(-1)^m\langle \alpha, \beta\rangle$. (Note in passing that the obvious choice for a substitute inverse of an element $\omega \in \Omega X$ is $\bar{\omega}$, the reversed loop. The compositions $\bar{\omega}\omega$ and $\omega\bar{\omega}$ are not completely trivial but they are connected to the base point of ΩX by canonical paths. We use all that in the explicit definition of the Samelson product on $\pi_*(\Omega X)$.)
Let us use cubical coordinates. Write $I = [0, 1]$. The element $[a, b]$ is represented by

$$f\colon\ \partial(I \times I^m \times I \times I^n) \longrightarrow \frac{I \times I^m}{\partial(I \times I^m)} \vee \frac{I \times I^n}{\partial(I \times I^n)}$$

$$f(s, x, t, y) \ = \ \begin{cases} (s, x) \text{ in the first wedge summand if } (t, y) \in \partial(I^n \times I) \\ (t, y) \text{ in the second wedge summand if } (s, x) \in \partial(I^m \times I). \end{cases}$$

Now it is tempting to write

$$f(s, x, t, y) = u_{x,y}(\gamma(s, t))$$

where

$$\gamma\colon \partial(I \times I) \longrightarrow \frac{I}{\partial I} \vee \frac{I}{\partial I}$$

[3]It is still hard. There is a beautiful proof in Chapter X (Thms 5.4, 3.10 and Cor 3.9) in the book by G Whitehead, *Elements of homotopy theory.*

is defined by $\gamma(s,t) = \begin{cases} s \text{ in the first wedge summand if } t \in \partial I \\ t \text{ in the second summand if } s \in \partial I \end{cases}$ and

$$u_{x,y}\colon \frac{I}{\partial I} \vee \frac{I}{\partial I} \longrightarrow \frac{I \times I^m}{\partial(I \times I^m)} \vee \frac{I \times I^n}{\partial(I \times I^n)}$$

is defined by $u_{x,y}(s) = (s,x)$ in the first wedge summand if s is in the first wedge summand $I/\partial I$, and $u_{x,y}(t) = (t,y)$ in the second wedge summand if t is in the second wedge summand $I/\partial I$. This is not exactly wrong but it is an incomplete description of f, because there are cases where $f(s,x,t,y)$ is defined but $\gamma(s,t)$ is not defined. This happens if $s,t \notin \partial I$ and either $x \in \partial(I^m)$ or $y \in \partial(I^n)$. If $y \in \partial(I^n)$ we can still write

$$f(s,x,t,y) = u_{x,y}(\gamma(s,0)) = u_{x,y}(\gamma(s,1)).$$

Similarly if $x \in \partial(I^m)$ we can write

$$f(s,x,t,y) = u_{x,y}(\gamma(0,t)) = u_{x,y}(\gamma(1,t)).$$

Now I hope the picture is becoming clearer. Let

$$e_1, e_2\colon I/\partial I \longrightarrow (I/\partial I) \vee (I/\partial I)$$

be the inclusions of the wedge summands. We can think of e_1 and e_2 as elements of

$$\Omega\big((I/\partial I) \vee (I/\partial I)\big).$$

Then γ is their commutator. (Think of $\partial(I \times I)$ as an interval of length 4 with endpoints glued together.) Consequently the map

$$I^m \times I^n \ni (x,y) \mapsto u_{x,y} \circ \gamma \in \Omega X$$

is the commutator of $(x,y) \mapsto u_{x,y} \circ e_1 = \alpha(x)$ and $(x,y) \mapsto u_{x,y} \circ e_2 = \beta(y)$. It is not trivial on the boundary of $I^m \times I^n$ but we have seen that the restriction to the boundary has a canonical nullhomotopy. Briefly, it seems that we do get $\langle \alpha, \beta \rangle$. What happened to the sign $(-1)^m$? We lost it because we did not pay attention to the orientation of $\partial(I \times I^m \times I \times I^n)$. The reader is invited to fill this in. (Alternative: read the proof in the G Whitehead book.) $\square$

Remark 5.3.3. To some extent the above proof wants to reduce the case of arbitrary m,n to the special case $m = n = 0$. Is the statement obvious for $m = n = 0$? In other words, is it obvious that γ represents the class $aba^{-1}b^{-1}$ in $\pi_1(S^1 \vee S^1)$? I believe yes. Nota bene: the multiplication on ΩX was defined just after the proof of theorem 5.1.2.

Finally the Samelson product is a key ingredient in the advanced formulation of the Hilton-Milnor theorem. To state this we need a more general form of the Samelson product. Again let K be a group-like topological monoid. So far the Samelson product was defined as a product on $\pi_* K$. We can set it up in more general terms. Suppose that L and M are based spaces which are homotopy equivalent to CW-spaces. Suppose that the inclusions of the base points, $* \hookrightarrow L$ and $* \hookrightarrow M$, are cofibrations. Then the inclusion of $L \vee M$ in $L \times M$ is a cofibration, and its cofiber is of course $L \wedge M$. Let $f\colon L \to K$ and $g\colon M \to K$ be based maps. We can define their Samelson product

$$\langle f, g \rangle\colon L \wedge M \longrightarrow K$$

(well defined up to based homotopy) as follows. Let $v, w\colon L \times M \to K$ be given by $v(x,y) = g(y)f(x)$ and $w(x,y) = f(x)g(y)$. They agree on $L \vee M$. Therefore we get $u\colon L \wedge M \to K$, unique up to based homotopy, such that $u \cdot v$ is homotopic to w relative to $L \vee M$. This map u is by definition the Samelson product $\langle f, g \rangle$.

Now let $K = J(X_1 \vee X_2 \vee \cdots \vee X_r)$ where $X_1, X_2, \ldots, X_r$ are based connected CW-spaces. We choose tags $a_1, a_2, \ldots, a_r$ corresponding to the X_i. Let $f_i : X_i \to K$ be the inclusion, for $i = 1, 2, \ldots, r$. The advanced Hilton-Milnor theorem states that a certain map of the form

$$\prod_{n\ge 1} \prod_{w=[a_{i_1},[a_{i_2},[a_{i_3},[\ldots,a_{i_n}]]\ldots]]]} J(X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_n}) \xrightarrow{g_w} J(X_1 \vee X_2 \vee \cdots \vee X_r)$$

is a homotopy equivalence. The product is taken over certain words involving Lie brackets, in the letters $a_1, a_2, \ldots, a_r$. Corresponding to a word

$$w = [a_{i_1}, [a_{i_2}, [a_{i_3}, [\ldots, a_{i_n}]]\ldots]]]$$

we have a map, an iterated Samelson product

$$\langle f_{i_1}, \langle f_{i_2}, \langle f_{i_3}, \langle \ldots, f_{i_n}\rangle\rangle \ldots\rangle\rangle\rangle$$

from $X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_n}$ to $J(X_1 \vee X_2 \vee \cdots \vee X_r)$. This extends uniquely to a (continuous) *homomorphism*

$$g_w : J(X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_n}) \longrightarrow J(X_1 \vee X_2 \vee \cdots \vee X_r).$$

Using the product in $J(X_1 \vee X_2 \vee \cdots \vee X_r)$ again, we can multiply these maps g_w pointwise (in an order which must be specified) to obtain a map from the product

$$\prod_{n\ge 1} \prod_{[a_{i_1},[a_{i_2},[a_{i_3},[\ldots,a_{i_n}]]\ldots]]]} J(X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_n})$$

to $J(X_1 \vee X_2 \vee \cdots \vee X_r)$. That map is claimed to be a homotopy equivalence.
It remains to be said how the words w can be found or listed, and also in which order they should be listed. Unfortunately I am not very well qualified to give an answer. My understanding is that the selected words must form a (graded) basis of the free Lie algebra (over the ground ring $\mathbb{Z}$) on generators $a_1, \ldots, a_r$, viewed as a graded free abelian group. (There could be other conditions.) The grading assigns n to a word in n letters such as

$$[a_{i_1}, [a_{i_2}, [a_{i_3}, [\ldots, a_{i_n}]]\ldots]]].$$

The ordering of the words could be relevant, but if so it is a long story.
Beware: this free Lie algebra $\mathscr{L}$ is understood to be strict, i.e., the law $[x, x] = 0$ is enforced (for all $x \in \mathscr{L}$). This implies $[x, y] = -[y, x]$, but it is stronger. More obviously the Jacobi relation $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ is enforced (for $x, y, z \in \mathscr{L}$). Wilhelm Magnus showed that $\mathscr{L}$ can also be obtained as the *lower central series* of the free group on generators $a_1, a_2, a_3, \ldots, a_r$. (The lower central series of a group always comes with the structure of a graded Lie algebra over $\mathbb{Z}$.)

(More details in *Elements of homotopy theory* by G.W.Whitehead, Springer 1978. The original article by Milnor is entitled *On the construction FK* and appears in a book edited by J F Adams, *Algebraic Topology – a student's guide*, LMS Lecture Note Series, Cambridge University Press, 1972. It seems to me that G.W.Whitehead proves a stronger theorem by making the connection with the Samelson product. But this comes at a price; he admits "the proof is long and complicated".)

Last not least ... my impression is that theorem 5.2.4 is not a cheap imitation of what I have called the advanced Hilton-Milnor theorem, but the generally accepted first (and most important) step towards it. To bridge the gap one should try to obtain an explicit description of the homotopy equivalence in theorem 5.2.4 in terms of Samelson products. The rest should be bookkeeping.

CHAPTER 6

Representability in the homotopy category of based spaces

6.1. The E. H. Brown representation theorem

Let $\mathscr{C}$ be the category of based connected CW-spaces. (There is an understanding that the base point in a based CW-space is always a 0-cell.) Let $\mathscr{H}o\mathscr{C}$ be the corresponding homotopy category. We look for a characterization of the *representable* contravariant functors from $\mathscr{H}o\mathscr{C}$ to sets.

Definition 6.1.1. A contravariant functor F from $\mathscr{H}o\mathscr{C}$ to sets is *half-exact* if it satisfies the conditions below.

(i) If X in $\mathscr{C}$ is the union of two based CW-subspaces A and B such that A, B and $A \cap B$ are again in $\mathscr{C}$, then for any $s \in F(A)$ and $t \in F(B)$ which determine the same element of $F(A \cap B)$, there exists $u \in F(X)$ restricting to $s \in F(A)$ and to $t \in F(B)$.

(ii) (*Wedge axiom.*) If X in $\mathscr{C}$ is a wedge, $X = \bigvee X_\alpha$, then the maps $F(X) \to F(X_\alpha)$ induced by the inclusions of X_α in X determine a bijection

$$F(X) \longrightarrow \prod_\alpha F(X_\alpha)\,.$$

In (ii), the case of an empty indexing set is allowed. In that case the condition means that $F(\star)$ has exactly one element.

Remark 6.1.2. If F is half-exact, then for every X in $\mathscr{C}$ the set $F(X)$ has a distinguished "zero" element. This is $g^*(z) \in F(X)$ for the unique map $g\colon X \to \star$ and the unique element $z \in F(\star)$.

Remark 6.1.3. Let $g\colon X \to Y$ be a based cellular map, where X and Y are in $\mathscr{C}$. Let $\mathrm{cone}(g)$ be the *reduced* mapping cone. (The reduced mapping cone is the quotient space obtained from the standard mapping cone by collapsing the copy of $[0,1] \times \star$ in the standard mapping cone to a single point. The reduced mapping cone of a based cellular map of CW-spaces is again a based CW-space in a preferred way.) Then the sequence

$$F(\mathrm{cone}(g)) \to F(Y) \xrightarrow{g^*} F(X)$$

(first arrow determined by the inclusion of Y in $\mathrm{cone}(g)$) is *exact*, i.e., an element of $F(Y)$ comes from $F(\mathrm{cone}(g))$ if and only if it maps to the zero element of $F(X)$. (*Exercise.*)

Example 6.1.4. Let Y be a based connected CW-space and let F_Y be the contravariant functor defined by $F_Y(X) = [X, Y]_\star$ (set of based homotopy classes of based maps from X to Y) for X in $\mathscr{C}$ or $\mathscr{H}o\mathscr{C}$. This F_Y is half-exact. Property (ii) in definition 6.1.1 is obviously satisfied. For property (i), the idea is as follows. Take $X = A \cup B$ etc.; replace

X by $X^\sharp$ which is the quotient of

$$A \sqcup \big([0,1] \times (A \cap B)\big) \sqcup B$$

by the relations $(0,x) \sim x \in A$ and $(1,x) \sim x \in B$ for $x \in A \cap B$, as well as $(t,\star) \sim (0,\star)$ for all t (where $\star$ is the base point in $A \cap B$). There is a projection map

$$X^\sharp \longrightarrow X$$

which forgets the extra coordinate in $[0,1]$ where applicable. It is an exercise to show that this projection map is a homotopy equivalence. (Use the HEP, homotopy extension property, for the inclusion of $A \cap B$ in A.) Now suppose given a map $f_0\colon A \to Y$ and a map $f_1\colon B \to Y$ such that $f_0|_{A\cap B} \simeq f_1|_{A\cap B}$. Choose a homotopy $(h_t)_{t\in[0,1]}$ from $f_0|_{A\cap B}$ to $f_1|_{A\cap B}$. Define $g\colon X^\sharp \to Y$ so that g agrees with f_0 on the copy of A in $X^\sharp$, agrees with f_1 on the copy of B in $X^\sharp$, and agrees with h_t on the copy of $\{t\} \times (A \cap B)$ in $X^\sharp$. Then the homotopy class of g, viewed as an element of $[X,Y]_\star = F_Y(X)$, satisfies $[g]|_A = [f_0]$ and $[g]|_B = [f_1]$. This confirms property (i) for F_Y, but I must apologize for changing labels: $s \rightsquigarrow f_0$, $t \rightsquigarrow f_1$, $u \rightsquigarrow g$.

More exotic examples will be given later.

Let F be half-exact and suppose that X in $\mathscr{C}$ is the monotone union of a sequence of CW-subspaces A_j where $j = 0,1,2,\dots$; so

$$X = \bigcup_j A_j$$

and $A_j \subset A_{j+1}$ for all j. Let $(v_j \in F(A_j))_{j\ge 0}$ be a sequence such that the map $F(A_{j+1}) \to F(A_j)$ induced by $A_j \hookrightarrow A_{j+1}$ takes v_{j+1} to v_j, for all $j \ge 0$.

LEMMA 6.1.5. *Then there exists $v_\infty \in F(X)$ which is taken to v_j under the map $F(X) \to F(A_j)$ induced by $A_j \hookrightarrow X$, for all $j \ge 0$.*

PROOF. Let's use the notation $\mathrm{cyl}(Y \to Z)$ for the mapping cylinder of a map $e\colon Y \to Z$. For the moment this is the unreduced mapping cylinder. In the standard description of the mapping cylinder we have a projection $\mathrm{cyl}(Y \to Z) \to [0,1]$ which takes the standard copy of Z to 0 and the standard copy of Y to 1. Here we parameterize this differently so that there is a projection $\mathrm{cyl}(Y \to Z) \to [0,2]$ which takes Y to 0 and Z to 2. More precisely, let $\mathrm{cyl}(Y \to Z)$ be (in this proof) the quotient of $\big([0,2] \times Y\big) \cup Z$ obtained by making identifications $(2,y) \sim e(y) \in Z$. If e is a based cellular map of based CW-spaces, then $\mathrm{cyl}(e) = \mathrm{cyl}(Y \to Z)$ is a CW-space in a preferred way, but here we use the CW-structure on $[0,2]$ with three 0-cells: the elements $0,1,2$ of $[0,2]$. The cylinder $\mathrm{cyl}(e)$ is not a based CW-space, but for later use we not that there is a copy of $[0,2] = \mathrm{cyl}(\star \to \star)$ inside $\mathrm{cyl}(e)$. This is clearly asking to be collapsed to a single point, but we have to delay that. — The *mapping telescope* of the diagram

$$A_0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow A_3 \hookrightarrow A_4 \hookrightarrow \cdots$$

is the union

$$\mathrm{cyl}(A_0 \to A_1) \cup_{A_1} \mathrm{cyl}(A_1 \to A_2) \cup_{A_2} \mathrm{cyl}(A_2 \to A_3) \cup_{A_3} \cdots$$

in self-explanatory notation. It is again a CW-space in an obvious way. Let's write $\mathrm{tel}(A_j \mid j \ge 0)$ for the telescope, just to have an abbreviation. There is a projection map

$$\mathrm{tel}(A_j \mid j \ge 0) \longrightarrow X$$

which on the piece $\mathrm{cyl}(A_j \to A_{j+1})$ agrees with the cylinder projection to A_{j+1}, followed by the inclusion $A_{j+1} \to X$. *Exercise*: show that this map is a homotopy equivalence.
There is a very useful continuous projection map

$$q\colon \mathrm{tel}(A_j \mid j \geq 0) \longrightarrow [0,2] \cup [2,4] \cup [4,6] \cup \cdots$$

which projects the piece $\mathrm{cyl}(A_j \to A_{j+1})$ to the interval $[2j, 2j+2]$ in the obvious way. Now let T_0 be the preimage under q of $[0,1] \cup [3,5] \cup [7,9] \cup \ldots$ and let T_1 be the preimage under q of $[1,3] \cup [5,7] \cup \ldots$. These are CW-subspaces of the telescope $\mathrm{tel}(A_j \mid j \geq 0)$ by construction. Then clearly

$$T_0 \simeq A_0 \sqcup A_2 \sqcup A_4 \sqcup A_6 \vee \ldots, \qquad T_1 \simeq A_1 \sqcup A_3 \sqcup A_5 \sqcup \ldots$$

whereas

$$T_0 \cap T_1 \cong A_0 \sqcup A_1 \sqcup A_2 \sqcup A_3 \sqcup \cdots .$$

But at this point we need to see reduced versions of these constructions. Let

$$\mathrm{tel}^\rho(A_j \mid j \geq 0) := \frac{\mathrm{tel}(A_j \mid j \geq 0)}{\mathrm{tel}(\star \to \star \to \star \to \cdots)}$$

and let T_0^ρ, T_1^ρ be the images of T_0 and T_1 in $\mathrm{tel}^\rho(A_j \mid j \geq 0)$. Then we have

$$T_0^\rho \simeq A_0 \vee A_2 \vee A_4 \vee A_6 \vee \ldots, \qquad T_1^\rho \simeq A_1 \vee A_3 \vee A_5 \vee \ldots$$

whereas

$$T_0^\rho \cap T_1^\rho \cong A_0 \vee A_1 \vee A_2 \vee A_3 \vee \cdots .$$

By the wedge axiom, $(v_0, v_2, v_4, \ldots)$ defines an element in $F(T_0^\rho)$. Similarly $(v_1, v_3, v_5, \ldots)$ defines an element in $F(T_1^\rho)$. By assumption on the sequence $(v_j)_{j \geq 0}$, these two elements determine the same element

$$(v_0, v_1, v_2, \ldots) \in F(T_0^\rho \cap T_1^\rho)$$

under the restriction maps $F(T_0^\rho) \to F(T_0^\rho \cap T_1^\rho)$ and $F(T_1^\rho) \to F(T_0^\rho \cap T_1^\rho)$. Therefore by half-exactness, there exists

$$w \in F(\mathrm{tel}^\rho(A_j \mid j \geq 0)) \cong F(\mathrm{tel}(A_j \mid j \geq 0)) \cong F(X)$$

which extends $(v_0, v_2, v_4, \ldots) \in F(T_0^\rho)$ and $(v_1, v_3, v_5, \ldots) \in F(T_1^\rho)$. The element w, viewed as an element of $F(X)$, is the answer to our prayers. □

LEMMA 6.1.6. *Let E be a half-exact functor, let X be an object of $\mathscr{C}$ and let $t \in E(X)$. There exist a based connected CW-space Y containing X as a CW-subspace, and an element $u \in E(Y)$ such that $u|_X = t$ and the map $\pi_k(Y) \to E(S^k)$ taking $[f\colon S^k \to Y]$ to $f^*(u)$ is bijective for all $k > 0$.*

As a preparation for the proof we take a closer look at the sets $E(S^k)$ for a half-exact functor E and $k \geq 1$. It turns out that these sets have a preferred group structure (abelian if $k \geq 2$). The reason is that S^k comes with a distinguished map $\kappa\colon S^k \to S^k \vee S^k$ which we used previously to define the group structure in homotopy groups π_k. Here we can use it to define a map

$$E(S^k) \times E(S^k) \longrightarrow E(S^k)$$

by writing $E(S^k) \times E(S^k) \cong E(S^k \vee S^k)$, wedge axiom for E, and then using $\kappa^*\colon E(S^k \vee S^k) \to E(S^k)$. This map makes $E(S^k)$ into a group (abelian if $k \geq 2$) because κ has the corresponding properties (which can be, should be and have been expressed in the homotopy category of based spaces). By the same reasoning, for X in $\mathscr{C}$ and $t \in F(X)$ the maps $\pi_k(X) \to E(S^k)$ taking $[f\colon S^k \to X]$ to $f^*(t)$ are group homomorphisms. (Here and in the following we write $\pi_k(X)$ instead of $\pi_k(X)$ on the understanding that X has a preferred base point.)

Proof of lemma 6.1.6. We construct $X \cup Y^n$ by induction on n, together with elements $u_n \in E(X \cup Y^n)$ such that the restriction of u_n to $X \cup Y^{n-1}$ is u_{n-1}. (The notation is a little informal; there is an understanding that $X \cap Y^n$ is X^n.) For the induction beginning set $u_0 := t$ and $X \cup Y_0 := X$, that is, $Y^0 = X^0$. For the first induction step, from $n = 0$ to $n = 1$, choose generators μ for the entire group $E(S^1)$. Define

$$X \cup Y^1 := X \vee \bigvee_\mu S^1 .$$

By the wedge axiom for E, we have

$$E(X \cup Y^1) = E(X) \times \prod_\mu E(S^1)$$

and we determine $u_1 \in E(X \cup Y^1)$ in such a way that the coordinate in $E(X)$ is $u_0 = t$, while the coordinate in the factor $E(S^1)$ corresponding to $\mu \in E(S^1)$ is exactly μ. By construction, the map $\pi_1(X \cup Y^1) \to E(S^1)$ taking $[f]$ to $f^*(u_1)$ is surjective. — For the remaining induction steps, suppose that $X \cup Y^n$ and $u_n \in E(X \cup Y^n)$ have already been constructed for a particular $n \geq 1$. Suppose that the homomorphisms $\pi_k(X \cup Y^n) \to E(S^k)$ taking $[f]$ to $f^*(u_n)$ are bijective for $1 \leq k < n$ and surjective for $k = n$. (This is part of the induction load.) Choose generators λ for the kernel of the homomorphism $\pi_n(X \cup Y^n) \to E(S^n)$, and for each λ, a based cellular map $f_\lambda \colon S^n \to X \cup Y^n$ in that homotopy class. Also choose generators μ for the entire group $E(S^{n+1})$. Define $X \cup Y^{n+1}$ to be

$$\mathrm{cone}\Big(\bigvee_\lambda f_\lambda \colon \bigvee S^n \longrightarrow X \cup Y^n\Big) \quad \vee \quad \bigvee_\mu S^{n+1}.$$

By half-exactness of E (see also remark 6.1.3) there is an element u_{n+1} of $E(X \cup Y^{n+1})$ such that the restriction to $X \cup Y^n$ is u_n and the restriction to the wedge summand S^{n+1} with label μ is precisely $\mu \in E(S^{n+1})$.
Now we have to show that the homomorphisms $\pi_k(X \cup Y^{n+1}) \to E(S^k)$ taking $[f]$ to $f^*(u_{n+1})$ are bijective for $1 \leq k \leq n$ and surjective for $k = n+1$. Surjectivity for $k = n+1$ is obvious from the construction. For the cases $k \leq n$ we look at the composition

$$\pi_k(X \cup Y^n) \to \pi_k(X \cup Y^{n+1}) \to E(S^k)$$

where the first arrow is induced by the inclusion $X \cup Y^n \to X \cup Y^{n+1}$. The first arrow is an isomorphism for $k < n$ by cellular approximation and the composition is an isomorphism for $k < n$ by inductive assumption, so the second arrow is also an isomorphism for $k < n$. For $k = n$ the first arrow is onto by cellular approximation, while the composite arrow is onto by inductive assumption and its kernel is contained in the kernel of the first arrow by construction. Therefore these two kernels must coincide as subgroups of $\pi_k(X \cup Y^n)$. It follows that the second arrow is again an isomorphism.
Now we have constructed $X \cup Y^n$ and u_n for all n. Let Y be the union (direct limit or colimit is a better expression) of the $X \cup Y^n$ for all n. By lemma 6.1.5, there exists $u \in E(Y)$ such that u restricted to $X \cup Y^n$ is u_n , for all $n \geq 0$. In particular, we have $u|_X = u_0 = t$. □

Theorem 6.1.7. (The Brown representation theorem.) *Any half-exact functor F from $\mathscr{H}o\mathscr{C}$ to sets is representable, i.e., there exist Y and $u \in F(Y)$ such that the map from $[X,Y]_\star$ to $F(X)$ given by $[f] \mapsto f^*(u)$ is bijective for every X in $\mathscr{C}$.*

Proof. By lemma 6.1.6 we can construct a based connected CW-space Y and an element $u \in F(Y)$ such that $u|_X = t$ and the map $\pi_k(Y) \to E(S^k)$ taking $[f]$ to $f^*(u)$ is bijective for all $k > 0$. We are going to show that the map

$$\alpha_X : [X,Y]_\star \longrightarrow F(X) \; ; \; [f] \mapsto f^*(u) \in F(X)$$

is bijective for every X in $\mathscr{C}$.
The idea is to construct a natural inverse $\beta_X : F(X) \to [X,Y]_\star$ for α_X using lemma 6.1.6 and the JHC Whitehead theorem. For $t \in F(X)$ we apply lemma 6.1.6 to the element $(t,u) \in F(X) \times F(Y) \cong F(X \vee Y)$. The outcome is that there is a CW-space Y' containing $X \vee Y$ as a CW-subspace and an element $u' \in F(Y')$ which extends $(t,u) \in F(X \vee Y)$ and has the property that the homomorphisms $\pi_k(Y') \to E(S^k)$ given by $[f] \mapsto f^*(u')$ are isomorphisms for all $k > 0$. Since the homomorphisms

$$\pi_k(Y) \to E(S^k)$$

given by $[f] \mapsto f^*(u)$ are also isomorphisms for all $k > 0$, it follows that the inclusion $Y \to Y'$ induces an isomorphism $\pi_k(Y) \to \pi_k(Y')$ for all $k > 0$. Therefore the JHC Whitehead theorem tells us that $Y \to Y'$ is a homotopy equivalence. We attempt to define $\beta_X(t) \in [X,Y]_*$ as the homotopy class of $X \hookrightarrow Y'$ followed by a based homotopy inverse for $Y \hookrightarrow Y'$. Now it remains to show (a) that β_X is well defined, (b) that $\alpha_X \beta_X = \mathrm{id}$ and (c) that $\beta_X \alpha_X = \mathrm{id}$.
(a) Suppose that we have selected Y' containing $X \vee Y$ and $u' \in F(Y')$ extending $(t,u) \in F(X \vee Y) \cong F(X) \times F(Y)$. Suppose that we have also selected Y'' containing $X \vee Y$ and $u'' \in F(Y'')$ extending $(t,u) \in F(X \vee Y) \cong F(X) \times F(Y)$. We are assuming that the homomorphisms $\pi_k(Y') \to E(S^k)$ and $\pi_k(Y'') \to E(S^k)$ given by $[f] \mapsto f^*(u')$ and $[f] \mapsto f^*(u'')$ respectively are isomorphisms. Then we can find a CW-space Y''' containing the union $Y' \cup_{X \vee Y} Y''$ (better described as a pushout) and an element $u''' \in F(Y''')$ such that the homomorphisms $\pi_k(Y''') \to E(S^k)$ given by $[f] \mapsto f^*(u''')$ are isomorphisms. Now we have three definitions of $\beta_X(t)$, corresponding to the selections Y', Y'' and Y'''. But it is clear that the first agrees with the third and the second agrees with the third, since $Y' \subset Y'''$ and $Y'' \subset Y'''$. So the first must agree with the second, as was to be shown.
(b) Let $t \in F(X)$ and suppose Y' containing $X \vee Y$ as well as $u' \in F(Y')$ extending $(t,u) \in F(X \vee Y)$ has been selected so that the composition $Y \hookrightarrow X \vee Y \hookrightarrow Y'$ is a homotopy equivalence. Then $\beta_X(t) \in [X,Y]_*$ is the composition of $X \hookrightarrow X \vee Y \hookrightarrow Y'$ with a homotopy inverse for $Y \to Y'$. Since that homotopy inverse $Y' \to Y$ will take $u \in F(Y)$ to $u' \in F(Y')$, it follows that $\alpha_X(\beta_X(t))$ is the restriction of $u' \in F(Y')$ to $X \subset Y'$. But that is $t \in F(X)$ by construction of u'.
(c) Let $[g] \in [X,Y]_\star$, so that $\alpha_X([g]) = g^*(u)$. To find out what $\beta_X(g^*(u))$ is we should construct Y' containing $X \vee Y$ and $u' \in F(Y')$ such that $u'|_X$ is $g^*(u)$. Then $\beta_X(g^*(u))$ is the composition $X \hookrightarrow X \vee Y \hookrightarrow Y' \simeq Y$. But we can take $Y' = \mathrm{cyl}(g : X \to Y)$. This contains a copy of $X \vee Y$. The cylinder projection $Y' \to Y$ is an explicit homotopy inverse for the inclusion $Y \to Y'$. For $u' \in F(Y')$ we can (must) take the unique element which restricts to $u \in F(Y)$; fortunately it is obvious that $u'|_X = g^*(u)$ for this choice of u'. Moreover the composition of $X \hookrightarrow X \vee Y \hookrightarrow Y'$ with the cylinder projection $Y' \to Y$ is exactly g. □

6.2. Eilenberg-MacLane spaces

Under construction.

LEMMA 6.2.1. *Let Y be a based connected CW-space and suppose that n is a positive integer such that $\pi_k(Y)$ is trivial for all $k > n$. Let X be any connected based CW-space. Then*

$$[X,Y]_* \cong \mathrm{im}\big[\,[X^{n+1},Y]_* \to [X^n,Y]_*\,\big].$$

More carefully stated: if a based map $X^n \to Y$ can be extended to a map $X^{n+1} \to Y$, then it can be extended to a map $X \to Y$, and the based homotopy class of such an extension is unique.

PROOF. □

COROLLARY 6.2.2. *Let X and Y be based connected CW-spaces, and let m,n be positive integers. Suppose that $\pi_k(Y)$ is trivial for all $k > n$ and suppose that $X^{m-1} = *$.*

- *If $m > n$, then $[X,Y]_*$ is trivial; in other words, all based maps $X \to Y$ are based nullhomotopic.*
- *If $m = n$, then the canonical map from $[X,Y]_*$ to the set of homomorphisms from $\pi_m(X)$ to $\pi_m(Y)$ is a bijection.*

PROOF. □

DEFINITION 6.2.3. Let n be a positive integer and let G be a group (abelian if $n > 1$). It is customary to write $K(G,n)$ for a connected based CW-space X which has $\pi_k(X)$ trivial for positive $k \neq n$ and $\pi_n(X)$ equipped with a group isomorphism to G. These spaces are also called *Eilenberg-MacLane spaces*.

If G is abelian, a space $K(G,n)$ can be obtained via Brown's representation theorem as a representing space for the functor

$$X \mapsto H^n(X;G).$$

Indeed, if Y is a representing space for that functor, then $[S^k,Y]_* \cong H^n(S^k;G)$, which is zero for $k \neq n$ and isomorphic (as a group) to G for $k = n$. (Little exercise here: $H^n(S^n;G)$ has two abelian group structures, one coming from the fact that S^n has a co-multiplication and the other one from the definition of H^n; but these agree.) This means that Y is a $K(G,n)$.

If $n = 1$ and G is not abelian (or if we do not wish to assume that G is abelian), the space $K(G,n) = K(G,1)$ can again be obtained via Brown's representation theorem as a representing space for the functor F defined by $F(X) = \hom(\pi_1(X),G)$ (set of group homomorphisms from $\pi_1(X)$ to G). To show that F is half-exact, we use the Seifert-vanKampen theorem. Namely if $X = A \cup B$ where A and B are connected CW-subspaces (containing $*$) and $A \cap B$ is also connected, then Seifert-vanKampen says that

$$\begin{tikzcd} \pi_1(A\cap B) \arrow[r] \arrow[d] & \pi_1(A) \arrow[d] \\ \pi_1(B) \arrow[r] & \pi_1(X) \end{tikzcd}$$

is a pushout square in the category of groups. It follows that

$$\begin{tikzcd} \hom(\pi_1(A\cap B),G) & \hom(\pi_1(A),G) \arrow[l] \\ \hom(\pi_1(B),G) \arrow[u] & \hom(\pi_1(X),G) \arrow[l] \arrow[u] \end{tikzcd}$$

is a pullback square in the category of sets. This implies that F is half-exact.

COROLLARY 6.2.4. *Suppose that X is a $K(G,m)$ and Y is a $K(J,n)$. If $m > n$, then $[X,Y]_\star$ has only one element, the homotopy class of the constant map. If $m = n$, then the evaluation map*

$$[X,Y]_\star \longrightarrow \mathrm{hom}(\pi_m(X), \pi_m(Y)) = \mathrm{hom}(G, J)$$

is bijective.

PROOF. ... □

Note that the corollary does not say much about the case $m < n$. Indeed that case is more difficult. There is some similarity with homotopy groups of spheres. We have $\pi_m(S^n)$ trivial if $m < n$, isomorphic to $\mathbb{Z}$ if $m = n \geq 1$, and difficult in the remaining cases. But note the difference: $\pi_m(S^n)$ trivial if $m < n$, whereas $[X,Y]_\star$ trivial if $m > n$.

COROLLARY 6.2.5. *Up to homotopy equivalence, there is a unique Eilenberg-MacLane space $K(G,n)$.*

PROOF. If X and Y both satisfy the conditions for being called $K(G,n)$, then by the above proposition there is a based map $X \to Y$ inducing an isomorphism $\pi_n(X) \to \pi_n(Y)$. That map is a homotopy equivalence by the JHC Whitehead theorem. □

CHAPTER 7

Homotopy colimits and homotopy limits

CHAPTER 8

The homotopy excision theorem

8.1. Blakers-Massey homotopy excision theorem (special form)

Situation: based CW-space X, two distinct cells attached (of dimensions m and n, respectively) to give CW-spaces $Y_1, Y_2 \supset X$:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Y_1 \cup Y_2 \end{tikzcd}$$

Therefore we have an inclusion $(Y_1, X) \to (Y_1 \cup Y_2, Y_2)$ of pairs. We want to look at the induced map

$$\pi_k(Y_1, X) \longrightarrow \pi_k(Y_1 \cup Y_2, Y_2) \tag{8.1.1}$$

(where $k > 0$) and we ask whether it is surjective, injective etc.

The plan here is to use general position arguments to answer this. There are three integer variables m, n, k in the question and the answer should be formulated in terms of them. Let $E_1 = Y_1 \smallsetminus X$ be that m-cell and let $E_2 \subset Y_2 \smallsetminus X$ be that n-cell, so E_1 open in Y_1 and E_2 open in Y_2. Choose $z_1 \in E_1$ and $z_2 \in E_2$. (These choices can be reconsidered in the following.) Let's try to show surjectivity in (8.1.1) first. Therefore we begin with

$$f\colon (D^k, S^{k-1}) \to (Y_1 \cup Y_2, Y_2). \tag{8.1.2}$$

Think of D^k as unit disk in $\mathbb{R}^k$ with $(1, 0, \dots, 0)$ as center, so that $(0, ..., 0)$ can take the role of the base point.
We can assume that f is smooth in the open subset $f^{-1}(U_1)$ and $f^{-1}(U_2)$ of D^k, where U_1 and U_2 are open neighborhoods of z_1 and z_2 in E_1 and E_2 , respectively. Then we can also assume that z_1 is a regular value for f (as in Sard's theorem, and by Sard's theorem). It follows that $A_1 := f^{-1}(z_1)$ is a smooth compact submanifold of $D^k \smallsetminus S^{k-1}$, without boundary. Let $A_2 = f^{-1}(z_2)$. It is important that $A_1 \cap A_2 = \varnothing$. It is important that $\dim(A_i) = k - m$. Neither A_1 nor A_2 contain the base point $(0, ..., 0)$ of D^k. In particular, since A_2 is compact, we can choose $\varepsilon > 0$ so that the small disk εD^k has empty intersection with A_2.
Now we try to move A_1 into the small disk εD^k without disturbing A_2. As a first attempt we try the homotopy (*isotopy* is a better word here)

$$\big(\varphi_t\colon A_1 \to D^k\big)_{t\in[\varepsilon,1]}$$

given by $\varphi_t(x) := (1+\varepsilon-t)x \in D^k$. *Isotopy* means that $\varphi_t\colon A_1 \to D^k$ is a smooth embedding for every $t \in [\varepsilon, 1]$. We have φ_ε = inclusion and $\varphi_1(A_1)$ is contained in εD^k minus boundary. We can also think of (φ_t) as a single smooth map

$$\varphi\colon A_1 \times [\varepsilon, 1] \longrightarrow D^k\,.$$

Since the plan was not to disturb A_2 , we ask whether $\mathrm{im}(\varphi)$ intersects A_2 . Better way to ask the question: whether z_2 is in the image of $f\varphi$. We can assume that z_2 is a regular value for the smooth map $f\varphi$. Since $\dim(A_1 \times [\varepsilon, 1]) = k - m + 1$ and the dimension of the target of $f\varphi$, as far as it is of interest here, is n, it follows that z_2 is not in the image of $f\varphi$ if

$$\boxed{k - m + 1 < n}\,.$$

This condition turns out to be the decisive one for surjectivity in (8.1.1).

LEMMA 8.1.3. (Special case of Thom's isotopy extension theorem.) *There exists a diffeotopy*

$$\big(\Phi_t\colon D^k \to D^k\big)_{t\in[\varepsilon,1]}\big)$$

which extends the isotopy (φ_t). *More precisely: each* $\Phi_t\colon D^k \to D^k$ *is a diffeomorphism and* Φ_t *agrees with the identity map on a neighborhood of* $S^{k-1} \cup A_2$ *, whereas* Φ_t *agrees with* φ_t *on* A_1. (This is the precise meaning of: moving A_1 into εD^k without disturbing A_2.)

We postpone the proof of the lemma, but we use it right away to finish the proof of surjectivity in (8.1.1). First we note that our map f in (8.1.2) is homotopic, as a map of pairs, to $g := f\Phi_1^{-1}$. The map g has the following convenient properties: $g^{-1}(z_1) \subset \varepsilon D^k$ whereas $g^{-1}(z_2) \cap \varepsilon D^k = \varnothing$. We now try the homotopy

$$g_t\colon D^k \to Y_1 \cup Y_2$$

where $t \in [\varepsilon, 1]$ and $g_t(z) = g((1 + \varepsilon - t)z)$. It is not guaranteed that $g_t(S^{k-1}) \subset Y_2$ but it is guaranteed that $g_t(S^{k-1}) \subset Y_1 \cup Y_2 \smallsetminus \{z_1\}$, and this is good enough for us since the inclusion

$$Y_2 \longrightarrow Y_1 \cup Y_2 \smallsetminus \{z_1\}$$

is a homotopy equivalence. Similarly, although it is not guaranteed that $g_1(D^k) \subset Y_1$, it is guaranteed that $g_1(D^k) \subset Y_1 \cup Y_2 \smallsetminus \{z_2\}$, and this is again good enough for us. Therefore we can conclude that $g = g_\varepsilon$ is homotopic, as a map of pairs from (D^k, S^{k-1}) to $(Y_1 \cup Y_2, Y_2)$, to a map from (D^k, S^{k-1}) to (Y_1, X). This establishes surjectivity in (8.1.1) under the condition $k + 1 < m + n$.

Next, we think about injectivity in (8.1.1). Therefore we should start by expressing the following situation: we have two elements in $\pi_k(Y_1, X)$ which determine the same element of $\pi_k(Y_1 \cup Y_2, Y_2)$. Note that this formulation does not assume group structures in π_k , with a view to the possibility that $k \le 2$. A very straightforward way to express the situation is then to say that we have a map

$$F\colon D^k \times [0, 1] \to Y_1 \cup Y_2$$

such that $F(D^k \times 0) \cup F(D^k \times 1) \subset Y_1$ and $F(S^{k-1} \times [0, 1]) \subset Y_2$. This takes $(0, ..., 0) \times [0, 1]$ to the base point of $Y_1 \cup Y_2$. We now reason with this F as we reasoned with f in (8.1.2) before.
We can assume that F is smooth in $F^{-1}(U_1)$ and $F^{-1}(U_2)$ and that z_1 is a regular value for F. Then we have a new $A_1 := F^{-1}(z_1)$, smooth submanifold of $D^k \times [0, 1]$. This has empty intersection with $S^{k-1} \times [0, 1]$ but it can have nonempty intersection with $D^k \times \partial[0, 1]$. More precisely, A_1 is a smooth submanifold with boundary of $D^k \times [0, 1]$

avoiding $S^{k-1}\times[0,1]$, of dimension $k+1-m$, and the boundary ∂A_1 is the transverse intersection of A_1 and $D^k\times\partial[0,1]$. We put $A_2 := F^{-1}(z_2)$. We choose $\varepsilon>0$ in such a way that $\varepsilon D^k\times[0,1]$ has empty intersection with A_2. Define a smooth isotopy

$$\big(\varphi_t\colon A_1\to D^k\times[0,1]\big)_{t\in[\varepsilon,1]}$$

by $\varphi_t(x,s) := ((1+\varepsilon-t)x,s)\in D^k\times[0,1]$. If

$$\boxed{k+1-m+1<n}$$

then by Sard's theorem (general position) we may assume or arrange that z_2 is a regular value of $F\varphi$ (where $\varphi\colon A_1\times[\varepsilon,1]\to D^k\times[0,1]$ is $(\varphi_t)_{t\in[\varepsilon,1]}$ reorganized). Then the image of each φ_t has empty intersection with A_2 .

Lemma 8.1.4. *There exists a diffeotopy*

$$\big(\Phi_t\colon D^k\times[0,1]\to D^k\times[0,1])_{t\in[\varepsilon,1]}\big)$$

which extends the isotopy (φ_t). *More precisely: each*

$$\Phi_t\colon D^k\times[0,1]\to D^k\times[0,1]$$

is a diffeomorphism and Φ_t *agrees with the identity map on a neighborhood of* $S^{k-1}\times[0,1]\ \cup A_2$, *whereas* Φ_t *agrees with* φ_t *on* A_1. (Moreover Φ_t takes $D^k\times\{0\}$ to itself and takes $D^k\times\{1\}$ to itself ... but this is automatic).

Again we postpone the proof (it is much like the postponed proof of lemma 8.1.3) and use the lemma to finish the proof of injectivity in (8.1.1). First we can replace $F=F\Phi_0$ by $G:=F\Phi_1^{-1}$, since $(F\Phi_t^{-1})_{t\in[\varepsilon,1]}$ is a homotopy from F to G respecting all the essential features. After that, we try the homotopy

$$G_t\colon D^k\times[0,1]\to Y_1\cup Y_2$$

where $t\in[\varepsilon,1]$ and $G_t(z,s)=G((1+\varepsilon-t)z,s)$. The details are as in the proof of surjectivity. Therefore we have shown:

Proposition 8.1.5. *The map* (8.1.1) *is surjective if* $k<m+n-1$ *and bijective if* $k<m+n-2$ (in addition to $k>0$).

Let's remark that surjectivity of (8.1.1) was already known to us in case $k<n$ (by cellular approximation) and also in case $k<m$ (because then both groups/sets are trivial, again by cellular approximation). So the interesting cases of the proposition, as far as surjectivity is concerned, are the cases where $\max\{m,n\}\le k<m+n-1$. Similarly injectivity of (8.1.1) was already known to us in case $k+1<n$ and in case $k<m$. So the interesting cases of the proposition, as far as injectivity is concerned, are the cases where $\max\{m,n+1\}\le k<m+n-2$.

Proof of lemma 8.1.3. *Under construction.* □

8.2. Blakers-Massey theorem, general form

8.3. The Freudenthal theorem

Under construction.
This is a special case of proposition 8.1.5. We take $X=S^{m-1}$ and Y_1 equal to the (closed) upper hemisphere S^m_+ of S^m, while Y_2 is equal to the closed lower hemisphere S^m_- of S^m.

8.4. The Serre theorems on homotopy groups of spheres

Under construction.

CHAPTER 9

Postnikov towers and obstruction theory

9.1. Postnikov tower and Postnikov-Moore factorization

Let Y be a based connected CW-space. The Postnikov tower of Y is a diagram of spaces

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \beta_3 Y \arrow[d] \\
 & \beta_2 Y \arrow[d] \\
 & \beta_1 Y \arrow[d] \\
Y \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & \beta_0 Y
\end{tikzcd}$$

where the map $Y \to \beta_k Y$ has the following property: isomorphism on π_m for all $m \le k$, whereas $\pi_m(\beta_k Y)$ is trivial for $m > k$. We also say that $\beta_k Y$ is obtained from Y by killing homotopy groups in dimensions $> k$. From the point of view of the Brown representation theorem, the spaces $\beta_k Y$ have a very pleasant definition. (A weakness of this point of view: it only constructs $\beta_k Y$ in the homotopy category and it only constructs the Postnikov tower as a diagram in the homotopy category.)

Definition 9.1.1. (in the style of Brown's representation theorem.) The space $\beta_k Y$ is a representing space for the half-exact functor

$$X \mapsto \mathrm{im}\Big[\ [X^{k+1}, Y]_\star \xrightarrow{\text{res}} [X^k, Y]_\star\ \Big].$$

This definition is rather terse. You can read it as follows: a homotopy class of based maps from X (variable) to $\beta_k Y$ is the same as a homotopy class of based maps $X^k \to Y$ which can be extended to (a homotopy class of) based maps $X^{k+1} \longrightarrow Y$. (The extension $X^{k+1} \to Y$ is not specified; it is only required to exist.)

The biggest puzzle with definition 9.1.1 is that it does not obviously describe a (contravariant) *functor* on $\mathscr{H}o\mathscr{C}$. We need to show that a homotopy class of based maps $g\colon W \to X$ determines a a map

$$\begin{tikzcd}
\mathrm{im}\Big[[X^{k+1}, Y]_\star \xrightarrow{\text{res}} [X^k, Y]_\star\Big] \arrow[d, dotted] \\
\mathrm{im}\Big[[W^{k+1}, Y]_\star \xrightarrow{\text{res}} [W^k, Y]_\star\Big].
\end{tikzcd}$$

If $g{:}W \to X$ is cellular, then it is clear how we can use it to define the dotted arrow: by precomposition with g, restricted to appropriate skeletons. But suppose that $f,g{:}W \to X$ are two cellular maps in the same homotopy class. Choose a cellular homotopy. The cellular homotopy restricts to a map from $W^k \times [0,1]$ to X^{k+1}. Using that observation, it is easy to see that it does not matter whether we define the dotted arrow using precomposition with g, or precomposition with f.

We should also verify that the functor described in definition 9.1.1 is half-exact as claimed. But the verification is unexciting. It is obvious that the functor satisfies the strong wedge axiom. Suppose now that $X = A \cup B$, where A and B are based connected CW-subspaces of the based CW-space X, and $A \cap B$ is also connected. We can replace X by $X^\sharp$ as in example 6.1.4. Suppose given based maps $f{:}A^k \to Y$ and $g{:}B^k \to Y$ such that the restrictions of f and g to $A^k \cap B^k$ are homotopic by a homotopy $(h_t)_{t\in[0,1]}$, and such that f extends to a map $\bar{f}{:}A^{k+1} \to Y$ while g extends to a map $\bar{g}{:}B^{k+1} \to Y$. Then we have a map from the $(k+1)$-skeleton of $X^\sharp$ to Y given by $\bar{f}$ on the copy of A^{k+1}, by $\bar{g}$ on the copy of B^{k+1}, and by h_t on the copy of $(A^k \cap B^k) \times \{t\}$. This map and its restriction to the k-skeleton of $X^\sharp$ constitute the solution to our problem.

DEFINITION 9.1.2. (in the style of Postnikov and Moore). The CW-space $\beta_k Y$ contains Y as a CW-subspace and is obtained from Y by attaching cells of dimension $> k+1$ to kill the homotopy groups in dimensions $> k$.

This calls for some explanations, too. There is the following more systematic definition, followed by an existence statement and a uniqueness statement.

DEFINITION 9.1.3. Let $f{:}X \to Y$ be a based map of based connected CW-spaces. A *Postnikov-Moore k-factorization* of f consists of a connected CW-space X' containing X as a CW-subspace and a based map $f_k{:}X' \to Y$ which extends f and has the following properties. The inclusion $X \to X'$ induces an isomorphism in π_j for $j \le k$ and a surjection for $j = k+1$, while f_k induces an injection in π_j for $j = k+1$ and an isomorphism in π_j for all $j > k+1$. (The homotopy groups of X' are then determined as follows: $\pi_j(X')$ is isomorphic to $\pi_j(X)$ when $j \le k$, isomorphic to the image of $f_*{:}\pi_j(X) \to \pi_j(Y)$ when $j = k+1$ and isomorphic to $\pi_j(Y)$ when $j \ge k+2$.)

I like to write $\beta_{f,k}X$ for X' , in view of existence and uniqueness statements below, but this is probably not standard notation.

PROPOSITION 9.1.4. *Let $f{:}X \to Y$ be a based map of based connected CW-spaces. A Postnikov-Moore k-factorization of f exists.*

PROOF. We construct inductively $\beta_{f,k,\ell}X$, a CW-space containing X, and a map

$$f_{k,\ell}{:}\beta_{f,k,\ell}X \to Y$$

extending f, such that the inclusion $X \to \beta_{f,k,\ell}X$ induces an isomorphism in π_j for $j \le k$ and a surjection for $j = k+1$, while $f_{k,\ell}$ induces an injection in π_j for $j = k+1$, an isomorphism in π_j for all j such that $k+2 \le j < \ell$ and a surjection for $j = \ell$. These conditions just start to make sense when $\ell = k+2$ and so the work begins with the construction of $f_{k,k+2}$ and $\beta_{f,k,k+2}X$. To construct $\beta_{f,k,k+2}X$ from X we choose a based map

$$u{:}\bigvee_{\lambda\in\Lambda} S^{k+1} \longrightarrow X$$

such that the image of the homomorphism in π_{k+1} determined by u is the kernel of the homomorphism in π_{k+1} determined by f. Choose also an extension of fu to a map

$$v: \bigvee_{\lambda\in\Lambda} D^{k+2} \longrightarrow Y$$

(such an extension exists by the construction of u). Choose also a based map

$$w: \bigvee_{\tau} S^{k+2} \longrightarrow Y$$

which induces a surjection in π_{k+2}. Let

$$\beta_{f,k,k+2}X := \mathrm{cone}(u) \vee \bigvee_{\tau} S^{k+2}$$

and define $f_{k,k+2}$ so that it agrees with w on the wedge of $(k+2)$-spheres, with f on the copy of X and with v when composed with the standard map from cone(source(u)) to cone(u). Note that the inclusion of X in the cone of u induces a surjection in π_{k+1} (by cellular approximation) whose kernel contains the kernel of $f_*: \pi_{k+1}(X) \to \pi_{k+1}(Y)$ by construction, but cannot be bigger (say why), so that cone(u) already has the correct π_{k+1}. By taking the wedge with many S^{k+2}-spheres we can make π_{k+2} bigger without changing π_j for $j \le k+1$. We do this in order to end up with a surjection from $\pi_{k+2}(\beta_{f,k,k+2}X)$ to $\pi_{k+2}(Y)$, as required.
The subsequent induction steps are like the first one. More precisely, if

$$f_{k,\ell}: \beta_{f,k,\ell}X \longrightarrow Y$$

has already been constructed, then we can declare $\beta_{f,k,\ell}X$ to be the new X and $f_{k,\ell}$ to be the new f and $\ell-1$ to be the new k, and repeat the procedure above. The outcome is $\beta_{f,k,\ell+1}X$ containing $\beta_{f,k,\ell}X$ and $f_{k,\ell+1}$ extending $f_{k,\ell}$. (Let me explain this in more detail. Following the instructions we get isomorphisms

$$\pi_j(\beta_{f,k,\ell}X) \longrightarrow \pi_j(\beta_{f,k,\ell+1}X)$$

induced by inclusion for $j \le \ell-1$, a surjection

$$(*) \qquad \pi_\ell(\beta_{f,k,\ell}X) \longrightarrow \pi_\ell(\beta_{f,k,\ell}X),$$

induced by inclusion, an injection

$$(**) \qquad \pi_\ell(\beta_{f,k,\ell+1}X) \longrightarrow \pi_\ell(Y)$$

induced by $f_{k,\ell+1}$ and a surjection

$$\pi_{\ell+1}(\beta_{f,k,k+3}X) \longrightarrow \pi_{\ell+1}(Y)$$

induced by $f_{k,\ell+1}$. But since the composition of $(**)$ and $(*)$ is surjective by construction (of $f_{k,\ell}$), it follows that $(**)$ is an isomorphism, which is what we want for $f_{k,\ell+1}$.) — When the induction is finished we define

$$\beta_{f,k}X := \bigcup_{\ell=k}^{\infty} \beta_{f,k,\ell}X$$

and we define f_k so that it agrees with $f_{k,\ell}$ on $\beta_{f,k,\ell}$. □

REMARK 9.1.5. The Postnikov-Moore k-factorization of a based map

$$f: X \to Y$$

of based CW-spaces has a uniqueness property. Suppose that

$$X \hookrightarrow X' \to Y, \qquad X \hookrightarrow X'' \to Y$$

are two Postnikov-Moore k-factorizations of f. That is, both compositions are equal to f, and we suppose that the inclusions $X \to X'$ and $X \to X''$ induce isomorphisms in π_j for $j \le k$ and a surjection in π_{k+1}, and that the maps $X' \to Y$ and $X'' \to Y$ induce an injection in π_{k+1} and an isomorphism in π_j for $j \ge k+2$. Form

$$X''' := X' \sqcup_X X'',$$

the union of X' and X'' along their common CW-subspace X (strictly speaking: the pushout or colimit of the diagram $X' \hookleftarrow X \hookrightarrow X''$). The maps $X' \to Y$ and $X'' \to Y$ that we started with agree on X and so define a map $X''' \to Y$. Make a Moore-Postnikov k-factorization for that:

$$X''' \hookrightarrow X'''' \to Y\ .$$

Exercise: Show that $X \hookrightarrow X'''' \to Y$ is also a Postnikov-Moore k-factorization of f. (This can be said to contain the other two that we started with. It follows that the inclusions $X' \hookrightarrow X''''$ and $X'' \hookrightarrow X''''$ are homotopy equivalences.)

Remark 9.1.6. Taking $Y = \star$, we have a unique $f\colon X \to Y$. We can define $\beta_k X$ to be the X' in a Postnikov-Moore k-factorization $X \hookrightarrow X' \to Y$ of this f, to make the connection with definition 9.1.1 at last.

Remark 9.1.7. Taking $X = \star$ is also a good idea! Let $\star \hookrightarrow X' \to Y$ be a Moore-Postnikov k-factorization of $f\colon \star \to Y$. In this case $\pi_j(X')$ is trivial for $j \le k+1$ and and $f_k\colon X' \to Y$ induces an isomorphism in π_j for $j \ge k+2$. In particular when $k = 0$ the map f_k has the homotopical properties of the universal covering of Y. More to the point, the diagram

$$\star \hookrightarrow \tilde{Y} \to Y$$

(where $\tilde{Y} \to Y$ is the universal covering) is an instance of a Postnikov-Moore 0-factorization.

Remark 9.1.8. Let $f\colon X \to Y$ be a based map of based connected CW-spaces, let

$$X \xrightarrow{e} X' \xrightarrow{f_k} Y$$

be a Postnikov-Moore k-factorization for f and let

$$X \xrightarrow{d} X'' \xrightarrow{e_\ell} X'$$

be a Postnikov-Moore ℓ-factorization for $e\colon X \to X'$, where $\ell > k$. Then

$$X \xrightarrow{d} X'' \xrightarrow{f_k e_\ell} Y$$

is a Postnikov-Moore ℓ-factorization for f. (*Exercise.*)

DEFINITION 9.1.9. Let $f\colon X \to Y$ be a based map of based connected CW-spaces. The Postnikov-Moore tower (or decomposition) of f is a commutative diagram

$$\begin{tikzcd} & \vdots \arrow[d] \\ & \beta_{f,2}X \arrow[d, "p_2"] \\ & \beta_{f,1}X \arrow[d, "p_1"] \\ & \beta_{f,0}X \arrow[d, "p_0"] \\ X \arrow[r, "f"'] \arrow[ur] \arrow[uur] \arrow[uuur] & Y \end{tikzcd}$$

where the lowest triangle is obtained by choosing a Postnikov-Moore 0-factorization of f, the triangle above that is obtained by choosing a Postnikov-Moore 1-factorization for $X \to \beta_{f,0}X$, the triangle above that is obtained by choosing a Postnikov-Moore 2-factorization for $X \to \beta_{f,1}X$, and so on. The naming of terms in the right-hand column is justified by remark 9.1.8; in particular

$$\begin{tikzcd} & \beta_{f,k}X \arrow[d, "p_0p_1\cdots p_k"] \\ X \arrow[r, "f"'] \arrow[ur] & Y \end{tikzcd}$$

is a Postnikov-Moore k-factorization of f.

REMARK 9.1.10. In the case where Y is a point we obtain the Postnikov tower of X. In that case it does not hurt to delete Y from the diagram and to write $\beta_k X$ instead of $\beta_{f,k}X$.

We continue with $f\colon X \to Y$ as in definition 9.1.9. Recall that the *relative homotopy groups of* f are defined as the homotopy groups of the pair $(\mathrm{cyl}(f), X)$ where $\mathrm{cyl}(f)$ is the reduced mapping cylinder of f. Since $\mathrm{cyl}(f) \simeq Y$, we can write the long exact sequence of homotopy groups of that pair in the form

$$\ldots \xrightarrow{\partial} \pi_m(X) \xrightarrow{f_*} \pi_m(Y) \longrightarrow \pi_m(f) \xrightarrow{\partial} \pi_{m-1}(X) \longrightarrow \ldots$$

The Postnikov-Moore factorization

$$X \longrightarrow \beta_{f,k}X \xrightarrow{f_k = p_0p_1\cdots p_k} Y$$

of f leads to a map of pairs $(\mathrm{cyl}(f), X) \to (\mathrm{cyl}(f_k), X)$. That map of pairs leads to a commutative diagram

$$\begin{tikzcd} \ldots \arrow[r, "\partial"] & \pi_m(X) \arrow[r, "f_*"] \arrow[d] & \pi_m(Y) \arrow[r] \arrow[d, equal] & \pi_m(f) \arrow[r, "\partial"] \arrow[d] & \pi_{m-1}(X) \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r, "\partial"] & \pi_m(\beta_{f,k}X) \arrow[r, "(f_k)_*"] & \pi_m(Y) \arrow[r] & \pi_m(f_k) \arrow[r, "\partial"] & \pi_{m-1}(\beta_{f,k}X) \arrow[r] & \ldots \end{tikzcd}$$

¿From that we can deduce:

LEMMA 9.1.11. *The arrow $\pi_m(f) \to \pi_m(f_k)$ is an isomorphism for $m \le k+1$ while $\pi_m(f_k)$ is trivial for $m > k+1$.* □

In the case where $Y = \star$ we have, rather obviously, $\pi_m(f) \cong \pi_{m-1}(X)$ and $\pi_m(f_k) \cong \pi_{m-1}(\beta_k X)$. So the lemma reduces in that case to something we already know: $\pi_{m-1}(\beta_k X) \cong \pi_{m-1}(X)$ for $m-1 \le k$ and $\pi_{m-1}(\beta_k X)$ is trivial for $m-1 > k$.

COROLLARY 9.1.12. *For the map $p_k \colon \beta_{f,k} X \to \beta_{f,k-1} X$ we have*

$$\pi_{k+1}(p_k) \cong \pi_{k+1}(f_k) \cong \pi_{k+1}(f)$$

and $\pi_m(p_k)$ is trivial for all $m \ne k+1$.

PROOF. If we take $m = k+1$ in the commutative diagram

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] & \pi_m(\beta_{f,k}X) \arrow[r, "(p_k)_*"] \arrow[d, equal] & \pi_m(\beta_{f,k-1}X) \arrow[r] \arrow[d] & \pi_m(p_k) \arrow[r, "\partial"] \arrow[d, "a"] & \pi_{m-1}(\beta_{f,k}X) \arrow[r] \arrow[d, equal] & \dots \\
\dots \arrow[r, "\partial"] & \pi_m(\beta_{f,k}X) \arrow[r, "(f_k)_*"] & \pi_m(Y) \arrow[r] & \pi_m(f_k) \arrow[r, "\partial"] & \pi_{m-1}(\beta_{f,k}X) \arrow[r] & \dots
\end{tikzcd}$$

then the five lemma tells us that the arrow labeled a is an isomorphism. (More precisely the two vertical arrows to the left of a are bijective, the one to the right of a is obviously bijective and the next one to the right is injective. There is enough group structure to go around, though perhaps not as much as we assume in the standard formulation of the five lemma; we can assume $k \ge 1$ and therefore $m \ge 2$.) If $m < k+1$ then $(p_k)_*$ is surjective in degree m, bijective in degree $m-1$, so that $\pi_m(p_k)$ is trivial by exactness of the top row. If $m > k+1$ then $(p_k)_*$ is bijective in degree m, injective in degree $m-1$, so that $\pi_m(p_k)$ is again trivial by exactness of the top row. □

9.2. One-step obstruction theory

This section is mostly about the problem indicated in the following diagram.

$$\begin{tikzcd}
 & E \arrow[d, "p"] \\
X \arrow[ur, dotted, "g"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

Here X, Y, E are based connected CW-spaces (until declared otherwise) and f, p are based maps, all given in advance. The problem is to find g. The diagram is meant to be commutative up to a specified homotopy $h = (h_t)$ from pg to f. It is convenient to formulate the problem in this generality, but we will get the most significant results when p satisfies a strong condition: *there is an integer $\ell \ge 1$ such that $\pi_k(p)$ is trivial whenever $k \ne \ell$.*

Let's begin with technical considerations. In the problem just formulated, the data f and p are given and by a *solution* of the problem we mean a pair (g, h) consisting of $g \colon X \to E$ and a based homotopy h from pg to f. The homotopy $h = (h_t)_{t \in [0,1]}$ is a map from $X \times I$ to Y, where $I := [0,1]$. It seems clear that we need to organize these pairs (g, h) into a *space*. More precisely we should first define $\mathrm{map}_\star(X, E)$, the space of based maps from

X to E, and also $\mathrm{map}_\star(X,Y)$, the space of based maps from X to Y. Then the space of solutions for our problem is

$$L(f;p) := \{(g,h) \in \mathrm{map}_\star(X,E) \times \mathrm{map}_\star(X \ltimes I, Y) \mid h_0 = pg,\ h_1 = f\},$$

where $X \ltimes I$ is improvised notation for the quotient of $X \times I$ by the subspace $\star \times I$. It is a subspace of $\mathrm{map}_\star(X,E) \times \mathrm{map}_\star(X \ltimes I, Y)$. The letter L is used to suggest *lift*; the solutions (g,h) are considered to be *lifts* of f (up to a specified homotopy).

Definition 9.2.1. Let V be a CW-space and let W be any space. Let $\mathrm{map}(V,W)$ be the set of all continuous maps from V to W. We use the compact-open topology to regard $\mathrm{map}(V,W)$ as a space. That is to say, a subset Q of $\mathrm{map}(V,W)$ is considered to be open if for every $e \in \mathrm{map}(V,W)$ there exists a non-negative integer ℓ, compact subsets $K_1,\dots,K_\ell \subset V$ and open subsets $U_1,\dots,U_\ell \subset W$ such that

- $e(K_j) \subset U_j$ for $j = 1,\dots,\ell$;
- all $e' \in \mathrm{map}(V,W)$ which satisfy $e'(K_j) \subset U_j$ for $j = 1,\dots,\ell$ also belong to Q.

Remark 9.2.2. Let P be a *compact* CW-space. There is a map (of sets)

$$\mathrm{map}(P \times V, W) \longrightarrow \mathrm{map}(P, \mathrm{map}(V,W))$$

given by adjunction. I hope it is an exercise to show that the map is a homeomorphism. This is good enough for many purposes. We are very interested in the cases where $P = S^n$ or $P = S^n \times I$ for some n.

The condition that P be compact can be dropped at a price: we must re-define the topology on $P \times V$ in such a way that a subset C of $P \times V$ is closed if and only if $C \cap (P' \times V')$ is closed for every choice of compact CW-subspaces $P' \subset P$ and $V' \subset V$. We might write $P \times_{CW} V$ for this modified product and call it the CW-product, etc. It does have the universal property of a product *in the category of CW-spaces and continuous maps.*

Remark 9.2.3. It is known that if V and W are CW-spaces, then $\mathrm{map}(V,W)$ is homotopy equivalent to a CW-space. This is shown in an old paper by John Milnor, *On spaces having the homotopy type of a CW-complex.* Beautifully written and highly recommended reading. But I will try not to use this fact.

In that connection, let's also keep the following in mind. If Z is *any* space, then there exists a CW-space $Z^\natural$ and a map $Z^\natural \to Z$ which is a weak equivalence. This is an easy consequence of the Brown representation theorem. (For any choice of base point in Z we have a half-exact functor $[-,Z]_\star$. The different path components of Z should be treated separately, and a base point should be selected in each of them.) We say that $Z^\natural$ is a CW-replacement for Z. This notion is of course applicable with $Z = \mathrm{map}(V,W)$, even if we don't know or don't want to know that $\mathrm{map}(V,W)$ is homotopy equivalent to a CW-space.

Example 9.2.4. Our definition of the space of lifts $L(f;p)$ for a diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

can be reformulated as follows:

$$L(f;p) = \mathrm{hofiber}_f(\mathrm{map}_\star(X,E) \xrightarrow{p\circ} \mathrm{map}_\star(X,Y)).$$

And here we define $\mathrm{map}_\star(X,Y)$ etc. as a subspace of $\mathrm{map}(X,Y)$.

Definition 9.2.5. The *homotopy pullback* of a diagram

$$\begin{tikzcd} & B \arrow[d, "g"] \\ C \arrow[r, "v"] & D \end{tikzcd}$$

is the space $\{\,(x,\omega,y)\in B\times \mathrm{map}(I,D)\times C \mid \omega(0)=g(x),\ \omega(1)=v(x)\,\}$. A commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, "v"] & D \end{tikzcd}$$

is a *weak homotopy pullback square* if the map from A to the homotopy pullback of g and v defined by $a\mapsto (u(a),\mathrm{const.path}, f(b))$ is a weak homotopy equivalence.

Remark 9.2.6. The homotopy pullback of g and v (notation as above) can also be defined as follows: replace g by a fibration $g^\sharp\colon B^\sharp\to D$ using the Serre construction. Then form the ordinary pullback of v and $g^\sharp$.

Exercise. Show that a commutative square of spaces

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, "v"] & D \end{tikzcd}$$

is a weak homotopy pullback square if and only if, for every $c\in C$, the map $\mathrm{hofiber}_c(f)\to \mathrm{hofiber}_{v(c)}(g)$ induced by the horizontal arrows u and v is a weak homotopy equivalence.

Exercise. Let $u\colon Y\to Z$ be a map of spaces. Let X be a CW-space. If u is a weak equivalence, then the map $u\circ : \mathrm{map}(X,Y)\to\mathrm{map}(X,Z)$ is again a weak equivalence.

Proposition 9.2.7. *Let*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, "v"] & D \end{tikzcd}$$

be a weak homotopy pullback square of spaces. Let X be a CW-space. Then

$$\begin{tikzcd} \mathrm{map}(X,A) \arrow[r, "u\circ"] \arrow[d, "f\circ"] & \mathrm{map}(X,B) \arrow[d, "g\circ"] \\ \mathrm{map}(X,C) \arrow[r, "v\circ"] & \mathrm{map}(X,D) \end{tikzcd}$$

is again a weak homotopy pullback square.

Proof. Let P be the homotopy pullback of $C\to D\leftarrow B$ and let Q be the homotopy pullback of $\mathrm{map}(X,C)\to\mathrm{map}(X,D)\leftarrow\mathrm{map}(X,B)$. We need to show that the comparison map from $\mathrm{map}(X,A)$ to Q is a weak equivalence. But Q can easily be identified with $\mathrm{map}(X,P)$. With that identification, the comparison map from $\mathrm{map}(X,A)$ to Q becomes the map from $\mathrm{map}(X,A)$ to $\mathrm{map}(X,P)$ induced by the comparison map $A\to P$. It is therefore a weak equivalence by one of the two exercises just above, since the comparison map $A \to P$ is a weak equivalence by assumption. □

PROPOSITION 9.2.8. *Suppose that, in a commutative diagram*

$$\begin{tikzcd} & E_0 \arrow[r] \arrow[d, "p_0"] & E_1 \arrow[d, "p_1"] \\ X \arrow[r, "f"] & Y_0 \arrow[r, "v"] & Y_1 \end{tikzcd}$$

of based spaces and based maps, the right-hand square is a weak homotopy pullback square, and X is a connected CW-space. Then the map

$$L(f;p_0) \longrightarrow L(vf;p_1)$$

determined by the horizontal arrows in the square is a weak equivalence.

PROOF. Follows from proposition 9.2.7 and the second of the two exercises. □

LEMMA 9.2.9. *Let*

$$\begin{tikzcd} E \arrow[d, "p"] \\ Y \end{tikzcd}$$

be a map of based connected CW-spaces. Suppose that Y is 1-connected. Suppose that there exists $\ell \geq 2$ such that $\pi_k(p)$ is trivial for all $k \neq \ell$; if $\ell = 2$, assume in addition that $\pi_\ell(p)$ is abelian. Then p is part of a weak homotopy pullback square of based connected CW-spaces

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"] & E_1 \arrow[d, "p_1"] \\ Y \arrow[r] & Y_1 \end{tikzcd}$$

where E_1 is contractible.

PROOF. We begin with the commutative square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"] & \mathrm{cone}(E) \arrow[d, "q"] \\ Y \arrow[r] & \mathrm{cone}(p) \end{tikzcd} \qquad (9.2.10)$$

where q is the inclusion $\mathrm{cone}(E) \to Y \sqcup \mathrm{cone}(E)$ followed by the quotient map from $Y \sqcup \mathrm{cone}(E)$ to $\mathrm{cone}(p)$. This square is a first approximation to the solution; the upper right-hand term is already good (because it is contractible), but the lower right-hand term is not. Now compose with the Postnikov approximation

$$u\colon \mathrm{cone}(p) \hookrightarrow \beta_\ell \mathrm{cone}(p).$$

(Recall that $\beta_\ell\mathrm{cone}(p)$ is obtained from $\mathrm{cone}(p)$ by attaching cells of dimension $> \ell+1$ to kill the homotopy groups of $\mathrm{cone}(p)$ in degrees $> \ell$.) In this way we get

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"] & \mathrm{cone}(E) \arrow[d, "q"] & \\ Y \arrow[r] & \mathrm{cone}(p) \arrow[r, "u"] & \beta_\ell\mathrm{cone}(p) \end{tikzcd}$$

and now deleting $\mathrm{cone}(p)$ gives a commutative square

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"] & \mathrm{cone}(E) \arrow[d, "uq"] \\ Y \arrow[r] & \beta_\ell\mathrm{cone}(p). \end{tikzcd} \tag{9.2.11}$$

It turns out that this is the solution. In other words, we will now show that (9.2.11) is a weak homotopy pullback square. Briefly: The Hurewicz theorem (relative case) implies that in the square (9.2.10), the Hurewicz homomorphism is an isomorphism

$$\pi_m(p) \to H_m(p)$$

for $m \le \ell$; only the case $m = \ell$ is really interesting. (Note, if it wasn't clear, that the homology of a *map* is by definition the reduced homology of its mapping cone.) Similarly the Hurewicz homomorphism

$$\pi_m(q) \to H_m(q)$$

is an isomorphism for $m \le \ell$. The map $\mathrm{cone}(p) \to \mathrm{cone}(q)$ induced by the horizontal arrows in square (9.2.10) is a homotopy equivalence (easy exercise) and so the horizontal arrows induce an isomorphism $H_\ell(p) \to H_\ell(q)$. Combining this with what we already know about the Hurewicz homomorphisms, we conclude that the horizontal arrows induce an isomorphism

$$\pi_m(p) \longrightarrow \pi_m(q) \cong \pi_m(\mathrm{cone}(p))$$

when $m \le \ell$. For $m > \ell$, we know that $\pi_m(p) = 0$ and we don't know much about about $\pi_m(\mathrm{cone}(p))$. But this gets much better if we look at square (9.2.11) instead. For $m \le \ell$ there is no need to distinguish between $\pi_m(q) \cong \pi_m(\mathrm{cone}(p))$ and $\pi_m(uq) \cong \pi_m(\beta_\ell\mathrm{cone}(p))$. Therefore the homomorphism $\pi_m(p) \to \pi_m(uq)$ is still an isomorphism for $m \le \ell$. But for $m > \ell$, the homotopy group $\pi_m(uq) \cong \pi_m(\beta_\ell\mathrm{cone}(p))$ is zero by definition or construction. Therefore the horizontal arrows in the square (9.2.11) induce an isomorphism

$$\pi_m(p) \longrightarrow \pi_m(uq)$$

for all m (in particular, for $m > \ell$ because any homomorphism between two groups which are zero is an isomorphism). It follows (by one of the exercises) that square (9.2.11) is a weak homotopy pullback square. $\square$

Remark 9.2.12. The space Y_1 in lemma 9.2.9 is an Eilenberg-MacLane space, with only one possibly nontrivial homotopy group in degree ℓ. That is so because

$$\pi_m(Y_1) \cong \pi_m(p_1)$$

since E_1 is contractible, and

$$\pi_m(p_1) \cong \pi_m(p)$$

since the square in the lemma is meant to be a homotopy pullback square, and $\pi_m(p)$ is trivial except possibly for $m=\ell$.

Let's return to our obstruction theory problem

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dotted, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$

stated at beginning of section, $\pi_m(p)$ nontrivial only for $m=\ell$, where ℓ is fixed. If Y happens to be 1-connected and $\ell\geq 3$ or $\ell=2$ and $\pi_2(p)$ is abelian, then proposition 9.2.8 and lemma 9.2.9 reduce this to the special situation where E is contractible and Y is consequently an Eilenberg-MacLane space. We now assume all that and write out the solution.

A first step is to reduce one more little step to the case where $p\colon E\to Y$ is the inclusion of the base point. (Justify this by applying proposition 9.2.8.) The map f then corresponds to a cohomology class

$$\kappa_f\in H^\ell(X;G)$$

where $G=\pi_\ell(Y)$. The space $L(f;p)$ is nonempty if and only if f is nullhomotopic, if and only if $\kappa_f=0\in H^\ell(X;G)$.

If $L(f;p)$ is nonempty and we select an element $z=(g,h)$ in it, then g is uninteresting and h is a nullhomotopy for f. It is not difficult at all to use the nullhomotopy h to make a homotopy equivalence from $L(f;p)$ to $L(e;p)$ where $e\colon X\to Y$ is the zero map.

Now $L(e;p)$ simplifies to $\Omega(\mathrm{map}_\star(X,Y))$. Here $\Omega Z\coloneqq \mathrm{map}_\star(S^1,Z)$, for a based space Z. (Name: *loop space* of Z.) In this way, since Y is still an Eilenberg-MacLane space,

$$\begin{array}{rcccl} \pi_m(L(e;p)) & = & [S^m,\Omega(\mathrm{map}_\star(X,Y)]_\star & \cong & [S^1\wedge S^m,\mathrm{map}_\star(X,Y)]_\star \\ \cong\ [S^{m+1}\wedge X,Y]_\star & \cong & \tilde H^\ell(S^{m+1}\wedge X;G) & \cong & \tilde H^{\ell-m-1}(X;G). \end{array}$$

In all, we have a fairly good understanding of $L(f;p)$.

Remark 9.2.13. The combined isomorphism

$$\pi_m(L(f;p))\cong\pi_m(L(e;p))\cong\tilde H^{\ell-m-1}(X;G)$$

which we have constructed depends on a choice of based homotopy h from f to the constant map e. Let's call it J_h therefore. If k is another homotopy from f to the constant based map, then the concatenation of h and the reverse of k is a based map from $S^1\wedge X$ to Y, where S^1 appears as a quotient of an interval such as $[0,2]$. This corresponds to an element $\delta(h,k)$ of $H^\ell(S^1\wedge X;G)\cong H^{\ell-1}(X;G)$. By inspection, the bijective map

$$\tilde H^{\ell-m-1}(X;G)\xrightarrow{J_h^{-1}}\pi_m(L(f;p))\xrightarrow{J_k}\tilde H^{\ell-m-1}(X;G)$$

is the identity when $m>0$ and is given by addition of $\delta(h,k)$ when $m=0$.

9.3. Many-step obstruction theory

This section consists only of a few words on how you are supposed to combine the science of the Postnikov tower (or the Postnikov-Moore factorization) with one-step obstruction theory to obtain something more generally applicable.

EXAMPLE 9.3.1. Suppose that X and Z are based connected CW-spaces. (The letter Z is used to avoid confusion with Y in the previous subsection, but it is not supposed to remind you of the set of integers.) Suppose that Z is simply connected, that is, $\pi_1(Z)$ is trivial. Suppose that X is finite-dimensional, $X = X^m$. We ask: what does the set $[X, Z]_\star$ look like. It is a good moment to remember the Postnikov tower of Z:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & \beta_3 Z \arrow[d] & \\
 & \beta_2 Z \arrow[d] & \\
 & \beta_1 Z \arrow[d] & \simeq \star \\
Z \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & \beta_0 Z & \simeq \star
\end{tikzcd}$$

This determines a diagram of sets (with base point)

$$\begin{tikzcd}
{[X, Z]} \arrow[d, equal] \\
{[X, \beta_m Z]_\star} \arrow[d] \\
{[X, \beta_{m-1} Z]_\star} \arrow[d] \\
{[X, \beta_{m-2} Z]_\star} \arrow[d] \\
\vdots \arrow[d] \\
{[X, \beta_3 Z]_\star} \arrow[d] \\
{[X, \beta_2 Z]_\star} \arrow[d] \\
{[X, \beta_1 Z]_\star} \arrow[d, equal] \\
\star
\end{tikzcd}$$

At the top of this diagram, we have $[X, Z]_\star = [X, \beta_m Z]_\star$ by the description of $\beta_m Z$ in definition 9.1.1. The obstruction to climbing from $[X, \beta_{k-1} Z]_\star$ to $[X, \beta_k Z]_\star$ is described by an exact sequence

$$[X, \beta_k Z]_\star \longrightarrow [X, \beta_{k-1} Z]_\star \longrightarrow H^{k+1}(X; \pi_k Z)$$

so that an element in $[X, \beta_{k-1} Z]_\star$ comes from $[X, \beta_k Z]_\star$ if and only if it goes to zero in $H^{k+1}(X; \pi_k Z)$. Also, if $g\colon X \to \beta_k Z$ is any based map, then we have a *surjection*

$$H^k(X; \pi_k Z) \longrightarrow \text{preimage of } [p_k g] \text{ under } [X, \beta_k Z]_\star \to [X, \beta_{k-1} Z]_\star\,.$$

We get this from the previous (sub)section by substituting $p_k\colon \beta_k Z \to \beta_{k-1} Z$ for what was called $p\colon E \to Y$ there. It is allowed because $p_k\colon \beta_k Z \to \beta_{k-1} Z$ has a single nonzero homotopy group, by the construction of the Postnikov tower. It sits in dimension $k+1$ and is isomorphic to $\pi_k(Z)$. The *surjection* above comes from writing $f := p_k g$. Then we have a bijection $\pi_0(L(f; p_k)) \cong H^k(X; \pi_k Z)$ as in the previous subsection. It should be clear that there is a forgetful map $\pi_0(L(f;p_k)) \to [X,\beta_k Z]$ whose image is exactly the preimage of the element $[p_k g]$ under the map $[X,\beta_k Z]_\star \to [X,\beta_{k-1}Z]_\star$. Unfortunately this surjective forgetful map is not always *bijective*.

EXAMPLE 9.3.2. Here is a long story to show that ... *this surjective map is not always bijective.* Maybe I am writing this mainly for my own education. It is a little over the top, to use a good English phrase.
Take $X = \mathbb{C}P^2$ and $Z = S^3$. This is a situation where $[X,Z]_\star$ can be determined without obstruction theory, although some knowledge of homotopy groups of spheres is required. It helps to know that $\pi_4(S^3) \cong \mathbb{Z}/2$ and it helps to know that the suspension homomorphism $\pi_3(S^2) \to \pi_4(S^3)$ is surjective. We will learn some of that in the next section.
Use the standard CW-structure on $X = \mathbb{C}P^2$ where $X^0 = X^1$ is a point, $X^2 = X^3 = \mathbb{C}P^1$ and $X^4 = \mathbb{C}P^2$. Use the standard structure on $Z = S^3$, too, where $Z^0 = Z^1 = Z^2$ is a point and $Z^3 = S^3$. Any based map from X to Z is homotopic to a cellular one, and a cellular map must have the form of a composition $X \to X/X^2 \to Z$, where $X \to X/X^2$ is the quotient map. Since $X/X^2 \cong S^4$, and since $[S^4,S^3]_\star = \pi_4(S^3)$ has only two elements, this means that $[X,Z]_\star$ has at most two elements. We get the two candidates by using representatives $S^4 \to S^3$ for the two elements of $\pi_4(S^3)$, and pre-composing with the quotient map $X \to X/X^2 = S^4$. But it turns out that the result is in both cases nullhomotopic as a based map $X \to Z$. (Idea for that: there is an exact sequence of pointed sets

$$[S^1 \wedge X^2, Z]_\star \longrightarrow [X/X^2, Z]_\star \longrightarrow [X,Z]_\star$$

where the map on the right is the one we have seen, induced by the quotient map $X \to X/X^2$. The other map is determined by the inclusion of X/X^2 in $\mathrm{cone}(X \to X/X^2)$ and uses the observation $\mathrm{cone}(X \to X/X^2) \simeq S^1 \wedge X^2$, a special case of a general fact. With the identifications $X/X^2 \cong S^4$ and $S^1 \wedge X^2 \cong S^3$ that map $X/X^2 \to S^1 \wedge X^2$ becomes the suspension of the attaching map $S^3 \to S^2 = X^2$ for the unique 4-cell of X; again a special case of a general fact. That attaching map is the Hopf map and its suspension is therefore *the* nontrivial element in $\pi_4(S^3)$. It follows that the left-hand arrow in the above "exact sequence" is onto and the other one is therefore trivial by exactness. The exact sequence is of course also a special case of a more general construction, called the *Barratt-Puppe* sequence.) Summarizing, $[X,Z]_\star$ has only one element, represented by the constant based map $X \to Z$.
But now let us try to compute $[X,Z]_\star$ using obstruction theory. We can begin with $[X,\beta_3 Z]_\star$. The space $\beta_3 Z$ is an Eilenberg-MacLane space $K(\mathbb{Z},3)$, so $[X,\beta_3 Z]_\star$ must be in bijection with $H^3(X;\mathbb{Z})$ which is however $= 0$ as a group. So $[X,\beta_3 Z]_\star$ has only one element. So the preimage P of that one element under the standard map

$$[X,\beta_4 Z]_\star \longrightarrow [X,\beta_3 Z]_\star$$

is all of $[X,\beta_4 Z]_\star$, and that means, it is all of $[X,Z]_\star$ since X is 4-dimensional. So that preimage P has only one element by the above calculation of $[X,Z]_\star$. BUT obstruction theory gives us a surjection from $H^4(X;\pi_4(Z))$ to P, and now $H^4(X;\pi_4(Z)) = H^4(X;\mathbb{Z}/2) \cong \mathbb{Z}/2$ has two elements. So that surjective map is not bijective.

CHAPTER 10

Spectral sequences

10.1. Definition and some general facts

Spectral sequences were invented by Jean Leray (mid- to late 1940s), and it is said that Jean-Pierre Serre made them prominent. They are not as bad as you have been told. It is not clear that the notion of *spectral sequence* comes with a genuine definition. But here is an attempt.

Definition 10.1.1. A *spectral sequence* is a sequence $\mathcal{E}^1, \mathcal{E}^2, \mathcal{E}^3, \dots$ of chain complexes (graded over $\mathbb{Z}$) together with specified isomorphisms

$$v_j^r \colon H_j(\mathcal{E}^r) \cong \mathcal{E}_j^{r+1}$$

for all $j \in \mathbb{Z}$ and $r \geq 1$.

(Notation: $\mathcal{E}_j^r$ is the j-th chain group of the chain complex $\mathcal{E}^r$.) In words: the homology of the chain complex $\mathcal{E}^r$ is identified with the graded abelian group obtained from the chain complex $\mathcal{E}^{r+1}$ by forgetting the differential. We will see that this definition does not give a sufficiently detailed picture. In most examples the chain complex $\mathcal{E}^r$ is a direct sum (again indexed by the integers) of chain subcomplexes and the isomorphisms v_j^r set up a complicated relationship between the preferred splitting of $\mathcal{E}^r$ and the preferred splitting of $\mathcal{E}^{r+1}$. But as a starting point, definition 10.1.1 is not all bad.

Spectral sequences usually arise in connection with a filtration of a space by subspaces, or a filtration of a chain complex by chain subcomplexes. Let's focus on chain complexes (of abelian groups) to begin with. A *filtration* of a chain complex C is an ascending sequence of chain subcomplexes

$$\dots C(-2) \subset C(-1) \subset C(0) \subset C(1) \subset C(2) \subset C(3) \subset \dots$$

with the properties

$$\bigcup_s C(s) = C\,, \qquad C(s) = 0 \text{ for some } s$$

(usually $C(s)$ is zero for $s < 0$). The task is, roughly speaking, to express the homology groups of C in terms of the homology groups of the subquotients $C(s)/C(s-1)$. That is what spectral sequences are good for.

Notation 10.1.2. $C(s,t) := C(s)/C(t)$ for $t \leq s$.

More precisely, we are dealing with two families of abelian groups. The first of these consists of the groups

$$\mathcal{E}_{s,t}^1 := H_{s+t}C(s,s-1) \qquad s,t \in \mathbb{Z}$$

and we pretend that we know it. The second family consists of the groups

$$\mathcal{E}_{s,t}^\infty := \frac{\operatorname{im}\bigl(H_{s+t}C(s) \to H_{s+t}C\bigr)}{\operatorname{im}\bigl(H_{s+t}C(s-1) \to H_{s+t}C\bigr)}$$

(where the arrows are induced by inclusion). These are subquotients of the homology groups of C. We pretend that we want to know them. If we did, we would know H_*C up to "extension problems". To repeat—the task is

express all of the groups $\mathcal{E}^\infty_{s,t}$ in terms of all of the groups $\mathcal{E}^1_{s,t}$.

We introduce some notation for some subgroups of $H_{s+t}C(s,s-1) = \mathcal{E}^1_{s,t}$:

$$\begin{aligned} Z^r_{s,t}: &\quad = \operatorname{im}\big(H_{s+t}C(s,s-r) \to H_{s+t}C(s,s-1)\big), \\ B^r_{s,t}: &\quad = \ker\big(H_{s+t}C(s,s-1) \to H_{s+t}C(s+r-1,s-1)\big) \\ &\quad = \operatorname{im}\big(\partial : H_{s+t+1}C(s+r-1,s) \to H_{s+t}C(s,s-1)\big) \end{aligned}$$

for $r > 0$ (allow $r = \infty$ also). Important exercise: show that

$$\cdots \subset B^r_{s,t} \subset B^{r+1}_{s,t} \cdots \subset B^\infty_{s,t} \subset Z^\infty_{s,t} \cdots \subset Z^{r+1}_{s,t} \subset Z^r_{s,t} \subset \dots$$

LEMMA 10.1.3. *There are preferred isomorphisms*

$$u: Z^r_{s,t}/Z^{r+1}_{s,t} \longrightarrow B^{r+1}_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1}.$$

PROOF. The idea is that we represent an element x of $Z^r_{s,t}/Z^{r+1}_{s,t}$ by an element x_1 of $Z^r_{s,t}$ which in turn we represent by an element $\bar{x}_1$ of $H_{s+t}C(s,s-r)$; see the definition of $Z^r_{s,t}$. Then we apply the boundary operator

$$\partial: H_{s+t}C(s,s-r) \longrightarrow H_{s+t-1}C(s-r,s-r-1) \tag{10.1.4}$$

associated with the short exact sequence of chain complexes

$$C(s-r,s-r-1) \to C(s,s-r-1) \to C(s,s-r). \tag{10.1.5}$$

We get $\partial\bar{x}_1 \in \operatorname{im}(\partial)$. Now we note that this $\operatorname{im}(\partial)$ is exactly $B^{r+1}_{s-r,\,t+r-1}$. In more detail, the source in (10.1.4) can also be written in the form

$$H_{(s-r)+(t+r-1)+1}C((s-r)+r,s-r)$$

and the target can be written as $H_{(s-r)+(t+r-1)}C(s-r,s-r-1)$. Therefore $\partial\bar{x}_1$ in $\operatorname{im}(\partial) = B^{r+1}_{s-r,\,t+r-1}$ represents an element

$$u(x) \coloneqq [\partial\bar{x}_1] \in B^{r+1}_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1}.$$

Now let us show that $u(x) \coloneqq [\partial\bar{x}_1]$ is well defined. By linearity of the construction, it is enough to show that if

$$x_1 \in Z^{r+1}_{s,t} \subset Z^r_{s,t}$$

then

$$\partial\bar{x}_1 \in B^r_{s-r,\,t+r-1} \subset B^{r+1}_{s-r,\,t+r-1}$$

as long as we follow the instructions for choosing $\bar{x}_1$. Indeed, $x_1 \in Z^{r+1}_{s,t}$ implies that

$$\bar{x}_1 = y + z \in H_{s+t}C(s,s-r)$$

where y comes from $H_{s+t}C(s,s-r-1)$ and z maps to zero in $H_{s+t}C(s,s-1)$, and therefore comes from $H_{s+t}C(s-1,s-r)$. The homomorphism (10.1.4) takes y to zero, by exactness of the long exact sequence associated with (10.1.5). The homomorphism (10.1.4) takes z to an element of

$$\operatorname{im}\big(\partial: H_{s+t}C(s-1,s-r) \longrightarrow H_{s+t-1}C(s-r,s-r-1)\big) = B^r_{s-r,\,t+r-1}.$$

This convinces us that u is well defined! Injectivity of u: if $\partial\bar{x}_1$ belongs to $B^r_{s-r,\,t+r-1}$, then it is in the image of

$$\partial: H_{s+t}C(s-1,s-r) \longrightarrow H_{s+t-1}C(s-r,s-r-1)$$

and so $\bar{x}_1 = y + z \in H_{s+t}C(s, s-r)$ where z comes from $H_{s+t}C(s-1, s-r)$ and y comes from $H_{s+t}C(s, s-r-1)$. Passing from $\bar{x}_1$ to $x_1 \in Z^r_{s,t}$, we see that the contribution of z is zero and the contribution of y lands in the subgroup $Z^{r+1}_{s,t}$. Therefore when we pass from x_1 to x, the element y also contributes zero. This establishes the injectivity of u. Surjectivity is an exercise. □

DEFINITION 10.1.6. Put $\mathcal{E}^r_{s,t} = Z^r_{s,t}/B^r_{s,t}$. The differential $d = d^r$ on $\mathcal{E}^r$ has the form $\mathcal{E}^r_{s,t} \longrightarrow \mathcal{E}^r_{s-r,t+r-1}$ and is defined as a composition

$$\begin{tikzcd} Z^r_{s,t}/B^r_{s,t} \arrow[r, "\text{proj}"] & Z^r_{s,t}/Z^{r+1}_{s,t} \arrow[d, "u", "\cong"'] & \\ & B^{r+1}_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1} \arrow[r, "\text{incl.}"] & Z^r_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1} \end{tikzcd}$$

Note that the first arrow in this composition of three is *surjective* and the other two are injective, so that the kernel of the composition is the kernel of the first arrow:

$$\ker\Big(d\colon Z^r_{s,t}/B^r_{s,t} \longrightarrow Z^r_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1}\Big) = Z^{r+1}_{s,t}/B^r_{s,t}.$$

Similarly, the last arrow in the composition of three is injective and the other two are surjective, so that the image of the composition is the image of the last arrow:

$$\operatorname{im}\Big(d\colon Z^r_{s,t}/B^r_{s,t} \longrightarrow Z^r_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1}\Big) = B^{r+1}_{s-r,\,t+r-1}/B^r_{s-r,\,t+r-1}.$$

On the basis of these little observations it is easy to verify that $d^r d^r = 0$, that is, the composition of

$$\mathcal{E}^r_{s+r,t-r+1} \xrightarrow{\ d\ } \mathcal{E}^r_{s,t} \xrightarrow{\ d\ } \mathcal{E}^r_{s-r,t+r-1} \tag{10.1.7}$$

is zero (because the image of the first arrow in this composition of two is contained in the kernel of the second arrow). Moreover, we can see immediately that kernel of the second arrow modulo image of the first arrow in (10.1.7) becomes

$$\frac{Z^{r+1}_{s,t}/B^r_{s,t}}{B^{r+1}_{s,t}/B^r_{s,t}} \;\cong\; Z^{r+1}_{s,t}/B^{r+1}_{s,t}. \tag{10.1.8}$$

This makes good on the promise expressed in definition 10.1.1 that the homology of $\mathcal{E}^r$ (with differential $d = d^r$) should be identified with $\mathcal{E}^{r+1}$ (without differential).

Note also that we now have two definitions of $\mathcal{E}^\infty_{s,t}$, one given in the first sentence of 10.1.6 and one given earlier, right after 10.1.2. They are however isomorphic (exercise). This will be very important in the coming paragraph.

Finally, it is clear from the definitions that $\mathcal{E}^{r+1}_{s,t}$ is a subquotient (quotient of subgroup) of $\mathcal{E}^r_{s,t}$, but what is the exact relationship between $\mathcal{E}^r_{s,t}$ and $\mathcal{E}^\infty_{s,t}$? From the definitions, $Z^r_{s,t}$ becomes independent of r for large r, in which case $\mathcal{E}^{r+1}_{s,t}$ is simply a quotient of $\mathcal{E}^r_{s,t}$. Furthermore $B^\infty_{r,t}$ is the union of the increasing sequence of abelian groups $B^r_{s,t}$. It follows that we can think of $\mathcal{E}^\infty_{s,t}$ as the direct limit (in the sense of category theory) of a sequence of surjective homomorphisms of abelian groups

$$\mathcal{E}^r_{s,t} \to \mathcal{E}^{r+1}_{s,t} \to \mathcal{E}^{r+2}_{s,t} \to \cdots$$

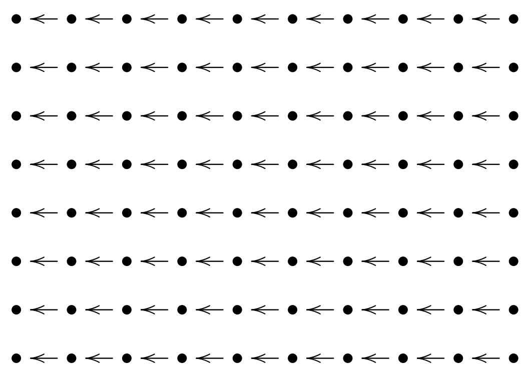
where r should be taken big enough so that $C(s-r)=0$. This is enough justification for saying that the spectral sequence *converges* to the homology C. A standard (but informal) way of writing this would be

$$\mathcal{E}^1_{s,t} = H_{s+t}C(s,s-1) \;\Rightarrow\; H_{s+t}C\,.$$

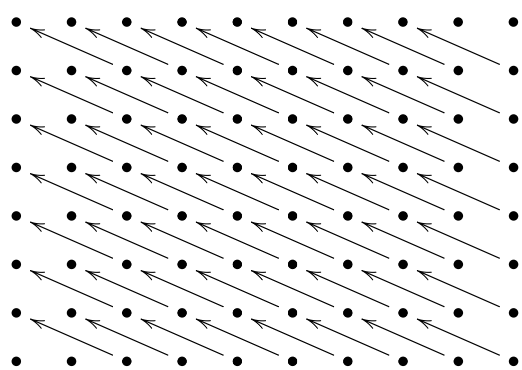
In our main example, $Z^r_{s,t}$ is independent of r if $r>s$ and $B^r_{s,t}$ is also independent of r as soon as $r>t+1$. In this situation we can say briefly that $\mathcal{E}^r_{s,t} = Z^r_{s,t}/B^r_{s,t}$ is *the same* as $\mathcal{E}^\infty_{s,t}$ for $r>\max\{s,t+1\}$. That is a much easier kind of convergence.

Time for some pictures:

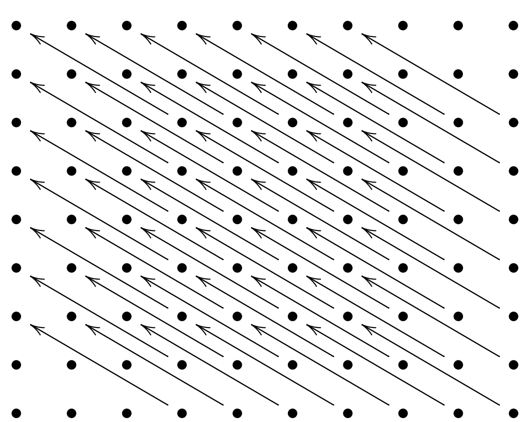
$\mathcal{E}^1_{**}$:



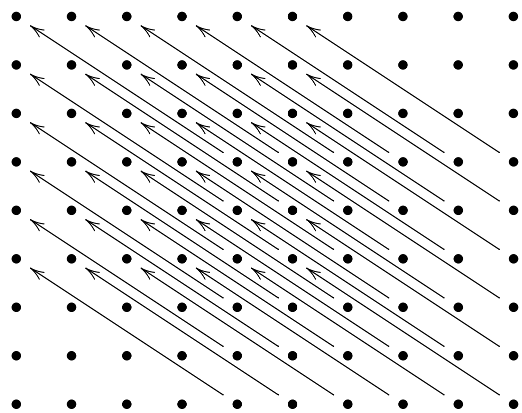
$\mathcal{E}^2_{**}$:



$\mathcal{E}^3_{**}$:



$\mathcal{E}^4_{**}$:



The following problem is important because it shows that what we have seen so far in this section is an enhanced version of the long exact homology sequence of a pair of chain complexes.

EXERCISE 10.1.9. Suppose that the filtration of C has only two *stages*; i.e., suppose $C(-1) = 0$ and $C(s) = C(1)$ for all $s \geq 1$. Then all we have is a chain complex $C(1)$ and a chain subcomplex $C(0) \subset C(1)$. What is $\mathcal{E}^1_{**}$, what is $\mathcal{E}^\infty_{**}$, what is the differential on $\mathcal{E}^1_{**}$, what is $\mathcal{E}^2_{**}$, what is the differential on $\mathcal{E}^2_{**}$, etc. ?

Unfortunately I just found out that the above discussion (spectral sequence of a filtered chain complex) is not abstract enough to be really useful for us, and so I have to generalize it slightly. A more general way to obtain a spectral sequence is to start with an *exact couple*. This is a very clever definition due to W. Massey. It's going to get a slightly sketchy treatment here, but it deserves better!

DEFINITION 10.1.10. An *exact couple* is a diagram of abelian groups

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

(not intended to be particularly commutative) which is *exact* at each vertex.

EXAMPLE 10.1.11. The standard example that one should have in mind is one that we know very well, as follows. For a filtered chain complex $C = C(\infty)$ with chain subcomplexes $C(s)$, as above, we set

$$A_{s,t} := H_{s+t}C(s), \quad E_{s,t} := \mathcal{E}^1_{s,t} = H_{s+t}C(s, s-1).$$

Let $A = \bigoplus_{s,t} A_{s,t}$ and $E := \bigoplus_{s,t} E_{s,t}$. Let $i: A \to A$ be given on the summand $A_{s,t}$ by the inclusion-induced map

$$A_{s,t} = H_{s+t}C(s) \to H_{s+t}C(s+1) = A_{s+1,t-1} \hookrightarrow A \ ,$$

let $j: A \to E$ be the map induced by the projections $C(s) \to C(s, s-1)$ and let k be the map induced by the boundary operators

$$\partial: H_*C(s, s-1) \to H_{*-1}C(s-1).$$

Returning to definition 10.1.10, we make the following observations. Firstly, there is a homomorphism $E \to E$ given by jk. This is a *differential* in the sense of $(jk)(jk) = 0$. Therefore E becomes a *differental abelian group* (like a chain complex without the grading). Next, an exact couple has a *derived* exact couple

$$\begin{tikzcd} A^\delta \arrow[rr, "i^\delta"] & & A^\delta \arrow[dl, "j^\delta"] \\ & E^\delta \arrow[ul, "k^\delta"] & \end{tikzcd}$$

where E^δ is the homology of E with differential jk, that is, $\ker(jk)/\mathrm{im}(jk)$, and $A^\delta = \mathrm{im}(i\colon A \to A)$. The new arrow k^δ is fairly obviously determined by the old k, but beware, the new j^δ is less obviously defined so that $j^\delta(i(a)) = j(a)$ (check that this is well defined). Showing that this is again a derived exact couple is an exercise. Therefore we can repeat this process as many times as we like. Writing $\mathcal{E}^1$ instead of E, then $\mathcal{E}^2$ instead of E^δ, then $\mathcal{E}^3$ instead of $(E^\delta)^\delta$ etc., we get a sequence $\mathcal{E}^1$, $\mathcal{E}^2$, $\mathcal{E}^3$ of differential abelian groups such that $\mathcal{E}^{r+1}$ is (exactly) the homology of $\mathcal{E}^r$. This is therefore a spectral sequence as in definition 10.1.1, except for the absence of a grading. (But we can add gradings as needed.)

If you take the exact couple of example 10.1.11, then you have a bi-grading on A and E and the spectral sequence that you get from the exact couple turns out to be identical with the spectral sequence that we constructed previously, more by hand. There are many little exercises concealed in this claim! In particular, we are led to the following definitions by comparing the exact couple construction of a spectral sequence with the earlier pedestrian construction for a filtered chain complex. Given an exact couple as in definition 10.1.10, let Z^r be the subgroup of E which is the pre-image under k of the image of the $(r-1)$-fold iteration of i (so that, for example, $Z^1 = E$). And let B^r be the subgroup of E which is the image under j of the kernel of the $(r-1)$-fold iteration of i (so that, for example, $B^1 = 0$). These definitions should be in agreement with our earlier definitions of $Z^r_{s,t}$ and $B^r_{s,t}$ ... and in particular it should be true, in the general setting of exact couples, that

$$\mathcal{E}^r \cong Z^r/B^r .$$

10.2. The spectral sequence associated with a fibration

We now come to the first serious example (and for this course, also the last) of a spectral sequence: the *Serre spectral sequence of a fibration.* There are actually two variants, one for homology and one for cohomology, but we begin with the homology variant. So let $p\colon X \to B$ be a fibration, and assume that B is a simply connected based CW-space. We make no special assumptions on the fibers. If you know something about singular homology: let C be the *singular* chain complex of the total space X, and let $C(s)$ be the singular chain complex of $p^{-1}(B^s)$, where B^s is the s-skeleton of the CW-space B. So you have a filtered chain complex, and you can put this into the machine which makes a spectral sequence out of a filtered chain complex. If you have another definition of homology in your head, not based on chain complexes, then you can proceed as follows: set up an exact couple (with bigrading) where $A_{s,t}$ is $H_{s+t}(p^{-1}(B^s))$ and $E_{s,t}$ is

$$\tilde{H}_{s+t}(p^{-1}(B^s) /\!\!/ p^{-1}(B^{s-1})).$$

(Remember that $/\!\!/$ is our notation for the mapping cone of an inclusion.)

Let's find out what the $\mathcal{E}^1_{**}$ term of this spectral sequence is. This amounts to calculating the homology of $p^{-1}(B^s)/\!\!/p^{-1}(B^{s-1})$ for all s. Let $F = p^{-1}(\star)$. Fix s and choose characteristic maps

$$\varphi_\lambda\colon (D^s, \partial D^s) \longrightarrow (B^s, B^{s-1})$$

for the s-cells of B. Let $z_\lambda = \varphi_\lambda(0)$ and $F_z = p^{-1}(z)$. Any choice of path $\gamma\colon[0,1] \to B$ from z_λ to the base point determines an invertible element of $[F_z, F]$. Indeed the homotopy lifting property guarantees that there is a homotopy $(g_t\colon F_z \to X)_{t\in[0,1]}$ such that g_0 is the inclusion and pg_t is constant with value $\gamma(t)$, for all $t \in [0,1]$. Therefore g_1 is essentially a map from F_z to F. (*Exercise*: show that this is well defined, i.e., depends only on γ but not on the lift (g_t).) Then, since B is simply connected, the element of $[F_z, F]$ so constructed does not depend on the path γ either. In the same way, we can use the HLP for the fibration $\varphi^*_\lambda X \to D^s$ (with contractible base space D^s) to show that the pair

$$(\varphi^*_\lambda X, \varphi^*_\lambda X|_{\partial D^s})$$

is homotopy equivalent to $(D^s \times F, \partial D^s \times F)$. By excision for mapping cones, we have

$$p^{-1}(B^s)/\!\!/p^{-1}(B^{s-1}) \;\simeq \bigvee_\lambda \varphi^*_\lambda X/\!\!/\varphi^*_\lambda X|_{\partial D^s} \;\simeq \bigvee_\lambda (D^s \times F)/\!\!/(\partial D^s \times F) \;\simeq \bigvee_\lambda \frac{S^s \times F}{\star \times F}.$$

Therefore the term $\mathcal{E}^1_{s,t}$ is identified with

$$\tilde{H}_{s+t}\left(\bigvee_\lambda \frac{S^s \times F}{\star \times F}\right) \cong \bigoplus_\lambda \tilde{H}_{s+t}\left(\frac{S^s \times F}{\star \times F}\right) \cong \bigoplus_\lambda H_t(F).$$

This proves the following:

LEMMA 10.2.1. *In the Serre spectral sequence for the fibration $p\colon X \to B$, the term $\mathcal{E}^1_{s,t}$ is identified with $C(B)_s \otimes H_t(F)$, where $C(B)$ is the* cellular *chain complex of B.*

This leads to a guess for the differential d^1.

LEMMA 10.2.2. *In the Serre spectral sequence for the fibration $p\colon X \to B$, the differential*

$$d^1\colon E^1_{s,t} \longrightarrow E^1_{s-1,t}$$

agrees with the standard differential $C(B)_s \otimes H_t(F) \to C(B)_{s-1} \otimes H_t(F)$, i.e., the differential in the cellular chain complex of B tensored with $H_t(F)$.

SKETCH PROOF. Choose a q-cell in B and a characteristic map

$$\varphi\colon (D^q, \partial D^q) \to (B^q, B^{q-1})$$

for that cell. Then we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \varnothing \arrow[r] \arrow[d] & \varphi^* X|_{\partial D^q} \arrow[r] \arrow[d] & \varphi^* X \arrow[r, "="] \arrow[d] & \varphi^* E \arrow[r, "="] \arrow[d] & {} \\
\cdots \arrow[r] & p^{-1}(B^{q-2}) \arrow[r] & p^{-1}(B^{q-1}) \arrow[r] & p^{-1}(B^q) \arrow[r] & p^{-1}(B^{q+1}) \arrow[r] & \cdots
\end{tikzcd}$$

In the lower row we see the filtration of X which we used to make the Serre spectral sequence for $p\colon X \to B$. In the upper row we see a filtration of $\varphi^* X$ which can also use to make a spectral sequence; let's write

$$\left(\mathscr{D}^r_{s,t}\right)_{r,s,t}$$

for that. Here we clearly have $\mathscr{D}^1_{s,t} = 0$ except when $s = q$ or $s = q - 1$; and we have

$$\mathscr{D}^1_{q,t} \cong H_t(F), \qquad \mathscr{D}^1_{q-1,t} \cong H_{q-1+t}(S^{q-1} \times F) \cong H_t(F) \oplus H_{q-1+t}(F)\,.$$

It is not hard to see that the d^1 differential

$$\mathscr{D}^1_{q,t} \longrightarrow \mathscr{D}^1_{q-1,t}$$

is given by the inclusion of $H_t(F)$ in the sum $H_t(F) \bigoplus H_{q-1+t}(F)$. By naturality, the diagram above induces a morphism of spectral sequences

$$\left(\mathscr{D}^r_{s,t}\right)_{r,s,t} \longrightarrow \left(\mathcal{E}^r_{s,t}\right)_{r,s,t}\,.$$

This takes $\mathscr{D}^1_{q,t} \cong H_t(F)$ isomorphically to the summand of

$$\mathcal{E}^1_{q,t} \cong C(B)_q \otimes H_t(F)$$

which corresponds to the cell λ. Therefore we can read off what the differential d^1 on $\mathcal{E}^1_{q,t}$ does on that summand. □

Corollary 10.2.3. *In the Serre spectral sequence for a fibration $p\colon X \to B$, the $\mathcal{E}^2$-term is given by*

$$\mathcal{E}^2_{s,t} \cong H_s(B; H_t(F)).$$

Example 10.2.4. (See also Fuks-Fomenko-Gutenmacher, *Homotopic Topology.*) For $n > 0$, let's try to calculate the homology of $SU(n)$ (the topological group of unitary complex $n \times n$-matrices with determinant 1). For this we observe that the evaluation map

$$p_n : SU(n) \longrightarrow S^{2n-1} \quad ; \quad p(A) = Ae_1 \in S^{2n-1} \subset \mathbb{C}^n$$

(where e_1 is the well-known standard basis vector) is a fiber bundle with fibers homeomorphic to $SU(n-1)$. (Proving this is an exercise. But it is clear that the fibers are as claimed: for $v \in S^{2n-1}$, the fiber $p^{-1}(v)$ consists of all unitary $n \times n$ matrices of determinant 1 sending e_1 to v.) We now try to use our spectral sequence and induction. The fibers of the fibration p_2 are homeomorphic to $SU(1)$, which is a point, so

$$SU(2) \cong \mathbb{S}^3$$

which in particular calculates the homology. Next we have

$$p_3 : SU(3) \longrightarrow S^5$$

with fibers homeomorphic to $SU(2) \cong S^3$. This means that the $\mathcal{E}^2_{**}$ term of the Leray-Serre spectral sequence for this fibration looks like this:

$$\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}$$

with the nonzero terms in positions $(0,0)$, $(0,3)$, $(5,0)$, $(5,3)$. It follows immediately that the differentials on $\mathcal{E}^2_{**}$ as well as those on $\mathcal{E}^3_{**}$, $\mathcal{E}^4_{**}$ etc. are zero, so that

$$\mathcal{E}^2_{**} \cong \mathcal{E}^\infty_{**}$$

(the spectral sequence *collapses*). We conclude that

$$H_*(SU(3)) \cong H_*(S^3 \times S^5)$$

(but it is not claimed that $SU(3) \simeq S^3 \times S^5$). Next we have

$$p_4 : SU(4) \longrightarrow S^7$$

with fibers homeomorphic to $SU(3)$. This means that the $\mathcal{E}^2_{**}$ term of the Leray-Serre spectral sequence for this fibration looks like this:

$$\begin{array}{cccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0
\end{array}$$

with the nonzero terms in positions $(0,0)$, $(0,3)$, $(0,5)$, $(0,8)$, $(7,0)$, $(7,3)$, $(7,5)$, $(7,8)$. Again you can easily convince yourself that none of the differentials on $\mathcal{E}^2_{**}$, $\mathcal{E}^3_{**}$, $\mathcal{E}^4_{**}$ etc. has a chance to be nonzero. Therefore

$$H_*(SU(4)) \cong H_*(SU(3) \times S^7) \cong H_*(S^3 \times S^5 \times S^7)\,.$$

One might hope that this will go on forever. Let's try one more time: The $\mathcal{E}^2_{**}$ term of the spectral sequence for p_5 looks like

$$\begin{array}{cccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&\mathbb{Z}&0&0&0&0&0&0&0&0&\mathbb{Z}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}$$

and we have a problem. Namely, there are two differentials in the spectral sequence which could be nonzero: they would be in the $\mathcal{E}^9_{**}$ term, from position $(9,0)$ to position $(0,8)$

and from position $(9,7)$ to position $(0,15)$. So our argument breaks down. All we know is that

$$H_*(U(n) \cong H_*(S^1 \times S^3 \times \cdots \times S^{2n-1}) \qquad \text{for } n \le 4\,.$$

For the cases $n > 4$, we need better equipment.

10.3. Some remarks on filtered spaces

As a preparation for the cohomology version of the Serre spectral sequence, we need to develop the elementary theory of filtered spaces. (I leave out some proofs for lack of time.) Generally speaking, if we have a definition of homology/cohomology (also generalized forms) in mind which does not rely very much on chain complexes, then it is no longer appropriate to pretend that spectral sequences arise *mainly* in connection with filtered chain complexes. Instead we can take the view that spectral sequences arise *mainly* in connection with filtered spaces and homology or cohomology theories.

Definition 10.3.1. A *filtered space* is a space X with a sequence of subspaces $X(s)$, where $s \in \mathbb{Z}$, such that $X(s) \subset X(s+1)$ for all s.
Let us say that a filtered space X (with distinguished subspaces $X(s)$ for $s \in \mathbb{Z}$) is *well-filtered* if the following conditions are satisfied:

- there is some $s \in \mathbb{Z}$ such that $X(s) = \varnothing$;
- the inclusion $X(s) \to X(s+1)$ is a cofibration, for all $s \in \mathbb{Z}$;
- $X = \bigcup_s X(s)$ and X has the direct limit topology with respect to the subspaces $X(s)$, so that a subset V of X is open in X if and only if $V \cap X(s)$ is open in $X(s)$ for every s.

Example 10.3.2. If X and Y are filtered spaces, with distinguished subspaces $X(s)$ and $Y(s)$ for $s \in \mathbb{Z}$, then $X \times Y$ has a preferred structure of a filtered space in the following way:

$$(X \times Y)(s) := \bigcup_{p+q \le s} X(p) \times Y(q)\,.$$

Suppose now that X and Y are well-filtered. Does it follow that $X \times Y$ with this preferred filtration structure is also well-filtered? That would be nice but I don't know.

Definition 10.3.3. Let X and Y be filtered spaces. A *morphism* (or *filtered map*) from X to Y is a continuous map f from X to Y such that

$$f(X(s)) \subset Y(s)$$

for all $s \in \mathbb{Z}$. Two filtered maps $f, g{:}\, X \to Y$ are *filtered homotopic* if there exists a homotopy $(h_t{:}\, X \to Y)_{t\in[0,1]}$ such that $h_0 = f$, $h_1 = g$ and each h_t is a filtered map from X to Y. A filtered map $f{:}\, X \to Y$ is a *filtered homotopy equivalence* if there exists a filtered map $g{:}\, Y \to X$ such that gf and fg are filtered homotopic to id_X and id_Y , respectively.

Example 10.3.4. Let $p{:}\, X \to B$ be a fibration, where B is a CW-space. Let B_T be the *telescope* associated with B and the filtration by skeletons. This is the space

$$[0,1] \times B^0 \cup [1,2] \times B^1 \cup [2,3] \times B^2 ...$$

which is also a CW-space in an obvious way. There is a projection $B_T \to B$. It is an exercise to show that this is a homotopy equivalence (for example by showing first that it is a weak homotopy equivalence). But the situation is a little better. Let q_B from B_T to $[0,\infty[$ be the obvious projection. Then B_T is filtered by subspaces $B_T(s)$, the preimage(s) of $[0,s]$ under q_B , where $s = 0,1,2,\dots$. The map $B_T \to B$ is then a *filtered* homotopy equivalence of *filtered* spaces. Similarly, let X_T be the telescope associated with X and the filtration by subspaces $X(s)$. This is the space

$$[0,1] \times X(0) \cup [1,2] \times X(1) \cup [2,3] \times X(2)...$$

where a subset U is considered open if its intersection with $[k,k+1] \times X(k)$ is open, for every k. Let q_X from X_T to $[0,\infty[$ be the obvious projection. Then X_T is filtered by subspaces $X_T(s)$, the preimage(s) of $[0,s]$ under q_X^{-1}. Moreover X_T is clearly a *well-filtered* space. I believe that the commutative square

$$\begin{tikzcd} X_T \arrow[r] \arrow[d] & X \arrow[d] \\ B_T \arrow[r] & B \end{tikzcd}$$

is a pullback square in the category of topological spaces. Therefore, using the homotopy lifting property, and the fact that the lower horizontal arrow is a homotopy equivalence of filtered spaces, we can easily deduce that the upper horizontal arrow is also a homotopy equivalence of *filtered* spaces. Conclusion: although there is not much evidence that X, with the filtration by subspaces $X(s) = p^{-1}(B^s)$, is well-filtered, we can say that X is filtered homotopy equivalent to X_T , a well-filtered space.

10.4. Cohomology version of the Serre spectral sequence

Let us now look at the cohomology version of the Serre spectral sequence for a fibration $p: X \to B$. As in the homology case, we take the view that it is about a space with a filtration to begin with, the space X with subspaces $X(s) = p^{-1}(B^s)$. Roughly as before, we set ourselves the task to compute $H^*(X)$ or something close to it, and we pretend that we know

$$H^*(X(s)/\!\!/X(s-1))$$

for all s. More precisely, we are interested in the filtration subquotients of the filtration of $H^q(X)$ given by subgroups $\mathrm{im}(H^q(X/\!\!/X(s-1)) \to H^q(X))$. Still more precisely:

$$\mathcal{E}_\infty^{s,t} = \frac{\mathrm{im}\big(H^{s+t}(X/\!\!/X(s-1)) \to H^{s+t}(X)\big)}{\mathrm{im}\big(H^{s+t}(X/\!\!/X(s)) \to H^{s+t}(X)\big)} = \frac{\ker\big(H^{s+t}(X) \to H^{s+t}(X(s-1))\big)}{\ker\big(H^{s+t}(X) \to H^{s+t}(X(s))\big)}$$

We can set up an exact couple

(10.4.1)
$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

as follows. Let $E^{s,t} = H^{s+t}(X(s)/\!\!/X(s-1))$ and put $E = \bigoplus_{s,t} E^{s,t}$. Let $A^{s,t} = H^{s+t}(X/\!\!/X(s-1))$ and put $A = \bigoplus A^{s,t}$. Note that this is a little surprising; you might have expected that we take $A^{s,t}$ to be $H^{s+t}(X(s))$, but no. The map i is the map

$$A^{s+1,t-1} = H^{s+t}(X/\!\!/X(s)) \longrightarrow H^{s+t}(X/\!\!/X(s-1)) = A^{s,t}$$

induced by the inclusion of mapping cones $X/\!\!/X(s-1) \to X/\!\!/X(s)$, and the map j is given by restrictions

$$A^{s,t} = H^{s+t}(X/\!\!/X(s-1)) \longrightarrow H^{s+t}(X(s)/\!\!/X(s-1)) = E^{s,t}.$$

The map k is a boundary map given by

$$\partial\colon E^{s,t} = H^{s+t}(X(s)/\!\!/X(s-1)) \longrightarrow H^{s+t+1}(X/\!\!/X(s)) = A^{s+1,t}$$

induced by $X/\!\!/X(s) \to \star/\!\!/X(s) \cong S^1 \wedge X(s) \hookrightarrow S^1 \wedge (X(s)/\!\!/X(s-1))$. Beware that the grading behavior is somewhat different from what we have seen before.

REMARK 10.4.2. If B is simply connected, then we have a preferred homotopy class of homotopy equivalences from $X(s)/\!\!/X(s-1)$ to $\bigvee_\lambda (S^s \times F)/(\star \times F)$, as noted before. It is also very useful to note that

$$H^{s+t}(X/\!\!/X(s-1)) = 0 \text{ if } t < 0\,.$$

To show this replace the filtered space X by X_T as in example 10.3.4. Given an element in $H^{s+t}(X_T/\!\!/X_T(s-1)) \cong H^{s+t}(X_T/X_T(s-1))$, represent by a mapping cycle (for example) and try to construct a nullhomotopy for it. Construct this on $X_T(s+k)/X_T(s-1)$, by induction on k. The obstruction in each step is an element of

$$\tilde{H}^{s+t}(X_T(s+k)/X_T(s+k-1)) \cong \tilde{H}^{s+t}(\bigvee_\lambda \frac{S^s \times F}{\star \times F}) \cong \prod_\lambda H^t(F) = 0,$$

so that there is no obstruction. (Most important point here: when the induction is completed, the partial nullhomotopies defined on $X_T(s+k)$ for all k define a nullhomotopy on X_T.)

Now we are in good shape for a discussion of convergence of the spectral sequence. Let $Z_r = k^{-1}(\mathrm{im}(i^{r-1})) \subset E$ as before (except for the positioning of the r in Z_r) and $B_r = j(\ker(i^{r-1})) \subset E$ as before ... and write $\mathcal{E}_r = Z_r/B_r$. Superscripts s,t can be added as needed. Since $X(s) = \varnothing$ for $s < 0$, it follows that $B_r^{s,t}$ is independent of r as soon as $r > s$, so that $\mathcal{E}_{r+1}^{s,t}$ is a subgroup of $\mathcal{E}_r^{s,t}$ for $r > s$. Since $H^{s+t}(X(s)/\!\!/X(s-1)) = 0$ for $t < 0$, it follows that $Z_r^{s,t} = E_{s,t} \cap \ker(k)$ for $r > t+1$, which is also independent of r. Therefore we can say that $\mathcal{E}_\infty^{s,t} = \mathcal{E}_r^{s,t}$ for $r > \max\{r, t+1\}$.

Now there is an additional problem: we want products. Massey wrote a paper on this (Annals of Mathematics, 1954), explaining what kind of additional structure we need on an exact couple to get a spectral sequence with products. His paper is about *internal* products, but I understand it better with *external* products. So I have adapted his arguments just a little. Let

$$\begin{tikzcd} A(\rho) \arrow[rr, "i"] & & A(\rho) \arrow[dl, "j"] \\ & E(\rho) \arrow[ul, "k"] & \end{tikzcd}$$

be an exact couple, where $\rho = 1,2,3$. The goal is to say what we mean by an external multiplication from exact couple number 1 times exact couple number 2 to exact couple number 3. To start with, the multiplication only relates $E(\rho)$ for $\rho = 1,2,3$. This is enough to give us external products relating the three associated spectral sequences. We assume a bi-grading in each of the three exact couples as in our example (10.4.1). We assume bilinear (bi-additive) maps

$$E(1)^{p,q} \times E(2)^{s,t} \longrightarrow E(3)^{p+s,q+t} \tag{10.4.3}$$

for which we write $(x,y) \mapsto x \cdot y$ where possible. Massey asks: what condition should we impose on these products to ensure that these bilinear maps induce similar maps on the derived exact couples? Here is his condition.

DEFINITION 10.4.4. The product (10.4.3) satisfies *condition* μ_n if, for $x \in E(1)^{p,q}$ and $y \in E(2)^{s,t}$ and $a \in A(1)^{p+n+1,q-n}$ and $b \in A(2)^{s+n+1,t-n}$ such that $k(x) = i^n(a)$ and $k(y) = i^n(b)$, there is $c \in A(3)^{p+s+n+1,q+t-n}$ such that

$$k(x \cdot y) = i^n(c) \quad \text{and} \quad j(c) = j(a) \cdot y + (-1)^{p+q} x \cdot j(b).$$

(Note: i^n is the n-fold iteration of the map i in the exact couples.) The product (10.4.3) is said to satisfy *condition* μ if it satisfies μ_n for all $n \geq 0$.

The case $n = 0$ is special. For $n = 0$ the letters a, b, c are superfluous: $a = k(x)$, $b = k(y)$ and $c = k(x \cdot y)$. So condition μ_0 just means

$$(jk)(x \cdot y) = (jk)(x) \cdot y + (-1)^{p+q} x \cdot (jk)(y).$$

In other words, condition μ_0 means that the *differential* jk in $E(3)$ behaves like a derivation for the product.

If the product satisfies condition μ_0 , then we can pass to homology, $\ker(jk)/\mathrm{im}(jk)$, to get a similar product on the derived exact couples:

$$\mathcal{E}(1)_2^{p,q} \times \mathcal{E}(2)_2^{s,t} \longrightarrow \mathcal{E}(3)_2^{p+s,q+t}. \tag{10.4.5}$$

(This is in the curly notation so that $\mathcal{E}(\rho)_r$ is the $(r-1)$-fold derived exact couple of $E(\rho) = \mathcal{E}(\rho)_1$.) Under these circumstances, if I understand him correctly, Massey claims that the product (10.4.3) satisfies μ_n if and only if the product (10.4.5) satisfies μ_{n-1}. And of course he claims that it is an exercise. Let's believe that. Therefore, if (10.4.3) satisfies μ, then (10.4.5) satisfies μ.

EXAMPLE 10.4.6. Let $X \to B(1)$ and $Y \to B(2)$ be fibrations. Then we have a fibration $Z \to B(3)$, where $Z = X \times Y$ and $B(3) = B(1) \times B(2)$. We assume that $B(1)$ and $B(2)$ are CW-spaces. For simplicity, assume that the number of cells in $B(1)$ and $B(2)$ is finite or countable. Then $B(3)$ with the product topology is also a CW-space. (This was mentioned, but not proved in full generality or detail, in the chapters on CW-spaces.) For each of the three fibrations, we obtain a cohomology spectral sequence as in (10.4.1), with the interpretation where for example $X(s) \subset X$ is the preimage of the skeleton $B(1)^s \subset B(1)$. Therefore our external product needs have the form

$$\begin{array}{c} \tilde{H}^{p+q}(X(p)/\!\!/X(p-1)) \times \tilde{H}^{s+t}(Y(s)/\!\!/Y(s-1)) \\ \big\downarrow \\ \tilde{H}^{p+s+q+t}(Z(p+s)/\!\!/Z(p+s-1)). \end{array}$$

By example 10.3.4 there are homotopy equivalences

$$X(p)/\!\!/X(p-1) \simeq X(p)/X(p-1),$$

and similarly for Y and Z. Then the external product that we need to invent can be given the alternative form

$$\begin{array}{c} \tilde{H}^{p+q}\Big(\dfrac{X(p)}{X(p-1)}\Big) \times \tilde{H}^{s+t}\Big(\dfrac{Y(s)}{Y(s-1)}\Big) \\ \big\downarrow \\ \tilde{H}^{p+s+q+t}\Big(\dfrac{Z(p+s)}{Z(p+s-1)}\Big). \end{array}$$

Now it emerges what it ought to be: the composition

$$\begin{array}{c} \tilde{H}^{p+q}\Big(\dfrac{X(p)}{X(p-1)}\Big) \times \tilde{H}^{s+t}\Big(\dfrac{Y(s)}{Y(s-1)}\Big) \\ \Big\downarrow \\ \tilde{H}^{p+s+q+t}\Big(\dfrac{X(p)}{X(p-1)} \wedge \dfrac{Y(s)}{Y(s-1)}\Big) \\ \Big\downarrow \\ \tilde{H}^{p+s+q+t}\Big(\dfrac{Z(p+s)}{Z(p+s-1)}\Big) \end{array}$$

where the first arrow is a standard external product in cohomology[1] and the other is induced by an obvious quotient map from $Z(p+s)/Z(p+s-1)$ to $X(p)/X(p-1) \wedge Y(s)/Y(s-1)$. — Now we should verify that these external products satisfy Massey's conditions μ_n for all $n \geq 0$. Here is a sketch of an argument. Suppose that we have $x \in E(1)^{p,q}$ and $y \in E(2)^{s,t}$ and $a \in A(1)^{p+n+1,q-n}$ and $b \in A(2)^{s+n+1,t-n}$ such that $k(x) = i^n(a)$ and $k(y) = i^n(b)$. What does it mean? It means

$$x \in \tilde{H}^{p+q}\Big(\frac{X(p)}{X(p-1)}\Big), \quad y \in \tilde{H}^{s+t}\Big(\frac{Y(s)}{Y(s-1)}\Big)$$

and

$$a \in \tilde{H}^{p+q+1}\Big(\frac{X}{X(p+n)}\Big), \quad b \in \tilde{H}^{s+t+1}\Big(\frac{Y}{Y(s+n)}\Big)$$

such that

$$a \mapsto \partial x \in \tilde{H}^{p+q+1}\Big(\frac{X}{X(p)}\Big), \quad b \mapsto \partial y \in H^{s+t+1}\Big(\frac{Y}{Y(s)}\Big).$$

Then there exist

$$\bar{x} \in \tilde{H}^{p+q}\Big(\frac{X(p+n)}{X(p-1)}\Big), \quad \bar{y} \in \tilde{H}^{s+t}\Big(\frac{Y(s+n)}{Y(s-1)}\Big)$$

such that $\bar{x}$ extends x and $\partial(\bar{x}) = a$, and $\bar{y}$ extends y and $\partial(\bar{y}) = b$. (Proof: ... I would say, Mayer-Vietoris.) Then we can form

$$\bar{x} \cdot \bar{y} \in \tilde{H}^{p+q+s+t}\Big(\frac{X(p+n)}{X(p-1)} \wedge \frac{Y(s+n)}{Y(s-1)}\Big)$$

and we can move from there to

$$\tilde{H}^{p+q+s+t}\Big(\frac{(X \times Y)(p+s+n)}{(X \times Y)(p+s+n-1)}\Big)$$

[1]Warning: here I am using the mapping cycle interpretation of cohomology. Readers who prefer the singular homology interpretation should probably not work with the filtered space X but instead with the singular chain complex C of X, filtered by chain subcomplexes $C(s)$ corresponding to $X(s)$, and with the chain complex $\hom(C,\mathbb{Z})$ etc. etc.

using a map

$$\begin{tikzcd} \dfrac{(X \times Y)(p+s+n)}{(X \times Y)(p+s+n-1)} \arrow[d] \\ \dfrac{X(p+n)}{X(p-1)} \wedge \dfrac{Y(s+n)}{Y(s-1)} \end{tikzcd}$$

which I hope is obvious! Therefore I take the liberty to write

$$\bar{x} \cdot \bar{y} \in \tilde{H}^{p+q+s+t}\Big(\frac{(X \times Y)(p+s+n)}{(X \times Y)(p+s+n-1)}\Big).$$

With that in mind we can write

$$c := \partial(\bar{x} \cdot \bar{y}) \in H^{p+q+s+t+1}\Big(\frac{(X \times Y)}{(X \times Y)(p+s+n)}\Big).$$