SEMINAR
"CONFIGURATION SPACE INTEGRALS AND DIFFEOMORPHISMS"
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Background. The expression *configuration space integral* describes a relatively new and peculiar tool in the classification of smooth compact manifolds and the study of their automorphism groups. The k-th (ordered) configuration space of a topological space X is the space of injective maps from the finite set $\underline{k} = \{1, 2, \ldots, k\}$ to X. For a smooth closed manifold M, the k-th configuration space of M is a smooth manifold in its own right. In a number of ways, it can be thought of as the interior of a compact smooth manifold with boundary and a complicated corner structure in the boundary (compactification of Fulton–Macpherson and Axelrod–Singer). More generally, this remains true if M is the interior of a compact smooth manifold $\bar{M}$.

In the early 1990s, Kontsevich developed some ideas, partly inspired by mathematical physics, indicating that the finer structure of the de Rham cochain algebras (of differential forms) of the ordered configuration spaces of M contains much useful and computable information relating to M itself or to $\bar{M}$, where applicable. Although this indication was rather obscure, it was clarified a few years later by Kuperberg and Thurston who championed the use of the *compactified* configuration spaces. Their aim and contribution was to develop new invariants of smooth 3-manifolds, especially 3-dimensional rational homology spheres. But their work also recovered known invariants in a simple manner, such as the Casson invariant.

In the 2000s and 2010s, Tadayuki Watanabe refined their methods and demonstrated that they were applicable to high-dimensional manifolds and also to bundles (a.k.a. families) of smooth manifolds, especially bundles of smooth rational homology spheres or homology disks. This confirmed what Kontsevich had already foreseen in his early 1990s article. But Watanabe also developed some new and highly original constructive methods to show that configuration space integrals could detect interesting phenomena in high dimensions. In particular, he constructed new classes in the rational (or de Rham) cohomology of $B\mathrm{Diff}_\partial(D^m)$, the classifying space for smooth bundles with fiber D^m and trivialized boundary. Initially this only worked for some odd dimensions $m \geq 5$, and cohomology in rather high degrees. In late 2018, Watanabe caused more of a stir by announcing that he could make it work for $m = 4$ and rational cohomology or rational homotopy in many degrees including low ones. Two consequences are the failure of the Smale conjecture in dimension 4 and the existence of smooth fiber bundles with 4-dimensional fiber and base S^2 that are smoothly non-trivial, but topologically trivial.

Themes.

(1) *Compactifications of ordered configuration spaces of smooth manifolds.* Apart from the original papers by Fulton-MacPherson and Axelrod-Singer, where the topic is not fantastically accessible, there are good overviews in various places such as [8] and [5]. It could be that the case of a smooth manifold M which is the interior of a compact $\bar{M}$ has not been treated systematically. In any case it would be good to emphasize this, since it appears to be essential in most applications.

(2) *Relations of* (1) *with operad theory, especially the little disk operads.* There is a mild variation of the Fulton-MacPherson and Axelrod-Singer compactification which is applicable to the ordered configuration spaces of $\mathbb{R}^n$. These compactified configuration spaces of $\mathbb{R}^n$ are prominent in an incarnation of the operad of little n-disks. This turns out to be useful both for operad theory and for the homotopy theory of compactified configuration spaces

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in general, with all their complicated boundary structure. Sinha [8] emphasizes this. To some extent it can also be seen in Göppl's thesis [2], though the official purpose of that is to develop methods for "computing" spaces of maps from one topological operad to another.

(3) *Invariants of 3-manifolds following Kuperberg–Thurston and the notes of Lescop.* Reading [3] and the more expository paper [5] seems to be a good "introductory" way to learn what configuration space integrals are and what they can teach us about 3-dimensional manifolds in particular. Watanabe persistently recommends [5]. Two more recent articles by Lescop could also be useful [6, 7].

(4) *Watanabe's constructive methods.* Although Watanabe's articles [9], [10] and [11] do contain an exposition of configuration space integrals in the style of Kuperberg–Thurston, they dedicate more energy to far-reaching and often unexpected reformulations making these integrals more accessible to computation.

(5) *Ramifications of all the above and related matters.* Configuration space integrals are typically indexed by some trivalent graphs, which should probably be seen as instruments for probing the structure of the de Rham cochain algebra of a configuration space. This suggests a relationship with formality theorems for configuration spaces, a theme famously initiated by Kontsevich and developed in more detail in [4]. Indeed the proofs of these formality theorems rely on a description of deRham cochain algebras of configuration spaces in terms of generators and relations, using various complicated graphs and graph complexes. Unfortunately [4], although readable, is a long article. In addition we have been told that the connection between [4] and configuration space integrals is not easy to make. Somebody suggested that we look for hints in an exposition of Kontsevich's ideas by Conant–Vogtman [1].

Organization. At the present stage (early July 2020), we organizers are less than familiar with the sources and consequently not in a very good position to make a schedule of talks. We will hold an organizational meeting towards the end of September. Those who are interested in participating are encouraged to contact us as early as possible.

Participants. We hope that the seminar will attract both Master and PhD students. It is open to all who are familiar with the basics of algebraic and geometric topology. If there is a demand, there should also be time for introductory talks.

References

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[9] Tadayuki Watanabe, *On Kontsevich's characteristic classes for higher dimensional sphere bundles. I. The simplest class*, https://link.springer.com/article/10.1007/s00209-008-0396-4

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[11] ______, *Some exotic nontrivial elements of the rational homotopy groups of* $\mathrm{Diff}(S^4)$, https://arxiv.org/pdf/1812.02448.pdf

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