SMSTC (2008/09)

Geometry and Topology

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Lecture 1: Smooth manifolds in euclidean space

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1.1 Topology of subsets of euclidean space

This section is for reference and revision.

Notation. We write $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ etc. for points in $\mathbb{R}^n$. The euclidean distance is defined by

$$d(x, y) = \|y - x\| = \left(\sum_{i=1}^{n} (y_i - x_i)^2\right)^{1/2} .$$

Let ε be a positive real number. The *open ball* of radius ε about $x \in \mathbb{R}^n$ is the set

$$B(x, \varepsilon) = \{y \in \mathbb{R}^n \mid \|y - x\| < \varepsilon\} .$$

Definition 1.1.1 A subset U of $\mathbb{R}^n$ is *open* if, for every $x \in U$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. A subset C of $\mathbb{R}^n$ is *closed* if its complement $\mathbb{R}^n \smallsetminus C$ is open.

Some examples: The empty set $\emptyset$ is open in $\mathbb{R}^n$. Also, $\mathbb{R}^n$ itself is open in $\mathbb{R}^n$. Any subset of $\mathbb{R}^n$ which is a union of (possibly many) open subsets is open. *Slightly less obvious*: For any $x \in \mathbb{R}^n$ and $\varepsilon > 0$, the open ball $B(x, \varepsilon)$ is an open subset of $\mathbb{R}^n$. (Prove it.) If U and V are open subsets of $\mathbb{R}^n$, then $U \cap V$ is open. (Prove it.)

Definition 1.1.2 Let X be a subset of $\mathbb{R}^n$. A subset V of X is *open in X* if, for every $x \in V$, there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) \cap X \subset V$. A subset E of X is *closed* in X if its complement $X \smallsetminus E$ is open in X. *Equivalently*: V is open in X if there exists an open subset U of $\mathbb{R}^n$ such that $V = X \cap U$, and E is closed in X if there exists a closed subset C of $\mathbb{R}^n$ such that $E = X \cap C$.

Alternative terminology: Some people say *open relative to X*, *closed relative to X*.

Some examples: The empty set $\emptyset$ is open in X. Also, X itself is open in X. Any subset of X which is a union of (possibly many) open subsets in X is open in X. *Slightly less obvious*: For any $x \in \mathbb{R}^n$ and $\varepsilon > 0$, the set $B(x, \varepsilon) \cap X$ is open in X. If U and V are open in X, then $U \cap V$ is open in X.

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Example 1.1.3 Here is a curious example of an open set in the unit interval $[0, 1]$. Choose a sequence $(a_i)_{i=0,1,2,...}$ of *rational* numbers such that every rational number appears in the sequence. Let $\varepsilon_i = 2^{-i-3}$ and let U be the union of the open balls $B(a_i, \varepsilon_i) \subset \mathbb{R}^1$ for $i = 0, 1, 2, \ldots$. Let $V = U \cap [0, 1]$. The set V is remarkable for having the following three properties. It is open in $[0, 1]$, it is *dense* in $[0, 1]$ (i.e., for every $x \in [0, 1]$ there exists a sequence in V converging to x), and it has *Lebesgue measure*, i.e. length, not greater than

$$\sum_{i=0}^{\infty} 2^{1-i-3} = 1/2.$$

The length estimate comes from the fact that each $B(a_i, \varepsilon_i)$ has length $2\varepsilon_i = 2^{-i-2}$.
(This example was bad news for the Riemann integration theory when it was discovered more than hundred years ago. It can be deduced from the three stated properties of V that the *indicator function* of V, the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 1$ if $x \in V$ and $f(x) = 0$ if $x \notin V$, is not Riemann integrable. But we need not go into that.)

Definition 1.1.4 Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. A map $f : X \to Y$ is *continuous* at $a \in X$ if, for every sequence $(x_i)_{i\in\mathbb{N}}$ in X converging to a, the sequence $(f(x_i))_{i\in\mathbb{N}}$ converges to $f(a)$. The map f is *continuous* if it is continuous at a for all $a \in X$.

Example 1.1.5 Let $X = \mathbb{R} \smallsetminus \{0\}$, a subset of $\mathbb{R}$. Let $Y = \mathbb{R}$, also a subset of $\mathbb{R}$. The map $f : X \to Y$ defined by $f(x) = 1$ if $x > 0$ and $f(x) = 0$ if $x < 0$ is continuous.

Lemma 1.1.6 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. A map $f : X \to Y$ is continuous at $a \in X$ if and only if, for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that $d(f(x), f(a)) < \epsilon$ for every $x \in X$ such that $d(x, a) < \delta$.*

Theorem 1.1.7 *A map $f : X \to Y$, with $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$, is continuous if and only if the following holds: For every U open in Y, the pre-image $f^{-1}(U)$ is open in X.*

Remarks. We emphasize that $f^{-1}(U)$ means: the set of all $x \in X$ such that $f(x) \in U$. This is (unfortunately) accepted and widely used notation. There is no assumption here that a map $f^{-1} : Y \to X$ exists which is inverse to f.
If this theorem is new to you, prove it as an exercise. (But if you have taken a course in metric spaces and/or topological spaces in the past, then you must have seen it there.)

Definition 1.1.8 Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. A continuous map $f : X \to Y$ is a *homeomorphism* if there exists a continuous map $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. If such a homeomorphism exists, we say that X and Y are *homeomorphic*.

Example 1.1.9 $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$. One direction is trivial and half-way through this course we should be able to prove the other direction, too ! On request, that is.

1.2 Calculus without coordinates

This section is also for reference and revision.
We assume that you are familiar with the concept of a *vector space* over the field of real numbers (or indeed any other field). Every vector space V over $\mathbb{R}$ admits a *basis*, i.e., a subset S such that every $v \in V$ can be written in a unique way as a finite linear combination of elements of S. A basis for V is in most cases not unique, but the number of elements (cardinality) which it has is well defined and is called the *dimension* of V. It can be infinite.

Let V and W be real vector spaces. A map $f : V \to W$ is *linear* if it satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in V$, and $f(ax) = af(x)$ for all $x \in V$ and $a \in \mathbb{R}$. In the case where V and W are finite dimensional and come equipped with *ordered* bases, there is a standard way to describe any linear map from V to W by a matrix of size $n \times m$, where $n = \dim(W)$ and $m = \dim(V)$. Unordered bases are also good enough, say S for V and T for W, provided you are happy to work with matrices which have one row for each element of T and one column for each element of S.

Matrix multiplication "corresponds" to composition of linear maps. If you are aware of this, cherish the fact, if not, sort it out and/or ask appropriate questions.

That was enough of linear maps for a little while. Now we introduce differentiable maps as maps which admit "good approximations" by linear maps.

Definition 1.2.1 Let U be an open set in $\mathbb{R}^m$. A continuous map $f : U \to \mathbb{R}^n$ is *differentiable* at a point $x \in U$ if there exists a linear map $A : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{y\to 0} \frac{f(x+y) - f(x) - A(y)}{\|y\|} = 0 \ .$$

In that case A is unique. We call it the (total) *differential* of f at x.
Notation: $A = Df(x)$ or $A = df(x)$ or $A = f'(x)$, and so on.

The point which we are trying to make here is that differentiating is all about making "linear approximations". We want an approximation of the form

$$f(x+y) \approx f(x) + A(y)$$

with an error which is small compared to $\|y\|$, provided $\|y\|$ itself is sufficiently small. That is exactly what the definition guarantees.
Note: the linear map $A = f'(x)$ can of course be described by an $n \times m$ matrix. This may be more familiar to you as the matrix of partial derivatives, with entry $\partial f_i/\partial x_j$ (evaluated at the point x) in row i and column j. But ... it is better to think *linear map*, not *matrix*.

Definition 1.2.2 Let K and L be finite dimensional normed[b] real vector spaces. Let U be an open set in K. A continuous map $f : U \to L$ is *differentiable* at a point $x \in U$ if there exists a linear map $A : K \to L$ such that

$$\lim_{y\to 0} \frac{f(x+y) - f(x) - A(y)}{\|y\|} = 0 \ .$$

A very important point to observe here is that, when you have to decide whether f is differentiable at x, it does not matter which norm functions on K and L you use. In particular, you may choose a vector space basis for K and a vector space basis for L, and use the coresponding "euclidean" norms.
This definition generalises to the case where K and L are arbitrary (possibly infinite dimensional) *Banach spaces*, i.e., normed real vector spaces which are complete for their norms.[c]

Example 1.2.3 Let K be the vector space of real $\ell \times \ell$ matrices. We have a continuous map

$$\det : K \longrightarrow \mathbb{R} \ .$$

Does it have a differential at I_ℓ , the identity matrix ? If so, that differential, $\det'(I_\ell)$, must be a linear map from K to $\mathbb{R}$. *Answer*: the differential exists and it is the trace function,

$$\mathrm{tr} : K \to \mathbb{R}$$

taking a matrix to the sum of its diagonal entries. To verify this, we should start by choosing a suitable norm function on K. We can take: $\|C\| = \max\{|c_{ij}|\}$ for $C \in K$ (remember that an element of K is a square matrix). Now, to verify this formula $\det'(I_\ell) = \mathrm{tr}$, we only have to show that

$$\det(I + C) \approx \det(I) + \mathrm{tr}(C)$$

with an error which is small compared to $\|C\|$ if $\|C\|$ itself is small. That is very easy: develop the determinant by any of your favourite methods and neglect all terms which involve more than one entry from C. Then you should see the light. The neglected terms are at most equal to $\|C\|^2$, which is indeed small compared to $\|C\|$ if $\|C\|$ itself is small.

Theorem 1.2.4 (Chain rule.) *Let J, K, L be finite dimensional real vector spaces. Let U, V be open sets in J, K respectively. Let $g : U \to V$ and $f : V \to L$ be continuous maps. If g is differentiable at $x \in U$ and f is differentiable at $y = g(x) \in V$, then $f \circ g$ is differentiable at x and we have*

$$D(f \circ g)(x) = D(f)(y) \circ D(g)(x) \ .$$

[b] i.e., equipped with a norm function. For the definition of a norm function on a vector space, see e.g. Dieudonné's book "Foundations of Analysis"

[c] A normed real vector space K is *complete* if every Cauchy sequence $(x_i)_{i=0,1,2,...}$ in K converges to some $x_\infty \in K$.

1.3 Smooth k-dimensional manifolds in $\mathbb{R}^n$

Let U and V be open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. A continuous map $f : U \to V$ is *differentiable* if it is differentiable at every $x \in U$.

Terminology. When we say that a map $f : U \to V$ is *smooth*, we mean that it is infinitely many times differentiable, i.e., all its partial derivatives of any order exist and are continuous. (Occasionally we may also write or say *differentiable* when we mean *smooth* ... unintentional but hard to avoid.) The following terminology is available where more precision is needed: a map $f : U \to V$ is C^r if all its partial derivatives up to order r exist and are continuous. For example, C^0 just means *continuous*, C^1 means continuously differentiable and C^∞ means smooth.

Definition 1.3.1 Let U and V be open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. A smooth map $f : U \to V$ is a *diffeomorphism* if there exists a smooth map $g : V \to U$ such that $g \circ f = \mathrm{id}_U$ and $f \circ g = \mathrm{id}_V$. If such a diffeomorphism exists, we say that U and V are *diffeomorphic.*

Let k and n be positive integers with $k \leq n$. In definition 1.3.2 below we think of $\mathbb{R}^k$ as a subset of $\mathbb{R}^n$,

$$\mathbb{R}^k = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i = 0 \text{ if } i > k\} \ .$$

Definition 1.3.2 Let M be a subset of $\mathbb{R}^n$. We say that M is a *k-dimensional smooth manifold in* $\mathbb{R}^n$ if the following holds. For every $x \in M$ there exist open sets U and V in $\mathbb{R}^n$ such that $x \in V$, and a diffeomorphism $\psi : U \to V$ such that

$$\psi(U \cap \mathbb{R}^k) = V \cap M \ .$$

Example 1.3.3 Special case $k = n$: An n-dimensional smooth submanifold in $\mathbb{R}^n$ is the same thing as an open subset of $\mathbb{R}^n$. (Prove it.)

Example 1.3.4 Here we take $k = n - 1$. The unit sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$$

is an $(n-1)$-dimensional differentiable manifold in $\mathbb{R}^n$. *Proof*: Fix $x = (x_1, \ldots, x_n) \in S^n$. At least one of the x_i is nonzero. Without loss of (much) generality, $x_n > 0$. We let $U = V = \{y \in \mathbb{R}^n \mid \sum_{i=1}^{n-1} y_i^2 < 1\}$ and we define $\psi : U \to V$ by

$$(y_1, \ldots, y_{n-1}, y_n) \mapsto \left(y_1, \ldots, y_{n-1}, y_n + \sqrt{1 - \textstyle\sum_{i=1}^{n-1} y_i^2}\ \right) \ .$$

Then $x \in V$, and $\psi(U \cap \mathbb{R}^{n-1})$ is precisely $V \cap S^{n-1}$.

Example 1.3.5 Fix positive integers ℓ and $r \leq \ell$. A real $\ell \times \ell$ matrix can be thought of as a "vector" with $\ell \cdot \ell$ coordinates. In this way, the real $\ell \times \ell$ matrices which have rank (also known as *dimension of the column space*) equal to r form a subset W of $\mathbb{R}^{\ell \cdot \ell}$. Let $k = \ell - r$. We are going to show that W is a differentiable manifold of dimension $\ell^2 - k^2$ in $\mathbb{R}^{\ell \cdot \ell}$.
So let $x \in W$. We think of x as an $\ell \times \ell$ matrix with entries x_{ij} , and we also write $x_{\bullet i}$ for the i-th column of x when necessary. We can find exactly r linearly independent columns in x, not more. Without loss of (much) generality, columns $1, 2, \ldots, r$ of x are linearly independent and each of the remaining columns is a linear combination of the first r columns. We choose an $\ell \times k$ matrix z such that the square matrix

$$Q_x = \begin{bmatrix} x_{\bullet 1} & x_{\bullet 2} & \cdots & x_{\bullet r} & z_{\bullet 1} & z_{\bullet 2} & \cdots & z_{\bullet k} \end{bmatrix}$$

is invertible (i.e., its columns make up a basis for $\mathbb{R}^\ell$). Let $V \subset \mathbb{R}^{\ell \cdot \ell}$ consist of all $\ell \times \ell$ matrices y such that

$$Q_y = \begin{bmatrix} y_{\bullet 1} & y_{\bullet 2} & \cdots & y_{\bullet r} & z_{\bullet 1} & z_{\bullet 2} & \cdots & z_{\bullet k} \end{bmatrix}$$

is invertible. Then it is clear that $x \in V$, and it is not hard to see that V is open in $\mathbb{R}^{\ell \cdot \ell}$. Put $U = V$ and define $\psi : U \to V$ by

$$\psi(y) = \begin{bmatrix} y_{\bullet 1} & y_{\bullet 2} & \cdots & y_{\bullet r} & y'_{\bullet r+1} & y'_{\bullet r+2} & \cdots & y'_{\bullet r+k} \end{bmatrix}$$

where $y' = Q_y\, y$. In words, the first r columns of $\psi(y)$ are the first r columns of y, and the last k columns of $\psi(y)$ are the last k columns of $Q_y\, y$. Note also that ψ is invertible. (If we know $\psi(y)$, then we can find Q_y because that is equal to $Q_{\psi(y)}$.)
We write $\mathbb{R}^{\ell\cdot\ell-k\cdot k} \subset \mathbb{R}^{\ell\cdot\ell}$ for the vector subspace of $\mathbb{R}^{\ell\cdot\ell}$ consisting of all $\ell\times\ell$ matrices c whose entries c_{ij} are zero for $i > r$ and $j > r$. It is easy to see that if $y \in U \cap \mathbb{R}^{\ell\cdot\ell-k\cdot k}$, then $\psi(y) \in V \cap W$, and if $y \notin \mathbb{R}^{\ell\cdot\ell-k\cdot k}$, then $\psi(y) \notin W$. Therefore

$$\psi(U \cap \mathbb{R}^{\ell\cdot\ell-k\cdot k}) = V \cap W\ .$$

1.4 Inverse function theorem and applications

Theorem 1.4.1 (Inverse function theorem.) *Let W be an open set in $\mathbb{R}^m$, let $\varphi : W \to \mathbb{R}^m$ be a smooth map and let $q \in W$. If the linear map $D\varphi(q) : \mathbb{R}^m \to \mathbb{R}^m$ is invertible, then there exists an open $W' \subset \mathbb{R}^m$ with $q \in W' \subset W$ such that $\varphi(W') \subset \mathbb{R}^m$ is open and the restriction $\varphi|W'$ is a diffeomorphism from W' to $\varphi(W')$.*

As a motivation for the next theorem, let's do an experiment with linear maps. Let $f : U \times V \to W$ be a linear map, where U, V and W are finite dimensional real vector spaces. Write f_1 and f_2 for the restrictions of f to $U \times 0$ and $0 \times V$, respectively. Suppose that $f_2 : V \to W$ is an isomorphism. How can we solve the equation

$$f(x, y) = 0$$

where $x \in U$ is "given" and $y \in V$ is "sought" ? We can reason $0 = f(x, y) = f_1(x) + f_2(y)$ which implies $f_2(y) = -f_1(x)$ and therefore $y = f_2^{-1}(f_1(x))$. Finished.

Theorem 1.4.2 (Implicit function theorem.) *Let U be an open set in $\mathbb{R}^k \times \mathbb{R}^\ell$, let $f : U \to \mathbb{R}^\ell$ be a smooth map and let $(p, q) \in U$ such that $f(p, q) = 0$. Write $D_1 f(p, q) : \mathbb{R}^k \to \mathbb{R}^\ell$ and $D_2 f(p, q) : \mathbb{R}^\ell \to \mathbb{R}^\ell$ for the linear maps obtained by restricting $Df(p, q) : \mathbb{R}^k \times \mathbb{R}^\ell \to \mathbb{R}^\ell$ to the factors $\mathbb{R}^k$ and $\mathbb{R}^\ell$, respectively. If $D_2 f(p, q)$ is invertible, then there exist an open ball $V_p = B(p, \delta)$ about $p \in \mathbb{R}^k$ and an open ball $V_q = B(q, \varepsilon)$ about $q \in \mathbb{R}^\ell$ such that the equation*

$$f(x, y) = 0$$

has a unique solution $y = g(x)$ in V_q for every x in V_p. The map g is smooth (if δ is sufficiently small). Its differential at p is

$$Dg(p) = -D_2 f(p, q)^{-1} \circ D_1 f(p, q)\ .$$

It is easy to deduce the implicit function theorem from the inverse function theorem. Start with U, f and (p, q) as in the hypotheses of the implicit function theorem. Put $W = U$ and define

$$\varphi(x, y) = (x, f(x, y)) \in \mathbb{R}^k \times \mathbb{R}^\ell$$

for $(x, y) \in W$. Apply the *inverse function theorem* to this φ. You get an open $W' \subset W$ with $(p, q) \in W'$ such that $\varphi(W')$ is open, and $\psi : \varphi(W') \to W'$ inverse to $\varphi|W'$. Then $\psi(x, y) = (x, \psi_2(x, y))$ for some smooth map ψ_2 from $\varphi(W')$ to $\mathbb{R}^\ell$ and all $(x, y) \in W'$. Now

$$\begin{aligned} & f(x, y) = 0 \\ \Leftrightarrow\quad & \varphi(x, y) = (x, 0) \\ \Leftrightarrow\quad & (x, y) = \psi(x, 0) \\ \Leftrightarrow\quad & y = \psi_2(x, 0)\ . \end{aligned}$$

Therefore we *must* have $g(x) = \psi_2(x, 0)$ and this is also smooth enough. (The formula for the differential $Dg(p)$ is a consequence of the equation $f(x, g(x)) = 0$ and the chain rule.)

It is even easier to deduce the inverse function theorem from the implicit function theorem. Start with a map $\varphi : W \to \mathbb{R}^m$ as in the inverse function theorem, let $U = \mathbb{R}^m \times W \subset \mathbb{R}^m \times \mathbb{R}^m$ and define $f : U \to \mathbb{R}^m$ by $f(x, y) = \varphi(y) - x$. Apply the *implicit function theorem* to this U and f and the point $(p, q) = (\varphi(q), q)$. You get V open in $\mathbb{R}^m$ containing $p = \varphi(q)$, and a smooth $g : V \to \mathbb{R}^m$ with $f(x, g(x)) = 0$, which means $\varphi(g(x)) - x = 0$, which means $\varphi(g(x)) = x$. So g is a right inverse for φ (where it is defined). Repeating the argument with g instead of φ, you can also find a right inverse for g, defined in a small open ball about q. (Then it is easy to show that this must agree with φ in a possibly smaller open ball about p.)

Corollary 1.4.3 *Let $W \subset \mathbb{R}^m$ be open, let $x \in W$, and let $f : W \to \mathbb{R}^\ell$ be a smooth map. If $f(x) = 0$ and the linear map $Df(x) : \mathbb{R}^m \to \mathbb{R}^\ell$ is surjective, then there exists an open set $W' \subset \mathbb{R}^m$ with $x \in W' \subset W$ such that $W' \cap f^{-1}(0)$ is a smooth manifold in $\mathbb{R}^m$, of dimension $m - \ell$.*

Proof Since $Df(x) : \mathbb{R}^m \to \mathbb{R}^\ell$ is surjective, it is easy to find a linear map $u : \mathbb{R}^m \to \mathbb{R}^k$ (with $k = m-\ell$) such that the map

$$y \mapsto (u(y), Df(x)(y))$$

from $\mathbb{R}^m$ to $\mathbb{R}^k \times \mathbb{R}^\ell$ is an invertible (linear) map. Define $\varphi : \mathbb{R}^m \to \mathbb{R}^k \times \mathbb{R}^\ell$ by the formula

$$\varphi(y) = (u(y), f(y)) \ .$$

The differential of φ at x is precisely the map $y \mapsto (u(y), Df(x)(y))$, so it is an invertible linear map by assumption. We may apply the inverse function theorem and obtain an open $W' \subset W$ containing x so that $\varphi(W')$ is open in $\mathbb{R}^k \times \mathbb{R}^l$ and $\varphi|W'$ is a diffeomorphism from W' to $\varphi(W')$. By construction, $\varphi|W'$ maps the set $W' \cap f^{-1}(0)$ bijectively to $\varphi(W') \cap \mathbb{R}^k$. (Here, as before, we treat $\mathbb{R}^k$ as a subspace of $\mathbb{R}^k \times \mathbb{R}^\ell$.) □

Example 1.4.4 A real $m \times m$ matrix A is *orthogonal* if $A^T A = I_m$, where A^T denotes the transpose matrix. This is equivalent to saying that the columns of A form an orthonormal basis of $\mathbb{R}^m$ (where "ortho-" means that the columns are pairwise perpendicular, and "-normal" means that they all have length one). The set of all orthogonal $m \times m$ matrices is denoted by $\mathrm{O}(m)$. The operation *matrix multiplication* makes it into a group. In particular, if $A, B \in \mathrm{O}(m)$ then $AB \in \mathrm{O}(m)$.
From the definition, $\mathrm{O}(m)$ is a subset of the vector space J of all real $m \times m$-matrices (which you can identify with $\mathbb{R}^{m \cdot m}$ if you wish). *We are going to show that $\mathrm{O}(m)$ is a smooth submanifold of J, of dimension $m(m-1)/2$.*
Let L be the vector space of real *symmetric* $m \times m$ matrices. Let $f : J \to L$ be given by $f(A) = A^T A - I_m$. Let's note that $\mathrm{O}(m) = f^{-1}(0)$. By corollary 1.4.3, if we can show that $Df(A) : J \to L$ is surjective for *every* $A \in \mathrm{O}(m) = f^{-1}(0)$, then we can be certain that $\mathrm{O}(m)$ is a smooth manifold in J, and its dimension will be $\dim(J) - \dim(L) = m(m-1)/2$. To find $Df(A)$ for a fixed A, let's try to find a linear approximation

$$f(A + B) \approx f(A) + \Lambda(B)$$

where $\Lambda(B) = Df(A)(B)$ depends linearly on B. This has to work when B is small. Writing out the left-hand side, we get

$$\begin{aligned} (A^T + B^T)(A + B) - I_m &= A^T A - I_m + A^T B + B^T A + B^T B \\ &= f(A) + (A^T B + B^T A) + \text{ negligible} \ . \end{aligned}$$

(Observe that $B^T B$ is indeed negligible when B is small. Compare example 1.2.3.) Therefore $\Lambda(B) = Df(A)(B) = A^T B + B^T A$, and this does indeed depend linearly on B.
Now that we have a formula for $Df(A)$, we can also show that $Df(A)$ is surjective if $f(A) = 0$. Given some $C \in L$ (a symmetric matrix), we can find a matrix $E \in J$ such that $C = E + E^T$. Let $B = AE$. We get

$$Df(A)(B) = A^T B + B^T A = A^T AE + E^T A^T A = E + E^T = C \ .$$

(We have used $A^T A = \mathrm{id}$, which is equivalent to $f(A) = 0$.) Since $C \in L$ was arbitrary, this shows that $Df(A)$ is surjective.

We often encounter smooth manifolds in euclidean space as solution sets of systems of equations, e.g., a system of polynomial equations as in example 1.4.4. The inverse function theorem and the implicit function theorem can help us to show that these solution sets are actually manifolds (in euclidean space). The next proposition is a consequence of the inverse function theorem which is specifically formulated for this purpose.

Proposition 1.4.5 *Let $W \subset \mathbb{R}^p$ be open and let $f : W \to \mathbb{R}^q$ be a smooth map. Suppose that, for every $x \in f^{-1}(0)$, there exist an open set $V_x \subset \mathbb{R}^p$ with $x \in V_x \subset W$, and a linear subspace $K_x \subset \mathbb{R}^q$ complementary to the image of the linear map $Df(x) : \mathbb{R}^p \to \mathbb{R}^q$, such that $f(V_x) \cap K_x = \{0\}$. Then for every non-negative integer $j \leq m$, the set*

$$\{x \in f^{-1}(0) \mid \mathrm{rank}(Df(x)) = j\}$$

is a smooth manifold in $\mathbb{R}^p$, of dimension $p - j$.

Proof Fix $x \in W$ with $f(x) = 0$. Let $L_x = \mathrm{im}(Df(x)) \subset \mathbb{R}^q$. Then we have $\mathbb{R}^q \cong L_x \times K_x$. Write $v : \mathbb{R}^q \to L_x$ for the projection to the first factor, so that $\ker(v) = K_x$. Now the composite linear map

$$\mathbb{R}^p \xrightarrow{Df(x)} \mathbb{R}^q \xrightarrow{\quad v \quad} L_x$$

is surjective. This is also the differential of the smooth map $v \circ f$ at x. By corollary 1.4.3, for a sufficiently small open set $W' \subset W$ containing x, the set

$$W' \cap (v \circ f)^{-1}(0)$$

is a smooth manifold in $\mathbb{R}^p$, of dimension $p - \dim(L_x) = p - j$. We may assume that $W' \subset V_x$. But then

$$W' \cap (v \circ f)^{-1}(0) = W' \cap f^{-1}(0)$$

because $y \in W'$ and $v(f(y)) = 0$ implies $f(y) \in f(V_x) \cap K_x$, so that $f(y) = 0$ by our assumptions. $\square$

Example 1.4.6 Here our goal is roughly the following. We fix strictly positive integers n and $i \leq n$ and we want to make the set of all i-dimensional linear (vector-)subspaces of $\mathbb{R}^n$ into a smooth manifold. In the next section we will achieve this "abstractly", but here we have to think of a suitable ambient euclidean space $\mathbb{R}^p$ (large p, presumably) and a way to "represent" each i-dimensional linear subspace of $\mathbb{R}^n$ by a single point or vector in $\mathbb{R}^p$.

This is not as hard as it looks. We observe that any i-dimensional linear subspace $E \subset \mathbb{R}^n$ determines an orthogonal projection to that subspace E. We think of that as a linear map $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ whose image is equal to E and whose kernel is equal to $E^\perp$, the orthogonal complement of E in $\mathbb{R}^n$. Two remarkable equations satisfied by α are

$$\alpha \circ \alpha = \alpha$$

(idempotence) and

$$\langle \alpha(v), w \rangle = \langle v, \alpha(w) \rangle$$

for all $v, w \in \mathbb{R}^n$ (self-adjointness). The first equation means that α restricts to the identity on $\mathrm{im}(\alpha)$ and the second implies that $\ker(\alpha)$ is perpendicular to the image space. Together, the two equations express the fact that α is an orthogonal projection. If we describe α by an $n \times n$ matrix A , then the first equation becomes $A^2 = A$ and the other turns into $A = A^T$. Summing up, instead of saying *set of i-dimensional linear subspaces of* $\mathbb{R}^n$ we now say *set of real symmetric idempotent $n \times n$-matrices of rank i*. That is a subset M of the real vector space J of symmetric $n \times n$ matrices. We can identify J with $\mathbb{R}^p$ where $p = n(n + 1)/2$. Now we want to show that M is a smooth submanifold in J. We are curious what its dimension might be.

Define $f : J \to J$ by

$$f(B) = B^2 - B$$

for $B \in J$. Then $f^{-1}(0)$ is the set of all orthogonal projections. Let's try to apply proposition 1.4.5 to this map f. Fix $A \in M \subset f^{-1}(0)$, so that A has rank i. Let $E = \mathrm{im}(A) \subset \mathbb{R}^n$ and $F = \ker(A) \subset \mathbb{R}^n$ so that $F = E^\perp$ in $\mathbb{R}^n$. To find $Df(A) : J \to J$ we do the linear approximation thing:

$$\begin{aligned} f(A+B) &= (A+B)^2 - (A+B) \\ &= f(A) + AB + BA + B^2 - B \\ &\approx f(A) + (AB + BA - B) + \text{ negligible} \end{aligned}$$

(where B^2 is negligible provided B itself has small entries). Therefore $Df(A)$ is the linear map

$$B \quad \mapsto \quad AB + BA - B$$

from J to J. If we think of B as a "block matrix" of linear maps

$$B = \begin{bmatrix} B_{EE} & B_{EF} \\ B_{FE} & B_{FF} \end{bmatrix} \qquad\qquad (\text{with } B_{EF} = B_{FE})$$

where B_{FE} for example means the composition

$$E \xrightarrow{\text{incl.}} \mathbb{R}^n \xrightarrow{\quad B \quad} \mathbb{R}^n \xrightarrow{\text{proj.}} F\ ,$$

then $AB + BA - B$ takes the form

$$\begin{bmatrix} B_{EE} & 0 \\ 0 & -B_{FF} \end{bmatrix} .$$

It follows immediately that the dimension of $\ker(Df(A))$ is $i(n-i)$. An obvious choice of a complementary linear subspace K_A for $\mathrm{im}(Df(A))$ in J is therefore as follows:

$$K_A = \{\, C \in J \ \mid \ C_{EE} = 0 \ , \ C_{FF} = 0 \,\}.$$

Now let's check that the condition of proposition 1.4.5 is satisfied for this choice of K_A. If this is the case, for arbitrary $A \in M$, then we know that M is a smooth manifold in J of dimension $i(n-i)$. We need to show that if $f(A) = 0$ and $f(A + B) \in K_A$ and $B \in J$ is small enough, then $f(A + B) = 0$. By the calculation above we have

$$f(A+B) = \begin{bmatrix} B_{EE} & 0 \\ 0 & -B_{FF} \end{bmatrix} + B^2$$

(but now we must not neglect B^2). Therefore $f(A + B) \in K_A$ holds if and only if

$$-B_{EE} = B_{EE}^2 + B_{EF}B_{FE} \ , \qquad B_{FF} = B_{FE}B_{EF} + B_{FF}^2 \ .$$

Assuming that B has small entries, we need to show that this forces $f(A + B) = 0$, which amounts to showing that

$$B_{FE}B_{EE} + B_{FF}B_{FE} = 0 \ , \quad B_{EE}B_{EF} + B_{EF}B_{FF} = 0 \ .$$

Gentle reader, do it !!!

SMSTC (2008/09)
Geometry and Topology
Lecture 2: Abstract smooth manifolds

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Contents

2.1 Charts, atlases and smooth maps

Definition 2.1.1 Let M be a set. A *smooth atlas* on M consists of a choice of non-negative integer m, which is the "dimension" of the atlas, and a set $\mathcal{A}$ of *charts*. Each chart is an injective map $\psi : U \to M$, where U is an open subset of $\mathbb{R}^m$. There are several conditions to be satisfied:

- The atlas covers all of M. That is, for each $z \in M$ there exists a chart $\psi : U \to M$ in $\mathcal{A}$ such that $z \in \psi(U)$.
- Changes of charts are smooth. More precisely, if $\psi_1 : U_1 \to M$ and $\psi_2 : U_2 \to M$ are any distinct charts in $\mathcal{A}$, then
  - the set $\psi_1^{-1}(\psi_2(U_2))$ is open in U_1 and hence in $\mathbb{R}^m$,
  - the set $\psi_2^{-1}(\psi_1(U_1))$ is open in U_2 and hence in $\mathbb{R}^m$,
  - the map $\psi_1^{-1}(\psi_2(U_2)) \longrightarrow \psi_2^{-1}(\psi_1(U_1))$ defined by $x \mapsto \psi_2^{-1}(\psi_1(x))$ is smooth (and consequently continuous).

Remarks. The "change of chart" map $\psi_1^{-1}(\psi_2(U_2)) \to \psi_2^{-1}(\psi_1(U_1))$ is actually a diffeomorphism. That's because the smoothness condition also holds for its inverse.

Example 2.1.2 Let $M = \mathbb{R} \cup \{\infty\}$. The following two charts make up a smooth atlas $\mathcal{A}$ for M, of dimension $m = 1$. Let $U_1 = \mathbb{R} = U_2$, and put

$$\begin{aligned} \psi_1 : U_1 \longrightarrow M \ , &\quad \psi_1(x) = x \\ \psi_2 : U_2 \longrightarrow M \ , &\quad \psi_2(x) = x^{-1} \end{aligned}$$

(where $x^{-1} = \infty$ if $x = 0$). Then $\psi_1^{-1}(\psi_2(U_2)) = \psi_2^{-1}(\psi_1(U_1)) = \mathbb{R} \setminus \{0\}$, and the two change-of-chart maps are given by the formula $x \mapsto x^{-1}$, which is clearly smooth. Both are maps from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R} \setminus \{0\}$.

Example 2.1.3 Let $M = \mathbb{R}^m \cup \{\infty\}$. Let $U_1 = \mathbb{R}^m = U_2$ and make a smooth atlas for M with the two charts

$$\begin{aligned} \psi_1 : U_1 \longrightarrow M \ , &\quad \psi_1(x) = x \\ \psi_2 : U_2 \longrightarrow M \ , &\quad \psi_2(x) = \|x\|^{-2}x \ . \end{aligned}$$

Then $\psi_1^{-1}(\psi_2(U_2)) = \psi_2^{-1}(\psi_1(U_1)) = \mathbb{R}^m \setminus \{0\}$. The two change-of-chart maps are given by the formula $x \mapsto \|x\|^{-2}x$, which is smooth.

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Example 2.1.4 Let M be the set of all lines through the origin in $\mathbb{R}^3$. We are going to make a smooth atlas for M, of dimension $m = 2$, with three charts $\psi_1 : \mathbb{R}^2 \to M$, $\psi_2 : \mathbb{R}^2 \to M$ and $\psi_3 : \mathbb{R}^2 \to M$. The resulting smooth manifold is known as the *projective plane*, in symbols $\mathbb{R}P^2$.
Let $\psi_1(x_1, x_2)$ be the line in $\mathbb{R}^3$ which passes through $(1, x_1, x_2) \in \mathbb{R}^3$ and the origin. Let $\psi_2(x_1, x_2)$ be the line in $\mathbb{R}^3$ which passes through $(x_1, 1, x_2)$ and the origin. Let $\psi_3(x_1, x_2)$ be the line which passes through $(x_1, x_2, 1)$ and the the origin. Note that $\psi_1(\mathbb{R}^2) \cup \psi_2(\mathbb{R}^2) \cup \psi_3(\mathbb{R}^2) = M$.
Let's check that the change of charts from ψ_1 to ψ_2 is smooth (the other two cases are similar). We have $\psi_1^{-1}(\psi_2(\mathbb{R}^2)) = \{x \in \mathbb{R}^2 \mid x_1 \neq 0\}$. The change of chart formula is

$$(x_1, x_2) \mapsto (x_1^{-1}, x_2 x_1^{-1}) \ .$$

This is indeed a smooth map from $\{x \in \mathbb{R}^2 \mid x_1 \neq 0\}$ to $\mathbb{R}^2$.

Example 2.1.5 Let M be a smooth manifold of dimension k in $\mathbb{R}^n$. There is a preferred way to make a smooth atlas for M, of dimension k. We start by looking for diffeomorphisms $\varphi : U \to V$, where U and V are open in $\mathbb{R}^n$, with the property

$$\varphi(U \cap \mathbb{R}^k) = V \cap M \ .$$

Such a φ might be called an *ambient chart* for M. By definition 1.3.2, these are in good supply: for every $x \in M$ there exists an ambient chart $\varphi : U \to V$ for M such that $x \in \varphi(U)$. Every ambient chart $\varphi : U \to V$ for M determines an "honest" chart for M by restriction:

$$\varphi | U \cap \mathbb{R}^k \ : \ U \cap \mathbb{R}^k \longrightarrow V \cap M \subset M \ .$$

Here $U \cap \mathbb{R}^k$ is an *open* subset of $\mathbb{R}^k$. Let $\mathcal{A}$ be the set of all the "honest" charts for M which can be obtained from ambient charts for M by restriction. Then $\mathcal{A}$ is a smooth atlas for M. Let's prove this. We have already observed that the first condition for an atlas is satisfied: for every $x \in M$ there exists a chart ψ in $\mathcal{A}$ such that x is in the image of ψ. To verify the second condition on smoothness of changes of chart, we can assume that two ambient charts are given:

$$\varphi_1 : U_1 \to V_1 \ , \qquad \varphi_2 : U_2 \to V_2 \ ,$$

(where U_1, U_2, V_1, V_2 are open in $\mathbb{R}^n$). By restriction, these determine two honest charts, $U_1 \cap \mathbb{R}^k \to V_1 \cap M$ and $U_2 \cap \mathbb{R}^k \to V_2 \cap M$. About these honest charts we have to show that $\varphi_1^{-1}(V_2 \cap M)$ is open in $U_1 \cap \mathbb{R}^k$, that $\varphi_2^{-1}(V_1 \cap M)$ is open in $U_2 \cap \mathbb{R}^k$, and that the map

$$\varphi_1^{-1}(V_2 \cap M) \longrightarrow \varphi_2^{-1}(V_1 \cap M)$$

defined by $x \mapsto \varphi_2^{-1}(\varphi_1(x))$ is smooth. Both assertions follow from the fact that

$$\varphi_1^{-1}(V_2 \cap M) = \varphi_1^{-1}(V_2) \cap \mathbb{R}^k \ , \varphi_2^{-1}(V_1 \cap M) = \varphi_2^{-1}(V_1) \cap \mathbb{R}^k \ .$$

The definition and the digression which follow are slightly premature, but you should welcome them if you were wondering what smooth atlases are "for".

Definition 2.1.6 Let M be a set with a smooth atlas $\mathcal{A}$ of dimension m, and let N be a set with a smooth atlas $\mathcal{B}$ of dimension n. A map

$$f : M \to N$$

is considered *smooth* if, for every chart $\varphi : U_1 \to M$ in $\mathcal{A}$ and every chart $\psi : U_2 \to N$ in $\mathcal{B}$, the set $\varphi^{-1}(f^{-1}(\psi(U_2)))$ is open in U_1 and the map

$$\varphi^{-1}(f^{-1}(\psi(U_2))) \longrightarrow U_2$$

defined by $x \mapsto \psi^{-1}(f(\varphi(x)))$ is smooth.

Definition 2.1.7 *Let M be a set with a smooth atlas $\mathcal{A}$ of dimension m, and let N be a set with a smooth atlas $\mathcal{B}$, also of dimension m. A smooth map $f : M \to N$ is a* diffeomorphism *if there exists a smooth map $g : N \to M$ such that $g \circ f = \mathrm{id}_M$ and $f \circ g = \mathrm{id}_N$.*

Remark. We have used the fact that a composition of two smooth maps (provided it is defined as a map of sets) is again a smooth map. The proof is left to you.

Digression 2.1.8 Several important branches of mathematics are concerned with the exploration of a particular *category*. A category, in the mathematical sense of the word, consists of a collection $\mathcal{C}$ of things called *the objects of* $\mathcal{C}$ and, for any two of these objects, say X and Y, a set $\mathrm{mor}(X,Y)$ whose elements are called *morphisms* from X to Y. There is a specified composition rule which to every $g \in \mathrm{mor}(X,Y)$ and every $f \in \mathrm{mor}(Y,Z)$ associates an element of $\mathrm{mor}(X,Z)$, usually denoted by $f \circ g$. The composition rule is associative (that's a condition) and it has "units" (another condition). That is, for every object X in $\mathcal{C}$, there exists an element $\mathrm{id}_X \in \mathrm{mor}(X,X)$ such that $f \circ \mathrm{id}_X = f$ for every object Y in $\mathcal{C}$ and $f \in \mathrm{mor}(X,Y)$, and also $\mathrm{id}_X \circ g = g$ for every object W in $\mathcal{C}$ and $g \in \mathrm{mor}(W,X)$.
The most basic example is the *category* $\mathcal{S}$ *of sets*. Every set qualifies as an object of $\mathcal{S}$. A morphism from a set X to a set Y is (by definition) just an ordinary map from X to Y. The composition rule is (by definition) ordinary composition of maps.
A more exciting example is the *category* $\mathcal{G}$ *of groups*. Every group qualifies as an object of $\mathcal{G}$. A morphism from a group G to a group H is a homomorphism from G to H. The composition rule is ordinary composition of homomorphisms.
Yet another example is the *category* $\mathcal{T}$ *of topological spaces*. Whether or not you know what a topological space is, you will understand that every topological space qualifies as an object of $\mathcal{T}$. Whether or not you know what a continuous map between topological spaces is, you will not be surprised to hear that a morphism from a topological space X to a topological space Y is, by definition, a continuous map from X to Y. The composition rule is ordinary composition of continuous maps.
In any category $\mathcal{C}$, the concept of *isomorphism* has a meaning. A morphism f from X to Y in $\mathcal{C}$ is an *isomorphism* if there exists a morphism g from Y to X in $\mathcal{C}$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In that case the objects X and Y are said to be *isomorphic*. Finding ways to decide whether or not two random objects of $\mathcal{C}$ are isomorphic tends to be one of the favourite activities for people who "explore" the category $\mathcal{C}$.
In many well-established categories other words are used for "isomorphism" and "isomorphic". In the category of topological spaces, for example, we say *homeomorphism* and *homeomorphic*. In the category of sets, we say *bijection* and *of the same cardinality*.
Using definitions 2.1.1 and 2.1.6, we can produce another very interesting example of a category. In this example, the objects are sets together with a smooth atlas. A morphism from one object $(M,\mathcal{A})$ to another object $(N,\mathcal{B})$ is a smooth map in the sense of definition 2.1.6. The isomorphisms are precisely the diffeomorphisms.
Except for some small changes which we will make in the next section, this category is the setting for a branch of topology called *differential topology*.

Example 2.1.9 This is a mild generalization of example 2.1.4. Fix a positive integer n and let M be the set of all lines through the origin in $\mathbb{R}^{n+1}$. We are going to make a smooth atlas for M, of dimension n, with $n+1$ charts $\psi_i : \mathbb{R}^n \to M$, where $i = 1,2,\dots,n+1$. The resulting smooth manifold is known as *projective n-space*, in symbols $\mathbb{R}P^n$.
Let $\psi_1(x_1,x_2,\dots,x_n)$ be the line in $\mathbb{R}^{n+1}$ which passes through the point $(1,x_1,x_2,\dots,x_n)$ and the origin.
Let $\psi_2(x_1,x_2,\dots,x_n)$ be the line in $\mathbb{R}^{n+1}$ which passes through the point $(x_1,1,x_2,\dots,x_n)$ and the origin.
Let $\psi_3(x_1,x_2,\dots,x_n)$ be the line in $\mathbb{R}^{n+1}$ which passes through the the point $(x_1,x_2,1,x_3,\dots,x_n)$ and the origin. And so on. Then $\bigcup_{i=1}^{n+1} \psi_i(\mathbb{R}^n) = M$.
To illustrate the smoothness check, let's assume $n = 10$ and let's look at the change of charts from ψ_3 to ψ_8. We have $\psi_3^{-1}(\psi_8(\mathbb{R}^n)) = \{x \in \mathbb{R}^{10} \mid x_7 \neq 0\}$ and $\psi_8^{-1}(\psi_3(\mathbb{R}^n)) = \{x \in \mathbb{R}^{10} \mid x_3 \neq 0\}$. The change of chart formula is

$$(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}) \mapsto (\frac{x_1}{x_7},\frac{x_2}{x_7},\frac{1}{x_7},\frac{x_3}{x_7},\frac{x_4}{x_7},\frac{x_5}{x_7},\frac{x_6}{x_7},\frac{x_8}{x_7},\frac{x_9}{x_7},\frac{x_{10}}{x_7}) \ .$$

Example 2.1.10 We finish this section with another example which is a serious generalisation of examples 2.1.4 and 2.1.9. To appreciate it fully try to block out example 1.4.6 from your memory. We fix positive integers p and m with $p \leq m$. Let M be the set of p-dimensional linear subspaces of $\mathbb{R}^m$. We are going to construct a smooth atlas on M, of dimension $p(m-p)$. The resulting smooth manifold is known as the *Grassmannian* of p-planes in $\mathbb{R}^m$. Suggested notation: $G_p(\mathbb{R}^m)$.
Let S be a subset of $\{1,2,\dots,m\}$ with p elements. The set S determines a linear isomorphism C_S from

$\mathbb{R}^m$ to $\mathbb{R}^p \times \mathbb{R}^{m-p}$ which moves the coordinates corresponding to elements of S to the "front" and the remaining ones to the "back". (For example, if $m = 10$, $p = 4$ and $S = \{2, 6, 7, 9\}$, then the formula for C_S is $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_10) \mapsto ((x_2, x_6, x_7, x_9), (x_1, x_3, x_4, x_5, x_8, x_{10})) \in \mathbb{R}^4 \times \mathbb{R}^6$.) Let $\hom(\mathbb{R}^p, \mathbb{R}^{m-p})$ be the vector space of linear maps from $\mathbb{R}^p$ to $\mathbb{R}^{m-p}$. You can identify it with the space of $(m - p) \times p$ matrices, and therefore also with $\mathbb{R}^{(m-p)p}$ if you wish. For $f \in \hom(\mathbb{R}^p, \mathbb{R}^{m-p})$ let $\Gamma(f)$ be the graph of f, a linear subspace of $\mathbb{R}^p \times \mathbb{R}^{m-p}$ of dimension p. We now define

$$\psi_S : \hom(\mathbb{R}^p, \mathbb{R}^{m-p}) \longrightarrow M$$

by $\psi_S(f) = C_S^{-1}(\Gamma(f))$.
Now it is being claimed that the charts ψ_S, as S runs through the subsets of $\{1, 2, 3, , \dots, m\}$ with p elements, constitute a smooth atlas for M. Let us just verify the first condition, which requires that every $L \in M$ be in the image of some ψ_S. Choose a vector space basis for L, necessarily with p elements. Writing these p vectors as elements of $\mathbb{R}^m$, you obtain an $m \times p$ matrix, say A. This has rank p because it has p linearly independent columns. It will therefore also have p linearly independent rows. Make a choice of p linearly independent rows in A. This amounts to choosing a subset S of $\{1, 2, 3, \dots, m\}$ with S elements. Then the composition

$$L \xrightarrow{\text{incl.}} \mathbb{R}^m \xrightarrow{C_S} \mathbb{R}^p \times \mathbb{R}^{m-p} \xrightarrow{\text{proj.}} \mathbb{R}^p$$

is a linear isomorphism. Therefore $C_S(L) = \Gamma(f)$ for some $f \in \hom(\mathbb{R}^p, \mathbb{R}^{m-p})$, and so $L = \psi_S(f)$.
The verification of the remaining conditions for a smooth atlas is left as an exercise.

2.2 Topological matters

The most fundamental concept in topology is the concept of a topological space. It was introduced essentially as we use it today by Felix Hausdorff in 1914. (We are hoping that you are somewhat familiar with it. If that is not the case, you should view this section as a crash course.) Hausdorff's goal in formulating the definition of a topological space was, presumably, to pin down and isolate the information that we need to have in order to talk about continuity. He was a passionate set-theorist and so it is not completely surprising that his concept of a topological space, and consequently his definition of *continuity*, is free of numbers (such as epsilons and deltas). That is what makes it hard to grasp, but that is also what makes it powerful.

For a set X, let $\mathcal{P}(X)$ be the power set of X, that is, the set whose elements are all the subsets of X.

Definition 2.2.1 A *topological space* consists of a set X and a subset $\mathcal{O} \subset \mathcal{P}(X)$ with the following properties.

- If $V \in \mathcal{O}$ and $W \in \mathcal{O}$, then $V \cup W \in \mathcal{O}$.
- For any subset $\mathcal{U}$ of $\mathcal{O}$, the union $\bigcup_{V\in\mathcal{U}} V$ is an element of $\mathcal{O}$.
- $\emptyset \in \mathcal{O}$ and $X \in \mathcal{O}$.

Terminology: A subset $\mathcal{O}$ of $\mathcal{P}(X)$ which satisfies these conditions is called a *topology* on X. Once a topology $\mathcal{O}$ on X has been specified, we use the expressions *open subset of* X to mean any subset of X which is an element of $\mathcal{O}$, and *closed subset of* X for any subset C of X such that $X \smallsetminus C$ is an element of $\mathcal{O}$.

Example 2.2.2 Let (X, d) be a metric space. In detail, this means the following:

- X is a set and d is a map from $X \times X$ to $\mathbb{R}$;
- $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;
- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

The map d is then a *metric* on X. We interpret $d(x, y)$ as the *distance* from x to y. For an x in X and a real number $\varepsilon > 0$, we often write $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$. This goes under the name *open ball of radius ε about x*.
The metric d determines a set $\mathcal{O} = \mathcal{O}(d)$ of subsets of X in the following way. A subset V of X is an element of $\mathcal{O}$ if and only if, for every $x \in V$, there exists $\varepsilon > 0$ such that the set $B(x, \varepsilon)$ (the open ball of radius ε about x) is a subset of V.
Then it is easy to verify that $\mathcal{O}$ satisfies the conditions in 2.2.1 for a topology. We call $\mathcal{O}$ the topology on X *induced by the metric d*.

Example 2.2.3 Let X be any set. One boring way of making a topology $\mathcal{O}$ on X is to let $\mathcal{O} = \mathcal{P}(X)$. This is the *discrete* topology on X. Another boring way of making a topology $\mathcal{O}$ on X is to let $\mathcal{O} = \{\emptyset, X\}$. This is the *indiscrete* topology on X.

Definition 2.2.4 Let $(X, \mathcal{O})$ and $(Y, \mathcal{W})$ be topological spaces (so X and Y are sets, $\mathcal{O}$ is a topology on X and $\mathcal{W}$ is a topology on Y). A map $f : X \to Y$ is *continuous* if, for every $U \in \mathcal{W}$, the pre-image $f^{-1}(U)$ is an element of $\mathcal{O}$. (In other words: f is continuous if and only if the preimage under f of every open subset of Y is an open subset of X.)

One important justification for this definition is that it "works" for metric spaces. If X is a set with a metric d_1 , and Y is a set with a metric d_2 , and $f : X \to Y$ is a map, then the following are equivalent.

- The map f is continuous in the ε and δ sense.
- The map f is continuous in the sense of Hausdorff's definition 2.2.4, with respect to the topology $\mathcal{O}$ on X induced by the metric d_1 (see example 2.2.2) and the topology $\mathcal{W}$ on Y induced by the metric d_2 .

The verification of this equivalence ... if you have not seen it, then you should do it as part of your crash course in topology.

Definition 2.2.5 Let $(X, \mathcal{O})$ be a topological space and let $x \in X$. A subset U of X is a *neighborhood* of x if there exists an open subset V of X such that $x \in V$ and $V \subset U$.

Definition 2.2.6 Let $(X, \mathcal{O})$ and $(Y, \mathcal{W})$ be topological spaces. Let $x \in X$. A map $f : X \to Y$ is *continuous at x* if, for every neighbourhood U of $f(x)$ in Y, the pre-image $f^{-1}(U)$ is a neighborhood of x in X.

Lemma 2.2.7 Let $(X, \mathcal{O})$ be a topological space. A subset U of X is open if and only if U is a neighbourhood of x for each $x \in U$.

Proof Exercise. □

Lemma 2.2.8 Let $(X, \mathcal{O})$ and $(Y, \mathcal{W})$ be topological spaces. A map $f : X \to Y$ is continuous if and only if it is continuous at every $x \in X$.

Proof Exercise. □

Remark. We have included definitions 2.2.5 and 2.2.6 because *neighbourhood* is a very useful concept. But there is also a historical reason: Hausdorff emphasized the neighborhoods more than the open sets.

Hausdorff's concept of a topological space is particularly useful for us because it turns out that a smooth atlas on a set M determines a *topology* on M, without giving us a preferred metric on M which would induce the topology.

Definition 2.2.9 Let M be a set with a smooth atlas $\mathcal{A}$ of dimension m. The atlas determines a topology $\mathcal{O}$ on the set M in the following way. A subset W of M is an element of $\mathcal{O}$ if and only if, for every chart $\varphi : U \to M$ in $\mathcal{A}$, the set $\varphi^{-1}(W)$ is open in U. (Here U is open in $\mathbb{R}^m$, etc.)

It is easy to verify that $\mathcal{O}$ satisfies the conditions for a topology.

Lemma 2.2.10 *Keep the notation of definition 2.2.9. Let $\varphi : U \to M$ be a chart in $\mathcal{A}$. Then φ is an* open *map, and so $\varphi(U)$ is homeomorphic to U.*

Proof Exercise. (A continuous map $f : X \to Y$ between topological spaces is *open* if, for every open $W \subset X$, the image $f(W)$ is open in Y.) □

Lemma 2.2.11 *Keeping the notation of definition 2.2.9, suppose that $(Y, \mathcal{W})$ is any topological space. A map $f : M \to Y$ is continuous (w.r.t. the topology $\mathcal{O}$ on M just defined, and the topology $\mathcal{W}$ on Y) if and only if, for every chart $\varphi : U \to M$ in $\mathcal{A}$, the composition $f \circ \varphi$ is continuous.*

Proof Suppose first that f is continuous as a map from $(M, \mathcal{O})$ to $(Y, \mathcal{W})$. Let W be an open set in Y. Let $\varphi : U \to M$ be any chart in $\mathcal{A}$. By definition of $\mathcal{O}$, the map φ is continuous. Hence $f \circ \varphi$ is continuous.
Suppose next that, for every chart $\varphi : U \to M$ in $\mathcal{A}$, the map $f \circ \varphi$ is continuous. Let W be an open set in Y. Then $\varphi^{-1}(f^{-1}(W))$ is open in U for every chart $\varphi : U \to M$. Hence $f^{-1}(W)$ is open in M by our definition of $\mathcal{O}$. Therefore f is continuous. □

Next we would like to test the topological spaces which we get from a smooth atlas on a set (as in definition 2.2.9) for various properties, such as connectedness, compactness, the Hausdorff separation property and "the second countability" property. Let's first recall informally what these properties mean.

A topological space $(X, \mathcal{O})$ is *connected* if the only subsets of X which are both open and closed are $\emptyset$ and X itself. Equivalently, X is connected if every continuous map from X to a *discrete* space (see example 2.2.3) is constant.

A topological space $(X, \mathcal{O})$ is *compact* if, for every subset $\mathcal{V}$ of $\mathcal{O}$ with the property $\bigcup_{V \in \mathcal{V}} V = X$, there exists a finite subset $\mathcal{U} \subset \mathcal{V}$ such that $\bigcup_{V \in \mathcal{U}} V = X$. (A subset $\mathcal{V}$ of $\mathcal{O}$ with the property $\bigcup_{V \in \mathcal{V}} V = X$ is an *open covering* of X.)

A topological space $(X, \mathcal{O})$ has the *Hausdorff separation property* if, whenever x and y are distinct elements of X, there exist open subsets U and V of X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. (Hausdorff included this condition in his definition of a topological space. It was later deleted from the list of axioms, presumably to legalise the "Zariski topologies" which arise in algebraic geometry.)

A topological space $(X, \mathcal{O})$ satisfies the *2nd countability axiom* if there exists a countable subset $\mathcal{W} \subset \mathcal{O}$ such that every $U \in \mathcal{O}$ can be written as a union of open sets from $\mathcal{W}$ (possibly many).

Lemma 2.2.12 *Let M be a nonempty set with a smooth atlas $\mathcal{A}$ of dimension m. Suppose that, for every chart $\varphi : U \to M$ in $\mathcal{A}$, the open set $U \subset \mathbb{R}^m$ is nonempty and connected. Then the following are equivalent:*

(i) *M is connected (with the topology $\mathcal{O}$ defined in 2.2.9)*

(ii) *it is impossible to decompose $\mathcal{A}$ into a disjoint union of nonempty subsets $\mathcal{A}_0$ and $\mathcal{A}_1$ such that, for every chart $\psi_0 : U_0 \to M$ in $\mathcal{A}_0$ and ever chart $\psi_1 : U_1 \to M$ in $\mathcal{A}_1$, the intersection $\psi_0(U_0) \cap \psi_1(U_1)$ is empty.*

Proof We show (ii)$\Rightarrow$(i) and leave the other direction as an exercise. Let $f : M \to X$ be a continuous map, where X is a discrete space. Let $x \in M$. We try to decompose $\mathcal{A}$ as a disjoint union, $\mathcal{A}_0 \cup \mathcal{A}_1$, as follows. For every chart $\varphi : U \to M$ in $\mathcal{A}$, the map $f \circ \varphi : U \to X$ is continuous, hence constant, since U is connected. If the constant value is equal to $f(x)$, the chart belongs to $\mathcal{A}_0$, otherwise to $\mathcal{A}_1$. It is clear that $\mathcal{A}_0$ is not empty, since every chart whose image contains x belongs to $\mathcal{A}_0$. But since we are assuming (ii), it follows that $\mathcal{A}_1 = \emptyset$ and so $\mathcal{A}_0 = \mathcal{A}$, and so f is constant. □

Lemma 2.2.13 *Let M be a set with a smooth atlas $\mathcal{A}$ of dimension m. Suppose that $\mathcal{A}$ is finite, with charts $\varphi_i : U_i \to M$ where $i = 1, 2, 3, \ldots, r$. Then the following are equivalent:*

(i) *M is compact (with the topology $\mathcal{O}$ defined in 2.2.9)*

(ii) *there exist compact subsets* $K_i \subset U_i$ *such that*

$$\bigcup_{i=1}^{r} \varphi(K_i) = M\ .$$

Proof We show (ii)⇒(i) and leave the other direction as an exercise. The map $\varphi : U_i \to M$ is continuous and so $\varphi(K_i)$ is compact in M (images of compact sets under continuous maps are compact). Therefore $\bigcup_{i=1}^{r} \varphi^{-1}(K_i) \ = \ M$ is compact (finite unions of compact sets are compact). □

Lemma 2.2.14 *Let M be a set with a smooth atlas $\mathcal{A}$ of dimension m. Then the following are equivalent:*

(i) *M has the Hausdorff separation property (with the topology $\mathcal{O}$ defined in 2.2.9)*

(ii) *whenever $\varphi_0 : U_0 \to M$ and $\varphi_1 : U_1 \to M$ are charts in $\mathcal{A}$, and $z^{(0)}, z^{(1)}, z^{(2)}, \ldots$ is a sequence in $\varphi_0^{-1}(\varphi_1(U_1)) \subset U_0$ converging to some point in U_0 but not in $\varphi_0^{-1}(\varphi_1(U_1))$, then the sequence in U_1 obtained by applying $\varphi_1^{-1} \circ \varphi_0$ to $z^{(0)}, z^{(1)}, z^{(2)}, \ldots$ "diverges" in U_1 (has no accumulation point).*

Proof We show (ii)⇒(i). Suppose for a contradiction that there are distinct points x and y in M which do not have disjoint neighborhoods in M. Choose a chart $\varphi : U \to M$ and another chart $\psi : V \to M$ such that $x = \varphi(x')$ and $y = \psi(y')$ for some $x' \in U$ and $y' \in V$. Note that x' is not in $\varphi^{-1}(\psi(V))$ because if it were we could choose disjoint neighborhoods of $\psi^{-1}(x)$ and $\psi^{-1}(y)$ in V, and transport them back to M using ψ. For every $\varepsilon \geq 0$ we have

$$\varphi(U \cap B(x', \varepsilon)) \ \cap \ \psi(V \cap B(y', \varepsilon)) \ \neq \ \emptyset$$

because $\varphi(U \cap B(x', \varepsilon))$ is an open neighbourhood for x in M and $\psi(V \cap B(y', \varepsilon))$ is an open neighbourhood for y in M. Let $\varepsilon_i = 2^{-i}$ and choose

$$a^{(i)} \in \varphi(U \cap B(x', \varepsilon_i)) \cap \psi(V \cap B(y', \varepsilon_i))$$

for each integer $i \geq 0$. Let $z^{(i)} = \varphi^{-1}(a^{(i)}) \in U$. Then the sequence consisting of the $z^{(i)}$ converges to $x' \in U$, and the sequence consisting of the $\psi(\varphi^{-1}(z^{(i)}))$ converges to $y' \in V$. But according to our assumptions the second sequence ought to diverge. □

Lemma 2.2.15 *Let M be a set with a smooth atlas $\mathcal{A}$ of dimension m. If $\mathcal{A}$ is countable, then M satisfies the 2nd countability axiom.*

Proof This follows from the fact that every open subset of $\mathbb{R}^m$ satisfies the 2nd countability axiom. □

Example 2.2.16 Let us show that M in example 2.1.4 is connected, compact, has the Hausdorff separation property and satisfies the 2nd countability axiom.
The atlas $\mathcal{A}$ has three charts $\psi_i : \mathbb{R}^2 \to M$, where $i = 1, 2, 3$. Certainly $\mathbb{R}^2$ is connected and nonempty, and we have $\psi_1(\mathbb{R}^2) \cap \psi_2(\mathbb{R}^2) \neq \emptyset$, $\psi_2(\mathbb{R}^2) \cap \psi_3(\mathbb{R}^2) \neq \emptyset$, $\psi_3(\mathbb{R}^2) \cap \psi_1(\mathbb{R}^2) \neq \emptyset$. Therefore M is connected by lemma 2.2.12.
Next, if $\ell \in M$ is a line (through the origin in $\mathbb{R}^3$) and (x_1, x_2, x_3) is a point on ℓ, distinct from the origin, then we can choose $i \in \{1, 2, 3\}$ so that $|x_i|$ is maximal. Then $\psi_i^{-1}(\ell)$ is contained in the square

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid \max\{|x_1|, |x_2|\} \leq 1\}$$

which is a compact subset K_i of $\mathbb{R}^2$. Therefore we have

$$M = \bigcup_{i=1}^{3} \psi_i(K_i)$$

and so M is compact by lemma 2.2.13.
Next we check that condition (ii) in lemma 2.2.14 is satisfied for the two charts $\psi_1 : \mathbb{R}^2 \to M$ and $\psi_2 : \mathbb{R}^2 \to M$ together. (We ought to make the same check for ψ_1 and ψ_3 together, and also for ψ_2 and ψ_3 together, but it is the same mechanism.) We have to imagine a sequence

$$z^{(0)}, z^{(1)}, z^{(2)}, \ldots$$

in $\mathbb{R}^2$ such that the first coordinates of all the $z^{(i)}$ are nonzero, but their limit as i tends to infinity is zero, and furthermore the limit of their second coordinates as i tends to infinity exists (as a real number). Then we apply the change of chart formula $(x_1, x_2) \mapsto (x_1^{-1}, x_2 x_1^{-1})$ to each $z^{(i)}$ in the sequence. It is clear that the resulting sequence in $\mathbb{R}^2$ will diverge because the second coordinates tend to infinity in absolute value.
Finally it is obvious that lemma 2.2.15 applies, so M satisfies the 2nd countability axiom.

Example 2.2.17 Here is a simple example of a set M with a smooth atlas $\mathcal{A}$ such that the resulting topology $\mathcal{O}$ on M fails to satisfy the Hausdorff separation condition. This is superficially rather similar to example 2.1.2, but in reality it is drastically different.
Let $M = \mathbb{R} \cup \{\infty\}$. The following two charts make up a smooth atlas $\mathcal{A}$ for M, of dimension $m = 1$:

$$\begin{array}{ll} \psi_1 : \mathbb{R} \longrightarrow M \ , & \psi_1(x) = x \\ \psi_2 : \mathbb{R} \longrightarrow M \ , & \psi_2(x) = x \text{ if } x \neq 0 \text{ and } \psi_2(0) = \infty \ . \end{array}$$

Then $\psi_1^{-1}(\psi_2(\mathbb{R})) = \psi_2^{-1}(\psi_1(\mathbb{R})) = \mathbb{R} \smallsetminus \{0\}$, and the two change-of-chart maps are given by the formula $x \mapsto x$, which is clearly smooth.
It is not possible to find two disjoint open sets in M containing, respectively, 0 and ∞.

We now return to definition 2.1.1 in order to complete the definition of an abstract smooth manifold. Essentially, an abstract smooth manifold is a set M together with a smooth atlas, as in definition 2.1.1. But is is customary to add two topological conditions:

> the topological space $(M, \mathcal{O})$, with the topology $\mathcal{O}$ determined by the atlas, as in definition 2.1.1, has the Hausdorff property and satisfies the 2nd countability axiom.

Topologists also like to suppress the random features that come with the choice of a particular atlas. The following definition helps with that.

Definition 2.2.18 Let M be a set. Two smooth atlases $\mathcal{A}$ and $\mathcal{A}'$ on M, of the same dimension m, are *equivalent* if $\mathcal{A} \cup \mathcal{A}'$ is also a smooth atlas.

(Another formulation of the same: the atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are equivalent if the identity map $M \to M$ is smooth, both as a map from M with atlas $\mathcal{A}$ to M with atlas $\mathcal{A}'$, and as a map from M with atlas $\mathcal{A}'$ to M with atlas $\mathcal{A}$. See definition 2.1.6. This alternative formulation makes it perhaps clearer that the proposed relation of equivalence is in fact an equivalence relation. It also implies more directly that equivalent atlases on M determine the same topology $\mathcal{O}$ on M.)

Definition 2.2.19 A smooth manifold of dimension m consists of a set M and an equivalence class of smooth atlases of dimension m on M, such that the resulting topology on M satisfies the Hausdorff separation axiom and the 2nd countability axiom.

Remark. We saw in example 2.1.5 that a smooth manifold M in $\mathbb{R}^n$ (definition 1.3.2), determines a smooth atlas $\mathcal{A}$ on M. The resulting topology on M (definition 2.2.9) agrees with the subspace topology (which in turn is induced by the usual metric, as in definition 1.1.2). It follows that this topology satisfies the Hausdorff condition and the 2nd countability axiom.

Remark. Most of the examples of smooth manifolds listed earlier have names. Example 2.1.2 is (diffeomorphic to) S^1, example 2.1.3 is (diffeomorphic to) the m-sphere S^m , example 2.1.4 is the real projective plane $\mathbb{R}P^2$, example 2.1.9 is the real projective space $\mathbb{R}P^n$ and example 2.1.10 is the Grassmann manifold $\mathrm{Gr}(p, m)$.

2.3 Open submanifolds and products

Example 2.3.1 Every open subset W of an (abstract) smooth manifold M of dimension m inherits the structure of an abstract smooth manifold of the same dimension. Namely, choose one of the allowed atlases for M, say $\mathcal{A}$. Then every chart $\psi : U \to M$ (with U open in $\mathbb{R}^m$) in $\mathcal{A}$ determines $\psi|U_W : U_W \to W$ where $U_W = \psi^{-1}(W) \subset U$. Here U_W is open in U and hence in $\mathbb{R}^m$, by our hypothesis on W. Therefore we obtain a smooth atlas $\mathcal{A}|W$ on W made up of charts of the form $\psi|U_W : U_W \to W$, with $\psi : U \to M$ in $\mathcal{A}$. The equivalence class of $\mathcal{A}|W$ depends only on the equivalence class of $\mathcal{A}$.

Example 2.3.2 Let M be a smooth manifold of dimension m and let N be a smooth manifold of dimension n. Then $M \times N$ (the product set) comes with a preferred structure of smooth manifold of dimension $m+n$. The corresponding topology on $M \times N$ is, as one might expect, the product topology (determined by the topologies on M and N, respectively, which we already have thanks to definition 2.2.9).
Details: Choose one of the allowed atlases for M, say $\mathcal{A}$, and choose one of the allowed atlases for N, say $\mathcal{B}$. For every chart $\varphi : U_1 \to M$ in $\mathcal{A}$ and every chart $\psi : U_2 \to N$ in $\mathcal{B}$ we obtain a map

$$U_1 \times U_2 \longrightarrow M \times N \; ; \qquad (x,y) \mapsto (\varphi(x), \psi(y))$$

which can serve as one chart in an atlas on $M \times N$. The equivalence class of that atlas depends only on the equivalence classes of $\mathcal{A}$ and $\mathcal{B}$.

2.4 Group actions and orbit manifolds

We recall the notion of a group action. Let G be a group. We write 1 for the neutral element in G, and $g_1 g_2$ or $g_1 \cdot g_2$ for the product of g_1 and g_2 in G.
An action of G on a set X is a map

$$\alpha : G \times X \to X$$

such that $\alpha(1,x) = x$ for all $x \in X$, and $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and all $x \in X$. It is customary to write gx instead of $\alpha(g,x)$.
Let $\mathrm{per}(X)$ be the group of permutations of X, that is, the group of all bijective maps from X to X, with composition of maps as the binary operation. An action α of G on X determines a group homomorphism $G \to \mathrm{per}(X)$ by the formula

$$g \mapsto (x \mapsto gx).$$

Conversely, a group homomorphism $\gamma : G \to \mathrm{per}(X)$ determines an action of G on X by the formula $\alpha(g,x) := \gamma(g)(x)$. In this way, actions of G on X correspond to group homomorphisms $G \to \mathrm{per}(X)$.
The orbits of an action (of G on X) are the equivalence classes of the relation "$\sim$" on X defined by: $x \sim y$ iff $x = gy$ for some $g \in G$. We sometimes write X/G for the set of orbits.

Example 2.4.1 Let the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers act on the set of all nonzero vectors in $\mathbb{C}^2$ by scalar multiplication. The set of orbits of this action is the set of 1-dimensional linear subspaces of $\mathbb{C}^2$.

Definition 2.4.2 Let M be a smooth manifold of dimension m. Let G be a group. An action α of G on M is *smooth* if, for every $g \in G$, the map from M to M defined by $x \mapsto gx = \alpha(g,x)$ is smooth.

A smooth action of G on M determines a homomorphism from G to the *diffeomorphism group* of M, that is, the group consisting of all diffeomorphisms $M \to M$.

Things get more complicated when the group G itself comes with the structure of a smooth manifold of dimension > 0, like $\mathrm{O}(m)$ in example 1.4.4. For the moment we want avoid this situation, so that our examples of groups tend to be "discrete".

Example 2.4.3 The group with two elements 1 and t acts smoothly on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ (example 1.3.4) by $tx = -x$ for $x \in S^{n-1}$. This action is often called the *antipodal action*, as $-x$ is the "antipode" of x in S^{n-1}.

Example 2.4.4 Let H be any finite subgroup of the special orthogonal group $\mathrm{SO}(3)$, the group of orthogonal 3×3-matrices with determinant $+1$. Then H acts smoothly on $\mathrm{SO}(3)$ by "translation", that is, $\alpha(A,B) = AB$ for $A \in H$ and $B \in \mathrm{SO}(3)$. The smooth manifold structure on $\mathrm{SO}(3)$ comes from example 1.4.4.
There are some interesting finite subgroups of $\mathrm{SO}(3)$ associated with the platonic solids (tetrahedron, cube, octahedron, dodecahedron, icosahedron). If we place one of these platonic solids in 3-space, with (bary)center at the origin, then the group of matrices $A \in \mathrm{SO}(3)$ which map the solid to itself is a finite subgroup H of $\mathrm{SO}(3)$. In the case of the icosahedron or the dodecahedron, it has 60 elements.

Example 2.4.5 Fix an integer $q > 1$. Let G be the set of q-th roots of unity in $\mathbb{C}$. This is a subset of $\mathbb{C}$ with q elements, and we can write it as

$$G = \{t^0, t^1, t^2, \ldots, t^{q-1}\}$$

where $t = \cos(2\pi/q) + i\sin(2\pi/q)$. Under complex multiplication G turns into a group (which is cyclic, with generator t). Now let S^{2n-1} be the unit sphere in the complex vector space $\mathbb{C}^n$,

$$S^{2n-1} = \{z \in \mathbb{C}^n \mid \sum_{r=1}^{n} |z|^2 = 1\}.$$

(Normally we say $S^{2n-1} \subset \mathbb{R}^{2n}$, but when you split each coordinate $z_r \in \mathbb{C}$ of $z \in \mathbb{C}^n$ into its real and imaginary part, you will see that it is much the same thing.) The group G acts smoothly on S^{2n-1} by

$$t^r \cdot (z_1, z_2, \ldots, z_n) = (t^r z_1, t^r z_2, \ldots, t^r z_n)$$

for $z = (z_1, z_2, \ldots, z_n) \in S^{2n-1} \subset \mathbb{C}^n$.

Example 2.4.6 Keeping the notation of the previous example, choose integers $(\ell_2, \ell_3, \ldots, \ell_n)$, all relatively prime to q. Then a smooth action of G on S^{2n-1} can be defined by

$$t^r \cdot (z_1, z_2, \ldots, z_n) = (t^r z_1, t^{r\ell_2} z_2, t^{r\ell_3} z_3, \ldots, t^{r\ell_n} z_n) \ .$$

Let's return to the general case of a group G acting smoothly on a smooth m-dimensional manifold M. We add a technical assumption:

> *Every x in M admits a neighbourhood W such that $gW \cap W = \emptyset$ whenever $g \in G$, $g \neq 1$.*
> (A smooth action with these properties is called *free and properly discontinuous.*)

In these circumstances the set of orbits M/G has a preferred structure of smooth m-dimensional manifold. This is not hard to understand. Choose one of the allowed atlases $\mathcal{A}$ for M. Let $p : M \to M/G$ be the projection. For every $x \in M$ choose an open neighborhood W_x of x in M such that $gW_x \cap W_x = \emptyset$ whenever $g \in G$, $g \neq 1$. Choose also a chart $\varphi_x : U_x \to M$ in the atlas $\mathcal{A}$ such that $x \in \varphi_x(U_x)$. Making the atlas larger, and making U_x and/or W_x smaller, we can assume $\varphi(U_x) = W_x$. Then the composition $\psi_x = p \circ \varphi_x$ is an injective map from $U_x \to M/G$. Taking all the ψ_x together, we have a smooth atlas for M/G. (Duplicate charts should be deleted.)

Lemma 2.4.7 *In the above circumstances, the projection $M \to M/G$ is smooth and locally diffeomorphic. That is, every $x \in M$ admits an open neighborhood W_x such that $p|W_x$ is a diffeomorphism from W_x to its image $p(W_x)$, an open set in M/G.*

This is obvious from the construction.

Remark. The smooth actions described in examples 2.4.3, 2.4.4, 2.4.5 and 2.4.6 are all free and properly discontinuous. The corresponding orbit manifolds M/G sometimes have "names": in example 2.4.3 the orbit manifold is projective $(n-1)$-space which we have met before, in examples 2.4.5 and 2.4.6 it is a *lens space* (please do not ask why). In the icosahedron case of example 2.4.4, the orbit manifold is the *Poincaré homology sphere*, for reasons which will perhaps be revealed in a later chapter. (In any case Poincaré was the first to think of this example, and it was important in the genesis of the Poincaré conjecture which was solved just recently.)

SMSTC (2008/09)
Geometry and Topology
Lecture 3: The tangent bundle of a smooth manifold

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Contents

3.1 Tangent vectors

We begin by developing the idea of "velocity vectors" of particles moving about in smooth manifolds.

Definition 3.1.1 Let M be a smooth manifold of dimension k in $\mathbb{R}^n$. Let $\varphi : U \to V$ be an ambient chart, so that U and V are open in $\mathbb{R}^m$, the map φ is a diffeomorphism, and $\varphi(U \cap \mathbb{R}^k) = V \cap M$. Let $y \in U$ and $z = \varphi(y) \in M$. The image of $\mathbb{R}^k \subset \mathbb{R}^n$ under the linear isomorphism $D\varphi(y) : \mathbb{R}^n \to \mathbb{R}^n$ is called the *tangent space* of M at z and denoted by T_zM. It is a k-dimensional linear subspace of $\mathbb{R}^n$. Elements of T_zM are called *tangent vectors* to M at z.

Remark. It does not matter which ambient chart $\varphi : U \to M$ with $z \in \varphi(U)$ you use to determine the linear subspace $T_zM \subset \mathbb{R}^n$. (Prove it.)

Example 3.1.2 Take $M = S^{n-1}$, the unit sphere in $\mathbb{R}^n$. For $z \in S^{n-1}$, we have

$$T_zS^{n-1} = \{v \in \mathbb{R}^n \mid \sum_{j=1}^{k} z_j v_j = 0\} \ .$$

Returning to the notation of definition 3.1.1, suppose that $J \subset \mathbb{R}$ is an open interval with $0 \in J$ and let $\gamma : J \to \mathbb{R}^n$ be a smooth curve such that $\gamma(t) \in M$ for all $t \in J$, and $\gamma(0) = z$. Let's write $\gamma'(0)$ for the velocity vector of γ at 0. (This is unfortunately not entirely consistent with section 1.2, where you were told that $\gamma'(0)$ is a linear map from $\mathbb{R}$ to $\mathbb{R}^n$.)

Lemma 3.1.3 *In this situation,* $\gamma'(0) \in T_zM$.

Proof For all t in a sufficiently short open interval $J_1 \subset J$ containing 0, we can write $\gamma(t) = \varphi(\beta(t))$. Here $\varphi : U \to V$ is an ambient chart with $z = \varphi(y)$ for some $y \in U$ and $\beta : J_1 \to U$ is a smooth curve in U. Then β runs in $U \cap \mathbb{R}^k$ and so $\beta'(0) \in \mathbb{R}^k \subset \mathbb{R}^n$. Also, $\beta(0) = y$. By the chain rule, $\gamma'(0) = D\varphi(y)(\beta'(0))$ which belongs to $D\varphi(y)(\mathbb{R}^k) = T_zM$. $\square$

Example 3.1.4 Let $\gamma : J \to \mathbb{R}^n$ be a smooth curve (defined on an open interval $J \subset \mathbb{R}$) such that $\|\gamma(t)\| = 1$ for all $t \in [a, b]$. Then by example 3.1.2 and lemma 3.1.3, the dot product $\gamma'(t) \cdot \gamma(t)$ is 0 for all $t \in [a, b]$. Another proof of this fact: differentiate the equation $\gamma(t) \cdot \gamma(t) = 1$.

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Next, we must try to imitate "all the above" for abstract smooth manifolds. So let M be an abstract smooth manifold of dimension k. Let $z \in M$.

Definition 3.1.5 A tangent vector to M at z is a rule v which to every chart $\varphi : U \to M$ (in any allowed atlas for M) with $z = \varphi(y) \in \varphi(U)$ assigns a vector $v_\varphi \in \mathbb{R}^k$, subject to the condition

$$v_\psi = D(\psi^{-1} \circ \varphi)(y)(v_\varphi).$$

(Here $\psi : U' \to M$ is another chart with $z \in \psi(U')$ and $D(\psi^{-1} \circ \varphi)(y)$ is the differential of $\psi^{-1} \circ \varphi$ at y, a linear isomorphism $\mathbb{R}^k \to \mathbb{R}^k$.)

Remark. You can "construct" a tangent vector v to M at z by choosing some chart $\varphi : U \to M$ such that $z \in \varphi(U)$, say $z = \varphi(y)$, and some vector in $\mathbb{R}^k$ to be called $v_\varphi \in \mathbb{R}^k$. For all those other charts $\psi : U' \to M$ with $z \in \psi(U')$, *define* v_ψ in terms of v_φ using that formula,

$$v_\psi = D(\psi^{-1} \circ \varphi)(y)(v_\varphi).$$

Example 3.1.6 Let $\gamma : J \to M$ be a smooth curve, where $J \subset \mathbb{R}$ is an open interval containing 0. (See definition 2.1.6.) Suppose that $\gamma(0) = z$. For every chart $\varphi : U \to M$ (in any allowed atlas for M) such that $z \in \varphi(U)$, let

$$v_\varphi = (\varphi^{-1} \circ \gamma)'(0) .$$

Then it is easy to verify, using the chain rule, that $v_\psi = D(\psi^{-1} \circ \varphi)(y)(v_\varphi)$, assuming $z = \varphi(y)$ and $\psi : U' \to M$ is another chart etc. Therefore the assignment $\varphi \mapsto v_\varphi$ constitutes a tangent vector v to M at z. We regard that as the velocity vector of γ at 0.

Proposition 3.1.7 *The tangent vectors to M at z form a real vector space whose dimension is equal to that of M.* (This is the abstract tangent space to M at z, denoted by T_zM.)

Proof Let v and w be tangent vectors to M at x. We define their sum by $(v + w)_\varphi = v_\varphi + w_\varphi$. Scalar multiplication with a real number c is defined by $(cv)_\varphi = c \cdot v_\varphi$. Hence the tangent vectors to M at x form a real vector space. To show that this has dimension $k = \dim(M)$, we fix a particular chart $\varphi : U \to M$ with $x \in U$. By "evaluating" a tangent vector to M at x on the chart φ, we obtain a linear map from the vector space of all those tangent vectors to $\mathbb{R}^k$. The map is bijective by the remark following definition 3.1.5. It follows that the two vector spaces have the same dimension. $\square$

Remark. In the case of a smooth manifold M in $\mathbb{R}^n$, we now have two definitions of T_zM for a point $z \in M$, because a smooth manifold in $\mathbb{R}^n$ is also an "abstract" smooth manifold according to 2.1.5. These two definitions are related by a preferred linear isomorphism. (Unravel this.)

3.2 The tangent bundle

Proposition 3.2.1 *Let M be a smooth manifold of dimension k in $\mathbb{R}^n$. Then the set*

$$TM = \{(z, w) \in \mathbb{R}^n \times \mathbb{R}^n \mid z \in M \ , \ w \in T_zM\}$$

is a smooth manifold of dimension $2k$ in $\mathbb{R}^n \times \mathbb{R}^n$. Note: this uses the definition 3.1.1 of T_zM as a linear subspace of $\mathbb{R}^n$.

Proof Let $\varphi : U \to V$ be an ambient chart for M in $\mathbb{R}^n$. So U and V are open subsets of $\mathbb{R}^m$ and φ is a diffeomorphism from U to V, taking $U \cap \mathbb{R}^k$ to $V \cap M$. Define $T\varphi : U \times \mathbb{R}^n \to V \times \mathbb{R}^n$ by

$$T\varphi(y, w) = (\varphi(y), D\varphi(y)(w))$$

where $D\varphi(y) : \mathbb{R}^n \to \mathbb{R}^n$ is the differential of φ at y, a linear isomorphism. We want to show that $T\varphi$ is an ambient chart for TM in $\mathbb{R}^n \times \mathbb{R}^n$, with $T\varphi(U \times \mathbb{R}^n)$ containing (obviously) all elements of TM of the form (z, w), for $z \in \varphi(U)$ and arbitrary $w \in T_xM$.
The map $T\varphi$ is a diffeomorphism, with smooth inverse defined by the formula

$$(y, v) \mapsto (\varphi^{-1}(y), D(\varphi^{-1})(y)(v)).$$

For $(y, v) \in U \times \mathbb{R}^n$ we have

$$T\varphi(y, v) \in TM \quad \Longleftrightarrow \quad y \in \mathbb{R}^k,\ v \in \mathbb{R}^k$$

so that $T\varphi$ takes $(U \times \mathbb{R}^n) \cap \mathbb{R}^k \times \mathbb{R}^k$ bijectively to $(V \times \mathbb{R}^n) \cap TM$. Here we regard $\mathbb{R}^k \times \mathbb{R}^k$ as a linear subspace of $\mathbb{R}^n \times \mathbb{R}^n$ using the inclusion $\mathbb{R}^k \to \mathbb{R}^n$ on each factor. $\square$

Proposition 3.2.2 *Let M be a smooth manifold of dimension k. Then the set*

$$TM = \{(z, w) \mid z \in M\ ,\ w \in T_z M\}$$

comes with a preferred structure of smooth manifold of dimension $2k$.

Proof We start by choosing an allowed smooth atlas $\mathcal{A}$ for M. Take a chart $\varphi : U \to M$ in $\mathcal{A}$. So U is open in $\mathbb{R}^k$. Let

$$TU = U \times \mathbb{R}^k$$

and define $T\varphi : TU \to TM$ so that $T\varphi(y, v_\varphi) = (z, v)$ for $y \in U$, $z = \varphi(y)$ and $v \in T_z M$. It is easy to check that the maps $T\varphi$ (as φ runs through the atlas $\mathcal{A}$) make up an atlas for TM. $\square$

Proposition 3.2.3 *The projection map $TM \to M$ defined by $(x, v) \mapsto x$ is smooth.*

Proof Take a chart $\varphi : U \to M$ for M and the corresponding chart $T\varphi : TU \to TM$ for TM (in the notation of the proof of 3.2.2). Then the composition

$$\begin{tikzcd}
TU \arrow[d, "T\varphi"'] & U \\
T\varphi(TU) \arrow[r, "\text{proj.}"] & \varphi(U) \arrow[u, "\varphi^{-1}"']
\end{tikzcd}$$

agrees with the projection from $TU = U \times \mathbb{R}^k$ to U , which is certainly a smooth map (from an open set in $\mathbb{R}^k \times \mathbb{R}^k$ to an open set in $\mathbb{R}^k$). $\square$

Remark. The projection map $TM \to M$ is called the *tangent bundle of* M. The proper expression for TM is *the total space of the tangent bundle of* M. That is often shortened to *the tangent bundle of* M (again).

Let now $f : M \to N$ be a smooth map, where $\dim(M) = m$ and $\dim(M) = n$. Let $x \in M$. We will see that f has a *differential* $Df(x)$ at x, which is a linear map from $T_x M$ to $T_{f(x)} N$.
Choose a chart $\varphi : U_1 \to M$ in an allowed atlas for M, with $x \in \varphi(U_1)$, and choose a chart $\psi : U_2 \to M$ in an allowed atlas for N, with $f(x) \in \psi(U_2)$. (So U_1 is open in $\mathbb{R}^m$, while U_2 is open in $\mathbb{R}^n$.) We define $Df(x)$ as the composition of the linear maps

$$\begin{tikzcd}
T_x M \arrow[d, "v \mapsto v_\varphi"'] & T_{f(x)} N \\
\mathbb{R}^m \arrow[r] & \mathbb{R}^n \arrow[u, "(w \mapsto w_\psi)^{-1}"']
\end{tikzcd}$$

where the left-hand vertical arrow is the linear isomorphism $T_x M \to \mathbb{R}^m$ associated with the chart φ, the right-hand vertical arrow is the inverse of the analogous isomorphism $T_{f(x)} N \to \mathbb{R}^n$ associated with the chart ψ, and the horizontal arrow is the differential of $\psi^{-1} \circ f \circ \varphi$ at $\varphi^{-1}(x)$. The chain rule implies that $Df(x)$ so defined is well defined, i.e., independent of the choice of charts φ and ψ.

Proposition 3.2.4 *The map $Tf : TM \to TN$ defined by $(x, v) \mapsto (f(x), Df(x)(v))$ is smooth in the sense of definition 2.1.6.*

Proof Let $\varphi : U \to M$ be a chart in an allowed smooth atlas for M and let $\psi : V \to M$ be a chart in an allowed smooth atlas for N. Then $T\varphi : TU \to TM$ and $T\psi : TV \to N$ are typical charts for TM and TN. Let $W = \varphi^{-1}(f^{-1}(\psi(V)))$, an open subset of U. Let $g : W \to V$ be defined by $x \mapsto \psi^{-1}(f(\varphi(x)))$. This is smooth by assumption on f. Then

$$(T\varphi)^{-1}((Tf)^{-1}((T\psi)(TV))) = W \times \mathbb{R}^m = TW\ ,$$

which is an open subset of TU. The map $TW \to TV$ for which we have to establish smoothness is just $Tg : TW \to TV$, in a formula, $Tg(y, v) = (g(y), Dg(y)(v))$. This is certainly smooth. $\square$

SMSTC (2008/09)
Geometry and Topology
Lecture 4: Mechanical systems and Lagrangian functions

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4.1 Mechanical systems and their configuration spaces

In the description of a mechanical system obeying Newton's laws of motion, a distinction is made between the possible *configurations* of the system and the possible *kinematical states* of the system. To describe a configuration of the system, at time t_0 say, we need to know where all the particles in the system are at time t_0. To describe a kinematical state of the system at time t_0, we need to know the positions *and the velocities* of all the particles in the system at time t_0. Knowledge of the configuration of the system at time t_0 is not enough to predict or reconstruct the configurations of the system at other times t. By contrast, knowledge of the kinematical state of the system at time t_0 *is enough* to predict and reconstruct the kinematical states (and hence the configurations) of the system at all other times t, or at least for all t in an open interval about t_0 (in the case of "explosive" evolution of the system). This may be regarded as an empirical truth, but mathematically speaking it is a consequence of the fact that Newton's second law of motion is a second order ODE, not a first order ODEs. In any case it indicates that the kinematical states of the system are "more important" than the configurations.
The possible configurations of the system make up the *configuration space* and the possible kinematical states make up what is called the *phase space* (or more precisely the *velocity phase space*). In most examples the configuration space is a smooth m-dimensional manifold M, for some m. The phase space is the TM. Therefore a study of some mechanical systems can give you a good feeling for what TM means.

Example 4.1.1 A weightless rod of length 1 metre connects two massive particles of masses 1kg and 2kg, respectively. Describe the possible motions of the rod in a gravitational field acting vertically downwards at 9.81m/s^2.

What is the configuration space of this system ? A configuration of the system is determined by the positions of the two massive particles. These two positions are subject to a condition, that of having distance 1 from each other. Therefore the configuration space is

$$\{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|x-y\| = 1\}.$$

This is not just a subspace of $\mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6$, but in fact a smooth manifold of dimension 5 in $\mathbb{R}^6$.

A different but equally correct answer can be given. To describe a configuration of the system we need to know the position of the particle of mass 1kg, and the unit vector which describes the direction of the rod. Therefore the configuration space is

$$\mathbb{R}^3 \times S^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$$

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where $S^2 \subset \mathbb{R}^3$ is the unit sphere.

Example 4.1.2 A pendulum consists of a taut string of length 1m, one end of which is fixed at the origin of $\mathbb{R}^2$, while the other end carries a bob of weight 1kg (and zero diameter). Gravity acts downwards at the usual 9.81 m/s^2. (Think of the x_1 axis as "horizontal" and of the x_2 axis as "vertical".)

The configuration space is just $S^1 \subset \mathbb{R}^2$.

Example 4.1.3 Similar to the previous example, but in addition the point where the pendulum is suspended now has a mass of 5kg and is allowed to move freely along the x_1 axis.

The configuration space is $\mathbb{R}^1 \times S^1$.

Example 4.1.4 Like example 4.1.2, but assume in addition that the taut string is suspended from a point which undergoes a (forced) periodic motion along the x_1 axis, so as to be $\sin(\omega t)$ to the right of the origin at time t (for a fixed $\omega > 0$).

The configuration space is again S^1.

Example 4.1.5 A stone flies through through the air under the influence of gravity. Assume that gravity acts vertically downwards at 9.81m/s^2, and that the stone consists of a homogeneous material (i.e., the specific mass is constant in the stone). Describe the possible motions of the stone, neglecting air resistance.

Here it is best to choose a coordinate system *within the stone*: origin at the centre of mass of the stone, and some orthonormal basis, fixed relative to the stone. The possible positions of the stone are then described by where the centre of mass is and by how those basis vectors (internal to the stone) are positioned. Therefore the configuration space of the system is

$$\mathbb{R}^3 \times \mathrm{SO}(3) \ .$$

We can think of that as a smooth 6-dimensional manifold in $\mathbb{R}^3 \times \mathbb{R}^9 = \mathbb{R}^{12}$.

4.2 Lagrangian functions

One of the fastest and best ways of modelling a mechanical system mathematically is to determine its configuration space, usually a smooth manifold M, and then to determine the *Lagrangian function* of the system,

$$L : TM \times \mathbb{R} \longrightarrow \mathbb{R} \ .$$

Here TM is of course the (total space of the) tangent bundle of M and the Lagrangian function typically has the form "kinetic energy minus potential energy". The factor $\mathbb{R}$ on the left is "time", allowing for the possibility that the expressions for the potential and kinetic energy depend on time. We shall see many examples in a moment. The solution curves of the system, i.e., smooth maps $\gamma : J \to M$ (where $J \subset \mathbb{R}$ is an open interval) which describe "physically realistic" evolutions of the system in time, are then characterised by an extremal property which involves the function L. Namely, $\gamma : J \to M$ is a solution curve if and only if it "minimises" the so-called action integral $\int L(\gamma(t), \gamma'(t), t)\ dt$. This is the content of Hamilton's *principle of least action*.
More precisely, we need to consider subintervals $[a, b] \subset J$ with a and b not too far apart, and smooth curves $\kappa : [a, b] \to M$ for which $\kappa(a) = \gamma(a)$ and $\kappa(b) = \gamma(b)$. On the "space" of such smooth curves κ we define a function Φ by

$$\Phi(\kappa) = \int_a^b L(\kappa(t), \kappa'(t), t)\ dt \ .$$

The function Φ turns out to be differentiable, in a sense which we need not make precise here. The correct way to state the "minimising" condition on γ in the interval $[a, b]$ is to say that the differential of Φ at $\gamma\big|[a, b]$ has to be zero. (Remember that we usually minimise values of differentiable functions by looking for critical points, i.e. for points where the differential of the function is zero.) A branch of mathematics called *variational calculus* converts this condition on γ into second order differential equations for γ.
We will not be concerned with the variational calculus. Instead we will just determine the Lagrangian functions attached to the mechanical systems listed in section 4.1 and discuss how the Lagrangian function of a mechanical system can be understood in local coordinates, i.e., in charts.

Example 4.2.1 In the case of example 4.1.1, the Lagrangian function is

$$L((x,y),(v,w),t) \quad = \quad 0.5(\|v\|^2 + 2\|w\|^2) - 9.81(x+2y) \ .$$

Here (x, y) is a point in the configuration manifold $M = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|x - y\| = 1\}$ and (v, w) is a "velocity vector" in the tangent space $T_{(x,y)}M$. The term $0.5(\|v\|^2 + 2\|w\|^2)$ is the kinetic energy, whereas $9.81(x + 2y)$ is the potential energy. We are thinking in metre/kilogramm/second units. The lagrangian function is time-independent.

Example 4.2.2 In the case of the planar pendulum, example 4.1.2 with configuration manifold $S^1 \subset \mathbb{R}^2$, the Lagrangian function is

$$L(x,v,t) \quad = \quad 0.5\|v\|^2 - 9.81x_2 \ .$$

Here $x \in S^1 \subset \mathbb{R}^2$ with second (vertical) coordinate x_2 , and $v \in T_xS^1$ is a "velocity vector". The Lagrangian function is time-independent.

Example 4.2.3 For the wandering pendulum, example 4.1.3 with configuration manifold $\mathbb{R}^1 \times S^1$ in $\mathbb{R}^1 \times \mathbb{R}^2$, the Lagrangian function is

$$L(x,y,v,w,t) \quad = \quad 3v^2 + 0.5\|w\|^2 + vw_1 - 9.81y_2 \ .$$

Here $x \in \mathbb{R}^1$ and $y \in S^1 \subset \mathbb{R}^2$ describe the configuration, while $v \in \mathbb{R}$ and $w \in T_yS^1 \subset \mathbb{R}^2$ describe the rate of change of x and y, respectively. The actual velocity of the pendulum bob is not $w = (w_1, w_2)$, but rather $(w_1 + v, w_2)$. That is why the total kinetic energy is

$$\frac{5v^2 + (w_1+v)^2 + w_2^2}{2} = \frac{6v^2 + \|w\|^2 + 2vw_1}{2}.$$

The Lagrangian function is time-independent.

Example 4.2.4 For the forced pendulum, example 4.1.4 with configuration manifold $S^1 \subset \mathbb{R}^2$, the Lagrangian function is

$$L(y,w,t) \quad = \quad 0.5\big(\|w\|^2 + 2\varepsilon w_1 \cos(\varepsilon t) + \varepsilon^2 \cos^2(\varepsilon t)\big) \ - \ 9.81y_2 \ .$$

The Lagrangian function is time-dependent. The complicated expression for the kinetic energy comes about in the following way. If the velocity vector of the bob relative to the configuration space S^1 is $w = (w_1, w_2) \in \mathbb{R}^2$ at time t, then the true velocity of the bob moving in $\mathbb{R}^3$ is $(w_1 + \cos(\varepsilon t), w_2)$.

Example 4.2.5 For the stone, example 4.1.5, the Lagrangian function is

$$L((x,A),(v,B),t) \quad = \quad \text{mass} \cdot \left(\iiint_{\text{stone}} \frac{1}{2\,\text{vol}} \ \|B(w)+v\|^2 \ dw_1dw_2dw_3 \quad - \quad 9.81x_3 \right) \ .$$

Here "mass" is the mass of the stone and "vol" is its volume. The data $x \in \mathbb{R}^3$ and $A \in \mathrm{SO}(3)$ describe the whereabouts of the stone in configuration space. In particular, A is a 3×3-matrix or a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$. The data v and B describe a tangent vector to $\mathbb{R}^3 \times \mathrm{SO}(3)$ in $\mathbb{R}^3 \times \mathbb{R}^{3\cdot 3}$. In particular $v \in \mathbb{R}^3$ is a velocity vector for the centre of mass of the stone and $B \in \mathbb{R}^{3\cdot 3}$ is a tangent vector to $\mathrm{SO}(3)$ at $A \in \mathrm{SO}(3)$, hence also a 3×3 matrix or a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$. The expression $B(w) + v$ describes the resulting velocity vector of a point in the stone, where $w = (w_1, w_2, w_3)$ are the coordinates of that point in the stone's own coordinate system. (In the expression $B(w)$, think of B as a linear map.) The quotient mass/vol is the specific mass of the stone. Therefore

$$\iiint_{\text{stone}} \frac{\text{mass}}{2\,\text{vol}} \|B(w)+v\|^2 \ dw_1dw_2dw_3$$

is the correct expression for the kinetic energy of the stone. More obviously, mass $\cdot\, 9.81x_3$ is the correct expression for the potential energy of the stone. The Lagrangian function is time-independent.
Some simplifications can be made in the kinetic energy expression. We have

$$\|B(w)+v\|^2 = \|B(w)\|^2 \ + \ 2B(w)\cdot v \ + \ \|v\|^2 \ .$$

When we integrate over the stone, the term $2B(w)\cdot v$ contributes zero because

$$\iiint_{\text{stone}} B(w)\cdot v \quad dw_1dw_2dw_3 \quad = \quad B\left(\iiint_{\text{stone}} w \quad dw_1dw_2dw_3\right)\cdot v \quad = B(0)\cdot v$$

because $\iiint w \quad dw_1dw_2dw_3$ is the centre of mass of the stone in the stone's own coordinate system. Therefore

$$\iiint \frac{1}{2\,\mathrm{vol}}\|B(w)+v\|^2 \; dw_1dw_2dw_3 \quad = \quad \iiint \frac{1}{2\,\mathrm{vol}}\|B(w)\|^2 \; dw_1dw_2dw_3 \quad + \quad \frac{1}{2}\|v\|^2 \;.$$

It is also useful to observe that $\|B(w)\|^2 = \|A^{-1}B(w)\|^2$ because A^{-1} is an orthogonal matrix. The advantage of that is that $A^{-1}B$ is now in the tangent space to $\mathrm{SO}(3)$ at $A^{-1}A = I_3$. So we get

$$L((x,A),(v,B),t) \quad = \quad \mathrm{mass}\cdot\left(\iiint_{\text{stone}} \frac{1}{2\,\mathrm{vol}}\|A^{-1}B(w)\|^2 \; dw_1dw_2dw_3 \quad + \quad \frac{1}{2}\|v\|^2 \quad - \quad 9.81x_3\right) \;.$$

Not finished yet ! Let's also observe that the complicated-looking integral expression is in fact a homogeneous quadratic polynomial (no constant terms, no linear terms) in the entries of $A^{-1}B$. So we can write it uniquely as $J_{\text{stn}}(A^{-1}B, A^{-1}B)$ where

$$J_{\text{stn}} : T_{I_3}\mathrm{SO}(3)\times T_{I_3}\mathrm{SO}(3) \longrightarrow \mathbb{R}$$

is a symmetric bilinear map. This is often called the *inertia tensor* (of the stone). Then

$$L((x,A),(v,B),t) \quad = \quad \mathrm{mass}\cdot\left(J_{\text{stn}}(A^{-1}B, A^{-1}B) \quad + \quad \frac{1}{2}\|v\|^2 \quad - \quad 9.81x_3\right) \;.$$

Choosing a basis for $T_{I_3}\mathrm{SO}(3)$ (with three elements), we also get a basis for the vector space of symmetric bilinear maps $T_{I_3}\mathrm{SO}(3)\times T_{I_3}\mathrm{SO}(3) \longrightarrow \mathbb{R}$ (with six elements). Then J_{stn} is given by six real numbers, which depend on the shape of the stone and also on our choice of an orthonormal basis in the stone. These six numbers are all we need to know about the stone in order to write down the Lagrangian function (except for the constant factor "mass" which can be dropped anyway). Consequently these six numbers also determine the way in which the stone moves.
It is an interesting exercise in linear algebra to show that for every stone there exists a rectangular brick, also equipped with an internal orthonormal coordinate system (which need not be lined up with the faces of the brick) such that $J_{\text{stn}} = J_{\text{brk}}$. Then the stone and the brick move in the same way.

Hamilton's principle of least action characterises the physically realistic evolutions of a mechanical system in a way which is "local". To be more precise, let M be the configuration space of the mechanical system, a smooth manifold of dimension k in $\mathbb{R}^n$. Let $J\subset\mathbb{R}$ be an open interval and let $\gamma: J\to M$ be a smooth curve. To test whether γ describes a realistic evolution of the system, we divide J into many short intervals of the form $[a,b]$ and ask whether each restriction $\gamma\big|[a,b]$ "minimises" the action integral

$$\int_a^b L(\gamma(t),\gamma'(t),t)\;dt$$

where L is the Lagrangian function of the system. (The word "minimises" must be interpreted generously, as explained earlier.) We are free to make the intervals $[a,b]$ quite short, and then we may also assume that the curve $\gamma\big|[a,b]$ proceeds in an open subset of M which has the form $V=\psi(U)$ for some chart $\psi: U\to M$ in an allowed atlas for M. So we are no longer concerned with M and $L: TM\times\mathbb{R}\to\mathbb{R}$, but rather with $V=\psi(U)$ and the restriction of L to $TV\times\mathbb{R}$. Furthermore it seems that, if V describes a large portion of the configuration space, large enough for our purposes, then U does that just as well and possibly even better. (Points in $V\subset M\subset\mathbb{R}^n$ would normally be described by n coordinates, but points in U have k coordinates only.) We only have to use ψ^{-1} and ψ to pass from V to U. That would mean that we are no longer interested in γ but rather in the curve $\beta=\psi^{-1}\circ\gamma$ in U, defined on the time interval $[a,b]$. But how can we recover the action integral

$$\int_a^b L(\gamma(t),\gamma'(t),t)\;dt$$

from the curve β ? This is clear in theory because $(\gamma(t), \gamma'(t)) \in TV$ and $(\beta(t), \beta'(t)) \in TU$ correspond to each other under the diffeomorphism

$$T\psi : TU \longrightarrow TV \ ; \ (y, v) \mapsto (\psi(y), D\psi(y)(v)).$$

(See the proof of proposition 3.2.2, and note also that $TU = U \times \mathbb{R}^k$ where k is the dimension of M.) Therefore we have:

Proposition 4.2.6 *Let $\beta = \psi^{-1} \circ \gamma$ on $[a, b]$. Then*

$$\int_a^b L(\gamma(t), \gamma'(t), t) \ dt = \int_a^b L_\psi(\beta(t)), \beta'(t), t) \ dt$$

where $L_\psi(y, v, t) = L(\psi(y), D\psi(y)(v), t)$.

This means that $L_\psi : TU \times \mathbb{R} \to \mathbb{R}$ is the correct Lagrangian function that we should use to test the curve $\psi^{-1} \circ \gamma : [a, b] \to U$.
While that may be clear in theory it is easily forgotten in practice. Perhaps the most important conclusion to draw is the following. Feel free to use any chart $\psi : U \to M$ you like *for the configuration space M*, as long as it is in an allowed atlas. Do not feel free to use any chart you like *for the phase space TM*, even if it is in an allowed atlas for TM. Only trust charts of the form $T\psi : TU \to TM$ determined by charts $\psi : U \to M$ for the configuration space, were $T\psi(y, v) = (\psi(y), D\psi(y)(v))$.
Let's look at a very simple example for illustration.

Example 4.2.7 In the case of the planar pendulum, example 4.1.2 with configuration manifold $S^1 \subset \mathbb{R}^2$, we can use a chart $\psi : U \mapsto S^1$ where U is the open interval from $-3\pi/2$ to $\pi/2$ and

$$\psi(\theta) = (\cos(\theta), \sin(\theta)) \in S^1 \subset \mathbb{R}^2 \ .$$

Then the resulting Lagrangian function on U is

$$L_\psi(\theta, \omega, t) \quad = \quad 0.5\omega^2 - 9.81 \sin(\theta)$$

where $\theta \in U$ and $\omega \in \mathbb{R}$. By contrast, if we use the chart $\varphi : W \to S^1$ where W is the open interval from -1 to $+1$ and

$$\varphi(y) = (y, -\sqrt{1 - y^2}\ ) \in S^1 \subset \mathbb{R}^2 \ ,$$

then the resulting Lagrangian function on W is

$$L_\varphi(y, z, t) = \frac{1}{1 - z^2} + 9.81\sqrt{1 - y^2}$$

where $y \in W$ and $z \in \mathbb{R}$.

SMSTC (2008/09)

Geometry and Topology

Lecture 5: Categories and homotopy

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5.1 The homotopy relation

Let X and Y be topological spaces. (If you wish, assume that X is a subspace of $\mathbb{R}^m$ and Y is a subspace of $\mathbb{R}^n$. *Subspace* means subset with the induced topology. Or if you wish, assume that X and Y are metric spaces.) Let f and g be continuous maps from X to Y. Let $[0,1]$ be the unit interval with the standard topology, a subspace of $\mathbb{R}$.

Definition 5.1.1 A *homotopy* from f to g is a continuous map $h : X \times [0,1] \to Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all $x \in X$. If such a homotopy exists, we say that f and g are *homotopic*, and write $f \simeq g$. We also sometimes write $h : f \simeq g$ to indicate that h is a homotopy from f to g.

Remark 5.1.2 If you did assume $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$, then you can also say $X \times [0,1] \subset \mathbb{R}^{m+1}$ and $Y \times [0,1] \subset \mathbb{R}^{n+1}$. If you made the less restrictive assumption that X and Y are metric spaces, then you should use the product metric on $X \times [0,1]$ and $Y \times [0,1]$, so that for example

$$d((x_1,t_1),(x_2,t_2)) := \max\{d(x_1,x_2), |t_1 - t_2|\ \}$$

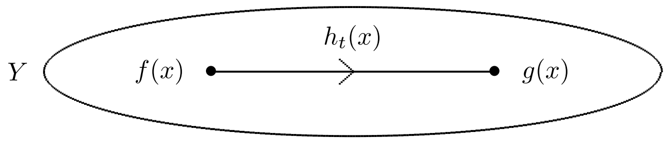
for $x_1, x_2 \in X$ and $t_1, t_2 \in [0,1]$. If you were happy with the assumption that X and Y are "just" topological spaces, then you need to know the definition of *product of two topological spaces* in order to make sense of $X \times [0,1]$ and $Y \times [0,1]$.

Remark 5.1.3 A homotopy $h : X \times [0,1] \to Y$ from $f : X \to Y$ to $g : X \to Y$ can be seen as a "family" of continuous maps

$$h_t : X \to Y\ ;\ h_t(x) = h(x,t)$$

such that $h_0 = f$ and $h_1 = g$. The important thing is that h_t depends continuously on $t \in [0,1]$.

Think of a homotopy as a single 'take' in a film, with h_t the position of the actors at time t, starting at $h_0 = f$ and ending at $h_1 = g$.



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Example 5.1.4 Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the identity map. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be the map such that $g(x) = 0 \in \mathbb{R}^n$ for all $x \in \mathbb{R}^n$. Then f and g are homotopic. The map $h : \mathbb{R}^n \times [0,1]$ defined by $h(x,t) = tx$ is a homotopy from f to g.

Example 5.1.5 Let $f : S^1 \to S^1$ be the identity map, $f(z) = z$. Let $g : S^1 \to S^1$ be the antipodal map, $g(z) = -z$. Then f and g are homotopic. Using complex number notation, we can define a homotopy by $h(z,t) = e^{\pi i t} z$.

Example 5.1.6 Let $f : S^2 \to S^2$ be the identity map, $f(z) = z$. Let $g : S^2 \to S^2$ be the antipodal map, $g(z) = -z$. Then f and g are *not* homotopic. We will prove this later in the course.

Example 5.1.7 Let $f : S^1 \to S^1$ be the identity map, $f(z) = z$. Let $g : S^1 \to S^1$ be the constant map with value 1. Then f and g are *not* homotopic. We will prove this quite soon.

Proposition 5.1.8 *"Homotopic" is an equivalence relation on the set of continuous maps from X to Y.*

Proof Reflexive: For every continuous map $f : X \to Y$ define the *constant homotopy* $h : X \times [0,1] \to Y$ by $h(x,t) = f(x)$.
Symmetric: Given a homotopy $h : X \times [0,1] \to Y$ from $f : X \to Y$ to $g : X \to Y$, define the *reverse homotopy* $\bar{h} : X \times [0,1] \to Y$ by $\bar{h}(x,t) = h(x, 1-t)$. Then $\bar{h}$ is a homotopy from g to f.
Transitive: Given continuous maps $e, f, g : X \to Y$, a homotopy h from e to f and a homotopy k from f to g, define the *concatenation homotopy* $k * h$ as follows:

$$(x,t) \mapsto \begin{cases} h(x, 2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ k(x, 2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \,. \end{cases}$$

$$k * h(x,0) = e(x) \xrightarrow[h(x,-) \text{ at twice the speed}]{} k * h(x,1/2) = f(x) \xrightarrow[k(x,-) \text{ at twice the speed}]{} k * h(x,1) = g(x)$$

Then $k * h$ is a homotopy from e to g. $\square$

Definition 5.1.9 The equivalence classes of the above relation "homotopic" are called *homotopy classes.* The homotopy class of a map $f : X \to Y$ is often denoted by $[f]$. The set of homotopy classes of maps from X to Y is often denoted by $[X, Y]$.

Proposition 5.1.10 *Let X, Y and Z be topological spaces. Let $f : X \to Y$ and $g : X \to Y$ and $u : Y \to Z$ and $v : Y \to Z$ be continuous maps. If f is homotopic to g and u is homotopic to v, then $u \circ f : X \to Z$ is homotopic to $v \circ g : X \to Z$.*

Proof Let $h : X \times [0,1] \to Y$ be a homotopy from f to g and let $w : Y \times [0,1] \to Z$ be a homotopy from u to v. Then $u \circ h$ is a homotopy from $u \circ f$ to $u \circ g$ and the map $X \times [0,1] \to Z$ given by $(x,t) \mapsto w(g(x), t)$ is a homotopy from $u \circ g$ to $v \circ g$. Because the homotopy relation is transitive, it follows that $u \circ f \simeq v \circ g$. $\square$

Definition 5.1.11 Let X and Y be topological spaces. A (continuous) map $f : X \to Y$ is a *homotopy equivalence* if there exists a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.
We say that X is *homotopy equivalent* to Y if there exists a map $f : X \to Y$ which is a homotopy equivalence.

Definition 5.1.12 If a topological space X is homotopy equivalent to a point, then we say that X is *contractible.* This amounts to saying that the identity map $X \to X$ is homotopic to a constant map from X to X.

Example 5.1.13 $\mathbb{R}^m$ is contractible, for any $m \geq 0$.

Example 5.1.14 $\mathbb{R}^m \smallsetminus \{0\}$ is homotopy equivalent to S^{n-1}.

Example 5.1.15 The general linear group of $\mathbb{R}^m$ is homotopy equivalent to the orthogonal group $\mathrm{O}(m)$. The Gram-Schmidt orthonormalisation process leads to an easy proof of that.

Digression 5.1.16 For a total appreciation of proposition 5.1.10 and definition 5.1.11 you need to go back to digression 2.1.8 and the definition of *category* given there.
We make a category $\mathcal{HT}$ as follows. As objects, we take all the (topological) spaces. As the set of morphisms from a space X to a space Y, we take $[X, Y]$, the set of homotopy classes of (continuous) maps from X to Y. The composition rule is defined by composing representatives of homotopy classes. Proposition 5.1.10 guarantees that this is well defined.
You might be tempted to infer that *homotopy equivalence* is just another expression for *isomorphism* in $\mathcal{HT}$, and *homotopy equivalent* is just another expression for *isomorphic* in $\mathcal{HT}$. Strictly speaking the second of these inferences is correct but the first is not. *Homotopy equivalence* is a designation which is normally applied to honest maps $f : X \to Y$ between spaces, not to homotopy classes of maps.
More about syntax: do not confuse *homotopic* and *homotopy equivalent*. "Homotopic" is a relation between maps, "homotopy equivalent" is a relation between spaces.

5.2 Based maps, based homotopies and homotopy groups

Definition 5.2.1 A *based space* (also *pointed space*) is a topological space X with a choice of distinguished point in X, the *base point*. The distinguished point is often denoted by $* \in X$.
Let X and Y be based spaces. A *based map* from X to Y is a (continuous) map $f : X \to Y$ such that $f(*_1) = *_2$ where $*_1$ and $*_2$ are the base points of X and Y, respectively.
Let f and g be based maps from X to Y. A *based homotopy* from f to g is a map

$$h : X \times [0, 1] \to Y$$

such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in X$, and moreover $h(*, t) = *$ for all t. In such a case each map $h_t : X \to Y$ defined by $h_t(x) = h(x, t)$ is of course a based map from X to Y.
If f and g are based maps from X to Y, and a based homotopy from f to g exists, then f and g are *based homotopic*. "Based homotopic" is an equivalence relation on the set of all maps from X to Y. The set of equivalence classes is denoted by $[X, Y]_*$.
We make a category $\mathcal{HT}_*$ as follows. As objects, we take all based (topological) spaces. As the set of morphisms from a based space X to a based space Y, we take $[X, Y]_*$, the set of homotopy classes of based maps from X to Y. The composition rule is defined by composing representatives of homotopy classes.

Definition 5.2.1 comes a little out of the blue. The deeper reason for considering based maps and based homotopy classes of maps is that we can often perform interesting "algebraic" operations with them. The following definition helps with that.

Definition 5.2.2 Let X and Y be based spaces, with base points $*_1$ and $*_2$ respectively. Suppose that X and Y are disjoint as sets. Let $\sim$ be the equivalence relation on $X \cup Y$ which makes $*_1$ equivalent to $*_2$ while everybody else is equivalent only to himself/herself. Topologists write

$$X \vee Y = \frac{X \cup Y}{\sim}$$

for the set of equivalence classes of this equivalence relation. This becomes a topological space with the quotient topology: a subset U of $X \vee Y$ is open iff $U \cap X$ and $U \cap Y$ are open in X and Y, respectively. (It is not quite correct to pretend that $X \subset X \vee Y$ and $Y \subset X \vee Y$, but there are obvious injective maps $X \to X \vee Y$ and $Y \to X \vee Y$.) The space $X \vee Y$ is the *wedge sum* of X and Y.

Remark 5.2.3 If you are not happy with topological spaces, you can assume that X and Y are metric spaces. Then you should use the following metric on $X \vee Y$. The distance from a to b is $d_X(a, b)$ or $d_Y(a, b)$ if a, b are both in X or both in Y. Otherwise it is

$$d_X(a, *_1) + d_Y(*_2, b)$$

if $a \in X$ and $b \in Y$, where $*_1$ and $*_2$ are the base points in X and Y.

Remark 5.2.4 If X and Y are *not* disjoint as sets, then you should fix that somehow, for example by using $X^\$ = X \times \{0\}$ and $Y^\$ = Y \times \{1\}$ instead. In such a case most topologists would still write $X \vee Y$ but they would mean $X^\$ \vee Y^\$$.

The topology on $X \vee Y$ is defined in such a way that a based map from $X \vee Y$ to another based space Z is exactly the same thing as a pair (f, g) consisting of a based map $f : X \to Z$ and a based map $g : Y \to Z$. For the same reasons, a based homotopy class of based maps from $X \vee Y$ to Z is the same thing as a pair, consisting of a based homotopy class of maps from X to Y and a based homotopy class of maps from Y to Z. Writing this in a formulaic way we have

Lemma 5.2.5 *The set* $[X \vee Y, Z]_*$ *is in bijective correspondence with* $[X, Y]_* \times [Y, Z]_*$ *by means of the map which takes the homotopy class* $[k]$ *of a based map* $k : X \vee Y \to Z$ *to the pair with first component* $[k|X]$ *and second component* $[k|Y]$.

Example 5.2.6 Let's make $S^1 \subset \mathbb{C}$ into a based space with base point 1. Let's pretend we know that $[S^1, S^1]_*$ is in bijection with $\mathbb{Z}$ (the integer m corresponds to the based homotopy class of the map $z \mapsto z^m$). Then $S^1 \vee S^1$ looks like a "figure eight" and you know from the above that there is a bijective map

$$[S^1 \vee S^1, S^1]_* \longrightarrow \mathbb{Z} \times \mathbb{Z}$$

in such a way that the pair of integers (m, n) corresponds to the map $S^1 \vee S^1 \to S^1$ which equals $z \mapsto z^m$ on the first wedge summand S^1 and $z \mapsto z^n$ on the second wedge summand S^1.

Remark 5.2.7 Lemma 5.2.5 has an interpretation in the language of categories. Let $\mathcal{C}$ be a category and let a, b be objects of $\mathcal{C}$. We say that the *product* of a and b in $\mathcal{C}$ exists if there exists an object c in $\mathcal{C}$ and morphisms $p_1 : c \to a$, $p_2 : c \to b$ such that the following is a bijection for *every* object y in $\mathcal{C}$:

$$\begin{aligned} \mathrm{mor}(y, c) &\mapsto \mathrm{mor}(y, a) \times \mathrm{mor}(y, b) \\ f &\mapsto (p_1 \circ f, p_2 \circ f). \end{aligned}$$

Notation:

$$c = a \Pi b \ .$$

Products will be familiar to you in many specific categories. They are usually called *products* in each of the specific categories. The morphisms p_1 and p_2 are usually called *projections* (from the product to its factors).

There is a "dual" notion of coproduct in a category. Let $\mathcal{C}$ be a category and let a, b be objects of $\mathcal{C}$. We say that the *coproduct* of a and b in $\mathcal{C}$ exists if there exists an object c in $\mathcal{C}$ and morphisms $j_1 : a \to c$, $j_2 : b \to c$ such that the following is a bijection for *every* object y in $\mathcal{C}$:

$$\begin{aligned} \mathrm{mor}(c, y) &\mapsto \mathrm{mor}(a, y) \times \mathrm{mor}(b, y) \\ f &\mapsto (f \circ j_1, f \circ j_2). \end{aligned}$$

Notation:

$$c = a \amalg b \ .$$

Copoducts may also be familiar to you from many specific categories. In the category of sets, and in the category of topological spaces, they are called *disjoint unions*. In the category of groups, they are called *free products*. In the category of left modules over a fixed ring, they are called *direct sums* (and they happen to agree with products).

What lemma 5.2.5 expresses is that $X \vee Y$ is the coproduct of X and Y in the category $\mathcal{HT}_\star$. It is also the coproduct of X and Y in $\mathcal{T}_*$, the category of based spaces and based maps.

Now we are in a position to make some very interesting algebraic structures out of certain homotopy sets $[X, Y]_*$. In the first example we choose $X = S^1 \subset \mathbb{C}$ with base point 1 as usual. Let

$$\kappa : S^1 \to S^1 \vee S^1$$

be the map given by

$$\kappa(z) = z^2 \text{ in the } \begin{cases} \text{second wedge summand if } z \text{ has imaginary part} \geq 0 \\ \text{first wedge summand if } z \text{ has imaginary part} < 0. \end{cases}$$

Then for any based space Y, we can define a "multiplication" on $[S^1, Y]_*$ by the following formula: given $[f] \in [S^1, Y]_*$ and $[g] \in [S^1, Y]_*$, their product $[f] \bullet [g]$ is the homotopy class of

$$(f \vee g) \circ \kappa \ .$$

Here $f \vee g$ denotes the map from $S^1 \vee S^1$ to Y which is equal to f on the first wedge summand and equal to g on the second wedge summand.

Theorem 5.2.8 *This multiplication makes $[S^1, Y]_*$ into a group, called the fundamental group of Y and denoted $\pi_1(Y)$.*

Proof This proof is actually a sketch which aims to show that the whole statement is much more a statement about S^1 than a statement about Y. We need the following facts about S^1.

(i) Writing $c : S^1 \to S^1$ for the constant based map, the compositions

$$S^1 \xrightarrow{\kappa} S^1 \vee S^1 \xrightarrow{\mathrm{id} \vee c} S^1 , \qquad S^1 \xrightarrow{\kappa} S^1 \vee S^1 \xrightarrow{c \vee \mathrm{id}} S^1$$

are based homotopic to the identity.

(ii) There exists a based map $\lambda : S^1 \to S^1$ such that the compositions

$$S^1 \xrightarrow{\kappa} S^1 \vee S^1 \xrightarrow{\mathrm{id} \vee \lambda} S^1 , \qquad S^1 \xrightarrow{\kappa} S^1 \vee S^1 \xrightarrow{\lambda \vee \mathrm{id}} S^1$$

are based homotopic to the constant based map c.

(iii) The composition

$$S^1 \xrightarrow{\kappa} S^1 \vee S^1 \xrightarrow{\kappa \vee \mathrm{id}} (S^1 \vee S^1) \vee S^1$$

is based homotopic to the composition

$$S^1 \xrightarrow{\kappa} S^1 \vee S^1 \xrightarrow{\mathrm{id} \vee \kappa} S^1 \vee (S^1 \vee S^1)$$

These are easy to show. (For λ, take the map $z \mapsto z^{-1}$ from $S^1 \to S^1$, in complex number notation.) Turning to $\pi_1(Y) = [S^1, Y]_*$ now, we note that the constant based map $c_Y : S^1 \to Y$ is a two-sided neutral element for the $\bullet$ product. For example $[f] \bullet [c_Y]$ is the homotopy class of

$$(f \vee c_Y) \circ \kappa = f \circ (\mathrm{id} \vee c) \circ \kappa \quad \simeq \quad f \circ \mathrm{id} = f \ .$$

Also, for any based map $f : S^1 \to Y$, the homotopy class of $f \circ \lambda$ is a right inverse for the $\bullet$ product to $[f]$ because

$$(f \vee (f \circ \lambda)) \circ \kappa = f \circ (\mathrm{id} \vee \lambda) \circ \kappa \quad \simeq \quad f \circ c = c_Y \ .$$

For a similar reason, $[f \circ \lambda]$ is left inverse to $[f]$. The associativity of the $\bullet$ product in $\pi_1(Y)$ follows from (iii) because, for elements $[e], [f], [g]$ in $\pi_1(Y)$, the element $([e] \bullet [f]) \bullet [g]$ is the homotopy class of

$$(e \vee f \vee g) \circ (\kappa \vee \mathrm{id}) \circ \kappa \quad \simeq \quad (e \vee f \vee g) \circ (\mathrm{id} \vee \kappa) \circ \kappa$$

and it is therefore equal to $[e] \bullet ([f] \bullet [g])$. □

Example 5.2.9 We shall see in the next few chapters that $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$, while $\pi_1(S^n)$ is a trivial group for $n > 1$. For the projective spaces we have $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$ if $n > 1$.
Fundamental groups are important to knot theorists. A *knot* is a smooth 1 manifold in $\mathbb{R}^3$ which, as a smooth manifold in its own right, is diffeomorphic to S^1. (More precisely a knot is an equivalence class of such things, two being equivalent if there is a diffeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ taking one to the other, and preferably with $f(x) = x$ for all x with large enough $\|x\|$.) The fundamental group of the complement of a knot says a lot about the knot.

The next example is a generalization of the last one. We look at $[X, Y]_\star$ taking X to be $S^n \subset \mathbb{R}^{n+1}$ with base point $(1, 0, \ldots, 0)$, assuming $n > 0$. We need a good map

$$\kappa : S^n \to S^n \vee S^n .$$

We give two descriptions of κ (they don't describe exactly the same map, but they are in the same based homotopy class). The first has the advantage of being explicit and the disadvantage of being rather fiddly. Take some $v \in S^n \subset \mathbb{R}^{n+1}$ and write it in the form

$$v = a \cdot (x_1, \cdots, x_n, 0) + b \cdot (0, \cdots, 0, 1)$$

where $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and $a \geq 0$ and $x_1, x_2, \cdots, x_n \in \mathbb{R}$ with $\sum_{i=1}^n x_i^2 = 1$. Determine $c, d \in \mathbb{R}$ such that $c + di = (a + bi)^2 \in \mathbb{C}$. Then put

$$\kappa(v) = c \cdot (1, 0, \cdots, 0) \pm d \cdot (0, x_1, \cdots, x_n),$$

where the whole expression is to be taken in the second wedge summand S^n if $b \geq 0$, and in the first wedge summand S^n if $b \leq 0$. The sign $\pm$ is minus if $b \geq 0$ and n is even, otherwise it is plus.
For the other description of κ, we need to make an "identification" of S^n with the space $I^n/\partial I^n$. Here $I = [0, 1]$ is the unit interval, I^n is the n-dimensional unit cube and ∂I^n is its boundary, consisting of all $(t_1, \ldots, t_n)$ for which at least one of the coordinates t_i is 0 or 1. The notation $I^n/\partial I^n$ means that we are taking a quotient space (all points in the boundary of the cube are declared to be the same point). A subset of $[0, 1]^n/\partial[0, 1]^n$ is declared to be open iff its preimage in the cube $[0, 1]^n$ is open. You identify S^n with $I^n/\partial I^n$ by choosing a homeomorphism between the two if you can (which should take the base point $(1, 0, \ldots, 0)$ to the point in $I^n/\partial I^n$ represented by all points in ∂I^n.) That being done, we define

$$\kappa : I^n/\partial I^n \longrightarrow (I^n/\partial I^n) \vee (I^n/\partial I^n)$$

by the formula

$$\kappa((t_1, \ldots, t_n)) = \begin{array}{ccc} (2t_1, t_2, \ldots, t_n) & \text{in second wedge summand} & \text{if } 2t_1 < 1 \\ (2t_1 - 1, t_2, \ldots, t_n) & \text{in first wedge summand} & \text{if } 2t_{\geq}1. \end{array}$$

Using any of the two descriptions of κ, we can define a "multiplication" on $[S^n, Y]_*$ by the following formula: given $[f] \in [S^1, Y]_*$ and $[g] \in [S^1, Y]_*$, their product $[f] \bullet [g]$ is the homotopy class of

$$(f \vee g) \circ \kappa \ .$$

Here $f \vee g$ denotes the map from $S^1 \vee S^1$ to Y which is equal to f on the first wedge summand and equal to g on the second wedge summand.

Theorem 5.2.10 *This multiplication makes $[S^n, Y]_*$ into a group, called the n-th homotopy group of Y and denoted $\pi_n(Y)$.*

Proof We need the following facts about S^n and κ.

(i) Writing $c : S^n \to S^n$ for the constant based map, the compositions

$$S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{\mathrm{id}\vee c} S^n \ , \qquad S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{c\vee \mathrm{id}} S^n$$

are based homotopic to the identity.

(ii) There exists a based map $\lambda : S^n \to S^n$ such that the compositions

$$S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{\mathrm{id}\vee\lambda} S^n \ , \qquad S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{\lambda\vee \mathrm{id}} S^n$$

are based homotopic to the constant based map c.

(iii) The composition

$$S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{\kappa\vee \mathrm{id}} (S^n \vee S^n) \vee S^n$$

is based homotopic to the composition

$$S^n \xrightarrow{\kappa} S^n \vee S^n \xrightarrow{\mathrm{id}\vee\kappa} S^n \vee (S^n \vee S^n)$$

These are easy to show using the $I^n/\partial I^n$ model for S^n. We can for example define λ by the formula $(t_1, \ldots, t_n) \mapsto (1 - t_1, t_2, \ldots, t_n)$. The remainder of the proof is formally identical with the end of the proof of theorem 5.2.8. $\square$

Theorem 5.2.11 *For any based space Y and any $n > 1$, the homotopy group $\pi_n(Y)$ is abelian.*

Proof It is enough to show that $\kappa : S^n \to S^n \vee S^n$ is homotopic to $\tau \circ \kappa$, where $\tau : S^n \vee S^n \to S^n \vee S^n$ interchanges the wedge summands. (In other words $\tau(z) = z^c$ where z^c is the clone of z in the other wedge summand.) This is left as an exercise. You may find the description of a homotopy from κ to $\tau \circ \kappa$ easier using the first (fiddly) description of κ. But it is also instructive to try it using the second description. $\square$

Example 5.2.12 Here is some info on $\pi_m(S^n)$, assuming $n > 0$.

(a) Isomorphic to $\mathbb{Z}$ if $m = n$.

(b) Trivial group (one element only) if $m < n$.

(c) Trivial group if $m > n$ and $n = 1$.

(d) Not always trivial if $m > n > 1$. The first example is $\pi_3(S^2)$ which is isomorphic to $\mathbb{Z}$. No "formula" describing all abelian groups $\pi_m(S^n)$ with $m > n > 1$ is known. This is a major open problem in homotopy theory. Even $\pi_m(S^2)$ is not well understood for all m.

(e) Known to be a finite (abelian) group if $m > n > 1$, except in the cases where n is even and $m = 2n - 1$. In those cases it has rank 1.

We will establish (a),(b),(c) soon, at least in outline.

Example 5.2.13 Take a "long knot" in $\mathbb{R}^3$. This is a smooth 1-dimensional manifold K in $\mathbb{R}^3$ which is connected and agrees with the x-axis outside a bounded set of $\mathbb{R}^3$. Then $\pi_m(\mathbb{R}^3 \smallsetminus K)$ is a trivial group for $m > 1$. This is a (corollary of a) deep theorem in 3-manifold topology.

SMSTC (2008/09)

Geometry and Topology

Lecture 6: Fundamental group and covering spaces

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Contents

Books

Allen Hatcher's downloadable book *Algebraic Topology* (see reference list on the SMSTC wiki) is an excellent introduction to algebraic topology. Whenever possible A.R. has included a page reference to the book, in the form [ATn].
A.R.'s own book *Algebraic and geometric surgery*

http : //www.maths.ed.ac.uk/ aar/books/surgery.pdf

describes the application of algebraic topology to the classification of manifolds. The reviews of foundational material it includes might be found useful.

6.1 Introduction: Invariantorama

How does one recognize topological spaces, and distinguish between them? In the first instance, it is not even clear whether the Euclidean spaces $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots$ are pairwise non-homeomorphic. Standard linear algebra shows that they are all non-isomorphic as vector spaces. It follows that $\mathbb{R}^m$ is diffeomorphic to $\mathbb{R}^n$ if and only if $m = n$, since every differential of a diffeomorphism is an isomorphism of vector spaces. In 1878 Cantor constructed bijections $\mathbb{R} \to \mathbb{R}^n$ for $n \geqslant 2$, which however were not continuous. In 1890 Peano constructed continuous surjections $\mathbb{R} \to \mathbb{R}^n$ for $n \geqslant 2$, the 'space-filling curves'. By analogy it seemed possible, at that time, that there might exist continuous injective maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $n > m$. In 1910, Brouwer was able to show that this was not the case.

Algebraic topology deals with topological invariants of spaces, that is, functions I which associate to a topological space X an object $I(X)$ which may be either a number or an algebraic structure such as a group. The essential requirement is that homeomorphic spaces X, Y have the same invariant $J(X) = J(Y)$, where $=$ means 'isomorphic to' for algebraic invariants. Thus if X, Y are such that $J(X) \neq J(Y)$ then X, Y are not homeomorphic. Here are some examples:

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- The dimension of a Euclidean space $\mathbb{R}^n$, $J(\mathbb{R}^n) = n$.
- The genus of an orientable surface Σ is an integer $g(\Sigma) \geqslant 0$ (1850's). [AT51]
- The Betti numbers of X (1860's). [AT130]
- The fundamental group $\pi_1(X)$ (Poincaré. 1895). [AT26]
- The homology groups $H_*(X)$ (1920's). [AT160]
- The cohomology ring $H^*(X)$ (1930's). [AT191]
- The higher homotopy groups $\pi_*(X)$ (1930's). [AT340]

Given a topological space X, the first thing one might ask about its topology is whether any two points can be joined by a path: given $x_0, x_1 \in X$ does there exist a continuous map $\alpha : I = [0,1] \to X$ from $\alpha(0) = x_0 \in X$ to $\alpha(1) = x_1 \in X$? Such a function is called a 'path' in X from x_0 to x_1. The relation defined on X by $x_0 \sim x_1$ if there exists a path from x_0 to x_1 is an equivalence relation. An equivalence class is called a 'path component' of X, and the set of path components is denoted by $\pi_0(X)$. The number of path-components in a space X

$$|\pi_0(X)| \in \{0, 1, 2, 3, \ldots, \infty\}$$

is perhaps the simplest topological invariant: if $m \neq n$ a space with m path-components cannot be homeomorphic to a space with n path-components. By definition, a space X is path-connected if $|\pi_0(X)| = 1$, i.e. if it is nonempty and for any $x_0, x_1 \in X$ there exists a path from x_0 to x_1.

Regard S^1 as the unit circle in the complex plane $\mathbb{C}$. A 'loop' in a space X at a point $x \in X$ is a continuous map $\omega : S^1 \to X$ such that $\omega(1) = x \in X$. The fundamental group $\pi_1(X, x)$ of X at $x \in X$ is defined geometrically to be the set of homotopy classes of loops $\omega : S^1 \to X$ at x, with the homotopies $\{\omega_t \,|\, 0 \leqslant t \leqslant 1\}$ required to be such that $\omega_t(1) = x$. If x is path-connected, then $\pi_1(X, x)$ is isomorphic to $\pi_1(X, y)$ for all $x, y \in X$. Often we assume that X is based, with base point $\star$. Then we write $\pi_1(X)$ instead of $\pi_1(X, \star)$.

Here are the key properties of the fundamental group:

- A (continuous) based map $f : X \to Y$ induces a group homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ which depends only on the based homotopy class of f.
- For any based space X the identity map from X to X induces the identity homomorphism from $\pi_1(X)$ to $\pi_1(X)$.
- For any continuous based maps $f : X \to Y$, $g : Y \to Z$

$$(gf)_* \;=\; g_* f_* \;:\; \pi_1(X) \to \pi_1(Z) \;.$$

- If $f : X \to Y$ is a based map which is a homotopy equivalence then f_* is an isomorphism. Thus two path-connected spaces with non-isomorphic fundamental groups cannot be homotopy equivalent, and a fortiori cannot be homeomorphic.

The isomorphism class of $\pi_1(X)$ is a topological invariant of a path-connected space X. A based space X is 'simply-connected' if it is path-connected and $\pi_1(X) = \{1\}$. This is equivalent to saying that X is path-connected and every continuous map $S^1 \to X$ is homotopic to a constant map.

In many cases it is actually possible to compute $\pi_1(X)$, and to use the fundamental group to make interesting statements about topological spaces. Here are some examples:

- The Euclidean spaces $\mathbb{R}^n$ $(n \geqslant 1)$ are all simply-connected, with $\pi_1(\mathbb{R}^n) = \{1\}$.
- The fundamental group of the circle S^1 is an infinite cyclic group. [AT29]
- Every loop $\omega : S^1 \to \mathbb{C}\backslash\{0\}$ is homotopic to the map $z \mapsto z^n$ for a unique $n \in \mathbb{Z}$ called the *winding number* of ω. Cauchy's theorem computes the winding number as a closed contour integral

$$\frac{1}{2\pi i} \oint\limits_{\omega} \frac{dz}{z} \;=\; n \;.$$

- The n-sphere S^n has $\pi_1(S^n) = \{1\}$ for $n \geqslant 2$. [AT35]
- The n-dimensional projective space $\mathbb{R}P^n$ has $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$ for $n \geqslant 2$. [AT74]
- The fundamental group of the closed orientable surface M_g of genus $g \geqslant 0$ has $2g$ generators and one relation
$$\pi_1(M_g) \;=\; \{a_1, b_1, \ldots, a_g, b_g \,|\, [a_1, b_1] \ldots [a_g, b_g]\}$$
with $[a, b] = a^{-1}b^{-1}ab$ the commutator of a, b. In particular, $M_0 = S^2$ is the sphere, with $\pi_1(M_0) = \{1\}$, and $M_1 = S^1 \times S^1$ is the torus with $\pi_1(M_1) = \mathbb{Z} \oplus \mathbb{Z}$, the free abelian group on 2 generators. Since the groups $\pi_1(M_g)$ $(g \geqslant 0)$ are all non-isomorphic, the surfaces M_g are non-homeomorphic. [AT51]
- For a knot $K \subset \mathbb{R}^3$ we have $\pi_1(\mathbb{R}^3 \smallsetminus K)$, a topological invariant of the knot. For example, if $K_0 \subset \mathbb{R}^3$ is the trivial knot and $K_1 \subset \mathbb{R}^3$ is the trefoil knot then
$$\pi_1(\mathbb{R}^3 \smallsetminus K_0) \;=\; \mathbb{Z} \;,\; \pi_1(\mathbb{R}^3 \smallsetminus K_1) \;=\; \{a, b \,|\, aba = bab\} \qquad \text{[AT55]}$$
These groups are not isomorphic (since one is abelian and the other one is not abelian), so that K_0, K_1 are essentially distinct knots. In particular, this algebra shows that the trefoil cannot be unknotted.
- Let $L \subset \mathbb{R}^3$ be a smooth 1-dimensional manifold diffeomorphic to a disjoint union of two copies of S^1 (we say that L is a link). Then $\pi_1(\mathbb{R}^3 \smallsetminus L)$ is a topological invariant of the link. For example, if $L_0 \subset S^3$ is the trivial link then $\pi_1(\mathbb{R}^3 \smallsetminus L_0) = \mathbb{Z} * \mathbb{Z}$ is the free nonabelian group on 2 generators, while if $L_1 \subset \mathbb{R}^3$ is the simplest non-trivial link then $\pi_1(\mathbb{R}^3 \smallsetminus L_1) = \mathbb{Z} \oplus \mathbb{Z}$. [AT24,47]

The **Seifert-van Kampen Theorem** states that the fundamental group of a union $X = X_1 \cup X_2$ of path-connected spaces X_1, X_2 , both open in X, and with the intersection $Y = X_1 \cap X_2$ path-connected, is isomorphic to the amalgamated free product $\pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$. [AT43]

- Example: The figure 8 has $\pi_1(8) = \mathbb{Z} * \mathbb{Z}$. [AT40,77]

Every group G is the fundamental group $G = \pi_1(X)$ of some path-connected space X, and every group morphism $\phi : G \to H$ is the induced morphism $\phi = f_*$ of a continuous map $f : X \to Y$ with $\pi_1(X) = G$, $\pi_1(Y) = H$. [AT89]

A *covering space* of a space X is a continuous map $p : Y \to X$ such that for each $x \in X$ there exist an open subset $U \subseteq X$ with $x \in U$, a set S (to be viewed as a discrete topological space) and a homeomorphism $\varphi : S \times U \to p^{-1}(U)$ such that $p(\varphi(a, u)) = u \in U$ for all $a \in S$ and $u \in U$.

One of the simplest and best examples of a covering space is the map $\mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi it}$, using complex number notation.

Content of this chapter. Let X be a *well-behaved* path connected and based space. (It will be clarified later what *well-behaved* means.) Our main result in this chapter is a correspondence between

- covering spaces of X
- sets with an action of $\pi_1(X)$.

Under this correspondence, covering spaces $p : Y \to X$ with simply-connected Y correspond to sets with a *free and transitive action of* $\pi_1(Y)$. It follows, broadly speaking, that if a covering space $p : Y \to X$ with simply connected Y is known, then $\pi_1(X)$ can be determined from it.

The correspondence between covering spaces of X and sets with an action of $\pi_1(X)$ leads to an easy proof of the Seifert-van Kampen theorem, at least in the cases where the spaces involved are well-behaved. This will be formulated as an exercise, with appropriate instructions.

6.2 Covering spaces

Definition 6.2.1 A *covering space* of a space X is a continuous map $p : Y \to X$ such that for each $x \in X$ there exist an open neighbourhood U of x in X, a set S (to be viewed as a topological space with the discrete topology) and a homeomorphism $\varphi : S \times U \to p^{-1}(U)$ such that $p(\varphi(a, u)) = u \in U$ for all $a \in S$ and $u \in U$.

A covering space $p : Y \to X$ is a "local homeomorphism": for each $y \in Y$ there exists an open neighbourhood V of y in Y such that $p|V$ as a map from V to $p(V)$ is a homeomorphism, and $p(V) \subset X$ is an open subset. The converse is not true, i.e., some local homeomorphisms are not covering spaces. The following example illustrates that.

Example 6.2.2 The map $p : \mathbb{R} \to S^1$ defined by $p(t) = e^{2\pi it} \in S^1 \subset \mathbb{C}$ is a covering projection.
Let $W \subset \mathbb{R}$ be a nonempty bounded open interval. Then $p|W : W \to S^1$ is not a covering projection, although it is still a local homeomorphism.

Example 6.2.3 Let G be a (discrete) group acting freely and discontinuously on a space Y. We recall what this means: every $y \in Y$ admits a neighbourhood U in Y such that $U \cap gU = \emptyset$ whenever $g \in G$ is not the neutral element. Let $X = Y/G$, the space of orbits. (A subset of X is open iff its preimage in Y is open in Y.) Then the projection $p : Y \to X$ is a covering space.

Example 6.2.4 Let $p : Y \to X$ be a covering space. Let B be another space and let $f : B \to X$ be any map. From this information we want to produce a covering space $q : A \to B$ such that, for every $b \in B$, the set $q^{-1}(b)$ is "identified" with the set $p^{-1}(f(b))$. A good way to define A and $q : A \to B$ is then

$$A = \{(b, y) \in B \times Y \mid f(b) = p(y)\}$$

(with the subspace topology) and $q(b, y) = b$ for $(b, y) \in A$. Now we want to to show: *the projection $q : A \to B$ is also a covering space.*
Let $b \in B$ be given. Choose a neighbourhood U of $f(b) \in X$ and a set S and a homeomorphism $\psi : S \times U \to p^{-1}(U)$ such that $p(\psi(s, u)) = u$ for all $(s, u) \in S \times U$. Then $V = f^{-1}(U)$ is a neighbourhood of b in B. We have a homeomorphism $S \times V \to q^{-1}(V)$ given by $(s, c) \mapsto (c, \psi(s, f(c))$. This completes the proof.
In the above situation it is customary to write $A = Y \times_B X$ and to say that the covering space $q : A \to B$ is the *pullback* of the covering space $p : Y \to X$ along $f : B \to X$.

Definition 6.2.5 A covering space $p : Y \to X$ is *trivial* if there exist a set S and a homeomorphism $\psi : S \times X \to Y$ such that $p(\psi(s, x)) = x$ for all $(s, x) \in S \times X$.

Lemma 6.2.6 Let $p : Y \to X$ be a covering space where $X = I^n$ is a cube. Then p is trivial.

Proof By contradiction (sketch): View the cube X as a union of two bricks such as $X_0 = I^{n-1} \times [0, a]$ and $X_1 = I^{n-1} \times [a, 1]$ where $a = 1/2$. The maps

$$p_0 : Y_0 \to X_0 \ , \quad p_1 : Y_1 \to X_1$$

obtained by restricting p to $Y_0 = p^{-1}(X_0)$ and $Y_1 = p^{-1}(X_1)$ are still covering spaces. It is easy to see that, if p is not trivial, then at least one of the covering spaces p_0 and p_1 must be nontrivial. Repeating the process many times, we see that if p is indeed nontrivial, then there exists a subcube or sub-brick $X_\$$ in X, with sidelengths as small as we please, such that

$$p_\$: Y_\$ \to X_\$$$

(the restriction of p to p^{-1} of $X_\$$) is a nontrivial covering space. This is however in contradiction with our assumption that p is a covering space. Namely, X does admit a (finite) open cover by open sets U_α such that the restriction of p to $p^{-1}(U_\alpha)$ is a trivial covering space with target U_α. The cube $X_\$$, if it is indeed small enough, will be contained in one of the U_α. $\square$

6.3 The path lifting property

Definition 6.3.1 Let $p : Y \to X$ be a covering space. A *lift* of a continuous map $f : Z \to X$ across p is a continuous map $f^\sharp : Z \to Y$ such that

$$p \circ f^\sharp = f \ .$$

Example 6.3.2 If $Y = S \times X$ for a set S, and $p : Y \to X$ is the projection, then every map $f : Z \to X$ admits a lift across p. We can choose some $s \in S$ and define $f^\sharp(z) = (s, f(z))$ for all $z \in Z$.

Theorem 6.3.3 (Path lifting property) *Let $p : Y \to X$ be a covering space. Let $x_0 \in X$, $y_0 \in Y$ be such that $p(y_0) = x_0 \in X$.*

- *Every path $\alpha : I \to X$ with $\alpha(0) = x_0$ has a unique lift across p to a path $\alpha^\sharp : I \to Y$ such that $\alpha^\sharp(0) = y_0$.*
- *For a family of paths $\alpha_t : I \to X$ with $\alpha_t(0) = x_0$, depending continuously on $t \in [0, 1]$, the lifted paths ${\alpha_t}^\sharp$ also depend continuously on t.*

Proof For the first claim we form $A = \{(t, y) \in I \times Y \mid \alpha(t) = p(y)\ \}$. Then we have a projection $q : A \to I$ and this is clearly a covering space, the pullback of $p : Y \to X$ along α. By lemma 6.2.6, $q : A \to I$ is a trivial covering space. Therefore we can find $\beta : I \to A$ such that $q \circ \beta = \mathrm{id}_I$ and $\beta(0) = (0, y_0) \in A$. Then the path $\alpha^\sharp : I \to Y$ defined by composing $\beta : I \to A$ with the projection $A \to Y$ solves our lifting problem. The same argument shows that $\alpha^\sharp$ is unique. More precisely, $\alpha^\sharp$ and β determine each other and it is clear that β is unique.
For the second claim, we can argue similarly, using I^2 instead of I. Indeed the family of paths α_t defines a map $I^2 \to X$ by the formula $(t, s) \mapsto \alpha_t(s) \in X$. □

Corollary 6.3.4 *Let $p : Y \to X$ be a covering space where X is based. Let S be the fiber of p over the base point, $S = p^{-1}(\star)$. The set S comes with a preferred action of the fundamental group $\pi_1(X)$.*

Proof Take $y \in S$ and $g \in \pi_1(X)$. We need to say what $gy \in S$ should be. Choose a loop γ representing the homotopy class g. More precisely, it will be convenient to view γ as a path, $\gamma : I \to X$ with $\gamma(0) = \gamma(1) = \star$. By the first part of theorem 6.3.3, there exists a unique path $\gamma^\sharp : I \to Y$ such that $\gamma^\sharp(0) = y$ and $p \circ \gamma^\sharp = \gamma$. For this path $\gamma^\sharp$ we clearly have $\gamma^\sharp(1) \in S = p^{-1}(\star)$. We define

$$gy := \gamma^\sharp(1) \in S \ .$$

By the second part of theorem 6.3.3, choosing another γ in the same homotopy class $g \in \pi_1(X)$ will give the same result, so that gy is well defined. The conditions for an action are easily verified. □

Remark 6.3.5 Let's make corollary 6.3.4 more precise using category language. What we have here is a relationship between two categories, $\mathcal{C}_X$ and $\mathcal{S}_\pi$ where π is short for $\pi_1(X)$. The first of these categories, $\mathcal{C}_X$, has as objects all the covering spaces

$$p : Y \to X$$

with fixed target X. A *morphism* from $p_1 : Y \to X$ to $p_2 : Y_2 \to X$ is, by definition, a (continuous) map $u : Y_1 \to Y_2$ such that $p_2 \circ u = p_1$. The other category $\mathcal{S}_\pi$ is simply the category of π-sets and π-maps (a π-set is the same thing as a set with an action of π).
The "rule" $\mathcal{F}$ which we have formulated in the corollary associates to each object $p : Y \to X$ of $\mathcal{C}_X$ an object $S = \mathcal{F}(p : Y \to X) = \mathcal{F}(p)$ of $\mathcal{S}_\pi$, given by $S = p^{-1}(\star)$ with the π-action described in the proof of the corollary. It is also true, and straightforward to verify, that a morphism u in $\mathcal{C}_X$ from $p_1 : Y_1 \to X$ to $p_2 : Y_2 \to X$ determines by restriction of u a map from

$$\mathcal{F}(p_1 : Y_1 \to X) = S_1 = p_1^{-1}(\star)$$

to

$$\mathcal{F}(p_2 : Y_2 \to X) = S_2 = p_2^{-1}(\star).$$

This is a π-map. We denote it by $\mathcal{F}(u) : \mathcal{F}(p_1) \to \mathcal{F}(p_2)$. Again it is easy to verify that when u, v are composable morphisms in $\mathcal{C}_X$, so that $u \circ v$ is defined, then $\mathcal{F}(u) \circ \mathcal{F}(v)$ is also defined and agrees with $\mathcal{F}(u \circ v)$. Finally if u is an identity morphism in $\mathcal{C}_X$, then $\mathcal{F}(u)$ is also an identity morphism in $\mathcal{S}_\pi$.
Briefly, $\mathcal{F}$ is a kind of *homomorphism* between categories. The official name for that is *functor*. So the rule described in the corollary is actually a functor $\mathcal{F}$ from $\mathcal{C}_X$ to $\mathcal{S}_\pi$, where $\pi = \pi_1(X)$.

Example 6.3.6 The above constructions show that $\pi_1(S^1)$ is not the trivial group. Consider the covering space $p : \mathbb{R} \to S^1$ with $p(t) = e^{2\pi it}$. By all the above, this determines a set $S = p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$ with an action of $\pi_1(S^1)$. It is clear by inspection that the element $[\mathrm{id}] \in \pi_1(S^1)$ acts nontrivially, taking $n \in S$ to $n + 1 \in S$. Therefore $[\mathrm{id}]$ is not the neutral element of $\pi_1(S^1)$.

6.4 Classification of covering spaces

Now we wish to show that the functor $\mathcal{F}$ described in corollary 6.3.4 and remark 6.3.4 is as "bijective" as can reasonably be expected. Some parts of this are easy and require only a mild hypothesis on X. Some other parts are harder and require stronger hypotheses on X. In particular it is not always true that there exists a covering space $p : Y \to X$ such that the action of $\pi_1(X)$ on $\mathcal{F}(p) = p^{-1}(\star)$ is free and transitive. In the cases where it exists, Y is simply connected. We then say that $p : Y \to X$ is a *universal covering* of X.

Lemma 6.4.1 *Let X be a based and path-connected space. Let $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ be covering spaces. If $u : Y_1 \to Y_2$ and $v : Y_1 \to Y_2$ are "morphisms" of covering spaces, and $\mathcal{F}(u) = \mathcal{F}(v)$, then $u = v$.*

Proof Saying that u and v are morphisms means that $p_2 \circ u = p_1$ and $p_2 \circ v = p_1$. Saying that $\mathcal{F}(u) = \mathcal{F}(v)$ means that $u(y) = v(y)$ for every $y \in Y_1$ which has $p_1(y)$ equal to the base point. Choose now an arbitrary $z \in Y_1$. We need to show $u(z) = v(z) \in Y_2$. We can choose a path $\omega : I \to X$ from $p_1(z)$ to $\star \in X$. This path lifts uniquely to a path $\omega^\sharp : I \to Y_1$ from z to some point y with $p_1(y) = \star \in X$. We form $u \circ \omega^\sharp : I \to Y_2$ and $v \circ \omega^\sharp : I \to Y_2$. Both of these paths are lifts of ω across p_2. Their endpoints also agree because $u(y) = v(y)$ because $p_1(y) = \star \in X$. Therefore by the unique path lifting property, applied to the covering space $p_2 : Y_2 \to X$, they are the same paths. In particular their starting points $u(z)$ and $v(z)$ are the same. □

Lemma 6.4.2 *Let $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ be covering spaces, where X is path-connected and based. Write $\pi = \pi_1(X)$. Every π-map v from $\mathcal{F}(p_1) = p_1^{-1}(\star)$ to $\mathcal{F}(p_2) = p_2^{-1}(\star)$ extends to a "morphism" $\bar{v} : Y_1 \to Y_2$ (so that $p_2 \circ u = p_1$).*

Proof Let's use the following notation. For a point $y \in Y_1$ and a path $\lambda : I \to X$ starting at $p_1(y)$, we write λy to mean *the endpoint of the unique path $\lambda^\sharp : I \to Y_1$ which lifts λ and starts at y.* This suggests that paths λ in X *act* on some points $y \in Y$, which is exactly what we want to express. Also, let's write λ^{-1} for the reverse of λ, so that $\lambda^{-1}(t) = \lambda(1 - t)$. Also, let's use the same kind of notation for points $y \in Y_2$ and paths in X starting at $p_2(y)$.
Given $y \in Y_1$ with $p_1(y) = x \in X$ we define

$$\bar{v}(y) = \gamma^{-1} v(\gamma y)$$

where $\gamma : I \to X$ is a path from x to the base point $\star \in X$. Note that γy is an element of $p_1^{-1}(\star)$, so that $v(\gamma y) \in Y_2$ is defined. Note also that $p_2(\bar{v}(y)) = x = p_1(y)$ so that $\bar{v}$ appears to be a "morphism" between covering spaces of X. But we still have to check that $\bar{v}(y)$ is well defined. As long as we are in doubt, let's write $\bar{v}_\gamma(y)$ for $\bar{v}(y)$ as written above. Now suppose we replace γ by another path μ from x to $\star$. Do we get $\bar{v}_\gamma(y) = \bar{v}_\mu(y)$?
Suppose first that μ is homotopic to γ "relative to start- and endpoints", i.e., there exists a continuous map $h : I \times I \to X$ such that $h(0, t) = \gamma(t)$, $h(1, t) = \mu(t)$ for all t, and $h(t, 0) = x$, $h(t, 1) = \star$ for all t. The homotopy determines a path from $u_\gamma(y)$ to $u_\mu(y)$ in the space $p_2^{-1}(x)$. As that space is discrete, we must have $\bar{v}_\gamma(y) = \bar{v}_\mu(y)$.
If μ is not homotopic to γ relative to start- and endpoints, then it will still be homotopic (relative to start- and endpoints) to a concatenated path $\omega \bullet \gamma$ where $\omega : I \to X$ has $\omega(0) = \star = \omega(1)$. Then, using the hypothesis that v is a π-map, we have

$$\bar{v}_\mu(y) = \gamma^{-1}\omega^{-1} v(\omega\gamma y) = \gamma^{-1}\omega^{-1}\omega v(\gamma y) = \gamma^{-1} v(\gamma y) = \bar{v}_\gamma(y).$$

□

Proposition 6.4.3 *Let X be a path-connected based space and $p : Y \to X$ a covering space. The following are equivalent:*

(i) *Y is simply connected;*

(ii) *the set $S = p^{-1}(\star)$ is nonempty and the action of $\pi_1(X)$ on it defined in corollary 6.3.4 is free and transitive.*

In these circumstances we say that $p : Y \to X$ is a universal covering space.

Proof Suppose that (i) holds. We first show that $S = p^{-1}(\star)$ is nonempty. As Y is nonempty, there exists $y \in Y$. Choose a path in X from $p(y)$ to $\star$. Lift it to a path in Y starting at y. This path will end at some point in $S = p^{-1}(\star)$. Hence S is nonempty.
Next, let $y_0, y_1 \in Y$ with $p(y_0) = p(y_1) = \star$. Since Y is path-connected, there exists a path $\gamma^\sharp : I \to Y$ with $\gamma^\sharp(0) = y_0$ and $\gamma^\sharp(1) = y_1$. Let $\gamma = p \circ \gamma^\sharp$ and let $g = [\gamma] \in \pi_1(X)$. Then $gy_0 = y_1$ by definition of the action of $\pi_1(X)$ on S. As y_0 and y_1 were arbitrary elements of S, this shows that the action is transitive.
Continuing with this notation, suppose we have some element $f \in \pi_1(X)$ represented by a path $\varphi : I \to X$ with $\varphi(0) = \star = \varphi(1)$, and that $fy_0 = y_1 = gy_0$. We need to show $f = g$ to establish that the action of $\pi_1(X)$ on S is free. Let $\varphi^\sharp : I \to Y$ be the unique path which lifts φ and has $\varphi^\sharp(0) = y_0$. By our assumption $fy_0 = y_1$ we must have $\varphi^\sharp(1) = y_1$. Therefore the concatenation of $\varphi^\sharp$ and the reverse of $\gamma^\sharp$ is a loop in Y. By assumption on Y, it is homotopic to a constant loop. Therefore (compose with p) the loop $\gamma \bullet \varphi^{-1}$ is nullhomotopic. Therefore $gf^{-1} = 1 \in \pi_1(X)$, so $f = g$. This shows that the action is free. Altogether, we have shown that (i) implies (ii).
Now suppose that (ii) holds. Transitivity of the action immediately implies that Y is path-connected. Choose some base point y_0 in Y such that $p(y_0) = \star \in X$. Let $\gamma^\sharp : I \to Y$ be any path having $\gamma^\sharp(0) = y_0 = \gamma^\sharp(1)$. Let $\gamma = p \circ \gamma^\sharp : I \to X$ and $g = [\gamma] \in \pi_1(X)$. Then $gy_0 = y_0$ by definition of the action of $\pi_1(X)$ on $S = p^{-1}(\star)$. As the action is free, it follows that $g = 1$, so γ is homotopic (relative to start- and endpoint) to the constant path at $\star \in X$. Now we can use the second part of the "path-lifting property", theorem 6.3.3, to deduce that $\gamma^\sharp$ is also homotopic relative to start- and endpoint to a path which proceeds entirely in $S = p^{-1}(\star)$. Such a path has to be constant since S is discrete. This shows that $\gamma^\sharp$ is the trivial element of $\pi_1(Y, y_0)$. As $\gamma^\sharp$ was arbitrary, the conclusion is that $\pi_1(Y, y_0)$ is the trivial group and so Y is simply connected. □

Lemma 6.4.4 *If the path-connected based space X admits a universal covering space $p : Y \to X$, then every $\pi_1(X)$-set is isomorphic to one of the form $\mathcal{F}(q) = q^{-1}(\star)$ for* some *covering space $q : Z \to X$.*

Proof (Sketch.) Let $\pi = \pi_1(X)$ and $S = p^{-1}(\star) \subset Y$. Choose a point y_0 in S. We are assuming that the action of π on S is free and transitive. So every point in S can be written in the form gy_0 for a unique $g \in \pi$. For $h \in \pi$ let $T_h : S \to S$ be the map defined by $gy_0 \mapsto (gh)y_0$ for $gy_0 \in S$. This is clearly a π-map. From lemma 6.4.2 we know that it extends uniquely to a "morphism"

$$\bar{T}_h : Y \to Y$$

(so that $p \circ \bar{T}_h = \bar{T}_h$). When $h = 1$ this is the identity map. When $h \neq 1$ we have $\bar{T}_h(y) \neq y$ for all $y \in Y$. (Sub-proof: if $\bar{T}_h(y) = y$ for some y, choose a path in Y from y to some point gy_0 in S, and use uniqueness of path-lifting to conclude $T_h(gy_0) = gy_0$, which means $ghy_0 = gy_0$, which implies $gh = g$ hence $h = 1$.) We also have

$$T_h \circ T_k = T_{kh}$$

and so $\bar{T}_h \circ \bar{T}_k = \bar{T}_{kh}$. Therefore the formula

$$yh := \bar{T}_h(y) \in Y$$

defines a right action of the group π on the space Y. We know already that the action is free. It is easy to show that it is free and properly discontinuous. (Explain.)
Now let K be any π-set (set with left action of π). Let

$$Y_K = Y \times_\pi K$$

which means: space of orbits of the left action of π on $Y \times K$ given by

$$g \cdot (y, k) = (yg^{-1}, gk)\ .$$

There is a projection map $q : Y_K \to X$ taking the equivalence class of (y, k) to $p(y)$. This projection map is a covering space of X. Then $q^{-1}(\star) = S \times_\pi K$ can identified with K by the formula

$$\text{class of } (y_0, k) \mapsto k$$

and the action of π on $q^{-1}(\star)$ defined by corollary 6.3.4 is given by

$$g \cdot (\text{class of } (y_0, k)) = \text{class of } (gy_0, k) = \text{class of } (y_0, gk)$$

so agrees with the prescribed action of π on K. □

Remark 6.4.5 Let X be a path-connected based space which admits a universal covering $p : Y \to X$. If you put lemmas 6.4.1 and 6.4.2 and 6.4.4 together, you will see that the functor $\mathcal{F}$ from $\mathcal{C}_X$ to $\mathcal{S}_\pi$ described in corollary 6.3.4 and remark 6.3.4 is indeed as "bijective" as can be expected. Firstly, for any covering spaces $q_1 : Z_1 \to X$ and $q_2 : Z_2 \to X$, the map

$$\text{mor}(q_1, q_2) \to \text{mor}(\mathcal{F}(q_1), \mathcal{F}(q_2))$$

which describes what $\mathcal{F}$ does on morphisms is a bijection. Secondly, any object in the target category $\mathcal{S}_\pi$ is isomorphic to one of the form $\mathcal{F}(q)$, for some covering space $q : Z \to X$.
A functor with these properties is called an *equivalence of categories.*

Now it remains to find conditions which ensure that X admits a universal covering space.

Definition 6.4.6 A path-connected space X is *well-behaved* if, for every $y \in X$ and every neighbourhood V of y, there exists a smaller neighbourhood U of y in X which is path-connected and such that the homomorphism $\pi_1(U, y) \to \pi_1(X, y)$ induced by the inclusion $U \to X$ is the trivial homomorphism.

Example 6.4.7 A popular example of a path-connected space which is not well-behaved is the *Hawaiian earring* E. This is the union of all circles in $\mathbb{R}^2$ with center $(0, k^{-1})$ and radius k^{-1} where $k = 1, 2, 3, \ldots$. Give it the subspace topology inherited from $\mathbb{R}^2$. The offending point in E is $(0, 0)$, which we can take as the base point of E. For every neighbourhood U of $(0, 0)$ in E there exist elements of $\pi_1(U)$ which have nontrivial image in $\pi_1(E)$. (You are already already qualified to show this using example 6.3.6.)
For another interesting example, take the same $E \subset \mathbb{R}^2$, view it as a subspace of $\mathbb{R}^3$ using the standard inclusion $\mathbb{R}^2 \to \mathbb{R}^3$, and draw all the straight line segments from points in E to the point $(0, 0, 1) \in \mathbb{R}^3$. Let X be the union of these line segments. Then X is contractible. This space X is well-behaved. Nevertheless, all sufficiently small neighbourhoods of $y = (0, 0, 0)$ have a frightfully large fundamental group.

Lemma 6.4.8 *Let X be a path-connected well-behaved based space. Then X admits a universal covering space $p : Y \to X$.*

Proof (This proof is both sketchy and fiddly. The most illuminating part is the definition of Y as a set. Try to sort out for yourself how Y is a topological space, because that is the fiddly part.)
A point in Y shall be an equivalence class of pairs (x, γ) where $x \in X$ and $\gamma : I \to X$ is a path from the base point $\star$ to x. We consider two such pairs (x_1, γ) and (x_2, ω) to be equivalent iff $x_1 = x_2$ and the loop obtained by first running through γ (from $\star$ to x_1) and then running through ω in reverse (from $x_1 = x_2$ to $\star$) represents the trivial element of $\pi_1(X)$. This defines Y as a set. There is an obvious projection map from Y to X, taking the equivalence class of (x, γ) to $x \in X$.
Next we wish to make Y into a topological space. It is intuitively clear how this should work. A point of Y represented by a pair (x_1, γ) as above is "close" to other points of Y represented by pairs (x_2, ω) where x_2 is close to x_1 in X and ω can be obtained by running first through γ and then through a short path μ from x_1 to x_2. The difficult thing is to say exactly and in full generality what we mean by a *short* path from x_1 to x_2. (In many specific examples it is not very difficult to say, for example when X is a smooth manifold.) We will say it in full generality using the hypothesis that X is well-behaved.
So, given a point y_1 in Y, represent it by a pair (x_1, γ) as above:

$$y_1 = \text{ class of } (x_1, \gamma) \; .$$

Choose a path-connected neighbourhood U of x_1 in X such that the inclusion $U \to X$ induces the trivial homomorphism $\pi_1(U, x_1) \to \pi_1(X, x_1)$. Let us adopt the informal idea that any $x_2 \in U$ is close to x_1 , and any path from x_1 to x_2 which proceeds in U is *short*. More to the point, we use this idea to construct a map $s : U \to Y$ with the property $s(x_1) = y_1$. Namely, given $x_2 \in U$ choose a path μ from x_1 to x_2 in U. Concatenate it with γ to obtain a path $\omega = \mu \bullet \gamma$ from $\star$ to x_2. The equivalence class of (x_2, ω) is by definition $s(x_2) \in Y$. It is well defined, i.e., independent of the choice of μ. (Explain.) We decree that $s(U)$ is a neighbourhood of y_1. We decree more precisely that a subset of Y containing the point y_1 is a neighbourhood of y_1 precisely if it contains $s(U)$ for some choice of U as above. And of course we decree that a subset of Y is open if it is a neighbourhood of all its elements. This defines a topology on Y.

The next thing to check is that the projection $p : Y \to X$ is a covering space. We omit this. It is more tedious than hard.

In any case it is clear that the set $S = p^{-1}(\star)$, also known as the fiber of p over the base point of X, is identified with $\pi_1(X)$. We claim that under this identification, the action of $\pi_1(X)$ on S defined in remark 6.3.4 is simply left translation. (Consequently it is free and transitive, and consequently p is a universal covering space.)

In order to show this we must recall how the action of $\pi_1(X)$ on S was defined. Given $y \in S$ and $g \in \pi_1(X)$, we obtain $gy \in S$ by choosing a path $\gamma : I \to X$ starting and ending at $\star$. Then we lift this to a path $\gamma^\sharp : I \to Y$ starting at y. That path will end at $gy \in S$. — To implement these instructions we start by making explicit what y is. It is of course an equivalence class, represented by a pair of the form $(\star, \omega)$ where $\omega : I \to X$ is a path from $\star$ to $\star$. Under our correspondence $S \cong \pi_1(X)$ the element y will then correspond to $w = [\omega] \in \pi_1(X)$. We also have $g = [\gamma] \in \pi_1(X)$. Using all this information, we want to define $\gamma^\sharp : I \to Y$ starting at y and lifting γ. This is easy: for $t \in I$ we define $\gamma^\sharp(t)$ to be the equivalence class of $(\gamma(t), \gamma_{[0,t]} \bullet \omega)$ where $\gamma_{[0,t]}$ is the restriction of γ from $[0, 1]$ to $[0, t]$, suitably reparametrised, and the bullet means concatenation. Now the endpoint of the path $\gamma^\sharp$ is the equivalence class of $(\star, \gamma \bullet \omega)$. That of course corresponds to $gw \in \pi_1(Y)$, which is what we wanted to show. $\square$

SMSTC (2008/09)

Geometry and Topology

Lecture 7: Maps between manifolds

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Contents

7.1 Introduction

Throughout this lecture we shall only be concerned with smooth manifolds and smooth atlases.

In this lecture we study smooth maps $f : M \to N$ between manifolds.

In Lecture 1 a *k-dimensional manifold M^k in $\mathbb{R}^n$* was defined (1.3.2) to be a subset $M \subseteq \mathbb{R}^n$ such that for every $x \in M$ there exist open sets $U, V \subseteq \mathbb{R}^n$ and a diffeomorphism $\psi : U \to V$ such that $\psi(U \cap \mathbb{R}^k) = V \cap M$, with

$$\mathbb{R}^k \ = \ \{(x_1, x_2, \ldots, x_k, 0, \ldots, 0) \,|\, x_j \in \mathbb{R}, 1 \leqslant j \leqslant k\} \subset \mathbb{R}^n \ .$$

In particular, M^k is a k-dimensional submanifold of the n-dimensional manifold $\mathbb{R}^n$. We shall call such manifolds *concrete*.

In Lecture 2 a *k-dimensional manifold* M^k was defined (2.1.1, 2.2.19) to be a set M with an equivalence class of k-dimensional atlases, such that the resulting topology $\mathcal{O}$ on M satisfies the Hausdorff separation axiom and the 2nd countability axiom.

Example 2.1.5 showed that a concrete k-dimensional submanifold $M \subseteq \mathbb{R}^n$ has a canonical equivalence class of k-dimensional atlases, so that it is a k-dimensional manifold.

Is every manifold M concrete? In other words, does there exist an embedding (= injective smooth map with injective differentials)

$$f \ : \ M \to \mathbb{R}^n \ ; \ x \mapsto (f_1(x), f_2(x), \ldots, f_n(x))$$

such that the given equivalence class of atlases on M coincides with the equivalence class M inherits from $\mathbb{R}^n$. The answer is yes, at least for compact M. The Whitney embedding theorem uses 'partitions of unity' to prove that every compact k-dimensional manifold M admits functions $f_1, f_2, \ldots, f_n : M \to \mathbb{R}$ such that $f = (f_1, f_2, \ldots, f_n)$ is an embedding, with $n \geqslant k$. (This is the principle of the CAT scan in medicine: given enough shadows it is possible to reconstruct the body).

Books

[Br] G. Bredon, *Topology and geometry*, Graduate Texts in Mathematics 139, Springer, 1993

[a] m.weiss@abdn.ac.uk

[GG] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Texts in Mathematics 14, Springer, 1973

[Mi] J. Milnor, *Topology from the differentiable view point*, Princeton University Press.

7.2 Partitions of unity

Definition 7.2.1 Let X be a topological space.
(i) A *covering* of X is a set $\mathcal{U}$ of subsets of X such that

$$\bigcup_{U \in \mathcal{U}} U = X \ .$$

The covering is *open* if each $U \in \mathcal{U}$ is open in X.
(ii) A covering $\mathcal{V}$ of X is a *refinement* of another covering $\mathcal{U}$ of X if every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$.
(iii) A covering $\mathcal{U}$ of X is considered finite if $\mathcal{U}$ is a finite set. (NB: in such a case, the finitely many elements of $\mathcal{U}$ are still subsets of X, and they can of course be infinite sets in their own right.)
(iv) X is *compact* if every open covering $\mathcal{U}$ of X admits a finite subcovering, i.e., there exists a finite subset $\mathcal{V} \subset \mathcal{U}$ such that $\mathcal{V}$ is still an open covering of X.
(v) A covering $\mathcal{U}$ of X is *locally finite* if every $x \in X$ has a neighbourhood W_x in X such that the set $\{U \in \mathcal{U} \mid W_x \cap U_\alpha \neq \emptyset\}$ is finite.
(vi) X is *paracompact* if for every open covering $\mathcal{U}$ of X there exists a locally finite open covering $\mathcal{V}$ which is a refinement of $\mathcal{U}$.

Remarks. (a) A covering in the sense of Definition 7.2.1 (i) is not the same as a covering space in the sense of Definition 4.3.1.
(b) A subcovering of a covering is a refinement, but a refinement need not be a subcovering.
(c) In many books and papers you will find notation like *... open covering* $\{U_\alpha | \alpha \in J\}$ *of* X. Here J is a set, and the author(s) is/are thinking of a map from J to the set of subsets of X, described by $\alpha \mapsto U_\alpha$.

Example 7.2.2 (i) The set of all open subsets of X is an open covering of X.
(ii) A compact space is paracompact.
(iii) Metric spaces are paracompact. (This is hard to prove, but we shall not need this fact).

Definition 7.2.3 A topological space X is *locally compact* if for every $x \in X$ there exist an open subset $U \subseteq X$ and a compact subset $K \subseteq X$ such that $x \in U \subseteq K$.

Example 7.2.4 (i) $\mathbb{R}^n$ is locally compact.
(ii) A manifold is locally compact.

Lemma 7.2.5 *A topological space which is Hausdorff, locally compact and 2nd countable is paracompact.*

Proof Proposition 4.2 of [GG]. □

Corollary 7.2.6 (i) *Every metric space is paracompact.*
(ii) *A manifold is Hausdorff, locally compact and 2nd countable, and hence paracompact.*

Definition 7.2.7 (i) The *support* of a function $f : X \to \mathbb{R}$ is

$$\mathrm{supp}(f) \ = \ \mathrm{closure}\{x \in X \,|\, f(x) \neq 0\} \subseteq X \ .$$

(ii) A *partition of unity* on a topological space X is a collection of continuous maps

$$\rho_\alpha \ : \ X \to \mathbb{R} \qquad (\alpha \in I)$$

such that

(a) every $x \in X$ admits a neighbourhood W_x such that the set $\{\alpha \in I \mid \mathrm{supp}(\rho_\alpha) \cap W_x \neq \emptyset\}$ is finite;

(b) $\rho_\alpha(x) \geqslant 0$ for every $\alpha \in I$, $x \in X$;

(c) For each $x \in X$ we have $\sum_{\alpha \in I} \rho_\alpha(x) = 1$.

If X is a smooth manifold and each ρ_α is smooth, we speak of a *smooth* partition of unity on X.
(iii) A partition of unity (ρ_α) is *subordinate* to an open covering $\mathcal{V}$ of X if for every $\alpha \in I$ there exists $V \in \mathcal{V}$ such that $\mathrm{supp}(\rho_\alpha) \subset V$.

Lemma 7.2.8 *Let $\mathcal{U}$ be a locally finite covering of a topological space X. If X is "second countable", then $\mathcal{U}$ (as a set in its own right) is countable.*

Proof "Second countable" means that there exists a countable basis $\{W_1, W_2, W_3, \dots\}$ for the topology of X. Then every open subset V of X is the union of the W_i contained in V. We can assume $W_i \neq \emptyset$ for $i = 1, 2, 3, \dots$, and we can assume $U \neq \emptyset$ for all $U \in \mathcal{U}$. Let S be the set of all pairs (U, i) where $U \in \mathcal{U}$ and $i \in \{1, 2, 3, 4, \dots\}$, subject to the condition $U \supset W_i$. There is a forgetful map

$$S \to \{1, 2, 3, \dots\} \ ; \ (U, i) \mapsto i$$

which is finite-to-one. Therefore S is countable. There is also a surjective map $S \to \mathcal{U}$ given by $(U, i) \mapsto U$. Therefore $\mathcal{U}$ is countable. $\square$

Lemma 7.2.9 *Let U be an open subset of $\mathbb{R}^k$ and let $K \subset U$ be compact. There exists a smooth function $\rho : U \to [0, 1]$ such that $\mathrm{supp}(\rho) \subset U$ is compact and $\rho(x) > 0$ for all $x \in K$.*

We will use this without proof.

Theorem 7.2.10 *Let M be a k-dimensional smooth manifold and let $\mathcal{U}$ be an open covering of M. Then there exists a smooth partition of unity $\{\rho_\beta \,|\, \beta \in J\}$ subordinate to $\mathcal{U}$. Moreover, if $\mathcal{U}$ is a locally finite covering, every $U \in \mathcal{U}$ has compact closure in M and every $U \in \mathcal{U}$ is the image of a chart*[b] *$\psi_U : V_U \to M$ with V_U open in $\mathbb{R}^k$, then we can take $J = \mathcal{U}$ and arrange that $\mathrm{supp}(\rho_U) \subset U$ for all $U \in \mathcal{U}$.*

Proof We start with the second part, so we assume that $\mathcal{U}$ is a locally finite covering, every $U \in \mathcal{U}$ has compact closure in M, and every $U \in \mathcal{U}$ is the image of a chart $\psi_U : V_U \to M$, where V_U is open in $\mathbb{R}^k$. By lemma 7.2.8, the covering $\mathcal{U}$ is countable. So we may suppose $\mathcal{U} = \{U_1, U_2, U_3, \dots\}$. Let

$$K_1 = M \smallsetminus (U_2 \cup U_3 \cup U_4 \cup \cdots) \ .$$

This is a closed subset of X, contained in U_1 , therefore also in the closure of U_1 which is compact. Hence K_1 is compact. By lemma 7.2.9, we may choose a smooth function $\psi_1 : M \to [0, 1]$ with compact support such that, moreover, $\mathrm{supp}(\psi_1) \subset U_1$ and $\psi_1(x) > 0$ for all $x \in K_1$. Let $U_1' = \{x \in M \mid \psi_1(x) > 0\}$.
We now continue with the locally finite open covering $\{U_1', U_2, U_3, \dots\}$ of X. Let

$$K_2 = M \smallsetminus (U_1' \cup U_3 \cup U_4 \cup U_5 \cup \cdots) \ .$$

This is a closed subset of M, contained in U_2 , therefore also in the closure of U_2 which is compact. Hence K_2 is compact. By lemma 7.2.9, we may choose a smooth function $\psi_2 : M \to [0, 1]$ with compact support such that, moreover, $\mathrm{supp}(\psi_2) \subset U_2$ and $\psi_2(x) > 0$ for all $x \in K_2$. Let $U_2' = \{x \in M \mid \psi_1(x) > 0\}$.
We now continue with the locally finite open covering $\{U_1', U_2', U_3, U_4, \dots\}$ of X. Let

$$K_3 = X \smallsetminus (U_1' \cup U_2' \cup U_4 \cup U_5 \cup \cdots) \ .$$

This is a closed subset of M, contained in U_3 , therefore also in the closure of U_3 which is compact. Hence K_3 is compact. By lemma 7.2.9, we may choose a smooth function $\psi_3 : M \to [0, 1]$ with compact support such that, moreover, $\mathrm{supp}(\psi_3) \subset U_3$ and $\psi_3(x) > 0$ for all $x \in K_3$. Let $U_3' = \{x \in M \mid \psi_1(x) > 0\}$.
And so on! Eventually we have a whole sequence of smooth functions $\psi_1, \psi_2, \psi_3, \dots$ on M. Each ψ_i has compact support, and $\mathrm{supp}(\psi_i) \subset U_i$ by construction. Also by construction, the open sets $U_i' = \{x \in M \mid \psi_i(x) > 0\}$ form an open covering of M. Therefore

$$\sum_{i=1}^{\infty} \psi_i(x) > 0$$

[b] In some allowed atlas for M.

for every $x \in M$. Now let

$$\rho_i(x) = \frac{\psi_i(x)}{\sum_{i=1}^{\infty} \psi_i(x)} \ .$$

Then $\{\rho_1, \rho_2, \rho_3, \dots\}$ is a partition of unity subordinate to the covering $\mathcal{U} = \{U_1, U_2, \dots\}$ of M.
Now let's look at the general case: $\mathcal{U}$ is just any open covering of M. It is easy to construct an open covering $\mathcal{V}$ of M which refines $\mathcal{U}$ and is such that each $V \in \mathcal{V}$ has compact closure in M, and is the image of some chart in some allowed smooth atlas for M. Then the paracompactness of M guarantees that we can find a locally finite open covering $\mathcal{W}$ of M which refines $\mathcal{V}$. Clearly each $W \in \mathcal{W}$ has compact closure in M, and is the image of some chart in some allowed smooth atlas for M. Therefore we can construct (as above) a smooth partition of unity subordinate to $\mathcal{W}$. This will also be subordinate to $\mathcal{U}$. □

Corollary 7.2.11 *A compact n-dimensional manifold M has a smooth partition of unity $(\rho_\beta)_{\beta \in J}$ where J is finite and each* $\mathrm{supp}(\rho_\beta)$ *is contained in the image of a smooth chart $\psi_\beta : V_\beta \to M$, with V_β open in $\mathbb{R}^n$.*

Proof Since M is compact, it has a finite atlas. □

7.3 The Whitney embedding theorem

Theorem 7.3.1 (Whitney, 1936) *Any compact k-dimensional manifold M is diffeomorphic to a smooth manifold in $\mathbb{R}^n$ for some $n \geqslant k$.*

Proof (Sketch.) By Corollary 7.2.11 we have a partition of unity $(\rho_i)_{i=1,2,3,\dots,s}$ subordinate to a finite covering $\{U_1, U_2, \dots, U_s\}$ of M, where each U_i is the image of a smooth chart $\varphi_i : V_i \to M$, with V_i open in $\mathbb{R}^k$. We can assume $\mathrm{supp}(\rho_i) \subset U_i$. For $x \in M$ let

$$\psi_i(x) = \rho_i(x) \cdot \varphi_i^{-1}(x) \in \mathbb{R}^k$$

if x is in the image of φ_i , and $\psi_i(x) = 0 \in \mathbb{R}^k$ otherwise. Then $\psi_i : M \to \mathbb{R}^k$ is a smooth map for every $i \in \{1, 2, 3, \dots, s\}$. We now define

$$f(x) = (\rho_1(x), \psi_1(x), \rho_2(x), \psi_2(x), \dots, \rho_s(x), \psi_s(x)) \in \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k \times \cdots \times \mathbb{R} \times \mathbb{R}^k \quad = \quad \mathbb{R}^{s(k+1)}$$

for $x \in M$. Let's show, at least in outline, that $f(M)$ is a smooth manifold in $\mathbb{R}^{s(k+1)}$ and that f, as a map from M to $f(M)$, is a diffeomorphism.
We start by showing that f is injective. If $x, y \in M$ and $f(x) = f(y)$, we can find i such that $\rho_i(x) \neq 0$. Then also $\rho_i(y) = \rho_i(x) \neq 0$ and so $x, y \in U_i$. Furthermore

$$\varphi_i^{-1}(x) = \frac{\psi^i(x)}{\rho_i(x)} = \frac{\psi^i(y)}{\rho_i(y)} = \varphi_i^{-1}(y)$$

and by applying φ_i , we get $x = y$. So f is injective. The same argument shows that f has a continuous inverse from $f(M)$ to M, that is, if $f(x)$ is "close" to $f(y)$ then x is "close" to y.
Keeping $x \in M$ as above, and the fixed i with $\rho_i(x) \neq 0$, we can therefore choose an open neighborhood W of $f(x) \in \mathbb{R}^{(k+1)s}$ such that $W \cap f(M) = f(U_i') = f(\varphi_i(V_i'))$ where

$$U_i' = \{x \in U_i \mid \rho_i(x) > 0\} \ , \quad V_i' = \{z \in V_i \mid \rho_i(\varphi_i(z)) > 0\} \ .$$

The smooth map

$$g : (t_1, v_1, t_2, v_2, \dots, t_s, v_s) \mapsto \frac{v_i}{t_i}$$

from W to $\mathbb{R}^k$ has a smooth "right inverse" in the shape of $f \circ \varphi_i$; more precisely, $g \circ f \circ \varphi_i | V_i'$ is the inclusion $V_i' \to \mathbb{R}^k$. It follows[c] that $f(\varphi_i(V_i')) = W \cap f(M)$ is a smooth manifold in $\mathbb{R}^{(k+1)s}$. Then it follows that $f(M)$ is a smooth manifold in $\mathbb{R}^{(k+1)s}$. □

Remark 7.3.2 Whitney's theorem has a stronger version: every compact k-dimensional manifold M admits an embedding $M \subset \mathbb{R}^{2k}$. See [Br,p.91] for the modification of the proof of Theorem 7.3.1 required to prove that there exists an embedding $M \subset \mathbb{R}^{2k+1}$.

[c] This needs a bit of elaboration.

7.4 Sard's Theorem

Let us start with some standard linear algebra. The *kernel* and *image* of a linear map $f : V \to W$ of vector spaces are the subspaces

$$\ker(f) \ = \ \{v \in V \,|\, f(v) = 0 \in W\} \subseteq V \ , \ \operatorname{im}(f) \ = \ \{f(v) \,|\, v \in V\} \subseteq W \ .$$

The inverse image of any $w \in \operatorname{im}(f)$ is an affine space, of the form

$$U \ = \ f^{-1}(w) \ = \ \{u + v \,|\, v \in \ker(f)\} \subseteq V$$

for any fixed $u \in U$. If f is surjective then

$$\dim(U) \ = \ \dim(V) - \dim(W) \ .$$

Roughly speaking, Sard's Theorem states that for a smooth map $f : M \to N$ of manifolds with $\dim(M) < \dim(N)$, the image $f(M) \subset N$ is of measure 0, while if $\dim(M) \geqslant \dim(N)$ there exists a subset $Z \subset N$ of measure zero such that for every $y \in N \backslash Z$ the inverse image $L = f^{-1}(y) \subset M$ is a submanifold of dimension

$$\dim(L) \ = \ \dim(M) - \dim(N) \ .$$

Definition 7.4.1 Let $f : V \to W$ be a linear map of finite-dimensional vector spaces.
(i) The *rank* of f is

$$\operatorname{rank}(f) \ = \ \dim(\operatorname{im}(f)) \ .$$

(ii) The *nullity* of f is

$$\operatorname{nullity}(f) \ = \ \dim(\ker(f)) \ .$$

The *rank theorem* states that $\operatorname{rank}(f) + \operatorname{nullity}(f) \ = \ \dim(V)$.
By the multivariable Taylor theorem, a differentiable map $f : \mathbb{R}^m \to \mathbb{R}^n$ can be approximated at each $x \in \mathbb{R}^m$ by the derivative linear map

$$Df(x) \ = \ (\partial f_j / \partial x_i) \ : \ T_x\mathbb{R}^m \ = \ \mathbb{R}^m \to T_{f(x)}\mathbb{R}^n \ = \ \mathbb{R}^n \ ,$$

in the sense that there exist an open neighbourhood $U \subseteq \mathbb{R}^m$ of x and a number $K \geqslant 0$ such that

$$\|f(x+h) - (f(x) + Df(x)(h))\| \leqslant K\|h\|^2 \ \ (x + h \in U) \ .$$

More generally, a smooth map $f : M \to N$ of manifolds can be approximated at any $x \in M$ by the linear map $Df(x) : T_xM \to T_xN$, working with charts at $x \in M$ and $f(x) \in N$.

Definition 7.4.2 Let $f : M \to N$ be a smooth map of manifolds.
(i) A point $x \in M$ is a *critical point* of f if the linear map $Df(x)) : T_xM \to T_{f(x)}N$ is *not* surjective. Let $C[f] \subset M$ be the set of critical points.
(ii) A point $y \in N$ is a *critical value* of f if $y \in f(C[f])$.
(iii) A point $x \in M$ is a *regular point* of f if $x \notin C[f]$.
(iv) A point $y \in N$ is a *regular value* of f if it is not a critical value, i.e. if $y \notin f(C[f])$.

Example 7.4.3 If $\dim(M) < \dim(N)$, then every $x \in M$ is a critical point of f and every element of $f(M)$ is a critical value of f. The regular values of f are precisely the elements of $N \setminus f(M)$.

The following is a rather obvious generalisation of *smooth manifold in* $\mathbb{R}^m$.

Definition 7.4.4 Let M be a smooth m-dimensional manifold. Let $\ell \leq m$. A subset $L \subset M$ is a *smooth ℓ-dimensional submanifold* of M if, for every $z \in L$, there exist an open set $U \subset \mathbb{R}^m$ and an open subset $V \subset M$ containing z, and a diffeomorphism

$$\varphi : U \to V$$

such that $\varphi(U \cap \mathbb{R}^\ell) = V \cap L$.

Example 7.4.5 Let m, n be positive integers with $m < n$. If you accept $\mathbb{R}^m \subset \mathbb{R}^n$, then you can probably also accept $\mathbb{R}P^{m-1} \subset \mathbb{R}P^{n-1}$. In that sense, $\mathbb{R}P^{m-1}$ is a smooth submanifold of $\mathbb{R}P^{n-1}$.

Theorem 7.4.6 *If $f : M \to N$ is a smooth map of manifolds and $y \in N$ is a regular value of f, then $L = f^{-1}(y) \subset M$ is a smooth submanifold of M. Its dimension is*

$$\dim(M) - \dim(N)$$

if that number is positive; otherwise L is empty.

Proof The case where $m = \dim(M)$ is less than $n = \dim(N)$ is easy: if y is a regular value of f, then $L = f^{-1}(y)$ is empty. In the case $m \geq n$ it may be assumed without loss of generality that $M = \mathbb{R}^m$, $N = \mathbb{R}^n$. This case is given by Corollary 1.4.3. □

Example 7.4.7 (i) For any integers $p, q \geqslant 0$ with $n = p + q \geqslant 1$ define the smooth map

$$f : \mathbb{R}^n \to \mathbb{R} \; ; \; x = (x_1, x_2, \dots, x_n) \mapsto \sum_{i=1}^{p} (x_i)^2 - \sum_{j=1}^{q} (x_{p+j})^2 \; .$$

The derivative linear map

$$Df(x) \; = \; (2x_1 \; \dots \; 2x_p \; -2x_{p+1} \; \dots \; -2x_n) \; : \; T_x\mathbb{R}^n \; = \; \mathbb{R}^n \to T_x\mathbb{R} \; = \; \mathbb{R}$$

is surjective for $x \neq 0 \in \mathbb{R}^n$, so

$$C[f] \; = \; \{0\} \; .$$

The set of critical values is $f(C[f]) = \{0\} \subset \mathbb{R}$, and the set of regular values is $\mathbb{R} \setminus \{0\}$. For any $y \neq 0 \in \mathbb{R} \setminus \{0\}$ the inverse image $L = f^{-1}(y) \subset \mathbb{R}^n$ is an $(n-1)$-dimensional submanifold.
(ii) The element $1 \in \mathbb{R}$ is a regular value for

$$f : \mathbb{R}^n \to \mathbb{R} \; ; \; (x_1, x_2, \dots, x_n) \mapsto (x_1)^2 + (x_2)^2 + \cdots + (x_n)^2 \; ,$$

with $S^{n-1} = f^{-1}(1) \subset \mathbb{R}^n$ an $(n-1)$-dimensional submanifold.

Where do regular values come from? By the following theorem, for any smooth map $f : M \to N$ the probability of an element $y \in N$ being a regular value of f is $99.99\ldots\%$:

Theorem 7.4.8 (Sard, 1942) *The set $f(C[f])$ of critical values of a smooth map $f : M \to N$ of smooth manifolds has measure 0.* (See the remark just below.)

Proof See pp. 34-36 of [GG]. Warning: [GG] use definitions of critical point, critical value etc., which differ slightly from the ones used here when $\dim(M) < \dim(N)$. □

Remark. A subset C of a smooth n-dimensional manifold N is said to have *measure zero* if, for every chart $\varphi : U \to N$ in an allowed smooth atlas for N, the set $\varphi^{-1}(C) \subset U \subset \mathbb{R}^n$ has Lebesgue measure zero.

7.5 Transversality

Imagine three smooth manifolds M_1, M_2, N and two smooth maps $f : M_1 \to N$, $g : M_2 \to N$. Write $m_1 = \dim(M_1)$, $m_2 = \dim(M_1)$, $n = \dim(N)$.

Definition 7.5.1 The maps f and g are *transverse* to each other, in symbols $f \pitchfork g$, if for every $x \in M_1$ and $y \in M_2$ with $f(x) = g(y) \in N$, the linear map

$$T_xM_1 \times T_yM_2 \longrightarrow T_{f(x)}N = T_{g(y)}N \; ; \qquad (v, w) \;\; \mapsto \;\; Df(x)(v) + Dg(y)(w)$$

is surjective.

Proposition 7.5.2 *If $f \pitchfork g$, then the set $L = \{(x,y) \in M_1 \times M_2 \mid f(x) = g(y)\}$ is a smooth submanifold of $M_1 \times M_2$, of dimension $m_1 + m_2 - \ell$. For $(x,y) \in L$, the tangent space*

$$T_{(x,y)}L \quad \subset \quad T_{(x,y)}(M_1 \times M_2) \;\cong\; T_xM_1 \times T_yM_2$$

is the linear subspace $\{(v,w) \in T_xM_1 \times T_yM_2 \mid Df(x)(v) = Dg(y)(w)\ \}$.

Proof Exercise. $\square$

Theorem 7.5.3 (a version of Thom's transversality theorem, early 1950's.) *Let smooth manifolds M_1, M_2, N and smooth maps $f : M_1 \to N$, $g : M_2 \to N$ be given. Then there exist an integer $q > 0$, an open subset U of $\mathbb{R}^q$ containing 0, a smooth map*

$$F : M_1 \times U \longrightarrow N \ ,$$

and a subset of measure zero $Z \subset U$ such that

- *$F(x,0) = f(x)$ for all $x \in M_1$;*
- *for any $a \in \mathbb{R}^q \smallsetminus Z$, the smooth map $M_1 \to N$ given by $x \mapsto F(x,a)$ is transverse to $g : M_2 \to N$.*

Remark. The maps $f^a : M_1 \to N$ given by $x \mapsto F(x,a)$, for fixed $a \in \mathbb{R}^k$, should be regarded as "small perturbations" of f, at least when a is close to 0. Each of these maps is (clearly) homotopic to f. In particular, $f^0 = f$.

Proof (Sketch.) We will cheat a little by assuming that N is compact. By theorem 7.3.1, we may then also assume that N is a smooth manifold in $\mathbb{R}^q$ for some q (alias smooth submanifold of $\mathbb{R}^q$). We choose an open set $V \subset \mathbb{R}^q$ containing N, and a smooth map

$$r : V \to N$$

such that $r|N = \mathrm{id}_N$. (*Exercise*: construct such a V and such a map r.) The linear map $Dr(x)$ from T_xV to $T_{r(x)}N$ is *surjective* if $x \in N \subset V$. Therefore, by making V smaller if necessary, we can arrange that $Dr(x) : T_xV \to T_{p(x)}N$ is surjective for all $x \in N$.
Next, choose an open neighbourhood U of $0 \in \mathbb{R}^q$ such that the map $M_1 \times U \to \mathbb{R}^q$ given by the formula $(x,a) \mapsto f(x) + a$ has image contained in V. Now define

$$F(x,a) = r(f(x) + a) \in N$$

for $x \in M_1$ and $a \in U$. Then clearly $F(x,0) = r(f(x)) = f(x)$ for all $x \in M_1$. Furthermore, the differential of F at any point $(x,a) \in M_1 \times U$ is a *surjective* linear map.[d] It follows (trivially) that $F : M_1 \times U \to N$ is transverse to $g : M_2 \to N$. Therefore, by proposition 7.5.2, the set

$$L = \{(x,a,y) \in M_1 \times U \times M_2 \mid F(x,a) = g(y)\}$$

is a smooth submanifold of $M_1 \times U \times M_2$, of dimension $m_1 + q + m_2 - n$. We have a smooth projection map

$$p : L \to U$$

given by $(x,a,y) \mapsto a$. Let $Z \subset U$ be the set of critical values of p. Then Z has measure zero by Sard's theorem. It remains to show that, for $a \in U \smallsetminus Z$, the map

$$f^a : M_1 \to N \ ; \quad x \mapsto F(x,a)$$

is transverse to $g : M_2 \to N$.
Suppose that we have $x \in M_1$ and $y \in M_2$ with $f^a(x) = g(y)$. Then $(x,a,y) \in L$ and, moreover, since $a \notin Z$, we know that (x,a,y) is a *regular* point of the projection $p : L \to U$. Hence the dimension of

$$\ker(Dp(x,a,y)) \quad = \quad T_{(x,a,y)}L \quad \cap \quad T_{(x,a,y)}(M_1 \times \{a\} \times M_2) \quad \subset \quad T_{(x,a,y)}(M_1 \times U \times M_2)$$

is equal to $\dim(L) - \dim(U) = m_1 + m_2 - n$. Now we can make an identification

$$T_{(x,a,y)}(M_1 \times \{a\} \times M_2) \;\cong\; T_xM_1 \times T_yM_2$$

and observe (using the formula for tangent spaces in proposition 7.5.2) that $T_{(x,a,y)}L \cap (T_xM_1 \times T_yM_2)$ is equal to $\{(v,w) \in T_xM_1 \times T_yM_2 \mid Df^a(x)(v) = Dg(y)(w)\ \}$. Since that has dimension $m_1 + m_2 - n$, the linear map $(v,w) \mapsto Df^a(x)(v) - Dg(y)(w)$ is surjective. Then $(v,w) \mapsto Df^a(x)(v) + Dg(y)(w)$ is also surjective. $\square$

[d] Use the chain rule, noting that F was defined as a composition of two smooth maps.

SMSTC (2008/09)
Geometry and Topology
Lecture 8: Applications of the transversality theorem

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Contents

The transversality theorem allows us to reach astonishing depths in differential topology for little work, somewhat like Galois theory in algebra. It will *not* be the dominating idea in our course, because it would take us too far from the metric and analytic aspects of geometry. This chapter at any rate is an excursion into the world of transversality.

8.1 Orientations

Definition 8.1.1 Let V be an n-dimensional real vector space, $n < \infty$. An *orientation* of V is a function which for every linear isomorphism $f : \mathbb{R}^n \to V$ selects a sign $s(f) \in \{-1, +1\}$, subject to the following condition:

$$s(f) = s(g) \cdot \text{ sign of } \det(f^{-1} \circ g)$$

(where $f, g : \mathbb{R}^n \to V$ are linear isomorphisms).

Lemma 8.1.2 Every finite dimensional real vector space has exactly two distinct orientations.

The proof is easy and you may find the result unremarkable. But it is worth noting that the lemma also holds in the case of a 0-dimensional vector space. This will be quite important for us.

Definition 8.1.3 Let V_1 and V_2 be finite dimensional real vector spaces. A choice of orientations s_1 and s_2 for V_1 and V_2 respectively determines an orientation $s = s_1 \oplus s_2$ for $V_1 \oplus V_2$ with the property

$$(s_1 \oplus s_2)\begin{pmatrix} f_1 & \\ & f_2 \end{pmatrix} = s(f_1) \cdot s(f_2) \ .$$

This simple-minded construction leads to a few more of the same ilk.

Definition 8.1.4 Let V be an n-dimensional real vector space, $W \subset V$ an m-dimensional linear subspace. A choice of orientations for two of the three vector spaces V, W and V/W determines an orientation for the third as follows. We choose a splitting

$$u : V/W \to V$$

of the projection $p : V \to V/W$, so that $p \circ u = \text{id}$. This allows us to identify V with $W \oplus V/W$. We now determine the missing orientation so that the equation $s_V = s_W \oplus s_{V/W}$ holds.

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Definition 8.1.5 Let V, W_1 and W_2 be finite dimensional real and let $f : W_1 \to V$, $g : W_2 \to V$ be linear maps. Suppose that the map

$$W_1 \times W_2 \longrightarrow V\ ; \quad (w', w'') \mapsto f(w') + g(w'')$$

is onto. Then any choice of orientations s_V, s_1 and s_2 for V, W_1 and W_2 respectively determines an orientation for the vector space

$$P = \{(w', w'') \in W_1 \times W_2 \mid f(w') = g(w'')\}$$

as follows. We use $s_1 \oplus s_2$ to orient $W_1 \times W_2 \cong W_1 \oplus W_2$. Now $P \subset W_1 \times W_2$ and we can identify V with $(W_1 \times W_2)/P$ using the map $(w', w'') \mapsto f(w') - g(w'')$ from $W_1 \times W_2$ to V. Then we apply the previous definition.

Definition 8.1.6 An *orientation* of a smooth manifold M is a choice of orientations for each of the tangent vector spaces T_xM , depending continuously on $x \in M$.

Lemma 8.1.7 *Let M_1 and M_2 be smooth manifolds. A choice of orientations for M_1 and M_2 determines an orientation of $M_1 \times M_2$.*

Proof This follows from definition 8.1.3. Note that the order of the factors matters. More precisely, if M_1 and M_2 are both odd-dimensional, then the diffeomorphism

$$M_1 \times M_2 \to M_2 \times M_1\ ; \qquad (x_1, x_2) \mapsto (x_2, x_1)$$

reverses orientations. In all other cases, it preserves orientations. $\square$

Lemma 8.1.8 *Let N, M_1 and M_2 be* oriented *smooth manifolds. Let $f : M_1 \to N$ and $g : M_2 \to N$ be smooth maps. Suppose that f is transverse to g. Then the smooth manifold*

$$L = \{(x_1, x_2) \in M_1 \times M_2 \ \mid\ f(x_1) = g(x_2)\}$$

has a preferred orientation.

Proof This follows from definition 8.1.5 and the description of the tangent spaces of L given in proposition 7.5.2. $\square$

8.2 Smooth manifolds with boundary

Write $\mathbb{R}^k_{\text{up}}$ for k-dimensional "upper half-space",

$$\mathbb{R}^k_{\text{up}} = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_1 \geq 0\}.$$

(For $k = 0$ let $\mathbb{R}^0_{\text{up}} = \mathbb{R}^0$.)

Definition 8.2.1 Let n and k be integers, $n \geq k \geq 0$. A subset M of $\mathbb{R}^n$ is a k-dimensional smooth manifold with boundary in $\mathbb{R}^n$ if, for each $x \in M$, there exists open subsets U and V of $\mathbb{R}^n$, with $x \in V$, and a diffeomorphism

$$\varphi : U \to V$$

such that $\varphi(U \cap \mathbb{R}^k_{\text{up}}) = V \cap M$.

In the above circumstances, we call $\varphi : U \to V$ an *ambient chart* about $x \in M$, as usual. Suppose that, for some ambient chart $\varphi : U \to V$ about $x \in M$, we have

$$\varphi^{-1}(x) \in \{y \in U \cap \mathbb{R}^k_{\text{up}} \mid y_1 = 0\}.$$

Then the same will be true for any other ambient chart about the same $x \in M$. (That's an exercise.) We call such an x a *boundary* point of M. The set of all boundary points in M is a subset ∂M of M, the *boundary* of M. Almost by definition, ∂M is a smooth manifold of dimension $k-1$ in $\mathbb{R}^n$. The difference $M \smallsetminus \partial M$ is a smooth manifold of dimension k in $\mathbb{R}^n$. (That's another exercise, but an easy one.) The boundary ∂M can of course be empty.[b]

[b]Therefore a smooth manifold in $\mathbb{R}^n$ is always a smooth manifold-with-boundary in $\mathbb{R}^n$, but strictly speaking a smooth manifold-with-boundary in $\mathbb{R}^n$ need not be a smooth manifold in $\mathbb{R}^n$.

Example 8.2.2 The disk $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\| \}$ is an n-dimensional smooth manifold with boundary in $\mathbb{R}^n$. Its boundary is, of course, the sphere $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\| \}$.

Example 8.2.3 The compact Moebius strip $M \subset \mathbb{R}^3$ can be described as the *image* of the map

$$f : [0, \pi] \times [-1, 1] \longrightarrow \mathbb{R}^3 \ ; \quad (\alpha, t) \mapsto \rho_{2\alpha}((2, 0, 0) + \sigma_\alpha(t, 0, 0))$$

where σ_α is the rotation about the x_2-axis by an angle α, and $\rho_{2\alpha}$ is the rotation about the x_3-axis by an angle 2α. Note that $f(0, t) = f(\pi, -t)$. In this description, the compact Moebius strip is a 2-dimensional smooth submanifold with boundary in $\mathbb{R}^3$. Its boundary is diffeomorphic to S^1.

Definition 8.2.4 Let M be a set. A *smooth atlas with boundary* for M consists of a choice of non-negative integer m, which is the “dimension” of the atlas, and a set $\mathcal{A}$ of *charts*. Each chart is an injective map $\psi : U \to M$, where U is an open subset of $\mathbb{R}^m_{\rm up}$. There are several conditions to be satisfied:

- The atlas covers all of M. That is, for each $z \in M$ there exists a chart $\psi : U \to M$ in $\mathcal{A}$ such that $z \in \psi(U)$.
- Changes of charts are smooth. More precisely, if $\psi_1 : U_1 \to M$ and $\psi_2 : U_2 \to M$ are any distinct charts in $\mathcal{A}$, then
  - the set $\psi_1^{-1}(\psi_2(U_2))$ is open in U_1 and hence in $\mathbb{R}^m_{\rm up}$,
  - the set $\psi_2^{-1}(\psi_1(U_1))$ is open in U_2 and hence in $\mathbb{R}^m_{\rm up}$,
  - the map $\psi_1^{-1}(\psi_2(U_2)) \longrightarrow \psi_2^{-1}(\psi_1(U_1))$ defined by $x \mapsto \psi_2^{-1}(\psi_1(x))$ is smooth (and consequently continuous).

Remarks. The “change of chart” map from $U_{1|2} = \psi_1^{-1}(\psi_2(U_2))$ to $U_{2|1} = \psi_2^{-1}(\psi_1(U_1))$ has a smooth inverse, because the smoothness condition also holds for its inverse. It follows that the change of chart map $U_{1|2} \to U_{2|1}$ takes $U_{1|2} \cap \mathbb{R}^{m-1}$ to $U_{2|1} \cap \mathbb{R}^{m-1}$, bijectively. (That's the same old exercise ...)
Consequently we can say that $x \in M$ is a *boundary point* of M if for some (hence any) chart $\psi : U \to M$ in $\mathcal{A}$ with $x \in \operatorname{im}(\psi)$, the element $\psi^{-1}(x) \in U$ belongs to $\{y \in U \mid y_1 = 0\}$. We write ∂M for the set of boundary points, so that $\partial M \subset M$. For each $\psi : U \to M$ in $\mathcal{A}$, the restriction

$$U \cap \{y \in \mathbb{R}^m \mid y_1 = 0\} \longrightarrow \partial M$$

of ψ can be regarded as a typical chart in an $(m-1)$-dimensional smooth atlas $\partial\mathcal{A}$ for the set ∂M.

Definition 8.2.5 A smooth m-dimensional manifold with boundary of consists of a set M and an equivalence class of smooth m-dimensional atlases with boundary on M, such that the resulting topology on M satisfies the Hausdorff separation axiom and the 2nd countability axiom.

Example 8.2.6 Let N be a smooth m-dimensional mfld, $f : N \to \mathbb{R}$ a smooth map and suppose that $0 \in \mathbb{R}$ is a regular value of f. Then

$$M = \{x \in N \mid f(x) \geq 0\}$$

is a smooth m-dim'l manifold with boundary, $\partial M = f^{-1}(0)$. (To get charts, start with diffeomorphisms $\psi : U \to V \subset N$ where U open in $\mathbb{R}^m$ and V open in N, and require $f(\psi(x)) \geq 0$ if $x_1 \geq 0$, $f(\psi(x)) \leq 0$ if $x_1 \leq 0$. Then use restrictions $\psi | U \cap \mathbb{R}^m_{\rm up}$.)

In the remainder of this (sub)section we describe somewhat informally how various concepts related to smooth manifolds generalise to smooth manifolds with boundary.

• Let M be a smooth m-dimensional manifold with boundary. Let $z \in M$. A *tangent vector* to M at z is a rule v which to every chart $\varphi : U \to M$ (in any allowed atlas for M) with $z = \varphi(y) \in \varphi(U)$ assigns a vector $v_\varphi \in \mathbb{R}^m$, subject to the condition

$$v_\psi = D(\psi^{-1} \circ \varphi)(y)(v_\varphi).$$

(Here $\psi : U' \to M$ is another chart with $z \in \psi(U')$ and $D(\psi^{-1} \circ \varphi)(y)$ is the differential of $\psi^{-1} \circ \varphi$ at y, a linear isomorphism $\mathbb{R}^m \to \mathbb{R}^m$.)

• The tangent vectors to M at z form an m-dimensional real vector space, which we denote by T_zM. This applies regardless of whether $z \in \partial M$ or $z \notin \partial M$. But if $z \in \partial M$, then we also have an $(m-1)$-dimensional tangent space $T_z\partial M$, and it is a good idea to think of that as a linear subspace of T_zM:

$$T_z\partial M \subset T_zM \ .$$

For $z \in M \smallsetminus \partial M$, any $v \in T_zM$ can be thought of as the velocity of a smooth curve $\gamma : J \to M$, where J is an open interval in $\mathbb{R}$ containing 0, and $\gamma(0) = z$. For $z \in \partial M$, any $v \in T_zM \smallsetminus T_z\partial M$ can be thought of as $\pm$ the velocity of a smooth curve $\gamma : L \to M$, where L is a half-open interval $[0, a)$ and $\gamma(0) = z$. (The sign, $+$ or $-$, is determined by v and tells us whether v points "inwards" or "outwards".)

• The (disjoint) union TM of all tangent spaces T_zM has a preferred structure of smooth manifold with boundary, of dimension $2m$. In fact any of the allowed atlases $\mathcal{A}$ with boundary for M determines an atlas for TM, made up of charts $T\psi : TU \to TM$ corresponding to charts $\psi : U \to M$ in $\mathcal{A}$. Here TU can be identified with $U \times \mathbb{R}^m$.

Definition 8.2.7 Let M be a smooth m-dimensional manifold with boundary. A subset L of M is a *neat* ℓ-dimensional smooth submanifold (with boundary) of M if the following holds. For every $x \in L$ there exists a chart $\varphi : U \to M$ in some allowed atlas for M, with U open in $\mathbb{R}^m_{\text{up}}$ and $x \in \varphi(U)$, and

$$\varphi^{-1}(L) = \mathbb{R}^\ell_{\text{up}} \cap U \ .$$

Here we think of $\mathbb{R}^\ell_{\text{up}}$ as $\{y \in \mathbb{R}^m_{\text{up}} \mid y_{\ell+1} = y_{\ell+2} = \cdots = y_m = 0\}$.

Definition 8.2.8 An orientation of a smooth m-dimensional manifold with boundary M is a choice of orientations for each of the vector spaces T_xM , depending continuously on $x \in M$.

Definition 8.2.9 An orientation of a smooth m-dimensional manifold with boundary M determines an orientation of ∂M in the following way. Let $x \in \partial M$. Choose a vector $u \in T_xM \smallsetminus T_x\partial M$ which points "out" of M. This determines a linear isomorphism

$$j : \mathbb{R} \oplus T_x\partial M \to T_xM \ : \quad t \oplus v \mapsto tu + v.$$

Therefore (see definition 8.1.3) every orientation s of $T_x\partial M$ determines an orientation $s_{\mathbb{R}} \oplus s$ of $T_xM \cong \mathbb{R} \oplus T_x\partial M$, and vice versa, with $s_{\mathbb{R}}$ equal to the standard orientation of $\mathbb{R}$. (This is an important convention. If you need a mantra for memorization, try *outward normal first*, abbreviated ONF.)

Example 8.2.10 The interval $[0, 1]$ is a smooth 1-dimensional manifold with boundary. It has a standard orientation which we get by identifying each tangent space T_xM with $\mathbb{R}$. Then $\partial[0, 1] = \{0, 1\}$. If we orient $\partial[0, 1]$ following the instructions in definition 8.2.9, then the point 1 is positively oriented, while 0 is negatively oriented.

Example 8.2.11 The disk $M = D^2 \subset \mathbb{R}^2$ is a smooth 2-dimensional manifold with boundary. It has a standard orientation which we get by identifying each tangent space T_xM with $\mathbb{R}^2$. The boundary ∂M is the circle S^1. If we orient this following the instructions in definition 8.2.9, then S^1 gets the "counterclockwise" orientation.

Finally we describe a mild generalization of the transversality theorem 7.5.3 to a situation "with boundary". The proof is not very different from the one we have seen in the case without boundaries.

Theorem 8.2.12 *Let smooth manifolds M_1, M_2, N and smooth maps $f : M_1 \to N$, $g : M_2 \to N$ be given. Assume that M_1 and N have empty boundary, but allow M_2 to have nonempty boundary. Then there exist an integer $q > 0$, an open subset U of $\mathbb{R}^q$ containing 0, a smooth map*

$$F : M_1 \times U \longrightarrow N \ ,$$

and a subset of measure zero $Z \subset U$ such that

- *$F(x, 0) = f(x)$ for all $x \in M_1$;*
- *for any $a \in \mathbb{R}^q \smallsetminus Z$, the smooth map $M_1 \to N$ given by $x \mapsto f_a(x) = F(x, a)$ is transverse to $g : M_2 \to N$ and also to the restriction $g|\partial M_2$.*

In this situation $L = \{(x_1, x_2) \in M_1 \times M_2 \ \mid \ f_a(x_1) = g(x_2) \ \}$ is a smooth manifold with boundary (actually a neat smooth submanifold of $M_1 \times M_2$). We have $n - \ell = (n - m_1) + (n - m_2)$ where n, m_1, m_2, ℓ are the dimensions of N, M_1, M_2 and L, respectively.

8.3 The bordism relation

Definition 8.3.1
- An unoriented m-dimensional compact smooth manifold M (without boundary) is said to be *nullbordant* if there exists an $(m+1)$-dimensional compact smooth manifold with boundary W such that ∂W is diffeomorphic to M.
- An oriented m-dimensional compact smooth manifold M (without boundary) is said to be *oriented nullbordant* if there exists an oriented $(m+1)$-dimensional compact smooth manifold with boundary W such that ∂W (with the orientation as in definition 8.2.9) admits an orientation preserving diffeomorphism to M.
- Two unoriented m-dimensional compact smooth manifolds M_0 and M_1 (without boundary) are *bordant* if $M_0 \amalg M_1$ is nullbordant.
- Two oriented m-dimensional compact smooth manifolds M_0 and M_1 (without boundary) are *oriented bordant* if $-M_0 \amalg M_1$ is oriented nullbordant. Here $-M_0$ is M_0 with the opposite of the specified orientation.

It is not very difficult to show that *bordant* and *oriented bordant* are equivalence relations. For symmetry, if W is a nullbordism for $M_0 \amalg M_1$, then it is also a nullbordism for $M_1 \amalg M_0$. (In the oriented setting, use $-W$). For reflexivity, $[0,1] \times M$ is a nullbordism of $M \amalg M$ in the unoriented setting (and a nullbordism of $-M \amalg M$ in the oriented setting). *Transitivity* is admittedly a little more difficult to establish. For example, in the unoriented setting, given a nullbordism W_1 for $M_0 \amalg M_1$ and another nullbordism W_2 for $M_1 \amalg M_2$, we would like to make a nullbordism W for $M_0 \amalg M_2$ by making identifications $f(x) \sim g(x)$ for $x \in M_1$, where

$$f : M_0 \amalg M_1 \to \partial W_1 \ , \qquad g : M_1 \amalg M_2 \to \partial W_2$$

are those diffeomorphisms which are given to us as part of the "nullbordism" data. Although it is clear how W is a topological space, it is not so clear how it is a smooth manifold. The following *collaring* lemma is needed:

Lemma 8.3.2 *Let M be an m-dimensional smooth manifold with boundary. Then there exist an open neighbourhood U of ∂M in M and a diffeomorphism $\psi : \partial M \times [0,1[\ \to U$, with $\psi(x,0) = x$ for $x \in \partial M$.*

A proof of this can be found in the Bröcker-Jaenich book. It uses partitions of unity. We will not reproduce it here.

The equivalence classes [...] of the bordism relation on n-dimensional (smooth, compact, unoriented, without boundary) manifolds form an abelian group $\mathfrak{N}_n$ with the operation

$$[M] + [M'] := [M \amalg M'] \ .$$

The neutral element of $\mathfrak{N}_n$ is $\emptyset$, which we are allowed to regard as an n-dimensional smooth manifold for any $n \in \mathbb{Z}$. Let's note that $\mathfrak{N}_n$ is actually a vector space over $\mathbb{Z}/2$ because $2[M] = [M \amalg M] = [\emptyset]$ because $M \amalg M$ is always nullbordant. Moreover the product of manifolds induces a multiplication operation

$$\mathfrak{N}_m \times \mathfrak{N}_n \longrightarrow \mathfrak{N}_{m+n}$$

which makes $\mathfrak{N}_* = (\mathfrak{N}_n)_{n \geq 0}$ into a commutative *graded ring* or more precisely a graded algebra over the field $\mathbb{Z}/2$. It is fairly clear that $\mathfrak{N}_0 \cong \mathbb{Z}/2$ and it is tempting to think that $\mathfrak{N}_n$ must be zero for all $n > 0$. But this is not the case. In fact, making spectacularly good use of his transversality theorem, René Thom was able to determine the structure of $\mathfrak{N}_\star$ in the mid 1950's. His result was that $\mathfrak{N}_\star$ is a polynomial algebra with generators x_n of degree n, one such for each integer $n > 0$ which is not of the form $2^i - 1$.

For similar reasons, the equivalence classes of the bordism relation on n-dimensional (smooth, compact, without boundary) *oriented* manifolds form an abelian group Ω_n with the operation

$$[M] + [M'] := [M \amalg M'] \ .$$

The product of oriented manifolds induces a multiplication operation

$$\Omega_m \times \Omega_n \longrightarrow \Omega_{m+n}$$

which makes $\Omega_* = (\Omega_n)_{n\geq 0}$ into a *graded commutative ring.* Note that "graded commutative" means

$$[M_1]\cdot[M_2] = (-1)^{m_1 m_2}[M_2]\cdot[M_1]$$

where m_1 and m_2 are the dimensions of M_1 and M_2 , respectively. The structure of Ω_* is more complicated than that of $\mathfrak{N}_*$. It was determined in the late 1950s by several people, especially J.Milnor and C.T.C.Wall. All we are going to use here is

$$\Omega_0 \cong \mathbb{Z}$$

which is an interesting exercise.

8.4 Homotopy groups that everybody should know

Theorem 8.4.1 *If $n > m > 0$ then $\pi_m(S^n)$ is trivial. If $n = m > 0$, then $\pi_m(S^n) \cong \mathbb{Z}$.*

An *idea* of the proof can be given in a few lines.

(i) We believe that every continuous based map $S^m \to S^n$ is homotopic to a smooth based map $S^m \to S^n$, and also that if two smooth based maps $S^m \to S^n$ are homotopic, then a smooth homotopy relating them can be found.

(ii) We observe that a smooth map $S^m \to S^n$ which avoids the point $c = (-1, 0, 0, \ldots, 0) \in S^n$ (the antipode of the base point) is based nullhomotopic. Therefore, given a smooth based map $f : S^m \to S^n$, we ask first of all: what is $f^{-1}(z)$?

(iii) Perturbing f slightly if necessary, we can assume that f is transverse to c (more precisely, that f is transverse to the inclusion $\{c\} \to S^n$). In that case, if $m < n$, the preimage $f^{-1}(c)$ must be empty and therefore f is nullhomotopic. In the case $m = n$, the preimage $f^{-1}(c)$ is an oriented 0-dimensional compact smooth manifold. It represents an element in the bordism group $\Omega_0 \cong \mathbb{Z}$. This is well defined, i.e., depends only on the based homotopy class of f.

(iv) Therefore $\pi_m(S^n)$ is trivial for $m < n$. If $m = n$, we have a homomorphism $\pi_m(S^n) \to \Omega_0$. It is clearly onto because the class [id] maps to a generator of $\Omega_0 \cong \mathbb{Z}$. We hope that it is also injective.

Unfortunately injectivity of the homomorphism in (iv) is not obvious. We need to reinforce (ii) above. This is done in the following lemma. Let X be a based compact Hausdorff space (compact metric space if you prefer).

Lemma 8.4.2 *Let $f : X \to S^n$ and $g : X \to S^n$ be based maps. If $f^{-1}(c) = g^{-1}(c)$ and $f|U = g|U$ for some open neighbourhood U of $f^{-1}(c)$ in X, then f and g are based homotopic.*

Proof The image under f of $X \smallsetminus U$ is compact. Therefore we can find a neighbourhood V of c in S^n such that $f^{-1}(V) \subset U$. Now it is easy to construct a based map $q : S^n \to S^n$, based homotopic to the identity, such that q takes the complement of V to the base point $\star \in S^n$. Then $q \circ f = q \circ g$, so that $q \circ f$ and $q \circ g$ are certainly homotopic. But then $f \simeq g$ because q is homotopic to the identity. □

Proof *of theorem* 8.4.1. Let $f : S^n \to S^n$ be a based map. We are mostly interested in its homotopy class. We may therefore assume that f is smooth and transverse to c. Then $Z = f^{-1}(c)$ is a finite set which we can enumerate: $Z = \{z(1), z(2), \ldots, z(r)\}$. We identify S^n with $\mathbb{R}^n \cup \infty$, letting the base point $\star$ correspond to ∞ and c to the origin 0. For each $z(i) \in Z$ we choose a little open cube Q_i about $z(i)$ and a diffeomorphism $\varphi_i : Q_i \to \mathbb{R}^n$ such that φ_i agrees with f in a neighbourhood of $z(i)$. This is possible (details omitted though) by the inverse function theorem, because f has invertible differential at $z(i)$. Now we have a new continuous map

$$g : S^n = \mathbb{R}^n \cup \infty \longrightarrow \mathbb{R}^n \cup \infty = S^n$$

given by $g|Q_i = \varphi_i$ and $g(x) = \infty$ if x is not in any of the open cubes Q_i. By lemma 8.4.2, the map g is based homotopic to f. We now subject g to further homotopies, initially by moving the points of Z around. If we move them so that they are lined up on the x_1-axis, they will be ordered and we can assume that our enumeration of them agrees with that ordering. Then we see that

$$[g] = [g_1] \bullet [g_2] \bullet \cdots \bullet [g_r] \in \pi_n(S^n)$$

where $g_i : S^n \cup \infty \to S^n \cup \infty$ is given by $g_i|Q_i = \varphi_i$ and $g(x) = \infty$ if x is not in Q_i. Here we have used the bullet symbol to describe the multiplication in the *group* $\pi_n(S^n)$.
Next we show that $[g_i]$ is equal to [id] if φ_i is orientation preserving. Choose some diffeomorphism $\psi_i : Q_i \to \mathbb{R}^n$ such that $\psi_i(x) = x - z(i)$ for all x in a small neighbourhood of the center of the cube Q_i. Then $\zeta_i = \varphi_i \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is an orientation preserving diffeomorphism. The formula

$$(x,t) \mapsto \begin{cases} t^{-1}\zeta_i(tx) & \text{if } t > 0 \\ D\zeta_i(0)(x) & \text{if } t = 0 \end{cases}$$

where $x \in \mathbb{R}^n$ and $t \in [0,1]$ describes, for every fixed $t \in [0,1]$, a diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$, equal to ζ_i when $t = 1$ and equal to the orientation preserving invertible linear map $D\zeta_i(0) : \mathbb{R}^n \to \mathbb{R}^n$ when $t = 0$. It follows that $[g_i]$ is equal to the homotopy class of the map $f_i : \mathbb{R}^n \cup \infty \to \mathbb{R}^n \cup \infty$ given by

$$D\zeta_i(0) \circ \psi_i$$

on Q_i and by $x \mapsto \infty$ for $x \notin Q_i$. Next we choose a continuous family of invertible linear maps $A_t : \mathbb{R}^n \to \mathbb{R}^n$ with $t \in [0,1]$ such that $A_0 = D\zeta_i(0)$ and $A_1 = \text{id}$. This is possible because the space of invertible real $n \times n$ matrices with positive determinant is path-connected. Then we use this to conclude that $[f_i]$ is equal to the homotopy class of the map given by ψ_i on Q_i and by $x \mapsto \infty$ for $x \notin Q_i$. Finally we use lemma 8.4.2 to show that the latter homotopy class is equal to [id].
We can similarly show that $[g_i]$ is equal to $[\text{id}]^{-1}$, the inverse of [id] in the group $\pi_n(S^n)$, in the case where $\varphi_i : Q_i \to \mathbb{R}^n$ is orientation reversing. In that case we start by choosing a diffeomorphism $\psi_i : Q_i \to \mathbb{R}^n$ such that $\psi_i(x_1, x_2, \dots, x_n) = (-x_1, x_2, \dots, x_n) - z(i)$ for all $x = (x_1, \dots, x_n)$ in a small neighbourhood of the center of the cube Q_i.
To sum up, we started with an arbitrary $[f] \in \pi_n(S^n)$. We showed that it was equal to some $[g]$, and that was found to be equal to a product $[g_1] \bullet [g_2] \bullet \cdots \bullet [g_r]$, where each $[g_i]$ is either equal to [id] or to $[\text{id}]^{-1}$. The conclusion to be drawn from this is that $\pi_n(S^n)$ is a cyclic group, with [id] as generator. It follows that our homomorphism $\pi_n(S^n) \to \Omega_0 \cong \mathbb{Z}$ is injective, because we know already that it is nonzero. □

Definition 8.4.3 Let M and N be smooth compact manifolds (without boundary) of the same dimension n, both oriented and both path-connected. Let $f : M \to N$ be any smooth map. The *degree* of f is an integer defined as follows. Choose a regular value $c \in N$ of f. Then $f^{-1}(c)$ is an oriented 0-dimensional compact smooth manifold, representing an element of $\Omega_0 \cong \mathbb{Z}$. This element is the degree of f. It is well defined, i.e., independent of the choice of a regular value c for f. (*Prove this.*) It also depends only on the homotopy class of f.

We have implicitly used this concept of *degree of a map* in proving theorem 8.4.1. Thus, homotopy classes of maps $S^n \to S^n$ are classified by their degree, which can be any integer. Although we calculated strictly speaking $\pi_n(S^n) = [S^n, S^n]_\star$, the set of based homotopy classes of based maps, we have also determined $[S^n, S^n]$, the set of homotopy classes of unbased maps. In fact it is easy to see that the forgetful map $[S^n\ S^n]_\star \to [S^n, S^n]$ is onto. But the degree function is defined on $[S^n, S^n]$, so that the forgetful map has to be injective, too.

8.5 The Euler number of the tangent bundle

Let M be an oriented smooth n-dimensional manifold, compact and without boundary. The Euler number of the tangent bundle TM of M is defined as follows. Let $\zeta : M \to TM$ be the standard inclusion, $\zeta(x) = 0_x \in T_xM \subset TM$. Choose another smooth map $\xi : M \to TM$ homotopic to ζ and transverse to ζ. Then

$$L = \{(x_1, x_2) \in M \times M \mid \zeta(x_1) = \xi(x_2)\ \}$$

is a 0-dimensional oriented compact (smooth) manifold. As such it defines an element in the oriented bordism group $\Omega_0 \cong \mathbb{Z}$.

Definition 8.5.1 That number is the Euler number of TM. We sometimes denote it by $\chi(M)$.

The Euler number of TM is well defined for the following reason. Suppose that $\xi_1 : M \to TM$ and $\xi_2 : M \to TM$ are *both* homotopic to ζ and transverse to ζ. Then we have

$$L_i = \{\ (x_1, x_2) \in M \times M \mid \zeta(x_1) = \xi_i(x_2)\ \}$$

for $i = 1, 2$. We must find an oriented bordism from L_1 to L_2 to show that the two represent the same element in Ω_0. To do so we choose a homotopy $h : M \times [0, 1] \to TM$ from ξ_1 to ξ_2. We can assume, after a suitable "perturbation", that it is smooth and transverse to ζ. (Here we use a little more than we have stated in chapter 7.5.3, because we need a perturbation which does not perturb h on $M \times \{0\}$ or on $M \times \{1\}$.) Then

$$K = \{\ (x_1, x_2, t) \in M \times M \times [0, 1] \mid \zeta(x_1) = h(x_2, t)\ \}$$

is an oriented compact 1-dimensional smooth manifold, and its oriented boundary is clearly diffeomorphic to $-L_1 \amalg L_2$. So K is a bordism from L_1 to L_2.

Remark 8.5.2 We have used the fact that an orientation of M determines an orientation of TM. But in fact the manifold TM always has a preferred orientation, even when M is not orientable. Think about that.

Remark 8.5.3 Typically we construct maps $\xi : M \to TM$ homotopic to ζ, and sometimes transverse to ζ, by choosing a *smooth tangent vector field* on M. A smooth tangent vector field on M is a map $\xi : M \to TM$ such that $p \circ \xi = \mathrm{id}_M$, where $p : TM \to M$ is the projection. What this means is that ξ selects, at every point $x \in M$, a tangent vector $\xi(x) \in T_xM$ which can be imagined as the velocity of some "fluid" or some gaseous substance moving about in the manifold. In such a case the points $x \in M$ where $\xi(x) = \zeta(x)$ are the points where $\xi(x)$ is the zero vector, i.e., the points where the vector field is "stationary". These points are important to us because we count them (with appropriate multiplicities) to get $\chi(M)$. In particular, if M admits a smooth tangent vector field which is nowhere zero, then $\chi(M) = 0$.

Proposition 8.5.4 $\chi(S^n) = \begin{cases} 2 & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd.} \end{cases}$

Proof Think of S^n as a submanifold of $\mathbb{R}^{n+1}$, so that TS^n becomes a submanifold of $\mathbb{R}^{2n+2}$. Define a vector field $\xi : S^n \to TS^n$ as follows: $\xi(x)$ is the orthogonal projection of $(0, 0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ to the tangent space $T_xS^n \subset \mathbb{R}^{n+1}$. This translates into

$$\xi(x) = (x_1, \dots, x_n, x_{n+1}, -x_{n+1}x_1, \dots, -x_{n+1}x_n, 1 - x_{n+1}x_{n+1}) \in \mathbb{R}^{2n+2}\ .$$

There are two points $x \in S^n$ where $\xi(x) = \zeta(x)$: they are $x = (0, \dots, 0, 1)$ and $x = (0, \dots, 0, -1)$. The differential of ξ at these points is the linearisation of the above formula for $\xi(x)$, that is

$$v = (v_1, \dots, v_n, 0) \mapsto (v_1, \dots, v_n, 0, -v_1, \dots, -v_n, 0)$$

for $x = (0, \dots, 0, 1)$ and

$$v = (v_1, \dots, v_n, 0) \mapsto (v_1, \dots, v_n, 0, v_1, \dots, v_n, 0)$$

for $x = (0, \dots, 0, -1)$. The tangent space of TS^n at $\xi(x) = \zeta(x)$ for these x is the linear space of all $w \in \mathbb{R}^{2n+2}$ with $w_{n+1} = 0 = w_{2n+2}$. The image of the differential $D\zeta(x)$ is the linear space of all $w \in \mathbb{R}^{2n+2}$ with $w_i = 0$ for $i = n+1, \dots, 2n+2$. It follows from all the above that ξ is transverse to ζ and that the point $x = (0, \dots, 0, -1)$ contributes $+1$ to the Euler number count, while $x = (0, \dots, 0, 1)$ contributes $(-1)^n$. □

Corollary 8.5.5 (Hairy ball theorem.) *For even n, every smooth tangent vector field ξ on S^n vanishes somewhere, i.e., there exists $x \in S^n$ such that $\xi(x) = 0 \in T_xM$.*

Proof If $\xi(x)$ is everywhere nonzero, then $\xi : M \to TM$ is certainly transverse to $\zeta : M \to TM$. There are no $x \in S^n$ with $\xi(x) = \zeta(x)$ and so $\chi(S^n) = 0$. This contradicts our calculation above. □

Now let's produce some generalisations. For a start, it turns out that $\chi(M)$ can be defined even when M is not orientable. A cheap way to do so is to use the *orientation covering* $p : N \to M$. This is a two-sheeted covering space, defined so that the two points of N lying above some $x \in M$ are the two possible orientations of the vector space T_xM. If $y \in N$ projects to $x \in M$, then $Dp(y) : T_yN \to T_xM$ is a linear isomorphism which we can use as an "identification", and so each tangent space T_yN has a tautological orientation (because y knows which of the two possible orientations it is). Therefore N is an oriented manifold. We now define

$$\chi(M) = \frac{\chi(N)}{2}$$

with the provisional justification that N looks twice as big as M. But there is a better justification, and this will also show us that $\chi(M)$ is an integer.
We have $\zeta_M : M \to TM$ (subscript added on for bookkeeping purposes) and we can certainly choose another smooth map $\xi_M : M \to TM$ homotopic to ζ_M and transverse to ζ_M. Let $u_M : M \times [0,1] \to TM$ be a homotopy from ζ_M to ξ_M. By the unique path lifting property, applied to the paths $t \mapsto u(x,t)$ for fixed $x \in M$, the homotopy u_M "lifts" uniquely to a homotopy

$$u_N : N \times [0,1] \to TN$$

from $\zeta_N : N \to TN$ to another smooth map $\xi_N : N \to TN$. Now ξ_N is transverse to ζ_N and we can use it to calculate $\chi(N)$. It is clear that every $x \in M$ where $\zeta_M(x) = \xi_M(x)$ gives rise to two points y_1, y_2 in N where ζ_N agrees with ξ_N. Moreover (perhaps less clear) these two points have the same orientation, either both positive or both negative. So $\chi(N)$ is even, and we can actually say that every $x \in M$ where $\zeta_M(x) = \xi_M(x)$ acquires a well defined orientation alias sign, either plus or minus. The Euler number $\chi(M)$ is then the total number of these points, counted with their signs. This counting convention does of course rely on a choice of homotopy u from ζ_M to ξ_M. (*End of better justification.*)

Example 8.5.6 Let n be even. Then $\chi(\mathbb{R}P^n) = 1$. This follows immediately from our calculation of $\chi(S^n)$ and the fact that the orientation covering of $\mathbb{R}P^n$ can be identified with the standard double covering map $S^n \to \mathbb{R}P^n$.

For another generalisation, suppose that M is a smooth compact n-dimensional manifold *with* boundary. We have the standard inclusion $\zeta : M \to TM$ as before, $\zeta(x) = 0_x \in T_xM \subset TM$. Let $p : TM \to M$ be the projection.

- Choose an *outward normal* vector field ν along ∂M ; in other words choose for every $x \in \partial M$ a vector $\nu(x) \in T_xM$ which is not in the linear subspace $T_x\partial M$ and points to the "outside" of M. This $\nu(x)$ is of course required to depend continuously and smoothly on $x \in \partial N$. The existence of such a normal field ν is guaranteed by the collaring lemma 8.3.2.
- Now choose another smooth map $\xi : M \to TM$, homotopic to ζ and transverse to ζ. More precisely a homotopy $h : M \times [0,1]$ from ζ to ξ must be selected such that $h(x,t) = t\nu(x)$ whenever x is in ∂M. It is also a good idea to arrange $p(\xi(x)) \notin \partial M$ whenever $x \notin \partial M$.
- Then $L = \{\ (x_1, x_2) \in M \times M \mid \zeta(x_1) = \xi(x_2)\ \}$ is a 0-dimensional oriented compact (smooth) manifold, contained in $M \times M$ but disjoint from $\partial M \times M \ \cup\ M \times \partial M$. As such, L defines an element in the oriented bordism group $\Omega_0 \cong \mathbb{Z}$. (The orientation of L is easier to understand in the case where M is oriented.)

Definition 8.5.7 That integer is the Euler number of TM. We sometimes denote it by $\chi(M)$.

Example 8.5.8 For all $n \geq 0$, we have $\chi(D^n) = 1$. To calculate this we can use the smooth vector field ξ on D^n given by $\xi(x) = x$. Note that this is indeed "outward normal" at points $x \in \partial D^n = S^{n-1}$, as required. The vector field has one stationary point which is $x = 0$. This contributes $+1$ to the Euler number count. There are no other contributions.

There is an obvious alternative to definition 8.5.7. Instead of choosing an *outward* normal vector field along ∂M, we can choose an *inward* normal vector field along ∂M, and proceed from there. The resulting alternative to $\chi(M)$ will be denoted (here) by $\bar{\chi}(M)$. It does not always agree with $\chi(M)$. Instead we have

Proposition 8.5.9 $\quad \bar{\chi}(M) = \chi(M) - \chi(\partial M)$.

Proof (Sketch.) It is enough to consider the orientable case. By the collaring lemma 8.3.2, we can find an open neighbourhood U of ∂M in M and a diffeomorphism $\psi : \partial M \times [0,1[\ \to U$ such that $\psi(x,0) = x$ for $x \in \partial M$. We choose some c with $0 < c < 1$ and put

$$K_c = \psi(\partial M \times [0,c]).$$

Let M_c be the closure of $M \smallsetminus K_c$ in M. Then M_c is diffeomorphic to M and K_c is diffeomorphic to $\partial M \times [0,c]$. The intersection $K_c \cap M_c$ in M is equal to ∂M_c.
Now we choose

$$\begin{array}{rcl} \xi_1 : \partial M & \to & T\partial M \\ \xi_2 : [0,c] & \to & T[0,c] \\ \xi_3 : M_c & \to & TM_c \end{array}$$

which are "perturbations" of the standard inclusion maps (*zero sections*)

$$\begin{array}{rcl} \zeta_1 : \partial M & \to & T\partial M \\ \zeta_2 : [0,c] & \to & T[0,c] \\ \zeta_3 : M_c & \to & TM_c \end{array}$$

and which are transverse to the latter. More precisely we construct ξ_2 following the instructions which apply when we need to determine $\bar{\chi}([0,c])$. We construct ξ_1 and ξ_3 following the instructions which apply when we need to determine $\chi(\partial M)$ and $\chi(M_c)$, respectively. Then we have a map

$$\partial M \times [0,c] \longrightarrow T\partial M \times T[0,c]\ ; \qquad (x,t) \mapsto (\xi_2(x), \xi_1(t))$$

which we can also regard as a map $\xi_4 : K_c \to TK_c$ and which we can use to calculate $\bar{\chi}(K_c)$. With a little extra care, this map agrees with $\xi_3 : M_c \to TM_c$ on the intersection $K_c \cap M_c = \partial M_c$. (The key observation here is that a tangent vector at some $x \in \partial M_c$ points "out" of M_c precisely if it points "into" K_c.) So the union of ξ_4 and ξ_3 is a map $\xi_5 : M \to TM$. If we use this to determine $\bar{\chi}(M)$, then we find

$$\bar{\chi}(M) = \bar{\chi}(K_c) + \chi(M_c) = \chi(\partial M) \cdot \bar{\chi}([0,c]) + \chi(M)$$

which simplifies to $\chi(M) - \chi(\partial M)$ because $\bar{\chi}([0,c]) = -1$. □

Remark 8.5.10 Our use of the notation $\chi(M)$ for the Euler number of TM is dreadfully premature. The number $\chi(M)$ has a more fundamental and official definition, and with that comes another name for it, *Euler characteristic of* M. The agreement between Euler characteristic of M and Euler number of the tangent bundle of M is a nontrivial theorem.

SMSTC (2008/09)
Geometry and Topology
Lecture 9: Differential forms in coordinates

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Contents

9.1 Exterior algebra

Exterior algebra is the brainchild of the "polymath" Hermann Grassmann (1809-1877) who received little recognition for it during his lifetime. It is a theory of volumes and measurement which generalises the folklore relationship between volumes and determinants. Let that be our starting point.

Theorem 9.1.1 *Let $v^{(1)}, v^{(2)}, \ldots, v^{(n)} \in \mathbb{R}^n$. The n-dimensional volume of the "parallelepipede"*

$$P(v^{(1)}, v^{(2)}, \ldots, v^{(n)}) \quad = \quad \left\{ \sum_{i=1}^{n} t_i v^{(i)} \,\middle|\, t_1, t_2, \ldots, t_n \in [0,1] \right\}$$

is equal to the absolute value of the determinant of the $n \times n$ matrix $\left(v_j^{(i)}\right)$.

Proof It has been observed (by Paul Halmos ?) that this theorem is often used, but rarely proved. An obvious difficulty in proving it is that a sound definition of n-dimensional volume is required before anything can be proved. Rather than developing measure theory, let's assume that we *have* a definition of volume (for parallelepipedes at least) which satisfies the following properties.

- If $v^{(1)}, v^{(2)}, \ldots, v^{(n)}$ are pairwise orthogonal, then the volume of $P(v^{(1)}, v^{(2)}, \ldots, v^{(n)})$ is
$$\|v^{(1)}\| \cdot \|v^{(2)}\| \cdot \|v^{(3)}\| \cdots \|v^{(n)}\| \ ;$$
- The volume of $P(v^{(1)}, v^{(2)}, \ldots, v^{(n)})$ does not change if one of the $v^{(i)}$ is replaced by $v^{(i)} + w$ where w is a linear combination of the $v^{(j)}$ for $j \neq i$. (This is just "Cavalieri's principle": volume doesn't change under shears.)

Assuming these properties, let's show by induction on k that the theorem holds if $v^{(1)}, \ldots, v^{(n-k)}$ are pairwise orthogonal. In the case $k = 0$, the volume of $P(v^{(1)}, v^{(2)}, \ldots, v^{(n)})$ is

$$\|v^{(1)}\| \cdot \|v^{(2)}\| \cdot \|v^{(3)}\| \cdots \|v^{(n)}\|$$

by one of our axioms. The determinant of $\left(v_j^{(i)}\right)$ is

$$\|v^{(1)}\| \cdot \|v^{(2)}\| \cdot \|v^{(3)}\| \cdots \|v^{(n)}\| \cdot \det\left(w_j^{(i)}\right)$$

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where $w^{(i)} = v^{(i)}/\|v^{(i)}\|$. (We can assume that the $v^{(i)}$ are all nonzero, otherwise there is nothing to prove.) Therefore it only remains, in this case, to show that

$$\det(A) = \pm 1 \ , \qquad \text{where } A = \big(w_j^{(i)}\big) \ .$$

But A is an orthogonal matrix, $A^T A = I_n$, so that $(\det(A))^2 = \det(A^T)\det(A) = \det(I_n) = 1$.
Fixing $k \geq 0$ now, and assuming that the theorem holds whenever $v^{(1)}, \dots, v^{(n-k)}$ are pairwise orthogonal, we must deal with the case where only $v^{(1)}, \dots, v^{(n-k-1)}$ are pairwise orthogonal. In such a case we can make a replacement

$$v^{(n-k)} \rightsquigarrow v^{(n-k)} + w$$

where w is a suitable linear combination of $v^{(1)}, \dots, v^{(n-k-1)}$, and $v^{(n-k)} + w$ is orthogonal to $v^{(i)}$ for $i = 1, 2, \dots, n-k-1$. That will not change the volume (by the Cavalieri axiom) and it will not change the determinant. This completes the induction step. □

Suppose that V is an arbitrary n-dimensional real vector space. One way to associate something like a volume (Grassmann might have said *extent*) to a list of n vectors

$$v^{(1)}, v^{(2)}, \dots, v^{(n)} \in V$$

is to choose, once and for all, a linear map $f : V \to \mathbb{R}^n$. Then we have the extent function

$$\big(v^{(1)}, v^{(2)}, \dots, v^{(n)}\big) \quad \mapsto \quad \det\big(f(v^{(i)})_j\big) \ .$$

Grassmann was also interested in associating an “extent” to a list of only k vectors

$$v^{(1)}, v^{(2)}, \dots, v^{(k)} \in V$$

where $k \leq n$. Here an obvious way to proceed is to choose, once and for all, a linear map $g : V \to \mathbb{R}^k$. Then we have the extent function

$$\big(v^{(1)}, v^{(2)}, \dots, v^{(k)}\big) \quad \mapsto \quad \det\big(g(v^{(i)})_j\big) \ .$$

The right-hand side is, up to sign, the volume of the parallelepipede $P\big(g(v^{(1)}), g(v^{(2)}), \dots, g(v^{(k)})\big)$ in $\mathbb{R}^k$. However, as Grassmann observed, more general extent functions can be obtained by choosing, once and for all, several linear maps $g_1, g_2, \dots, g_q : V \to \mathbb{R}^k$, which leads to

$$\big(v^{(1)}, v^{(2)}, \dots, v^{(k)}\big) \quad \mapsto \quad \sum_{\alpha=1}^{q} \det\big(g_\alpha(v^{(i)})_j\big) \ .$$

Such functions are nowadays called, less romantically, *alternating k-forms* on V. Let's develop their properties in abstracto.

Definition 9.1.2 Let V be an n-dimensional real vector space, $n < \infty$. For an integer $k > 0$, an *alternating k-form* on V is a map

$$\omega : \underbrace{V \times V \times \cdots \times V}_{k \text{ factors}} \longrightarrow \mathbb{R}$$

with the following properties: ω is $\mathbb{R}$-linear in each of the k variables when the remaining variables are kept fixed, and

$$\omega(v^{(1)}, v^{(2)}, \dots, v^{(k)}) = 0$$

when $v^{(r)} = v^{(s)}$ for some $r \neq s$.
The alternating k-forms on V form a real vector space denoted by $\mathrm{alt}^k(V)$. For $k = 0$ we set $\mathrm{alt}^0(V) = \mathbb{R}$.

Example 9.1.3 Let $t \mapsto A_t$ be a continuous curve in $\hom(V, \mathbb{R}^k)$, where $t \in [0, 1]$. Define

$$\omega(v^{(1)}, v^{(2)}, \dots, v^{(k)}) = \int_0^1 \det\big((A_t(v^{(i)}))_j\big)\ dt \ .$$

Then ω is an alternating k-form on V.

Lemma 9.1.4 *Let $\omega \in \mathrm{alt}^k(V)$ and let σ be a permutation of $\{1, 2, 3, \ldots, k\}$. Then*

$$\omega(v^{(1)}, v^{(2)}, \ldots, v^{(k)}) = \mathrm{sign}(\sigma) \cdot \omega(v^{(\sigma(1))}, v^{(\sigma(2))}, \ldots, v^{(\sigma(k))}) \ .$$

Proof Every permutation of $\{1, 2, \ldots, k\}$ is a product of transpositions, i.e., permutations which interchange just two of the numbers $1, 2, \ldots, k$. It is therefore enough to prove the formula when σ is a transposition, say the transposition interchanging r and s where $r < s$. In that case $\mathrm{sign}(\sigma) = -1$. Let

$$u^{(i)} = \begin{cases} v^{(i)} & \text{when } i \neq r, s \\ v^{(r)} + v^{(s)} & \text{otherwise.} \end{cases}$$

Then

$$\omega(u^{(1)}, u^{(1)}, \ldots, u^{(k)}) = 0$$

because $u^{(r)} = u^{(s)}$. Expand $\omega(u^{(1)}, u^{(1)}, \ldots, u^{(k)})$ as a sum of four terms, using linearity of ω in the slots number r and s. Two of these terms are zero because they have identical inputs in slots number r and s. Deleting these leaves us with the equation

$$\omega(v^{(1)}, v^{(2)}, \ldots, v^{(k)}) + \omega(v^{(\sigma(1))}, v^{(\sigma(2))}, \ldots, v^{(\sigma(k))}) = 0.$$

$\square$

Corollary 9.1.5 *Let $n = \dim(V)$. The dimension of $\mathrm{alt}^k(V)$ is $\binom{n}{k}$. In particular $\mathrm{alt}^n(V)$ has dimension 1, and for $k > n$ we have $\mathrm{alt}^k(V) = 0$.*

Proof Choose an ordered basis $v^{(1)}, v^{(2)}, \ldots, v^{(n)}$ for V. Let $\omega \in \mathrm{alt}^k(V)$. Using the multilinearity of ω and the previous lemma, it is easy to see that ω is determined by its values

$$\omega(v^{(g(1))}, v^{(g(2))}, \ldots, v^{(g(k))})$$

where $g : \{1, 2, 3, \ldots, k\} \to \{1, 2, 3, \ldots, n\}$ runs through all order-preserving injective maps. These values can also be prescribed arbitrarily and independently in $\mathbb{R}$. Therefore $\mathrm{alt}^k(V)$ has a vector space basis with one element for each order-preserving injective map from $\{1, 2, 3, \ldots, k\}$ to $\{1, 2, 3, \ldots, n\}$. The number of such maps is equal to $\binom{n}{k}$. $\square$

Remark. It is useful to have an explicit formula for the basis elements in $\mathrm{alt}^k(V)$, depending of course on a choice of a basis for V. Each order-preserving injective map $g : \{1, 2, 3, \ldots, k\} \to \{1, 2, 3, \ldots, n\}$ determines a linear map

$$A_g : V \to \mathbb{R}^k$$

given by extracting the coordinates, w.r.t that basis of V, numbered $g(1), g(2), \ldots, g(k)$. The basis element of $\mathrm{alt}^k(V)$ corresponding to g is the alternating k-form given by

$$(w^{(1)}, w^{(2)}, \ldots, w^{(k)}) \quad \mapsto \quad \det\left((A_g(w^{(i)}))_j\right) \ .$$

where the $w^{(i)}$ are arbitrary elements of V.

The next topic is a "product" $\mathrm{alt}^k(V) \times \mathrm{alt}^\ell(V) \to \mathrm{alt}^{k+\ell}(V)$.

Definition 9.1.6 A permutation σ of the set $\{1, 2, \ldots, k+\ell\}$ is a (k, ℓ)-*shuffle* if it preserves the natural order of the elements $1, 2, 3, \ldots k$ and the natural order of the elements $k+1, k+2, \ldots, k+\ell$, that is, $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+\ell)$. Let

$$\mathrm{shuf}(k, \ell)$$

be the set of these shuffles. It is a subset of $\mathrm{per}(k+\ell)$, the group of permutations of $\{1, 2, \ldots, k+\ell\}$. The number of (k, ℓ)-shuffles is $\binom{k+\ell}{k}$.

Sometimes it is useful to note that every element of $\mathrm{per}(k+\ell)$ can be uniquely written in the form $\sigma\lambda$, where σ is a (k, ℓ)-shuffle and $\lambda \in \mathrm{per}(k) \times \mathrm{per}(\ell) \subset \mathrm{per}(k+\ell)$. Here we think of $\mathrm{per}(k) \times \mathrm{per}(\ell)$ as a subgroup of $\mathrm{per}(k+\ell)$, the subgroup consisting of the permutations which permute the first k elements among themselves (and consequently also permute the last ℓ elements among themselves).

Definition 9.1.7 Let $\omega_1 \in \mathrm{alt}^k(V)$ and $\omega_2 \in \mathrm{alt}^\ell(V)$. Their *wedge product* $\omega_1 \wedge \omega_2 \in \mathrm{alt}^{k+\ell}(V)$ is defined (for $k, \ell > 0$) by the formula

$$\begin{aligned} & (\omega_1 \wedge \omega_2)(u^{(1)}, u^{(2)}, \ldots, u^{(k+\ell)}) \\ = & \sum_{\sigma \in \mathrm{shuf}(k,\ell)} \mathrm{sign}(\sigma) \cdot \omega_1(u^{(\sigma(1))}, \ldots, u^{(\sigma(k))}) \cdot \omega_2(u^{(\sigma(k+1))}, \ldots, u^{(\sigma(k+\ell))}). \end{aligned}$$

If $k = 0$, then ω_1 is a number $c \in \mathbb{R}$ and $\omega_1 \wedge \omega_2$ is defined to be $c\omega_2$. Similarly, if $\ell = 0$, then ω_2 is a number c and $\omega_1 \wedge \omega_2$ is defined to be $c\omega_1$.

Example 9.1.8 Let $A : V \to \mathbb{R}^k$ and $B : V \to \mathbb{R}^\ell$ be linear maps. Associated with these we have an alternating k-form ω_A and an alternating ℓ-form ω_B , defined by

$$\begin{aligned} \omega_A(u^{(1)}, u^{(2)}, \ldots, u^{(k)}) &= \det\left((A(u^{(i)}))_j\right) \\ \omega_B(u^{(1)}, u^{(2)}, \ldots, u^{(\ell)}) &= \det\left((B(u^{(i)}))_j\right) . \end{aligned}$$

Then $(\omega_A \wedge \omega_B)(u^{(1)}, u^{(2)}, \ldots, u^{(k+\ell)}) = \det\left((C(u^{(i)}))_j\right)$ where $C : V \to \mathbb{R}^{k+\ell}$ is the linear map defined by $v \mapsto (A(v), B(v)) \in \mathbb{R}^k \times \mathbb{R}^\ell \cong \mathbb{R}^{k+\ell}$.

Lemma 9.1.9 *The wedge product is bilinear, associative and has* $1 \in \mathrm{alt}^0(V)$ *as a multiplicative unit. It is also "graded" commutative, i.e., for* $\omega_1 \in \mathrm{alt}^k(V)$ *and* $\omega_2 \in \mathrm{alt}^\ell(V)$ *we have*

$$\omega_1 \wedge \omega_2 = (-1)^{k\ell} \omega_2 \wedge \omega_1 \ .$$

Proof Bilinearity is obvious from the definition. For the "graded commutative" property, the definition of the wedge product implies that

$$(\omega_1 \wedge \omega_2)(u^{(1)}, u^{(2)}, \ldots, u^{(k+\ell)}) = (\omega_2 \wedge \omega_1)(u^{(\tau(1))}, u^{(\tau(2))}, \ldots, u^{(\tau(k+\ell))})$$

where $\tau \in \mathrm{shuf}(k, \ell)$ is the shuffle which moves the first k elements to the back end. Now we can use lemma 9.1.4 and the observation that $\mathrm{sign}(\tau) = (-1)^{k\ell}$.
To prove associativity, let $\omega^1 \in \mathrm{alt}^k(V)$, $\omega_2 \in \mathrm{alt}^\ell(V)$, $\omega_3 \in \mathrm{alt}^m(V)$. We can assume $k, \ell, m > 0$. Looking at the summands making up $(\omega_1 \wedge \omega_2) \wedge \omega_3$ and $\omega_1 \wedge (\omega_2 \wedge \omega_3)$, it is clearly enough to produce a bijection

$$b : \mathrm{shuf}(k, \ell) \times \mathrm{shuf}(k + \ell, m) \longrightarrow \mathrm{shuf}(k, \ell + m) \times \mathrm{shuf}(\ell, m)$$

preserving signs, so that if $b(\sigma, \tau) = (\mu, \lambda)$ then $\mathrm{sign}(\sigma)\mathrm{sign}(\tau) = \mathrm{sign}(\mu)\mathrm{sign}(\lambda)$. But this is easy because both $\mathrm{shuf}(k, \ell) \times \mathrm{shuf}(k + \ell, m)$ and $\mathrm{shuf}(k, \ell + m) \times \mathrm{shuf}(\ell, m)$ are identified with $\mathrm{shuf}(k, \ell, m)$, the set of permutations ζ of $\{1, 2, 3, \ldots, k + \ell + m\}$ which satisfy

$$\begin{aligned} & \zeta(1) < \zeta(2) < \cdots < \zeta(k) \ , \\ & \zeta(k + 1) < \zeta(k + 2) < \cdots < \zeta(k + \ell) \ , \\ & \zeta(k + \ell + 1) < \zeta(k + \ell + 2) < \cdots < \zeta(k + \ell + m). \end{aligned}$$

□

Example 9.1.10 Let $V = \mathbb{R}^n$. Let $q_i : \mathbb{R}^n \to \mathbb{R}$ be the linear projection which singles out the i-th coordinate. Then $q_i \in \mathrm{alt}^1(V)$ for $i = 1, 2, \ldots, n$. By (the proof of) corollary 9.1.5 , the vector space $\mathrm{alt}^k(V)$ has a basis consisting of all expressions

$$q_{i_1} \wedge q_{i_2} \wedge \cdots \wedge q_{i_k}$$

where $i_1, i_2, \ldots, i_k$ are distinct elements of $\{1, 2, \ldots, n\}$ listed in their natural order. To multiply such basis elements (in dimensions k and ℓ, say), use associativity for wedge products and the relations

$$q_i \wedge q_j = -q_j \wedge q_i \ ,$$

in particular $q_i \wedge q_i = 0$, which are special cases of the graded commutativity.

Definition 9.1.11 Returning to the case of an arbitrary finite dimensional real vector space V, we shall often write $\mathrm{alt}^*(V)$ for the collection of the vector spaces $\mathrm{alt}^k(V)$ where $k \in \mathbb{N} = \{0, 1, 2, \dots\}$. This is viewed as a *graded ring* with the wedge product (more precisely, a *graded algebra* over $\mathbb{R}$, to express the fact that the real numbers act by scalar multiplication in every dimension). Occasionally the asterisk is also used as a placeholder for a k that wants to remain anonymous.
In general, a *graded ring* is a collection of abelian groups R_i where $i \in \mathbb{Z}$ or $i \in \mathbb{N}$, together with multiplication maps $R_i \times R_j \to R_{i+j}$ satisfying appropriate distributive and associativity laws, and a neutral element for the multiplication, $1 \in R_0$. If all the R_i are vector spaces over $\mathbb{R}$, and the multiplication maps are bilinear, then we speak of a graded algebra over $\mathbb{R}$.

Example 9.1.12 Let V and W be finite dimensional real vector spaces. Any linear map $g : V \to W$ determines a homomorphism of graded rings, $g^* : \mathrm{alt}^*(W) \to \mathrm{alt}^*(V)$, by

$$(g^*\omega)(v^{(1)}, \dots, v^{(k)}) = \omega(g(v^{(1)}), \dots, g(v^{(k)}))$$

for $\omega \in \mathrm{alt}^k(W)$ and arbitrary $v^{(1)}, \dots, v^{(k)}$ in V.

Remark. As this example illustrates, the asterisk $*$ has, unfortunately, many uses. Apart from being used as an "unspecified dimension" indicator, it is often used to indicate that one map is "induced" (determined) by another map. For example $g : V \to W$ determines/induces a map $\mathrm{alt}^*(W) \to \mathrm{alt}^*(V)$ which we denote by g^*. In cases like this, we use the *subscript* position for the asterisk if arrow directions are preserved and the *superscript* position if arrow directions are reversed. (In our case g^* is a map from $\mathrm{alt}^k(W) \to \mathrm{alt}^k(V)$, which justifies the choice of the superscript position for the asterisk. We saw other examples in earlier chapters: a map $f : X \to Y$ of pointed spaces induces $f_* : \pi_1(X) \to \pi_1(Y)$, preserving arrow directions.)

9.2 Differential forms on open subsets of $\mathbb{R}^n$

Definition 9.2.1 Let U be open in $\mathbb{R}^n$. A differential k-form on U is a smooth map $U \to \mathrm{alt}^k(\mathbb{R}^n)$. The set of all differential k-forms on U is denoted by $\Omega^k(U)$.

Remark. For a differential k-form ω on U and a point $x \in U$, the value $\omega(x) \in \mathrm{alt}^k(\mathbb{R}^n)$ will almost always be regarded as an alternating k-form on the *tangent space* of U at x. The tangent space happens to be identified with $\mathbb{R}^n$.

Example 9.2.2 A differential 0-form on U is just a smooth function from U to $\mathbb{R}$.

Example 9.2.3 Let $f : U \to \mathbb{R}$ be a smooth function. This determines a smooth map

$$df : U \to \mathrm{alt}^1(\mathbb{R}^n)$$

whose value at $x \in U$ is $df(x)$, the differential of f at x. (Instead of $df(x)$ we also wrote $Df(x)$ in earlier chapters. At any rate $df(x)$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$.) We can now write

$$df \in \Omega^1(U)\ .$$

Important special cases: In the case where f is the function $x \mapsto x_i$ which extracts the i-th coordinate, we write dx_i for df. (The person who long ago introduced this notation probably felt that x_i is a very good name for the function $x \mapsto x_i$. Discuss.)

The differential forms on U make up a graded ring $\Omega^*(U)$ which is again graded commutative. In more detail, there are multiplication maps $\Omega^k(U) \times \Omega^\ell(U) \to \Omega^{k+\ell}(U)$ taking (ω_1, ω_2) to $\omega_1 \wedge \omega_2$. Here $\omega_1 \wedge \omega_2$ is of course defined pointwise,

$$(\omega_1 \wedge \omega_2)(x) = \omega_1(x) \wedge \omega_2(x) \in \mathrm{alt}^{k+\ell}(\mathbb{R}^n)$$

for $x \in U$. In particular, starting with the differential 1-forms dx_i and using the multiplication, we can construct more differential forms such as

$$dx_{i_1} \wedge dx_{i_2} \wedge \dots dx_{i_k} \quad \in \Omega^k(U)$$

where $i_1, i_2, \dots, i_k \in \{1, 2, \dots, n\}$.

Definition 9.2.4 For a smooth function $g : U \to \mathbb{R}$ and $\omega \in \Omega^k(U)$, we define $g\omega \in \Omega^k(U)$ by

$$(g\omega)(x) = g(x)\omega(x) \in \mathrm{alt}^k(\mathbb{R}^n) \ ,$$

using the vector space structure on $\mathrm{alt}^k(\mathbb{R}^n)$. (This is strictly speaking exactly the same thing as $g \wedge \omega$, once we admit that $g \in \Omega^0(U)$. But the custom is to write $g\omega$, not $g \wedge \omega$.)

Lemma 9.2.5 *Every $\omega \in \Omega^k(U)$ can be written uniquely in the form*

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} g_{\,i_1,i_2,\ldots,i_k}\ dx_{i_1} \wedge dx_{i_2} \wedge \ldots dx_{i_k}$$

where the $g_{i_1,i_2,\ldots,i_k}$ are smooth functions from U to $\mathbb{R}$.

Proof For $y \in U$, the element $\omega(y) \in \mathrm{alt}^k(U)$ can be written uniquely as a linear combination of terms

$$q_{i_1} \wedge q_{i_2} \wedge \ldots q_{i_k}$$

(in the notation of example 9.1.10). Let $g_{\,i_1,i_2,\ldots,i_k}(y)$ be the coefficient of $q_{i_1} \wedge q_{i_2} \wedge \ldots q_{i_k}$ in that linear combination. Note that $q_i = dx_i(y)$. □

Example 9.2.6 Let $f : U \to \mathbb{R}$ be smooth. Let us see how $df \in \Omega^1(U)$ can be (uniquely) written in the form given by lemma 9.2.5,

$$df = g_1 dx_1 + g_2 dx_2 + \cdots + g_n dx_n \ ,$$

for appropriate smooth functions g_i on U which we have to determine. This is not hard:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

and so $g_i = \partial f / \partial x_i$.
At this point we can also see that there are "usually" many elements in $\Omega^1(U)$ which are *not* of the form df for any $f \in \Omega^0(U)$. For simplicity, suppose $n = 2$ and $U \neq \emptyset$. If we have $\omega \in \Omega^1(U)$, we can write it uniquely as $g_1 dx_1 + g_2 dx_2$. If $\omega = df$, then by the above we must have $g_1 = \partial f / \partial x_1$ and $g_2 = \partial f / \partial x_2$ and consequently

$$\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1}$$

by the symmetry property of second derivatives. (That's a condition which you will remember if you have been exposed to "conservative systems" in elementary mechanics). The condition is clearly not satisfied for *all* choices of g_1 and g_2.

Definition 9.2.7 Let $f : U \to V$ be a smooth map, where U is open in $\mathbb{R}^m$ and V is open in $\mathbb{R}^n$. We define $f^* : \Omega^*(V) \to \Omega^*(U)$ by

$$(f^*\omega)(x) = (Df(x))^*\omega(f(x))$$

for $\omega \in \Omega^k(V)$, $x \in U$ and consequently $f(x) \in V$.

This is rather convoluted, so let's unravel it. We start with $\omega \in \Omega^k(V)$ and we want to describe $f^*\omega$ in $\Omega^k(U)$. We take some $x \in U$ and try to say what $(f^*\omega)(x) \in \mathrm{alt}^k(\mathbb{R}^m)$ should be. The remark just after definition 9.2.1 is foremost in our mind(s). We remember therefore that f has a differential at x, which is a linear map $Df(x) : \mathbb{R}^m \to \mathbb{R}^n$, but in reality, a linear map from the tangent space T_xU to the tangent space $T_{f(x)}V$. Now we also remember example 9.1.12 and conclude that $Df(x) : \mathbb{R}^m \to \mathbb{R}^n$ determines a homomorphism of graded rings,

$$(Df(x))^* : \mathrm{alt}^*(\mathbb{R}^n) \to \mathrm{alt}^*(\mathbb{R}^m) \ .$$

We apply this to $\omega(f(x)) \in \mathrm{alt}^k(\mathbb{R}^n)$. The "output" is an element of $\mathrm{alt}^k(\mathbb{R}^m)$ which we call $(f^*\omega)(x)$. That constitutes the definition of $f^*\omega$. But if you find this too lofty, then you might like the formula

$$(f^*\omega)(x)(u^{(1)}, \ldots, u^{(k)}) = \omega(f(x))(Ju^{(1)}, \ldots, Ju^{(k)})$$

where $J = \big(\partial f_i / \partial x_j\big)$ is the Jacobi matrix of f at x. *Note*: an element ω of $\Omega^0(V)$ is simply a smooth function on V, and $f^*\omega \in \Omega^0(U)$ is just $\omega \circ f$. (That should be a special case of all the above, but if you cannot agree to that, take it as the definition of $f^*\omega$ for $\omega \in \Omega^0(V)$.)

9.3 Integration of differential forms

Let U be an open set in $\mathbb{R}^n$ and let $\omega \in \Omega^n(U)$. Suppose for simplicity that ω has compact support.[b] By lemma 9.2.5, we can write

$$\omega = f \cdot dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

for a unique smooth function $f : U \to \mathbb{R}$ with compact support.[c]

Definition 9.3.1 The integral $\int_U \omega = \int_U f \cdot dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ is defined to be the (Riemann) integral

$$\int_U f(x)\ dx_1 dx_2 \ldots dx_n\ .$$

Remark. There is a danger of confusion here which is easiest to illustrate when $n = 2$. The definition

$$\int_U f\ dx_1 \wedge dx_2 = \int_U f(x)\ dx_1 dx_2$$

implies that $\int_U f\ dx_2 \wedge dx_1 = -\int_U f(x)\ dx_2 dx_1$. The reason is that $dx_2 \wedge dx_1 = -dx_1 \wedge dx_2$ whereas $\int_U f(x)\ dx_1 dx_2 = \int_U f(x)\ dx_2 dx_1$ by definition of the Riemann integral.

In the remainder of this section we shall struggle to extend the first-year calculus "substitution rule" to higher dimensions, i.e. to the setting of definition 9.3.1. In a fairly general formulation, the substitution rule says that given a smooth function $g : [a, b] \to \mathbb{R}$ and a smooth function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\int_{g(a)}^{g(b)} f(x)\ dx = \int_a^b f(g(x)) \cdot g'(x)\ dx\ .$$

To generate some ideas, let's note that if we write the left-hand side integrand as a differential 1-form, namely $\omega = f\ dx \in \Omega^1(\mathbb{R})$, then the right-hand side integrand turns into $g^*\omega$, rather pleasantly. (See definition 9.2.7, then you will see it.) So we may write the 1-dimensional substitution rule in the form

$$\int_{g(J)} \omega = \pm \int_J g^*\omega$$

where $J = [a, b]$ and $\omega \in \Omega^1(\mathbb{R})$. The $\pm$ is a "minus" sign if $g(a) > g(b)$, otherwise it is a "plus" sign.
In generalising this to higher dimensions, let us concentrate for now on the case where the substitution is *reversible*. In the 1-dimensional "model" case, that would mean that g is a diffeomorphism from $[a, b] = J$ to $g(J)$. Then it is enough to have f (or $\omega = f\ dx$) defined on $g(J)$.
To be prepared for sign issues, let's observe that a diffeomorphism $g : U \to V$ between *connected* open sets in $\mathbb{R}^n$ will either have $\det(Dg(x)) > 0$ for all $x \in U$, or $\det(Dg(x)) < 0$ for all $x \in U$. In the first case, we say that g is *orientation preserving*, in the second case, that it is *orientation-reversing.*

Theorem 9.3.2 *Let U and V be connected open sets in $\mathbb{R}^n$. Let $g : U \to V$ be a diffeomorphism and let ω be a differential n-form on V with compact support. Then*

$$\int_V \omega = \pm \int_U g^*\omega\ .$$

The sign is "+" if g is orientation preserving, and "−" if g is orientation-reversing.

Remark. This is equivalent to the statement

$$\int_V f(x)\ dx_1 dx_2 \ldots dx_n \quad = \quad \int_U f(g(x)) \cdot |\det(Dg(x))|\ dx_1 dx_2 \ldots dx_n$$

for a smooth function $f : V \to \mathbb{R}$ with compact support. Smoothness of f is not essential ; in the proof below we only use that f is continuous.

[b] The support of ω is the closure in U of the set $\{x \in U \mid \omega(x) \neq 0\}$.
[c] The support of f is the closure in U of the set $\{x \in U \mid f(x) \neq 0\}$.

Proof We can write $\omega = f \cdot dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$, so that $\int_V \omega = \int_V f(x)\, dx_1 \cdots dx_n$. Choose a compact $K \subset U$ such that $f \circ g$ is zero outside K. Choose $\varepsilon > 0$. Find $\delta > 0$ such that $B(x, 2\delta) \subset U$ for every $x \in K$ and

$$\|g(x+v) - g(x) - Dg(x)(v)\| \quad < \quad \varepsilon\|v\| \qquad\qquad (*)$$

whenever $x \in K$ and $v \in \mathbb{R}^n$, $\|v\| \leq \delta$. This is possible by the mean value theorem.[d] Making δ smaller still if necessary, we can also arrange that

$$|f(g(x+v)) - f(g(x))| \quad < \quad \varepsilon \qquad\qquad (**)$$

whenever $x \in K$ and $\|v\| \leq \delta$. This is possible because continuous functions are uniformly continuous on compact sets. Now imagine a compact cube L inside U, for example

$$L = \{\, p + v \mid v_1, \ldots, v_n \in [0, \delta] \,\}$$

where p is some point in $K \subset U$. Then by the above inequality $(*)$, the region $g(L) \subset V$ differs only very little from the parallelepipede $(Dg(p))(L)$, in the sense that $g(L)$ contains a slightly smaller parallelepipede and is contained in a slightly larger paralellepipede. Better still, if we use $(**)$ also, we see that the integral of f over $g(L)$ differs from $f(g(p)) \cdot \mathrm{vol}(Dg(p)(L)) = f(g(p)) \cdot |\det(Dg(p))| \cdot \delta^n$ only by a small amount. That is,

$$\left| \int_{g(L)} f(x)\, dx_1 \cdots dx_n \quad - \quad f(g(p)) \cdot |\det(Dg(p))| \cdot \delta^n \right| \quad \leq \quad s \cdot \varepsilon \cdot \delta^n \qquad\qquad (***)$$

for some constant $s > 0$ depending only on g and f. (Finding such an s is an exercise for you.) This being done, choose a collection of little compact cubes L_j of sidelength δ, like L above, enough to cover K. More precisely, each cube L_j should have one vertex p_j in K, which implies $L_j \subset U$, and we want $K \subset \bigcup_j L_j$, and the cubes should not intersect except in faces of dimension $< n$. Then we have

$$\int_V f(x)\, dx_1 \cdots dx_n = \sum_j \int_{g(L_j)} f(x)\, dx_1 \cdots dx_n$$

because the union $\bigcup_j g(L_j)$ contains all points where f is not zero.[e] Therefore by $(***)$,

$$\left| \int_V f(x)\, dx_1 \cdots dx_n \quad - \quad \sum_j f(g(p_j)) \cdot |\det(Dg(p_j))| \cdot \delta^n \right| \quad \leq \quad s \cdot \varepsilon \sum_j \delta^n \,.$$

Now let ε and δ tend to 0. Then the expression

$$\sum_j f(g(p_j)) \cdot |\det(Dg(p_j))| \cdot \delta^n$$

converges to the Riemann integral

$$\int_U f(g(x)) \cdot |\det(Dg(x))|\, dx_1 \cdots dx_n \,.$$

Meanwhile $\sum_j \delta^n$ remains bounded (because the union of the cubes L_j is always contained in a subset of U whose diameter is at most the diameter of K plus 2δ). Therefore $s \cdot \varepsilon \sum_j \delta^n$ tends to 0. Therefore we have

$$\int_V f(x)\, dx_1 \cdots dx_n \quad = \quad \int_U f(g(x)) \cdot |\det(Dg(x))|\, dx_1 \cdots dx_n$$

which translates into $\displaystyle\int_V \omega = \pm \int_U g^*\omega$. □

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[d] If the inequality is violated, then $|g_i(x+v) - g_i(x) - Dg_i(x)(v)| > (\varepsilon/n)\|v\|$ for some $i \in \{1, 2, \ldots, n\}$, some x and some v. By the mean value theorem applied to the function $t \mapsto g_i(x+tv)$, there exists $t \in [0,1]$ such that $g_i(x+v) - g_i(x) = Dg_i(x+tv)(v)$. Then we conclude $Dg_i(x+tv)(v) - Dg_i(x)(v) > (\varepsilon/n)\|v\|$ for the same t. But thanks to the continuity of the first derivatives of g_i, we can make δ so small that this will not happen. Note $\|tv\| \leq \delta$ by assumption.

[e] This is not as obvious as it appears at first because of the possible overlaps. Let ∂L_j be the boundary of L_j. We need to know that $g(\partial L_j)$ has measure zero. This follows from Sard's theorem, e.g., because ∂L_j is a union of finitely many smooth submanifolds of $\mathbb{R}^n$ of dimension $< n$.

Remark. The "compact support" assumption in theorem 9.3.2 can often be weakened. For example, if $\omega = f\ dx_1 \wedge \ldots dx_n$ where $f \geq 0$ and $\int_V f(x)\ dx_1 dx_2 \ldots dx_n < \infty$ (but f does not have compact support), then the equation $\int_U g^*\omega \ = \ \int_V \omega$ still holds. To prove this, choose a nondecreasing sequence of non-negative smooth functions from V to $\mathbb{R}$ with compact support, converging uniformly to f on compact subsets of V.

Example 9.3.3 We calculate

$$\int\limits_{\substack{x\in\mathbb{R}^2\\ x_2>1}} \|x\|^{-4}\ dx_1 dx_2\ .$$

Let's write $\mathbb{C}$ instead of $\mathbb{R}^2$ when that seems helpful. For example the region of integration, U, consists of all complex numbers with imaginary part > 1. Let $V \subset \mathbb{C}$ consist of all complex numbers having modulus < 1. There is a well-known diffeomorphism

$$g : U \to V\ ; \qquad z \mapsto \frac{z-2i}{z}$$

(in complex number notation). The derivative, in the complex number sense, is $g'(z) = 2iz^{-2}$, which implies that

$$\det(Dg(x)) = 4\|x\|^{-4}$$

in the real number sense, where $x = (x_1, x_2) \in \mathbb{R}^2$ corresponds to $z = x_1 + ix_2 \in \mathbb{C}$. Therefore

$$\int_U \|x\|^{-4}\ dx_1 dx_2 \quad = \quad \int_U \frac{1}{4}|\det(Dg(x))|\ dx_1 dx_2 \quad = \quad \int_V \frac{1}{4}\ dx_1 dx_2 = \frac{\pi}{4}\ .$$

Remark. Let ω be a differential k-form on an open set $U \subset \mathbb{R}^n$. If $k < n$, we cannot "integrate" ω on U to obtain a number. But we can still use ω to produce numbers from smooth maps $f : W \to U$ where W is open in $\mathbb{R}^k$. For example, if $f^*\omega$ has compact support, then $\int_W f^*\omega$ is defined. Moreover, if $f(W)$ is a connected smooth manifold in $\mathbb{R}^n$ and f induces a diffeomorphism $W \to f(W)$, then $|\int_W f^*\omega|$ depends only on $f(W)$, not directly on W and f. This follows directly from theorem 9.3.2.

SMSTC (2008/09)
Geometry and Topology
Lecture 10: Differential forms on smooth manifolds

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Contents

10.1 The exterior derivative in coordinates

Theorem 10.1.1 *Let U be open in $\mathbb{R}^n$. There exist unique $\mathbb{R}$-linear maps $d_k : \Omega^k(U) \to \Omega^{k+1}(U)$, where $k = 0, 1, 2, 3, \ldots$, such that the following holds.*

(i) *d_0 agrees with d of example 9.2.3*

(ii) *$d_{k+1} \circ d_k = 0$ for all k*

(iii) *for $\omega \in \Omega^k(U)$ and $\lambda \in \Omega^\ell(U)$ we have* $\quad d_{k+\ell}(\omega \wedge \lambda) \;=\; d_k(\omega) \wedge \lambda \;+\; (-1)^k \omega \wedge d_\ell(\lambda)$.

Proof We prove uniqueness first, i.e., we assume that maps d_k with these properties exist and try to nail them down. Let $\omega \in \Omega^k(U)$. As in lemma 9.2.5, there are unique functions $g_{i_1,i_2,\ldots,i_k}$ such that

$$\omega \;=\; \sum_{1\le i_1<i_2<\cdots<i_k\le n} g_{i_1,i_2,\ldots,i_k} \wedge d_0x_{i_1} \wedge d_0x_{i_2} \wedge \cdots \wedge d_0x_{i_k}$$

The first "$\wedge$" in each term translates into "ordinary multiplication", and we have written d_0 for d in accordance with rule (i). By linearity and rule (iii), we must have

$$\begin{aligned} d_k(\omega) \;&=\; \sum_{1\le i_1<i_2<\cdots<i_k\le n} d_0 g_{i_1,i_2,\ldots,i_k} \wedge (d_0x_{i_1} \wedge d_0x_{i_2} \wedge \cdots \wedge d_0x_{i_k}) \\ &\quad + \sum_{1\le i_1<i_2<\cdots<i_k\le n} g_{i_1,i_2,\ldots,i_k} \wedge d_k(d_0x_{i_1} \wedge d_0x_{i_2} \wedge \cdots \wedge d_0x_{i_k}). \end{aligned}$$

Rules (iii) and (ii) can now be used to show that

$$d_k(d_0x_{i_1} \wedge d_0x_{i_2} \wedge \cdots \wedge d_0x_{i_k}) = 0\ .$$

Therefore the only possible definition of d_k for $k > 0$ is

$$\begin{aligned} d_k(\omega) \;&=\; \sum_{1\le i_1<i_2<\cdots<i_k\le n} d_0 g_{i_1,i_2,\ldots,i_k} \wedge (d_0x_{i_1} \wedge d_0x_{i_2} \wedge \cdots \wedge d_0x_{i_k}) \\ &=\; \sum_{1\le i_1<i_2<\cdots<i_k\le n} d g_{i_1,i_2,\ldots,i_k} \wedge (dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}). \end{aligned}$$

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This takes care of uniqueness. Now for existence: If we use this formula to define d_k, and rule (i) to define d_0, then we need not worry about rule (i). For rule (ii), it is enough (by linearity) to consider the case where

$$\omega = g\ dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = g \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

for some smooth function $g : U \to \mathbb{R}$. Then

$$d_k(\omega) = dg \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = \sum_{r=1}^{n} \frac{\partial g}{\partial x_r}\ dx_r \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

(using example 9.2.3). Repeating this argument gives

$$d_{k+1}(d_k(\omega)) = \sum_{r,s=1}^{n} \frac{\partial^2 g}{\partial x_s \partial x_r}\ dx_s \wedge dx_r \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

This expression is zero because $dx_s \wedge dx_r = -dx_r \wedge dx_s$ and second partial derivatives of smooth functions have certain symmetry properties.
For the verification of rule (iii), we can assume

$$\omega = f \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}\ , \qquad \lambda = g \wedge dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_k}\ .$$

The verification then boils down rather quickly to showing that $d(f \cdot g) = g \cdot df + f \cdot dg$. But this is the ordinary product rule for derivatives. $\square$

Definition 10.1.2 Standard usage has d instead of d_k, as in $d : \Omega^k(U) \to \Omega^{k+1}(U)$. Terminology: d is the *exterior derivative.*

Example 10.1.3 Take $n = 3$, so U open in $\mathbb{R}^3$. After some translation work, we shall see that the maps $d : \Omega^k(U) \to \Omega^{k+1}(U)$ turn into grad when $k = 0$, curl when $k = 1$ and div when $k = 2$.
From lemma 9.2.5 we get the following identifications:

$$\begin{array}{lcl} \Omega^0(U) & = & \text{set of smooth maps } U \to \mathbb{R} \\ \Omega^1(U) & = & \text{set of smooth vector fields on } U \\ \Omega^2(U) & = & \text{set of smooth vector fields on } U \\ \Omega^3(U) & = & \text{set of smooth maps } U \to \mathbb{R}. \end{array}$$

To be quite precise, a smooth vector field $v = (v_1, v_2, v_3)$ on U determines a differential 1-form

$$v_1 dx_1 + v_2 dx_2 + v_3 dx_3$$

on U, and also a differential 2-form

$$v_1\ dx_2 \wedge dx_3 - v_2\ dx_1 \wedge dx_3 + v_3\ dx_1 \wedge dx_2\ .$$

(Note the minus sign.) Similarly, a smooth map $g : U \to \mathbb{R}$ determines a differential 3-form

$$g\ dx_1 \wedge dx_2 \wedge dx_3\ .$$

These correspondences are all bijective by lemma 9.2.5, despite the unexpected minus sign. For $f \in \Omega^0(U)$ the differential 1-form

$$df = \sum_{i=1}^{3} (\partial f / \partial x_i) dx_i$$

in $\Omega^1(U)$ corresponds to the vector field $(\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3) = \mathrm{grad}(f)$. For a vector field $v = (v_1, v_2, v_3)$ on U, corresponding to $v_1 dx_1 + v_2 dx_2 + v_3 dx_3$ in $\Omega^1(U)$, we obtain

$$\begin{array}{rcl} d(v_1 dx_1 + v_2 dx_2 + v_3 dx_3) & = & \displaystyle\sum_{i=1}^{3} \sum_{j=1}^{3} ((\partial v_i/\partial x_j)\ dx_j) \wedge dx_i \\ & = & (\partial v_3/\partial x_2 - \partial v_2/\partial x_3)\ dx_2 \wedge dx_3 \\ & & +(\partial v_3/\partial x_1 - \partial v_1/\partial x_3)\ dx_1 \wedge dx_3 \\ & & +(\partial v_2/\partial x_1 - \partial v_1/\partial x_2)\ dx_1 \wedge dx_2 \end{array}$$

in $\Omega^2(U)$, which corresponds to the vector field

$$(\partial v_3/\partial x_2 - \partial v_2/\partial x_3,\ -\partial v_3/\partial x_1 + \partial v_1/\partial x_3,\ \partial v_2/\partial x_1 - \partial v_1/\partial x_2) \quad = \quad \mathrm{curl}(v)\ .$$

For a vector field $v = (v_1, v_2, v_3)$ on U, corresponding to $v_1\ dx_2 \wedge dx_3 - v_2\ dx_1 \wedge dx_3 + v_3\ dx_1 \wedge dx_2$ in $\Omega^2(U)$, we have

$$d(v_1\ dx_2 \wedge dx_3 - v_2\ dx_1 \wedge dx_3 + v_3\ dx_1 \wedge dx_2) \quad = \quad \sum_{i=1}^{3} (\partial v_i/\partial x_i)\ dx_1 \wedge dx_2 \wedge dx_3$$

in $\Omega^3(U)$, which corresponds to the function $\sum_{i=1}^{3}(\partial v_i)/\partial x_i) = \mathrm{div}(v)$.

Proposition 10.1.4 *Let $f : U \to V$ be a smooth map, where U is open in $\mathbb{R}^m$ and V is open in $\mathbb{R}^n$. Then $d(f^*(\omega)) = f^*(d\omega)$ for every $\omega \in \Omega^*(V)$.*

Proof We start with a special case. If $\omega \in \Omega^0(V)$, then ω is a smooth function from V to $\mathbb{R}$ and $f^*\omega = \omega \circ f$ by definition of f^*. Therefore $d(f^*(\omega)) = d(\omega \circ f)$, so that $d(\omega \circ f)(x)$ for $x \in U$ is the differential of $\omega \circ f$ at x. By the chain rule, this is equal to the composition of $d\omega(f(x))$ with $Df(x)$, the differential of f at x. But that was also the definition of $f^*(d\omega)(x)$. Therefore $d(f^*(\omega)) = f^*(d\omega)$ in the case where $\omega \in \Omega^0(V)$.
The general case follows from the special case above. Using lemma 9.2.5 we may suppose that

$$\omega = g\ dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = g \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

for some $g \in \Omega^0(V)$. Then $d\omega = dg \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ and so

$$\begin{aligned} f^*(d\omega) &= f^*(dg \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) \\ &= f^*(dg) \wedge f^*(dx_{i_1}) \wedge f^*(dx_{i_2}) \wedge \ldots f^*(dx_{i_k}) \\ &= d(f^*g) \wedge d(f^*x_{i_1}) \wedge d(f^*x_{i_2}) \wedge \cdots \wedge d(f^*x_{i_k}) \\ &= d(f^*g \wedge d(f^*x_{i_1}) \wedge d(f^*x_{i_2}) \wedge \cdots \wedge d(f^*x_{i_k})) \\ &= d(f^*g \wedge f^*(dx_{i_1}) \wedge f^*(dx_{i_2}) \wedge \ldots f^*(dx_{i_k})) \\ &= d(f^*\omega)\ . \end{aligned}$$

(Here you really have to believe that x_i is the name of a function from V to $\mathbb{R}$, otherwise f^*dx_i has no meaning.) □

10.2 Differential forms on smooth manifolds

The definition of a differential form on a smooth manifold is suggested by the remark following definition 9.2.1 and by definition 9.2.7. We start with a rough formulation and discuss the details afterwards:

Definition 10.2.1 Let M be a smooth manifold of dimension m. A differential k-form on M is a *smooth map* ω which for every $x \in M$ selects an element $\omega(x) \in \mathrm{alt}^k(T_xM)$.

Question. What does the expression *smooth map* mean in this context ?

Answer 1. A map should first of all have a source and a target. In our case the source of ω is clearly M and the target appears to be the (disjoint) union

$$\bigcup_{x \in M} \mathrm{alt}^k(T_xM) =: \mathrm{alt}^k(TM).$$

But since ω is required to be a smooth map, a structure of smooth manifold on $\mathrm{alt}^k(TM)$ is needed. We can produce this by imitating the proof of proposition 3.2.2. Let $\varphi : U \to M$ be a chart for M, with U open in $\mathbb{R}^m$. Then we obtain a map

$$\varphi_! : \mathrm{alt}^k(TU) \longrightarrow \mathrm{alt}^k(TM)$$

which identifies $\mathrm{alt}_k(T_yU)$ for $y \in U$ with $\mathrm{alt}^k(T_{f(y)}M)$, using the differential $D\varphi(y) : T_yU \to T_{f(y)}M$, a linear isomorphism. We note that T_yU is identified with $\mathbb{R}^m$, for every $y \in U$, and so

$$\mathrm{alt}^k(TU) = U \times \mathrm{alt}^k(\mathbb{R}^m).$$

This is an open set in the vector space $\mathbb{R}^m \times \mathrm{alt}^k(\mathbb{R}^m)$. It is then easy to show that the maps $\varphi_!$ make up a smooth atlas for $\mathrm{alt}^k(TM)$.

Answer 2. It is not important to specify what the target of the "map" ω should be, since we understand perfectly well that ω selects an element $\omega(x) \in \mathrm{alt}^k(T_xM)$ for every $x \in M$. But we need to decide what the word *smooth* should mean. Suppose that $\varphi : U \to M$ is a chart for M, with U open in $\mathbb{R}^m$. Then for every $y \in U$ we obtain an element $\varphi^*(\omega)(y)$ in $\mathrm{alt}^k(\mathbb{R}^m) = \mathrm{alt}^k(T_yU)$, the element corresponding to

$$\omega(\varphi(y)) \in \mathrm{alt}^k(T_{\varphi(y)}M)$$

under the linear isomorphism $D\varphi(y) : \mathbb{R}^m \to T_{\varphi(y)}M$. (See example 9.1.12.) The obvious smoothness condition to impose on ω is that the map

$$y \mapsto \varphi^*(\omega)(y)$$

from U to $\mathrm{alt}^k(\mathbb{R}^m)$ so defined be a *smooth* map. That should hold for every chart $\varphi : U \to M$ in a smooth atlas for M. It is easy to verify that only the equivalence class of the atlas matters. (See definition 2.2.18.)

Remark. The two answers are equivalent in the sense that they lead to the same concept of differential k-form on a smooth manifold M. If you prefer answer 1, you may need to remind yourself that a differential k-form ω on M is not just *any* smooth map from M to $\mathrm{alt}^k(TM)$. We require that $\omega(x) \in \mathrm{alt}^k(T_xM)$ for all $x \in M$.

Definition 10.2.2 The vector space of all differential k-forms on M is denoted by $\Omega^k(M)$. Together, the vector spaces $\Omega^k(M)$ for $k \geq 0$ form a graded ring $\Omega^*(M)$ with the wedge product, defined by $(\omega \wedge \lambda)(x) = \omega(x) \wedge \lambda(x)$ for $\omega \in \Omega^k(M)$, $\lambda \in \Omega^\ell(M)$ and $x \in M$.
A smooth map $f : M \to N$ induces a graded ring homomorphism $f^* : \Omega^*(N) \to \Omega^*(M)$ by

$$(f^*\omega)(x)(u_1, \dots, u_k) = \omega(y)(v_1, \dots, v_k)$$

where $\omega \in \Omega^k(N)$ and $x \in M$ and $y = f(x)$ and $v_i = Df(x)(u_i)$ for $i = 1, 2, \dots, k$. Here $Df(x)$ is the differential of f at x, a linear map $T_xM \to T_yN$.

The following "algebraic" lemma will help us to generalise theorem 10.1.1 to the manifold setting. It is useful in many other situations.

Lemma 10.2.3 *Let U be open in $\mathbb{R}^m$. Every $\omega \in \Omega^k(U)$ with compact support is a (finite) sum of elements of the form $p \wedge dq_1 \wedge dq_2 \wedge \cdots \wedge dq_k$ where $p, q_1, \dots, q_k \in \Omega^0(U)$ also have compact support.*

Proof By lemma 9.2.5, we can write ω as a sum of elements of the form $p \wedge dq_1 \wedge dq_2 \wedge \cdots \wedge dq_k$ where $p, q_1, \dots, q_k \in \Omega^0(U)$, and the support of p is contained in the support of ω. We choose[b] an $f \in \Omega^0(U)$ which has compact support and such that $f \equiv 1$ on the support of ω. Then ω is a sum of elements of the form $p \wedge d(fq_1) \wedge d(fq_2) \wedge \cdots \wedge d(fq_k)$ and here $p, fq_1, \dots fq_k$ all have compact support. □

Theorem 10.2.4 *There exist unique $\mathbb{R}$-linear maps $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$, where $k = 0, 1, 2, 3, \dots$, such that the following holds.*

(i) *$d_0(\omega)$ is the derivative of ω, for $\omega \in \Omega^0(M)$*

(ii) *$d_{k+1} \circ d_k = 0$ for all k*

(iii) *for $\omega \in \Omega^k(M)$ and $\lambda \in \Omega^\ell(M)$ we have* $\quad d_{k+\ell}(\omega \wedge \lambda) = d_k(\omega) \wedge \lambda + (-1)^k \omega \wedge d_\ell(\lambda).$

[b]For some instructions on how to choose, try the notes for lecture 9, heading *partitions of unity.*

Proof We start with the existence part this time. Fix $\omega \in \Omega^k(M)$. Choose a smooth chart $\varphi : U \to M$ with U open in $\mathbb{R}^m$. We want to define $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ in such a way that the diagram of maps

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{\varphi^*} & \Omega^k(U) \\ \big\downarrow{\scriptstyle d_k} & & \big\downarrow{\scriptstyle d_k} \\ \Omega^{k+1}(M) & \xrightarrow{\varphi^*} & \Omega^{k+1}(U) \end{array}$$

is *commutative*, i.e., such that $d_k \circ \varphi^* = \varphi^* \circ d_k$. Note that the right-hand side d_k in the diagram is already defined thanks to theorem 10.1.1. The commutativity requirement determines $(d_k\omega)(y)$ for every $y \in \varphi(U)$, because knowing $(\varphi^*(d_k\omega))(x)$ for $x \in U$ amounts to knowing $(d_k\omega)(y)$ for $y = \varphi(x)$. (These two correspond to each other under the isomorphism $\mathrm{alt}^k(T_yM) \to \mathrm{alt}^k(\mathbb{R}^m)$ determined by the differential of φ at x, a linear isomorphism $\mathbb{R}^m \to T_yM$.) We need to ensure that this provisional definition of $(d_k\omega)(y)$ is unambiguous, i.e., does not depend on the choice of a smooth chart $\varphi(U) \to M$ with $y \in \varphi(U)$. Fortunately that follows from the chain rule and proposition 10.1.4, specialised to the case of a diffeomorphism between two open sets in $\mathbb{R}^m$ (for us, a "change of chart" diffeomorphism).
To prove uniqueness, we start with $\omega \in \Omega^k(M)$ and $y \in M$. Let $\psi : U \to M$ be a smooth chart for M such that $y \in \psi(U)$. Let $V = \psi(U)$. Choose a smooth function $g : M \to \mathbb{R}$ with compact support in V, and such that $g \equiv 1$ in a neighborhood of $y \in V$. Then $g\omega$ also has compact support contained in V. We lose no information on $g\omega$ by viewing it as a differential k-form on V with compact support. By lemma 10.2.3, that differential k-form on V can be written as a sum of elements of the form

$$p \wedge dq_1 \wedge dq_2 \wedge \cdots \wedge dq_k$$

where $p, q_1, \ldots, q_k \in \Omega^0(V)$ all have compact support in V. We can and we shall view $p, q_1, \ldots, q_k$ as elements of $\Omega^0(M)$ with compact suppport contained in V. Then we have

$$g \wedge \omega = p \wedge dq_1 \wedge dq_2 \wedge \cdots \wedge dq_k \ ,$$

an equation in the graded ring $\Omega^*(M)$. Here $dq_i \in \Omega^1(M)$ for $i = 1, 2, \ldots, k$ is just the derivative of q_i. Then we must have

$$d_k(g \wedge \omega) = dp \wedge dq_1 \wedge dq_2 \wedge \cdots \wedge dq_r$$

by rules (ii) and (iii), and again $dp = d_0p$ is the derivative of p in accordance with rule (i). But $d_k(g \wedge \omega) = d_0g \wedge \omega + g \wedge d_k\omega$ by rule (iii). This agrees with $d_k\omega$ in a neighbourhood of y, since $d_0g = dg \equiv 0$ in a neighborhood of y. Therefore

$$d_k\omega \equiv dp \wedge dq_1 \wedge dq_2 \wedge \cdots \wedge dq_r$$

in a neighborhood of y, and this is enough to determine $(d_k\omega)(y)$. □

Definition 10.2.5 Once again, it is customary to write d instead of d_k, as in $d : \Omega^k(M) \to \Omega^{k+1}(M)$. Terminology: d is the *exterior derivative* or sometimes the *coboundary operator*.

Proposition 10.2.6 *Let $f : M \to N$ be a smooth map between smooth manifolds. Then the graded ring homomorphism $f^* : \Omega^*(N) \to \Omega^*(M)$ "commutes" with the exterior derivative, $d(f^*(\omega)) = f^*(d\omega)$ for every $\omega \in \Omega^*(N)$.*

Proof Let $\varphi : U \to M$ be a chart with $x \in \varphi(U)$ and let $\psi : V \to N$ be a chart with $f(x) \in \psi(V)$. We can assume that $f(\varphi(U)) \subset V$. Then we have the following commutative square of rings and ring homomorphisms:

$$\begin{array}{ccc} \Omega^*(N) & \xrightarrow{f^*} & \Omega^*(M) \\ {\scriptstyle \psi^*}\big\downarrow & & \big\downarrow{\scriptstyle \varphi^*} \\ \Omega^*(V) & \longrightarrow & \Omega^*(U) \end{array}$$

The two vertical ones commute with d and the lower horizontal one, induced by the map $x \mapsto \psi^{-1}(f(\varphi(x)))$ also does, by proposition 10.1.4. It follows immediately that $d(f^*(\omega))(y) = f^*(d\omega)(y)$ for $\omega \in \Omega^*(N)$ and every $y \in \varphi(U)$. □

10.3 DeRham cohomology of smooth manifolds

Definition 10.3.1 A *cochain complex* C^* is a sequence of abelian groups C^k for $k \in \mathbb{Z}$, together with group homomorphisms

$$\cdots \xrightarrow{d_{-3}} C^{-2} \xrightarrow{d_{-2}} C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots$$

such that $d_{k+1} \circ d_k = 0$ for all $k \in \mathbb{Z}$. The homomorphisms d_k are more commonly denoted by the single letter d. Together they constitute the "coboundary operator" of C^*. Elements in $\ker(d_k)$ are called *cocycles* and elements in $\mathrm{im}(d_{k+1})$ are called *coboundaries.* The k-th *cohomology group* of the cochain complex C^* is the abelian group

$$H^k(C^*) = \ker(d_k)/\mathrm{im}(d_{k-1}) \ .$$

Definition 10.3.2 For a smooth manifold M there is the cochain complex $\Omega^*(M)$ with the coboundary operator defined and characterised in theorem 10.2.4

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots$$

The "missing" negative terms, $\Omega^k(M)$ for $k < 0$, should be defined as 0. The cohomology groups of this cochain complex are the *de Rham cohomology groups of M*. We abbreviate

$$H^k(\Omega^*(M)) = H^k_{\mathrm{dR}}(M) = H^k(M)$$

depending on the level of precision required. Here each $\Omega^k(M)$, in addition to being an abelian group, comes with a structure of real vector space and the coboundary operator is $\mathbb{R}$-linear. It follows immediately that the abelian groups $H^k_{\mathrm{dR}}(M)$ are also real vector spaces.

The deRham cohomology groups $H^*_{\mathrm{dR}}(M)$ are important algebraic "manifestations" of the topological complexity of the manifold M. Our main business in this chapter and the next few chapters is to develop tools for calculating these vector spaces (e.g., determining their dimensions).

Example 10.3.3 Let M be a point, $M = \mathrm{pt}$. This is a 0-dimensional smooth manifold. Clearly $\Omega^0(\mathrm{pt}) = \mathbb{R}$ and $\Omega^k(\mathrm{pt}) = 0$ for $k \neq 0$, so that $H^0_{\mathrm{dR}}(\mathrm{pt}) = \mathbb{R}$ and $H^k_{\mathrm{dR}}(\mathrm{pt}) = 0$ for $k \neq 0$.

Example 10.3.4 Let $M = S$ be a finite or countably infinite set. This is again a 0-dimensional smooth manifold. Then $\Omega^0(S) = \mathbb{R}^S$ (the vector space of all maps from S to $\mathbb{R}$) and $\Omega^k(S) = 0$ for $k \neq 0$. Therefore $H^0_{\mathrm{dR}}(S) = \mathbb{R}^S$ and $H^k_{\mathrm{dR}}(S) = 0$ for $k \neq 0$.

Example 10.3.5 Let $M = \mathbb{R}$. By lemma 9.2.5, we can identify both $\Omega^0(\mathbb{R})$ and $\Omega^1(\mathbb{R})$ with the vector space of smooth functions from $\mathbb{R}$ to $\mathbb{R}$. Indeed, a smooth function $f : \mathbb{R} \to \mathbb{R}$ *is* an element of $\Omega^0(\mathbb{R})$ and also *determines* an element $f\ dx$ of $\Omega^1(\mathbb{R})$. In these terms, the differential $\Omega^0(\mathbb{R}) \to \Omega^1(\mathbb{R})$ is given by the derivative, $f \mapsto f'$. That is, a smooth function f has $df = f' \cdot dx$, which incidentally "confirms" the dubious notation df/dx for f'. It follows that

$$d : \Omega^0(\mathbb{R}) \to \Omega^1(\mathbb{R})$$

is surjective (because smooth functions can be integrated) but not injective (because all constant functions have derivative 0). The kernel is 1-dimensional, and is identified with $\mathbb{R}$. Therefore

$$H^0_{\mathrm{dR}}(\mathbb{R}) \cong \mathbb{R}$$

while all other deRham cohomology groups of $\mathbb{R}$ are zero.

Example 10.3.6 A slightly more exciting example is $M = S^1$. To unravel this we can use the map $p : \mathbb{R} \to S^1$ defined by $p(x) = (\cos x, \sin x)$. The induced map

$$p^* : \Omega^*(S^1) \to \Omega^*(\mathbb{R})$$

is easily seen to be injective. The image of $p^* : \Omega^0(S^1) \to \Omega^0(\mathbb{R})$ consists of the smooth functions $f : \mathbb{R} \to \mathbb{R}$ which are periodic with 2π as a period. The image of $p^* : \Omega^1(S^1) \to \Omega^1(\mathbb{R})$ consists of the differential 1-forms $f\ dx$ where f is again periodic with 2π as a period. Therefore both $\Omega^0(S^1)$ and $\Omega^1(S^1)$

can be identified with the vector space of smooth functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$. In these terms, the differential $\Omega^0(S^1) \to \Omega^1(S^1)$ is given by the derivative, $f \mapsto f'$. It is *not* surjective. The point is that a periodic function $f : \mathbb{R} \to \mathbb{R}$ need not have a periodic (indefinite) integral. More precisely, if $f : \mathbb{R} \to \mathbb{R}$ is smooth with 2π as a period, then the indefinite integral $\int f(x)\ dx$ will have 2π as a period if and only if

$$\int_0^{2\pi} f(x)\ dx = 0\ .$$

Therefore the quotient of $\Omega^1(S^1)$ by the image of $d : \Omega^0(S^1) \to \Omega^1(S^1)$ has dimension 1, and is again identified with $\mathbb{R}$. The kernel of $d : \Omega^0(S^1) \to \Omega^1(S^1)$ is also identified with $\mathbb{R}$, as in the previous example. Therefore

$$H^0_{\mathrm{dR}}(S^1) \cong \mathbb{R}\ , \qquad H^1_{\mathrm{dR}}(S^1) \cong \mathbb{R}$$

while all other deRham cohomology groups of S^1 are zero.

Example 10.3.7 For any smooth manifold M, the group $H^0_{\mathrm{dR}}(M)$ is the kernel of $d : \Omega^0(M) \to \Omega^1(M)$. This kernel clearly consists of all the smooth functions $M \to \mathbb{R}$ which have zero (total) derivative, i.e., which are constant on every connected component of M. The connected components of M agree with the path components of M. (That's an interesting exercise for you.) Therefore we may write

$$H^0_{\mathrm{dR}}(M) \cong \mathbb{R}^{\pi_0(M)}.$$

10.4 Homotopy invariance of deRham cohomology

Definition 10.4.1 Let C^* and D^* be cochain complexes. A *cochain map* $f : C^* \to D^*$ is a sequence of homomorphisms $f_k : C^k \to D^k$ such that $d \circ f_k = f_{k-1} \circ d$ for all $k \in \mathbb{Z}$. Such a cochain map f induces homomorphisms

$$H^k(C^*) \to H^k(D^*)$$

for every k, by the rule $[x] \mapsto [f(x)]$.

Remark. The definition of the induced maps $H^k(C^*) \to H^k(D^*)$ calls for some explanations. We write $[x] \in H^k(C^*)$. We mean $x \in C^k$ with $d(x) = 0$ and we use the square brackets to indicate the coset $x + \mathrm{im}(d : C^{k-1} \to C^k)$. Then we have to show, first of all, that $d(f(x)) = 0$, so that $[f(x)]$ is a meaningful element of $H^k(D^*)$. This is correct because $d(f(x)) = f(d(x)) = f(0) = 0$, by assumption on f. Next we have to show that $[f(x)]$ is well defined as a function of $[x]$. Indeed, if we change x to $x + d(y)$ where $y \in C^{k-1}$, then $f(x + d(y)) = f(x) + f(d(y)) = f(x) + d(f(y))$, by our assumption on f again. Hence $[f(x + d(y))] = [f(x)] \in H^k(D^*)$.

Example 10.4.2 Let $f : M \to N$ be a smooth map. Then $f^* : \Omega^*(N) \to \Omega^*(M)$ is a cochain map by proposition 10.2.6. It induces therefore

$$f^* : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$$

Here we still write f^*, although for consistency we should probably write something like $(f^*)_*$. The "reversal of arrow direction" happens in passing from the map $M \to N$ to the induced map of chain complexes $\Omega^*(N) \to \Omega^*(M)$.

Definition 10.4.3 Let C^* and D^* be cochain complexes. Let $f, g : C^* \to D^*$ be two cochain maps. A *cochain homotopy* h from f to g is a sequence of homomorphisms $h_k : C^k \to D^{k-1}$ such that

$$d \circ h_k + h_{k+1} \circ d = g_k - f_k$$

for all $k \in \mathbb{Z}$. If such a cochain homotopy exists, we say that f and g are (cochain) *homotopic.* If in addition $f = 0$, so that $d \circ h_k + h_{k+1} \circ d = g_k$ for all k, then we say that g is (cochain) nullhomotopic.

Lemma 10.4.4 *Let $f, g : C^* \to D^*$ be two cochain maps. Suppose that they are cochain homotopic. Then for every $k \in \mathbb{Z}$, the homomorphisms $H^k(C^*) \to H^k(D^*)$ induced by f and g are the same.*

Proof Fix $[x] \in H^k(C^*)$, represented by $x \in C^k$ with $dx = 0$. Then

$$[g(x)] - [f(x)] = [g(x) - f(x)] = [dh_k(x) + h_{k+1}(dx)] = [dh_k(x)] = 0 \in H^k(D^*).$$

□

Cochain homotopies are algebraic counterparts of homotopies between maps (between spaces), as the following discussion illustrates.

Let M be a smooth manifold. We will play around with some interesting operators[c] on $\Omega^*(M \times \mathbb{R})$. For $s \in \mathbb{R}$ there is the shift operator $a_s : \Omega^*(M \times \mathbb{R}) \to \Omega^*(M \times \mathbb{R})$ defined by $(a_s\omega)(x,t) = \omega(x, t+s)$ for $\omega \in \Omega^*(M \times \mathbb{R})$ and $(x,t) \in M \times \mathbb{R}$. There is the operator $b : \Omega^*(M \times \mathbb{R}) \to \Omega^{*-1}(M \times \mathbb{R})$ defined by

$$(b\omega)(x,t)(v_1, v_2, \ldots, v_{k-1}) = \omega(x,t)(1_{\mathbb{R}}, v_1, v_2, \ldots, v_{k-1})$$

for $\omega \in \Omega^k(M \times \mathbb{R})$, where $1_{\mathbb{R}} \in T_{(x,s)}(M \times \mathbb{R})$ is the velocity 1 vector in the $\mathbb{R}$ direction and $v_1, \ldots v_{k-1}$ are arbitrary elements of $T_{(x,s)}(M \times \mathbb{R})$.

Lemma 10.4.5 *In this situation,* $d \circ b + b \circ d = \left.\dfrac{d}{dt}\right|_{t=0} a_t = \lim\limits_{t \to 0} \dfrac{a_t - a_0}{t}$.

Proof For a fixed ω in $\Omega^k(M \times \mathbb{R})$, we have to show

$$d(b\omega) + b(d\omega) = \lim_{t \to 0} \frac{a_t\omega - a_0\omega}{t} \ .$$

This has to be established *at* every (x,t) in $M \times \mathbb{R}$. We can therefore work in local coordinates near (x,t). So there is no loss of generality in assuming that M is an open set in $\mathbb{R}^m$. We write $x_1, \ldots, x_m$ for the coordinates in M and t for the "time" coordinate. By lemma 9.2.5 and linearity, we may assume that either

$$\omega = f \ dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \qquad \text{(type 1)}$$

with $1 \le i_1 < i_2 < \ldots i_k \le m$, or

$$\omega = f \ dt \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{k-1}} \qquad \text{(type 2)}$$

with $1 \le i_1 < i_2 < \ldots i_{k-1} \le m$. In both cases f is a smooth function from $M \times \mathbb{R}$ to $\mathbb{R}$. In the type 1 situation, writing λ for $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, we get $b\omega = 0$ and

$$b(d\omega) \ = \ b(df \wedge \lambda) \ = \ b\Big((\partial f/\partial t) \ dt \wedge \lambda \ + \ \textstyle\sum_{j=1}^m (\partial f/\partial x_j) \ dx_j \wedge \lambda\Big) \ = \ (\partial f/\partial t) \ \lambda$$

so that $d(b\omega) + b(d\omega)$ is $(d/dt)_{|t=0} \ a_t(\omega)$. In the type 2 situation, writing λ for $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{k-1}}$, we get

$$b(d\omega) \ = \ b(df \wedge dt \wedge \lambda) \ = \ b\Big(\textstyle\sum_{j=1}^m (\partial f/\partial x_j) \ dx_j \wedge dt \wedge \lambda\Big) \ = \ -\textstyle\sum_{j=1}^m (\partial f/\partial x_j) \ dx_j \wedge \lambda$$

whereas

$$d(b\omega) \ = \ d\big(f\lambda\big) \ = \ \textstyle\sum_{j=1}^m (\partial f/\partial x_j) \ dx_j \wedge \lambda \ + \ (\partial f/\partial t) \ dt \wedge \lambda \ .$$

Therefore, again, $b(d\omega) + d(b\omega) = (d/dt)_{|t=0} \ a_t(\omega)$. □

Corollary 10.4.6 *For $s \in \mathbb{R}$ let $b_s = b \circ a_s$, in the notation of lemma 10.4.5. Then*

$$d \circ b_s + b_s \circ d = \left.\frac{d}{dt}\right|_{t=s} a_t(\omega) \ .$$

Proof It is easy to verify that $a_s \circ d = d \circ a_s$. Therefore

$$d \circ b_s + b_s \circ d = d \circ (b \circ a_s) + (b \circ a_s) \circ d = (d \circ b + b \circ d) \circ a_s = \left.\frac{d}{dt}\right|_{t=0} a_t(\omega) \circ a_s = \left.\frac{d}{dt}\right|_{t=s} a_t(\omega) \ .$$

The last equation is a consequence of $a_{s+t} = a_s \circ a_t$ (for all $s, t \in \mathbb{R}$). □

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[c] *Operator* is an informal word for a map from a set to itself.

Corollary 10.4.7 *Let* $B = \int_0^1 b_t \; dt$ *, in the notation of corollary 10.4.6. Then*

$$d \circ B + B \circ d = a_1 - a_0 \; .$$

Proof It is easy to see that $d \circ B = \int_0^1 d \circ b_t \; dt$. Therefore

$$d \circ B + B \circ d = \int_0^1 d \circ b_s + b_s \circ d \; ds = \int_0^1 \left(\frac{d}{dt}\Big|_{t=s} a_t \right) ds \;=\; a_1 - a_0 \; .$$

□

In the "cochain homotopy" language, we can restate this last corollary by saying that B is a cochain homotopy from $a_0 = \mathrm{id}$ to a_1. Unravelling the construction of B, we have

$$\begin{aligned} B : \Omega^*(M \times \mathbb{R}) \quad &\longrightarrow \quad \Omega^{*-1}(M \times \mathbb{R}) \\ (B\omega)(x,t)(v_1, v_2, \ldots, v_{k-1}) \quad &= \quad \int_0^1 \omega(x, t+s)(1_{\mathbb{R}}, v_1, v_2, \ldots, v_{k-1}) \; ds \end{aligned}$$

for $\omega \in \Omega^k(M \times \mathbb{R})$ and elements $v_1, v_2, \ldots, v_{k-1}$ in the tangent space $T_{(x,t)}(M \times \mathbb{R})$, which we also view as elements of $T_{(x,t+s)}(M \times \mathbb{R})$.

Theorem 10.4.8 *Let M and N be smooth manifolds. If two smooth maps $f, g : M \to N$ are homotopic, then they induce the same homomorphism from $H^*_{\mathrm{dR}}(N)$ to $H^*_{\mathrm{dR}}(M)$.*

Proof The following fact will be used without proof: If f and g are homotopic, then there exists a *smooth* map $h : M \times \mathbb{R} \to N$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in M$. The emphasis is on *smooth*. It is always easy to extend a continuous map $M \times [0, 1] \to N$ to a continuous map $M \times \mathbb{R} \to N$. Assume therefore that we have a smooth $h : M \times \mathbb{R} \to N$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in M$. Define $e : M \to M \times \mathbb{R}$ by $e(x) = (x, 0)$. The composition

$$\Omega^*(N) \xrightarrow{\;h^*\;} \Omega^*(M \times \mathbb{R}) \xrightarrow{\;a_0^*\;} \Omega^*(M \times \mathbb{R}) \xrightarrow{\;e^*\;} \Omega^*(M)$$

is $f^* : \Omega^*(N) \to \Omega^*(M)$. The composition

$$\Omega^*(N) \xrightarrow{\;h^*\;} \Omega^*(M \times \mathbb{R}) \xrightarrow{\;a_1^*\;} \Omega^*(M \times \mathbb{R}) \xrightarrow{\;e^*\;} \Omega^*(M)$$

is $g^* : \Omega^*(N) \to \Omega^*(M)$. The two are cochain homotopic by means of the composition

$$\Omega^*(N) \xrightarrow{\;h^*\;} \Omega^*(M \times \mathbb{R}) \xrightarrow{\;B\;} \Omega^{*-1}(M \times \mathbb{R}) \xrightarrow{\;e^*\;} \Omega^{*-1}(M)$$

where B comes from corollary 10.4.7. An explicit formula for this cochain homotopy (let's call it $h_\$$) is

$$h_\$\omega(x)(v_1, \ldots, v_{k-1}) = \int_0^1 (h^*\omega)(x, s)(1_{\mathbb{R}}, v_1, \ldots, v_{k-1}) \; ds$$

where $\omega \in \Omega^k(N)$ and $h_\$\omega \in \Omega^{k-1}(M)$ and $x \in M$ and $v_1, \ldots, v_{k-1} \in T_xM$. It is not easy to prove directly from this formula that $d \circ h_\$ + h_\$ \circ d = g^* - f^*$, but you are very welcome to try. □

Remarks on history. The method that we have used to prove theorem 10.4.8 appears to have been developed in essence by Vito Volterra (1889). It was formulated and used in a special case, the case where $M = N = \mathbb{R}^m$, the map f is the identity and g is the constant map $x \mapsto 0$. The homotopy h was $h(x, t) = tx$ for $x \in \mathbb{R}^m$ and $t \in [0, 1]$ or, more generally $t \in \mathbb{R}$. The result in this special case is usually called the *Poincaré lemma.* It is true that Poincaré stated it without proof some years before Volterra proved it.

Corollary 10.4.9 *Let M and N be smooth manifolds. If M is homotopy equivalent to N, then $H^k_{\mathrm{dR}}(M)$ is isomorphic to $H^k_{\mathrm{dR}}(N)$, for every $k \in \mathbb{Z}$.*

Proof Every continuous map $M \to N$ is homotopic to a smooth map $M \to N$. (We will use this without proof.) Therefore, assuming that M and N are homotopy equivalent, we can find *smooth* maps $f : M \to N$ and $g : N \to M$ such that $g \circ f$ is homotopic to the identity id_M and $f \circ g$ is homotopic to the identity id_N. Then we have $f^* : H^*(N) \to H^*(M)$ and $g^* : H^*(M) \to H^*(N)$. The composition

$$H^*(N) \xrightarrow{f^*} H^*(M) \xrightarrow{g^*} H^*(N)$$

agrees with the map induced by $f \circ g : N \to N$ (by inspection). Therefore it must be the identity map of $H^*(N)$ by theorem 10.4.8, as $f \circ g \simeq \mathrm{id}_N$. Similarly the composition

$$H^*(M) \xrightarrow{f^*} H^*(N) \xrightarrow{g^*} H^*(M)$$

is the identity of $H^*(M)$. $\square$

Corollary 10.4.10 $H^k_{\mathrm{dR}}(\mathbb{R}^n) \cong 0$ *for all* $k \neq 0$*, and* $H^0_{\mathrm{dR}}(\mathbb{R}^n) = 0$.

Proof The manifolds $\mathbb{R}^n$ and pt are homotopy equivalent. $\square$

SMSTC (2008/09)
Geometry and Topology
Lecture 11: The Mayer-Vietoris sequence

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Contents

11.1 Short exact sequences and long exact sequences

Definition 11.1.1 Let C, D, E be cochain complexes (definition 10.3.1) and let $f : C \to D$, $g : D \to E$ be cochain maps (definition 10.4.1). We say that

$$C \xrightarrow{\ f\ } D \xrightarrow{\ g\ } E$$

is a *short exact sequence* of cochain complexes if $f_k : C^k \to D^k$ is injective, $g_k : D^k \to E^k$ is surjective and $\ker(g_k) = \operatorname{im}(f_k)$, for all $k \in \mathbb{Z}$.

Our chief example is the following:

Example 11.1.2 Let M be a smooth manifold, V and W open subsets of M such that $V \cup W = M$. Write $j_V : V \to M$ and $j_W : W \to M$ and $e_V : V \cap W \to V$ and $e_W : V \cap W \to W$ for the various inclusion maps. Then we have chain maps

$$\Omega^* M \xrightarrow{\begin{pmatrix} j_V^* \\ j_W^* \end{pmatrix}} \Omega^* V \oplus \Omega^* W \xrightarrow{\begin{pmatrix} e_V^* & -e_W^* \end{pmatrix}} \Omega^*(V \cap W) \; .$$

These chain maps form a short exact sequence. Writing g for the right-hand map and f for the left-hand map, it is indeed clear that $\ker(g) = \operatorname{im}(f)$ and that f is injective. To prove that g is surjective, we choose a *partition of unity* (chapter 5) subordinate to the open covering $\{V, W\}$ of M. This consists of two smooth functions $\varphi_V : M \to [0,1]$ and $\varphi_W : M \to [0,1]$ with support in V and W, respectively, such that $\varphi_V(x) + \varphi_W(x) = 1$ for all $x \in M$. Now if $\lambda \in \Omega^k(V \cap W)$, then $-\varphi_V \cdot \lambda \in \Omega^k(V \cap W)$ extends to an element in $\Omega^k(W)$ which is zero outside V, and $\varphi_W \cdot \lambda \in \Omega^k(V \cap W)$ extends to an element in $\Omega^k(V)$ which is zero outside W. Therefore $\lambda = \varphi_V \cdot \lambda + \varphi_W \cdot \lambda$ is in the image of g.

Definition 11.1.3 Keeping the notation and assumptions of definition 11.1.1, we define a homomorphism

$$\partial : H^k(E) \longrightarrow H^{k+1}(C)$$

by $\partial[z] \;\; = \;\; [f^{-1}(dz')]$, where $z' \in D^k$ is selected so that $g(z') = z$.

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Comments. In the formula for $\partial[z]$ we are assuming that $z \in E^k$ is a cocycle, which means $dz = 0$. The corresponding cohomology class is denoted by $[z]$. We select $z' \in D^k$ with $g(z') = z$. Such a z' exists because g is surjective. Then we have $dz' \in \ker(g)$ because

$$g(dz') = d(g(z')) = dz = 0 \; .$$

Therefore dz' is in the image of f, so that $f^{-1}(dz') \in C^{k+1}$ exists. Note the following:

- $f^{-1}(dz')$ is a cocycle. Indeed by injectivity of f, it is enough to show that dz' is a cocycle, but that follows from $d \circ d = 0$.
- Every choice of z' with $g(z') = z$ gives the same class $[f^{-1}(dz')] \in H^{k+1}(C)$. Indeed, two choices of z' differ by some element in D^k which is in $\ker(g)$ and so has the form $f(y)$ for some $y \in C^k$. Then the two versions of $f^{-1}(dz')$ differ by dy, so they define the same cohomology class.

Example 11.1.4 In the setting of example 11.1.2, the operator

$$\partial : H^k_{\mathrm{dR}}(V \cap W) \longrightarrow H^{k+1}_{\mathrm{dR}}(M)$$

is given by $[\lambda] \mapsto [\omega]$ where $\omega|W = d(\varphi_V \lambda)$ and $\omega|V = -d(\varphi_W \lambda)$. *Prove it.*

Theorem 11.1.5 *Let* $C \xrightarrow{f} D \xrightarrow{g} E$ *be a short exact sequence of cochain complexes. Then the sequence of cohomology groups and homomorphisms*

$$\cdots \xrightarrow{\partial} H^k(C) \xrightarrow{f_*} H^k(D) \xrightarrow{g_*} H^k(E) \xrightarrow{\partial} H^{k+1}(C) \xrightarrow{f_*} H^{k+1}(E) \xrightarrow{g_*} \cdots$$

is exact, *i.e., the kernel of each homomorphism in the sequence agrees with the image of the previous homomorphism.*

Proof Exactness at $H^k(D)$: It is clear that $g_* \circ f_* = 0$ because $g \circ f = 0$. If $z \in D^k$ is a cocycle and $g_*[z] = 0$, then $\exists\, y \in E^{k-1}$ with $dy = g(z)$. Choose $y' \in D^{k-1}$ with $g(y') = y$. Then $z - dy'$ is in $\ker(g) = \mathrm{im}(f)$ since $g(z - dy') = g(z) - d(g(y')) = g(z) - dy = 0$. Therefore $[z] = [z - dy']$ is in the image of $f_* : H^k(C) \to H^k(D)$.
Exactness at $H^k(E)$: It follows directly from the definition of ∂ that $\partial \circ g_* = 0$. Suppose that $z \in E^k$ is a cocycle with $\partial[z] = 0$. Choose $z' \in D^k$ such that $g(z') = z$. Then $[f^{-1}(dz')] = \partial[z] = 0$, so $\exists y \in C^k$ with $dy = f^{-1}(dz')$, which means $f(dy) = d(f(y)) = dz'$. So $z' - f(y) \in D^k$ is a cocycle. Clearly $g_*[z' - f(y)] = [z]$.
Exactness at $H^{k+1}(C)$: It follows directly from the definition of ∂ that $f_* \circ \partial = 0$. Suppose that $z \in C^{k+1}$ is a cocycle with $f_*[z] = 0$. Then $f(z) = dy$ for some $y \in D^k$. Then $d(g(y)) = g(dy) = g(f(z)) = 0$. Hence $[g(y)] \in H^k(D)$ is defined. We have $\partial[g(y)] = [f^{-1}(dy)] = [z]$. $\square$

Terminology. The sequence of cohomology groups in theorem 11.1.5 is called the *long exact sequence* (LES) of cohomology groups associated with the short exact sequence of cochain complexes $C \to D \to E$.

Remark. If the cochain complexes C, D, E are cochain complexes of real vector spaces, with $\mathbb{R}$-linear differentials, then the long exact sequence of cohomology groups is in fact a long exact sequence of real vector spaces and $\mathbb{R}$-linear maps.

Example 11.1.6 In the situation of example 11.1.2, theorem 11.1.5 delivers what is called the long exact *Mayer-Vietoris* sequence of the open covering $\{V, W\}$ of M: a long exact sequence of deRham cohomology groups

$$\cdots \xrightarrow{\partial} H^k_{\mathrm{dR}}(M) \xrightarrow{a} H^k_{\mathrm{dR}}(V) \oplus H^k_{\mathrm{dR}}(W) \xrightarrow{b} H^k_{\mathrm{dR}}(V \cap W) \xrightarrow{\partial} H^{k+1}_{\mathrm{dR}}(M) \xrightarrow{a} \cdots$$

where $a = \begin{pmatrix} j_V^* \\ j_W^* \end{pmatrix}$ and $b = \begin{pmatrix} e_V^* & -e_W^* \end{pmatrix}$.

11.2 De Rham cohomology of spheres

Theorem 11.2.1 *The deRham cohomology groups of* S^n *(for* $n > 0$*) are*

$$H^k_{\mathrm{dR}}(S^n) \cong \begin{cases} \mathbb{R} & \text{if } k = n \\ \mathbb{R} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof We proceed by induction on n. The induction beginning is the case $n = 1$ which we have already dealt with in example 10.3.6. For the induction step, suppose that $n > 1$. We use the Mayer-Vietoris sequence for S^n and the open covering $\{V, W\}$ with $V = S^n \smallsetminus \{p\}$ and $W = S^n \smallsetminus \{q\}$ where $p, q \in S^n$ are the north and south pole, respectively. We will also use the homotopy invariance of deRham cohomology, theorem 10.4.8 and its corollary 10.4.9. This gives us

$$H^k_{\mathrm{dR}}(V) \cong H^k_{\mathrm{dR}}(W) \cong \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

because V and W are homotopy equivalent to a point, and

$$H^k_{\mathrm{dR}}(V \cap W) \cong \begin{cases} \mathbb{R} & \text{if } k = n-1 \\ \mathbb{R} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

by the induction hypothesis, since $V \cap W$ is homotopy equivalent to S^{n-1}. Furthermore it is clear what the inclusion maps $V \cap W \to V$ and $V \cap W \to W$ induce in deRham cohomology: an isomorphism in H^0_{dR} and (necessarily) the zero map in H^k_{dR} for all $k \neq 0$. Thus the homomorphism

$$H^k_{\mathrm{dR}}(V) \oplus H^k_{\mathrm{dR}}(W) \xrightarrow{\ b\ } H^k_{\mathrm{dR}}(V \cap W)$$

from the Mayer-Vietoris sequence takes the form

$$\mathbb{R} \oplus \mathbb{R} \xrightarrow{(1\ \ -1)} \mathbb{R}$$

when $k = 0$, and

$$0 \longrightarrow \mathbb{R}$$

when $k = n - 1$. In all other cases, its source and target are both zero. Therefore the exactness of the Mayer-Vietoris sequence implies that $H^0_{\mathrm{dR}}(S^n) \cong \mathbb{R}$ and $H^n_{\mathrm{dR}}(S^n) \cong \mathbb{R}$, while $H^k_{\mathrm{dR}}(S^n) = 0$ for all other $k \in \mathbb{Z}$. $\square$

Theorem 11.2.2 *Let $f : S^n \to S^n$ be the antipodal map. Then $f^* : H^n_{\mathrm{dR}}(S^n) \to H^n_{\mathrm{dR}}(S^n)$ is multiplication by $(-1)^{n+1}$.*

Proof We proceed by induction again. For the induction beginning, we take $n = 1$. The antipodal map $f : S^1 \to S^1$ is homotopic to the identity, so that $f^* : H^1_{\mathrm{dR}}(S^1) \to H^1_{\mathrm{dR}}(S^1)$ has to be the identity, too. For the induction step, we use the setup and notation from the previous proof. Exactness of the Mayer-Vietoris sequence for S^n and the open covering $\{V, W\}$ shows that

$$\partial : H^{n-1}_{\mathrm{dR}}(V \cap W) \to H^n_{\mathrm{dR}}(S^n)$$

is an isomorphism. The diagram

$$\begin{array}{ccc} H^{n-1}_{\mathrm{dR}}(V \cap W) & \xrightarrow{\ \partial\ } & H^n_{\mathrm{dR}}(S^n) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ H^{n-1}_{\mathrm{dR}}(V \cap W) & \xrightarrow{\ \partial\ } & H^n_{\mathrm{dR}}(S^n) \end{array}$$

is "meaningful" because f takes $V \cap W$ to $V \cap W$. But the diagram is not commutative (i.e., it is not true that $f^* \circ \partial$ equals $\partial \circ f^*$). The reason is that f interchanges V and W, and it does matter in the Mayer-Vietoris sequence which of the two comes first. Therefore we have instead

$$f^* \circ \partial = -\partial \circ f^*$$

in the above square. By the inductive hypothesis, the f^* in the left-hand column of the square is multiplication by $(-1)^n$, and therefore the f^* in the right-hand column of the square must be multiplication by $(-1)^{n+1}$. $\square$

11.3 The usual applications

The results in this section can also be obtained using transversality arguments and the definition of "degree" as in chapter 8.

Theorem 11.3.1 (Brouwer's fixed point theorem, smooth version). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map such that $\sup_{x\in\mathbb{R}^n} \|f(x)\| < \infty$. Then f has a fixed point, i.e., there exists $y \in \mathbb{R}^n$ such that $f(y) = y$.*

Proof Suppose for a contradiction that f does not have a fixed point. Choose $c > 0$ such that $\|f(x)\| \leq c$ for all $x \in \mathbb{R}^n$. Let S be the sphere of radius c about the origin in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, let $g(x)$ be the point where the ray (half-line) from $f(x)$ to x intersects S. Then g is a smooth map from $\mathbb{R}^n$ to S, and we have $g|S = \mathrm{id}_S$. Summarising, we have

$$S \xrightarrow{\quad j \quad} \mathbb{R}^n \xrightarrow{\quad g \quad} S$$

where j is the inclusion, $g \circ j = \mathrm{id}_S$. Therefore we get

$$H^{n-1}_{\mathrm{dR}}(S) \xleftarrow{\quad j^* \quad} H^{n-1}_{\mathrm{dR}}(\mathbb{R}^n) \xleftarrow{\quad g^* \quad} H^{n-1}_{\mathrm{dR}}(S)$$

with $j^* \circ g^* = (g \circ j)^* = \mathrm{id}$. Thus the vector space $H^{n-1}_{\mathrm{dR}}(S)$ is isomorphic to a direct summand of the vector space $H^{n-1}(\mathbb{R}^n)$. But from our calculations above, we know that this is not true. If $n > 1$ we have $H^{n-1}(\mathbb{R}^n) = 0$, while $H^{n-1}(S)$ is nonzero. If $n = 1$ we have $H^{n-1}(\mathbb{R}^n) = \mathbb{R}$, while $H^{n-1}(S) = \mathbb{R} \oplus \mathbb{R}$. □

Remark. The standard formulation of Brouwer's fixed point theorem is as follows: *Every continuous map $f : D^n \to D^n$ has a fixed point.* Here $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. The standard proof of the standard formulation is very much like the proof just given, but it uses a version of cohomology which is better attuned to continuous (rather than smooth) maps.

Let $f : S^n \to S^n$ be a smooth map, $n > 0$. The induced linear map $f^* : H^n_{\mathrm{dR}}(S^n) \to H^n_{\mathrm{dR}}(S^n)$ is multiplication by some number $n_f \in \mathbb{R}$, since $H^n_{\mathrm{dR}}(S^n)$ is isomorphic to $\mathbb{R}$.

Definition 11.3.2 The number n_f is the *degree* of f.

Remark. It turns out that this definition of degree, for a map $S^n \to S^n$, agrees with the definition of degree given in section 8.4. (In particular n_f is always an integer.) We are not in a good position to prove this right now, but will return to the matter in the next chapter.

Remark. The degree n_f of $f : S^n \to S^n$ is clearly an invariant of the homotopy class of f.

Example 11.3.3 According to theorem 11.2.2, the degree of the antipodal map $S^n \to S^n$ is $(-1)^{n+1}$.

Lemma 11.3.4 *Let $f : S^n \to S^n$ be a smooth map. If $f(x) \neq x$ for all $x \in S^n$, then f is smoothly homotopic to the antipodal map. If $f(x) \neq -x$ for all $x \in S^n$, then f is smoothly homotopic to the identity map.*

Proof Let $g : S^n \to S^n$ be the antipodal map, $g(x) = -x$ for all x. Assuming that $f(x) \neq x$ for all x, we show that f is (smoothly) homotopic to g. We think of S^n as the unit sphere in $\mathbb{R}^{n+1}$, with the usual notion of distance. We can make a homotopy $(h_t : S^n \to S^n)_{t\in[0,1]}$ from f to g by "sliding" along the unique minimal geodesic arc from $f(x)$ to $g(x)$, for every $x \in S^n$. In other words, $h_t(x) \in S^n$ is situated $t \cdot 100$ percent of the way from $f(x)$ to $g(x)$ along the minimal geodesic arc from $f(x)$ to $g(x)$. (The important thing here is that $f(x)$ and $g(x)$ are not "antipodes" of each other, by our assumptions. Therefore that minimal geodesic arc is unique.)
Next, assume $f(x) \neq -x$ for all $x \in S^n$. Then, for every x, there is a unique minimal geodesic from x to $f(x)$, and we can use that to make a homotopy from the identity map to f. □

Corollary 11.3.5 (Hairy ball theorem). *Let ξ be a smooth tangent vector field* (explanations follow) *on S^n. If $\xi(z) \neq 0$ for every $z \in S^n$, then n is odd.*

Comments. In general, a smooth tangent vector field on a smooth manifold M is a smooth map ζ from M to TM such that $p \circ \zeta = \mathrm{id}_M$, where $p : TM \to M$ is the usual projection. In the case where M is a smooth manifold in $\mathbb{R}^i$ for some i, we have a simpler definition: a smooth tangent vector field on M is a smooth map

$$\xi : M \to \mathbb{R}^i$$

such that $\xi(z) \in T_z M \subset \mathbb{R}^i$ for each $z \in M$. In the case where $M = S^n \subset \mathbb{R}^{n+1}$, this boils down to a smooth map $\xi : S^n \to \mathbb{R}^{n+1}$ such that $\xi(z) \perp z$, for all $z \in S^n$.

Proof Define $f : S^n \to S^n$ by $f(x) = \xi(x)/\|\xi(x)\|$. Clearly $f(x) \neq x$ and $f(x) \neq -x$ for all $x \in S^n$, since $f(x)$ is always perpendicular to x. Therefore f is homotopic to the antipodal map, and also homotopic to the identity. It follows that the antipodal map is homotopic to the identity. Hence the antipodal map has degree $+1$. Therefore by theorem 11.2.2, n is odd. $\square$

11.4 A finiteness theorem

Theorem 11.4.1 *Let M be a smooth manifold, V open in M. Let $j : V \to M$ be the inclusion. If there exists a compact $K \subset M$ containing V, then the image of $j^* : H^*_{\mathrm{dR}}(M) \longrightarrow H^*_{\mathrm{dR}}(V)$ is finite dimensional.*

Proof To begin with we assume that M is an open subset of some euclidean space $\mathbb{R}^n$. We can find finitely many open "rectangles" $W_1, W_2, \ldots, W_r$ where

$$W_i = \prod_{\ell=1}^{n} U_{i\ell} \quad \subset \quad \mathbb{R}^n$$

for suitable open intervals $U_{i\ell} \subset \mathbb{R}$, such that

$$K \subset \bigcup_{i=1}^{r} W_i \subset M \ .$$

(This uses the compactness of K.) It is enough to show that $H^*_{\mathrm{dR}}(\bigcup_i W_i)$ is finite dimensional, because $j : V \to M$ is a composition of inclusions $V \to \bigcup_i W_i \to M$ and therefore j^* from $H^*_{\mathrm{dR}}(M)$ to $H^*_{\mathrm{dR}}(V)$ can be written as a composition

$$H^*_{\mathrm{dR}}(M) \longrightarrow H^*_{\mathrm{dR}}(\bigcup_i W_i) \longrightarrow H^*_{\mathrm{dR}}(V) \ .$$

In fact the dimension of $H^*_{\mathrm{dR}}(\bigcup_i W_i)$ is at most $2^r - 1$, as we will now show by induction on r. Let

$$W = \bigcup_{i=1}^{r} W_r \ , \quad W' = \bigcup_{i=1}^{r-1} W_i \ , \quad W'' = W_r \ .$$

The total dimension of $H^*_{\mathrm{dR}}(W')$ is at most $2^{r-1} - 1$ by inductive hypothesis and the total dimension of $H^*_{\mathrm{dR}}(W'')$ is clearly 1. The total dimension of $H^*_{\mathrm{dR}}(W' \cap W'')$ is also at most $2^{r-1} - 1$, because $W' \cap W''$ is a union of $r - 1$ rectangles $W_i \cap W_r$ (with $i = 1, 2, \ldots, r - 1$). Now the Mayer-Vietoris long exact sequence

$$\cdots \xrightarrow{\partial} H^k_{\mathrm{dR}}(W) \longrightarrow H^k_{\mathrm{dR}}(W') \oplus H^k_{\mathrm{dR}}(W'') \longrightarrow H^k_{\mathrm{dR}}(W' \cap W'') \xrightarrow{\partial} H^{k+1}_{\mathrm{dR}}(W) \longrightarrow \cdots$$

reveals that the total dimension of $H^*_{\mathrm{dR}}(W)$ is at most equal to the sum of the total dimensions of $H^*_{\mathrm{dR}}(W')$, $H^*_{\mathrm{dR}}(W'')$ and $H^*_{\mathrm{dR}}(W' \cap W'')$. But that sum, according to the estimates above, is not greater than $2^{r-1} - 1 + 2^{r-1} - 1 + 1 = 2^r - 1$.
Next we must look at the general case where M is an arbitrary smooth manifold, not necessarily an open subset of $\mathbb{R}^n$. We start by choosing finitely many open subsets $W_1, W_2, \ldots, W_r$ (recycled names) of M such that each W_i is diffeomorphic to an open subset of $\mathbb{R}^n$ and

$$K \subset \bigcup_i W_i \ .$$

(We can do so because K is compact.) We now proceed by induction on r. If $r=1$, then $V\subset K\subset W_1$ and since W_1 is diffeomorphic to an open subset of euclidean space, we know from the first part of the (ongoing) proof that the inclusion-induced map from $H^*_{\mathrm{dR}}(W_1)$ to $H^*_{\mathrm{dR}}(V)$ has finite dimensional image. *A fortiori*, the inclusion-induced map from $H^*_{\mathrm{dR}}(M)$ to $H^*_{\mathrm{dR}}(V)$ has finite dimensional image. In the case $r>1$ we introduce abbreviations

$$W=\bigcup_{i=1}^{r}W_i\ ,\quad W'=\bigcup_{i=1}^{r-1}W_i\ ,\quad W''=W_r\ .$$

We choose open V',V'' and compact K',K'' in M such that $V=V'\cup V''$ and $K=K''\cup K''$ and $V'\subset K'\subset W'$ and $V''\subset K''\subset W''$. [This can be done as follows: For the open covering of W by the two open sets W' and W'', , choose a subordinate partition of unity, consisting of smooth maps $\varphi',\varphi'':W\to[0,1]$. See chapter 5 for details. Let $K'=K\cap\mathrm{supp}(\varphi')$ and let $K''=K\cap\mathrm{supp}(\varphi'')$. Let V' be the interior of $V\cap K'$ and let V'' be the interior of $V\cap K''$.] The two Mayer-Vietoris sequences for $W=W'\cup W''$ and $V=V'\cup V''$ arrange themselves nicely in a commutative diagram[b]

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H^k_{\mathrm{dR}}(W) \arrow[r] \arrow[d] & H^k_{\mathrm{dR}}(W')\oplus H^k_{\mathrm{dR}}(W'') \arrow[r] \arrow[d, "u"] & H^k_{\mathrm{dR}}(W'\cap W'') \arrow[r, "\partial"] \arrow[d, "v"] & H^{k+1}_{\mathrm{dR}}(W) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r, "\partial"] & H^k_{\mathrm{dR}}(V) \arrow[r] & H^k_{\mathrm{dR}}(V')\oplus H^k_{\mathrm{dR}}(V'') \arrow[r] & H^k_{\mathrm{dR}}(V'\cap V'') \arrow[r, "\partial"] & H^{k+1}_{\mathrm{dR}}(V) \arrow[r] & \cdots
\end{tikzcd}$$

By our inductive assumptions, the arrows labelled u and v have finite-dimensional images (for all $k\in\mathbb{Z}$, and of course the images are zero if $k<0$ or $k>n$). [More precisely, $V'\subset K'\subset W'$ where W' is covered by $r-1$ open sets $W_1,\dots,W_{r-1}$ which are diffeomorphic to open subsets of $\mathbb{R}^n$. Similarly $V'\cap V''\subset K'\cap K''\subset W'\cap W''$ where $W'\cap W''$ is covered by $r-1$ open sets $W_r\cap W_1,\dots,W_r\cap W_{r-1}$ which are diffeomorphic to open subsets of $\mathbb{R}^n$. The case of $V''\subset K''\subset W''$ is easier, since W'' itself is diffeomorphic to an open subset of $\mathbb{R}^n$.] The exactness of the two rows in the diagram now implies that the unlabelled vertical arrows in the diagram also have a finite-dimensional image. *A fortiori*,

$$j^*:H^k_{\mathrm{dR}}(M)\to H^k_{\mathrm{dR}}(V)$$

has finite-dimensional image for all $k\in\mathbb{Z}$, because $V\subset W\subset M$. □

Corollary 11.4.2 *Let M be a compact smooth manifold. Then $H^*_{\mathrm{dR}}(M)$ is finite dimensional.*

Proof Take $V=K=M$ in the previous theorem. □

[b] A diagram of sets and maps is *commutative* if any two compositions $f_1\circ f_2\circ\cdots\circ f_k$ and $g_1\circ g_2\circ\cdots\circ g_\ell$ of arrows in the diagram which start at the same place (set) in the diagram and end at the same place (set) in the diagram are equal.

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Geometry and Topology
Lecture 12: Stokes' theorem

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Contents

12.1 Oriented smooth manifolds and integration of forms

We had a glimpse of *oriented manifolds* in definition 8.1.6, in connection with the notion of *degree*. Another place where orientations appeared was in theorem 9.3.2 on integration by substitution. The time has come for a more thorough treatment.

Definition 12.1.1 Let M be a smooth n-dimensional manifold, $n > 0$, with (representative) atlas $\mathcal{A}$. We say that $\mathcal{A}$ is an *oriented atlas* if, for any two charts $\varphi : U \to M$ and $\psi : V \to M$ in $\mathcal{A}$, and arbitrary $x \in U$ with $\varphi(x) \in \mathrm{im}(\psi)$, the differential of $\psi^{-1} \circ \varphi$ at x has *positive* determinant.

(*Unravelling:* U and V are open in $\mathbb{R}^n$ and the differential of $\psi^{-1} \circ \varphi$ at x is an invertible linear map $\mathbb{R}^n \to \mathbb{R}^n$. As such it will automatically have a nonzero determinant.)

Lemma 12.1.2 *An oriented atlas for M determines an orientation for M.*

Proof For $z \in M$, choose a chart $\varphi : U \to M$ in the oriented atlas such that $z = \varphi(x)$ for some $x \in U$. The differential of φ at x is a linear isomorphism $\mathbb{R}^n \to T_zM$, and so provides an ordered basis for T_zM. This ordered basis depends of course on the choice of a chart φ and x with $\varphi(x) = z$, but the orientation of T_zM which it determines does not. Indeed if $\psi : V \to M$ is another chart from that oriented atlas and $y \in V$ is such that $\psi(y) = z$, then the differential of $\psi^{-1} \circ \varphi$ at x is orientation preserving and so the orientations of T_zM determined by $D\varphi(x)$ and $D\psi(y)$ are the same. $\square$

Lemma 12.1.3 *For $n > 0$, any orientation of a smooth n-dimensional manifold M can be "realised" by an oriented atlas for M (in the specified equivalence class of smooth atlases).*

Proof It is easy to make an atlas $\mathcal{A}$ for M (in the specified equivalence class) having only charts $\varphi : U \to M$ where U is *connected.* If $\varphi : U \to M$ is such a chart, then the differentials of φ at points of U will either be all orientation-preserving, or all orientation-reversing. Accordingly, we say that φ is orientation-preserving or orientation-reversing. We make a new atlas $\mathcal{A}'$ by

- keeping all the orientation-preserving charts in $\mathcal{A}$

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- replacing each orientation-reversing chart $\varphi : U \to M$ by $\varphi \circ \rho : U' \to M$ where
$$U' = \{(x_1, x_2, \dots, x_n) \mid (-x_1, x_2, \dots, x_n) \in U\}$$
and $\rho : U' \to U$ is given by $\rho(x_1, x_2, \dots, x_n) = (-x_1, x_2, \dots, x_n)$.

The new atlas $\mathcal{A}'$ is oriented. The orientation on M which it determines according to lemma 12.1.2 is clearly the "given" orientation of M. □

Let M be a smooth *oriented* n-dimensional manifold. Let $\omega \in \Omega^n(M)$ be a differential n-form with *compact* support. We come to a definition of
$$\int_M \omega$$
generalising definition 9.3.1. Choose an allowed oriented atlas $\mathcal{A}$ for M, compatible with the given orientation. Choose a smooth partition of unity $\{\rho_j \mid j \in J\}$ on M and, for each $j \in J$, a chart $\varphi_j : U_j \to M$ in $\mathcal{A}$ such that the support of ρ_j is contained in $\mathrm{im}(\varphi_j)$. We try
$$\boxed{\int_M \omega := \sum_{j \in J} \int_{U_j} \varphi_j^*(\rho_j \cdot \omega)}\ .$$

Each term in the sum on the right-hand side is already defined (9.3.1). Let's also note that the sum has only finitely many nonzero terms. (Since ω has compact support, $\rho_j \cdot \omega$ can only be nonzero for finitely many j.) A provisional "excuse" for this provisional definition is that we have good reasons to expect
$$\int_M \omega = \sum_{j \in J} \int_M \rho_j \cdot \omega$$
(additivity property of the integral), and furthermore
$$\int_M \rho_j \cdot \omega = \int_{\varphi_j(U_j)} \rho_j \cdot \omega$$
because $\rho_j \cdot \omega$ has support in $\varphi_j(U_j)$, and finally
$$\int_{\varphi_j(U_j)} \rho_j \cdot \omega \ = \ \int_{U_j} \varphi_j^*(\rho_j \cdot \omega)$$
because we ought to believe in "integration by substitution", as seen in theorem 9.3.2. See also definition 10.2.2 if the meaning of φ_j^* is not clear.

Lemma 12.1.4 *$\int_M \omega$ is well defined (by means of the boxed equation).*

Proof Suppose that we have to choices of oriented (allowed) atlas $\mathcal{A}$ and $\mathcal{B}$ for M, two partitions of unity $\{\rho_j \mid j \in J\}$ and $\{\zeta_k \mid k \in K\}$, and choices of charts $\varphi_j : U_j \to M$ in $\mathcal{A}$ for all $j \in J$, and $\psi_k : V_k \to M$ in $\mathcal{B}$ for all $k \in K$, such that $\mathrm{supp}(\rho_j) \subset \mathrm{im}(\varphi_j)$ and $\mathrm{supp}(\zeta_k) \subset \mathrm{im}(\psi_k)$ for all j and k. Then we have to show
$$\sum_{j \in J} \int_{U_j} \varphi_j^*(\rho_j \cdot \omega) = \sum_{k \in K} \int_{V_k} \psi_k^*(\zeta_k \cdot \omega)\ .$$
To that end we introduce another partition of unity
$$\{\rho_j \cdot \zeta_k \mid (i, k) \in J \times K\}\ .$$

We have $\mathrm{supp}(\rho_j \cdot \eta_k) \subset \mathrm{im}(\varphi_j) \cap \mathrm{im}(\psi_k)$. Now we can write

$$\begin{aligned}
& \sum_{j\in J} \int_{U_j} \varphi_j^*(\rho_j \cdot \omega) \\
= & \sum_{j\in J}\sum_{k\in K} \int_{U_j} \varphi_j^*(\rho_j \cdot \zeta_k \cdot \omega) \\
= & \sum_{j\in J}\sum_{k\in K} \int_{U_j \cap \varphi_j^{-1}(\psi_k(V_k))} \varphi_j^*(\rho_j \cdot \zeta_k \cdot \omega) \\
= & \sum_{k\in K}\sum_{j\in J} \int_{V_k \cap \psi_k^{-1}(\varphi_j(U_j))} \psi_k^*(\rho_j \cdot \zeta_k \cdot \omega) \qquad \text{(using theorem 9.3.2)} \\
= & \sum_{k\in K}\sum_{j\in J} \int_{V_k} \psi_k^*(\rho_j \cdot \zeta_k \cdot \omega) \\
= & \sum_{k\in K} \int_{V_k} \psi_k^*(\zeta_k \cdot \omega).
\end{aligned}$$

□

Sometimes we use the (boxed) definition of $\int_M \omega$ also when ω does not have compact support. Whether that is meaningful, well-defined etc. must be decided on a case-by-case basis.

Example 12.1.5 Let M be a (nonempty) smooth k-dimensional manifold in $\mathbb{R}^n$, where $k > 0$. Suppose that M is also oriented as a smooth manifold in its own right. Let's define a "preferred" element $\omega \in \Omega^k(M)$ in the following way. For $x \in M$ and $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$ in T_xM put

$$\omega(x)(v^{(1)}, v^{(2)}, \ldots, v^{(k)}) = \pm\sqrt{\det(A^TA)} \qquad \text{where } A = (v_j^{(i)}) \ .$$

The sign is positive if the list $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$ is an oriented basis of T_xM and negative if it is a non-oriented basis. If $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$ are linearly dependent, then $\det(A^TA) = 0$ and the sign is not an issue.

It is well-known that the square root of $\det(A^TA)$ is the k-dimensional volume of the k-dimensional parallel-epipedon in $\mathbb{R}^n$ spanned by the vectors $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$. Therefore the number

$$\int_M \omega$$

defined by the boxed formula above is what you should regard as the k-dimensional volume of M. It is well-defined. All the summands in the formula will be positive, but some may be infinite and even if they are not, the sum may be infinite. But if M is compact and nonempty, then $\int_M \omega$ is a positive real number.

Let's note in passing that $\omega(x) \neq 0$ for any $x \in M$. In general, a differential k-form λ on a smooth k-dimensional manifold which has $\lambda(x) \neq 0$ for all x is called a *volume form.* So the above $\omega \in \Omega^k(M)$ is an example of a volume form.

Example 12.1.6 We calculate the 2-dimensional volume (also known as area) of the unit sphere $S^2 \subset \mathbb{R}^3$. *The purpose of this exercise is to illustrate the definitions just given. It is very far from being "smart".* Let's choose an orientation first by declaring that two linearly independent tangent vectors $v^{(1)}, v^{(2)}$ at $y \in S^2$ form an oriented basis iff the list $y, v^{(1)}, v^{(2)}$ (in that order) is an oriented basis of $\mathbb{R}^3$. Next, let's observe that the area of S^2 is equal to the area of $S^2 \smallsetminus \{P\}$ where $P = (0, 0, -1)$ is the south pole. The advantage of working with $S^2 \smallsetminus \{P\}$ is that we can get away with a single chart, e.g., the inverse of *stereographic projection.* Therefore let

$$\psi : \mathbb{R}^2 \to S^2 \smallsetminus \{P\}$$

be defined by

$$\psi(x) = \frac{1}{\|x\|^2 + 1}(2x_1, 2x_2, 1 - \|x\|^2)$$

for $x = (x_1, x_2) \in \mathbb{R}^2$. That is a diffeomorphism. It is also orientation-preserving for the above choice of orientation of S^2. We have the standard volume (=area) form ω on S^2 , as defined in example 12.1.5. What we need to calculate now is

$$\int_{S^2} \omega \quad = \quad \int_{S^2 \smallsetminus \{P\}} \omega \quad = \quad \int_{\mathbb{R}^2} \psi^* \omega \ .$$

The main challenge here is clearly to find out what $\psi^*\omega$ is, and that boils down mostly to finding out what the differentials of ψ look like. I seem to get

$$\begin{aligned} \frac{\partial \psi}{\partial x_1} &= \frac{-2x_1}{(\|x\|^2+1)^2}(2x_1, 2x_2, 1-\|x\|^2) + \frac{1}{\|x\|^2+1}(2, 0, -2x_1) \\ \frac{\partial \psi}{\partial x_2} &= \frac{-2x_2}{(\|x\|^2+1)^2}(2x_1, 2x_2, 1-\|x\|^2) + \frac{1}{\|x\|^2+1}(0, 2, -2x_2) \ . \end{aligned}$$

Suppose now that $x_2 = 0$. Then it is easy to check that these two vectors (for fixed x_1) are perpendicular to each other. A semi-tedious calculation shows that they both have the same length

$$\frac{2}{\|x\|^2+1}$$

so that the area of the parallelogram (square) which they span is

$$\frac{4}{(\|x\|^2+1)^2} \ .$$

In the general case, when x_2 is arbitrary, a symmetry argument (using the fact that ψ has rotational symmetry) shows that the area of the parallelogram spanned by the vectors

$$\frac{\partial \psi}{\partial x_1} \ , \ \frac{\partial \psi}{\partial x_2} \quad \in \mathbb{R}^3$$

depends only on $\|x\|$. Therefore we know what it is and we get

$$\psi^*\omega = g \cdot dx_1 \wedge dx_2 \quad \in \Omega^2(\mathbb{R}^2)$$

where $g(x) = \dfrac{4}{(\|x\|^2+1)^2}$. It follows that

$$\int_{\mathbb{R}^2} \psi^*\omega \quad = \quad \int_{\mathbb{R}^2} \frac{4}{(\|x\|^2+1)^2} \ dx_1 dx_2 \ .$$

Conversion to polar coordinates shows that this is equal to

$$4\pi \int_0^\infty \frac{2x}{(x^2+1)^2} \ dx = 4\pi \int_1^\infty \frac{1}{u^2} \ du \quad = \quad 4\pi \ .$$

In the case of a 0-dimensional smooth oriented manifold M, a 0-dimensional oriented atlas may not exist, or perhaps we should agree that the concept has no clear meaning. In any case we need an entirely new definition of

$$\int_M \omega$$

for a differential 0-form with compact support on M. Here it is.

Definition 12.1.7 Let M be a 0-dimensional smooth oriented manifold and let ω be a differential 0-form on M with compact support. Then we let

$$\int_M \omega = \sum_{x \in M} \varepsilon(x)\omega(x)$$

where $\varepsilon(x)$ is $+1$ or -1 depending on whether the orientation of M is positive or negative at x. (Although M can be an infinite set, the sum has only finitely many nonzero terms because we assumed that ω has compact support.)

Let's conclude with an important observation: integration of differential forms is *invariant under orientation preserving diffeomorphisms.* So if M and N are smooth oriented m-manifolds, and $f : M \to N$ is an orientation-preserving diffeomorphism, and $\omega \in \Omega^m(N)$ has compact support, then

$$\int_M f^*\omega = \int_N \omega \ .$$

This generalises the "integration-by-substitution" formula, theorem 9.3.2, and it also follows from that same formula (and the definitions).

12.2 Stokes' theorem

A special case of Stokes' theorem states that

$$\int_M d\omega \ = \ 0$$

for a smooth oriented m-dimensional manifold M and a differential form $\omega \in \Omega^{m-1}(M)$ with compact support. In order to state Stokes' theorem in full generality, we need to allow differential forms on manifolds *with boundary.*

The definition of a differential k-form on a smooth m-dimensional manifold M with boundary, by analogy with definition 10.2.1, is now rather obvious. We therefore have

$$\Omega^*(M) \ ,$$

a graded ring (with the wedge product), and the exterior derivative $d : \Omega^*(M) \to \Omega^{*+1}(M)$, satisfying $d \circ d = 0$ and $d(\omega \wedge \lambda) = d\omega \wedge \lambda + (-1)^k \omega \wedge d\lambda$ by construction, for all $\omega \in \Omega^k(M)$ and $\lambda \in \Omega^\ell(M)$. We also have as before the deRham cohomology groups

$$H^k_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))} \ .$$

The main results of chapters 7 and 8 (homotopy invariance, long exact Mayer-Vietoris sequence) carry over without difficulty to the case of a smooth manifold with boundary.

Theorem 12.2.1 *Let M be a smooth oriented m-dimensional manifold with boundary, $m > 0$. Let ∂M have the orientation induced from M. Let ω be a differential $(m-1)$-form on M with compact support. Write $e : \partial M \to M$ for the inclusion. Then*

$$\int_{\partial M} e^*\omega \quad = \quad \int_M d\omega \ .$$

Proof To keep notation under control, we will write $\int_{\partial M} \omega$ for $\int_{\partial M} e^*\omega$, and so on.

Step 1. We assume $M = \mathbb{R}^m_{\mathrm{up}}$, so that $\partial M = \mathbb{R}^{m-1}$. (Let's continue to write M and ∂M.) If M has the standard orientation, and m is odd, then ∂M also inherits the standard orientation from M. If m is even, then ∂M gets the non-standard orientation.
We know from lemma 9.2.5 that

$$\omega = \sum_{i=1}^m g_i \cdot \lambda_i$$

where λ_i is the wedge product (taken in the natural ordering) of the dx_j for $j \in \{1, 2, 3, \ldots, m\}$ with $j \neq i$, and the g_i are smooth functions (with compact support) on M. Therefore

$$d\omega = \sum_i dg_i \wedge \lambda_i \ .$$

We can calculate directly

$$\int_{\partial M} \omega = \int_{\partial M} g_m \cdot \lambda_m = (-1)^{m-1} \int_{\partial M} g_m \ \ dx_1 dx_2 \cdots dx_{m-1} \ ,$$

while

$$\begin{aligned}\int_M d\omega &= \sum_i \int_M dg_i \wedge \lambda_i \\ &= (-1)^{i-1} \sum_i \int_M (\partial g_i/\partial x_i) \;\; dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m \\ &= \sum_i (-1)^{i-1} \int_M (\partial g_i/\partial x_i) \;\; dx_1 dx_2 \cdots dx_m \; .\end{aligned}$$

By the fundamental theorem of calculus, and by Fubini's theorem, and because the g_i have compact support, only one of the summands in the last sum has a chance to be nonzero. That one has the value

$$(-1)^{m-1} \int_{\partial M} g_m \;\; dx_1 dx_2 \cdots dx_{m-1} \; .$$

Step 2. Here M is an open set in $\mathbb{R}^m_{\text{up}}$, and $\partial M = M \cap \mathbb{R}^{m-1}$. This case reduces easily to the previous one: since ω has compact support in M, it can be extended to a differential $(m-1)$-form on $\mathbb{R}^m_{\text{up}}$ by the declaration $\omega(x) = 0$ for $x \in \mathbb{R}^m_{\text{up}} \smallsetminus M$.

Step 3. This is the general case. We choose a smooth partition of unity $\{\rho_j \mid j \in J\}$ on M such that every ρ_j has support contained in the image of some chart $\psi_j : U_j \to M$ in some atlas for M. Here U_j is open in $\mathbb{R}^m_{\text{up}}$. We start with the observation

$$\sum_j d\rho_j \cdot \omega = (\sum_j d\rho_j) \cdot \omega = d(\sum_j \rho_j) \cdot \omega = 0 \; .$$

It follows that

$$d\omega = \sum_{j \in J} \rho_j \cdot d\omega = \sum_{j\in J} (\rho_j \cdot d\omega + d\rho_j \cdot \omega) = \sum_{j \in J} d(\rho_j \cdot \omega) \; .$$

Therefore

$$\int_M d\omega = \sum_j \int_M d(\rho_j \cdot \omega) = \sum_j \int_{U_j} \psi_j^*(d(\rho_j \cdot \omega)) = \sum_j \int_{U_j} d(\psi_j^*(\rho_j \cdot \omega)) \; .$$

By step 2 of the proof, we have

$$\sum_j \int_{U_j} d(\psi_j^*(\rho_j \cdot \omega)) \;\; = \;\; \sum_j \int_{\partial U_j} \psi_j^*(\rho_j \cdot \omega) \; .$$

In the right-hand side of the last equation, we may replace ψ_j and ρ_j by appropriate restrictions to ∂M without making any difference, and so

$$\sum_j \int_{\partial U_j} \psi_j^*(\rho_j \cdot \omega) = \int_{\partial M} \omega$$

from the definition of $\int_{\partial M} \omega$. Putting the equations together, we obtain

$$\int_M d\omega = \int_{\partial M} \omega \; .$$

$\square$

12.3 Degrees of maps and integration of forms

In chapter 8, we saw a definition of the degree of a smooth map f from S^n to itself. This relied on Sard's theorem and (some) transversality. Let us take a different approach here using integration of differential forms.

A smooth map $f : S^n \to S^n$ (where $n > 0$) induces a linear map $f^* : H^n_{\text{dR}}(S^n) \to H^n_{\text{dR}}(S^n)$. Since $H^n_{\text{dR}}(S^n)$ is a 1-dimensional real vector space (chapter 11), this linear map is given by multiplication with a real number a_f.

Definition 12.3.1 The number $a_f \in \mathbb{R}$ is the *degree of* f.

(That definition of degree competes with the chapter 8 definition, but let's try to forget about chapter 8 for the moment.)

By Stokes' theorem, integration of differential n-forms gives a well-defined map

$$H^n_{\mathrm{dR}}(S^n) \longrightarrow \mathbb{R} \;\; ; \;\; [\omega] \mapsto \int_{S^n} \omega \ .$$

Here $\omega \in \Omega^n(S^n)$ is a differential n-form, with $d\omega = 0$ automatically. The reason for *well-defined* is that if $\omega, \lambda \in \Omega^n(S^n)$ represent the same deRham cohomology class, then

$$\omega = \lambda + d\psi$$

for some $\psi \in \Omega^{n-1}(S^n)$. Consequently

$$\int_{S^n} \omega = \int_{S^n} \lambda + d\psi = \int_{S^n} \lambda + \int_{S^n} d\psi = \int_{S^n} \lambda$$

where the last equality uses Stokes.
The integration map $H^n_{\mathrm{dR}}(S^n) \longrightarrow \mathbb{R}$ is nonzero (hence a linear isomorphism) because we have "volume forms" on S^n , as in example 12.1.6. Therefore definition 12.3.1 can be reformulated as follows.

Definition 12.3.2 For smooth $f : S^n \to S^n$, there exists a unique number $a_f \in \mathbb{R}$ such that

$$\int_{S^n} f^*\omega \quad = \quad a_f \cdot \int_{S^n} \omega$$

holds for every $\omega \in \Omega^n(S^n)$. This number a_f is the degree of f.

These "new" definitions of the degree of $f : S^n \to S^n$ make it very clear that the degree is a smooth homotopy invariant, but they do not tell us that the degree is always an integer. That is a good excuse for trying to make the connection with the chapter 8 definition of degree, after all.

Suppose therefore that $f : S^n \to S^n$ is smooth and has $P \in S^n$ as a regular value. Without loss of generality, P is the north pole. Choose a small open ball U about P in S^n. By the inverse function theorem, $f^{-1}(U)$ is a disjoint union of finitely many connected open sets V_i , each of which is mapped diffeomorphically to U by f (if U is small enough). Let $\epsilon_i = \pm 1$, depending on the orientation behaviour of $f|V_i$. Then the "chapter 5" degree of f is $\sum_i \epsilon_i$.
Now choose a map $g : S^n \to S^n$ which

- is smoothly homotopic to the identity
- maps U diffeomorphically (and preserving orientation) to $S^n \smallsetminus Q$, where Q is the south pole
- maps the entire complement of U to the south pole Q.

Let $f_1 = g \circ f$. By construction, $f_1^{-1}(S^n \smallsetminus Q)$ is the disjoint union of the open sets V_i , each of which is mapped diffeomorphically to $S^n \smallsetminus Q$ by f_1. All points of S^n which are not in any of the V_i are mapped to the south pole Q by f_1. Therefore

$$\int_{S^n} f_1^*\omega \;\; = \sum_i \int_{V_i} f_1^*\omega = \sum_i \left(\epsilon_i \int_{S^n \smallsetminus Q} \omega \right) = \sum_i \left(\epsilon_i \int_{S^n} \omega \right) = \left(\sum_i \epsilon_i \right) \int_{S^n} \omega \ .$$

This tells us that a_{f_1} equals $\sum_i \epsilon_i$, which is the degree of f in the chapter 5 sense. But f_1 is smoothly homotopic to f, so that $a_f = a_{f_1}$ and consequently

$$\boxed{a_f = \sum_i \epsilon_i} \ .$$

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Geometry and Topology
Lecture 13: Poincaré duality for oriented manifolds

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Contents

13.1 Ring structure on deRham cohomology

Definition 13.1.1 For a smooth manifold M, with or without boundary, there is a graded ring structure on $H^*_{\mathrm{dR}}(M)$ defined as follows:

$$[\omega] \cdot [\lambda] = [\omega \wedge \lambda] \ .$$

Here $\omega \in \Omega^k(M)$ and $\lambda \in \Omega^\ell(M)$, with $d\omega = 0$ and $d\lambda = 0$, so that $[\omega] \in H^k_{\mathrm{dR}}(M)$ and $[\lambda] \in H^\ell_{\mathrm{dR}}(M)$.

The product is well defined thanks to the formula

$$d(\gamma \wedge \sigma) = d\gamma \wedge \sigma \pm \gamma \wedge d\sigma$$

which we have from the definition of the exterior derivative d. In more detail, suppose that $\omega, \omega' \in \Omega^k(M)$ satisfy $d\omega = 0 = d\omega'$ and $[\omega] = [\omega'] \in H^k_{\mathrm{dR}}(M)$. Then $\omega' = \omega + d\psi$ for some $\psi \in \Omega^{k-1}(M)$ and so, for any $\lambda \in \Omega^\ell(M)$ we have

$$\omega' \wedge \lambda = (\omega + d\psi) \wedge \lambda = \omega \wedge \lambda + d\psi \wedge \lambda = \omega \wedge \lambda + d(\psi \wedge \lambda) \mp \psi \wedge d\lambda \ .$$

It follows that, if $d\lambda = 0$, then $\omega' \wedge \lambda = \omega \wedge \lambda + d(\psi \wedge \lambda)$ and so

$$[\omega' \wedge \lambda] = [\omega \wedge \lambda] \in H^{k+\ell}_{\mathrm{dR}}(M) \ .$$

That proves one half of "well defined" and the other half is similar.
The product is also bilinear, associative and graded commutative. This follows directly from the properties of the wedge product. To emphasize the bi*linearity* we can say that the graded ring $H^*_{\mathrm{dR}}(M)$ is a graded *algebra* over $\mathbb{R}$. (That simply means that the underlying graded abelian group comes with a structure of graded vector space over $\mathbb{R}$, and the product is bilinear.) Moreover the ring $H^*_{\mathrm{dR}}(M)$ has a unit, $1 \in H^0_{\mathrm{dR}}(M)$, represented by the constant function with value 1 on M.

Example 13.1.2 Let M and N be compact oriented smooth manifolds, of dimensions m and n, respectively, both without boundary. Let $\omega \in \Omega^m(M)$ and $\lambda \in \Omega^n(N)$. Let $p_1 : M \times N \to M$ and $p_2 : M \times N \to N$ be the projections. Then we have, almost from the definitions,

$$\int_{M\times N} (p_1^*\omega \wedge p_2^*\lambda) \quad = \quad \int_M \omega \ \cdot \ \int_N \lambda \ .$$

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Therefore, if $\int_M \omega \neq 0$ and $\int_N \lambda \neq 0$, then $\int_{M\times N}(p_1^*\omega \wedge p_2^*\lambda) \neq 0$ and so $[p_1^*\omega]\cdot[p_2^*\lambda] \neq 0$ by Stokes' theorem. Examples of ω and λ with $\int_M \omega \neq 0$ and $\int_N \lambda \neq 0$ can easily be constructed as *volume forms*. See examples 12.1.5 and 12.1.6.

13.2 Products and integration of forms

The most basic formulation of Poincaré duality is for a compact smooth oriented m-dimensional manifold M without boundary. It is a statement about the graded ring or graded algebra $H^*_{\mathrm{dR}}(M)$ *together* with the homomorphism

$$\mathcal{I} : H^m_{\mathrm{dR}}(M) \to \mathbb{R} \; ; \quad \mathcal{I}(\,[\omega]\,) = \int_M \omega \; .$$

Before stating it, let's introduce some vocabulary related to bilinear maps.

Definition 13.2.1 Let V and W be real vector spaces over $\mathbb{R}$. A bilinear map $g : V \times W \to \mathbb{R}$ is *nonsingular on the left* if the linear map

$$V \to \hom_{\mathbb{R}}(W, \mathbb{R})$$

defined by $v \mapsto (w \mapsto g(v, w))$ is a linear isomorphism. Similarly, g is *nonsingular on the right* if the linear map $W \to \hom_{\mathbb{R}}(V, \mathbb{R})$ defined by $w \mapsto (v \mapsto g(v, w))$ is a linear isomorphism.

If, in the preceding definition, V and W are both finite dimensional, then *nonsingular on the left* forces *nonsingular on the right* and vice versa. Then we say simply that g is *nonsingular*. Also, in the case where V and W are finite dimensional, we can choose ordered bases for V and W, respectively, and so express the bilinear map $g : V \times W \to \mathbb{R}$ as a matrix. It is easy to show that g is nonsingular if and only if that matrix is a square matrix (i.e., number of columns equal to number of rows) and its determinant is nonzero.

In these terms, the statement of Poincaré duality in the compact-without-boundary case is as follows.

Corollary 13.2.2 *(of theorem 13.2.3 below:) For a compact smooth oriented m-dimensional manifold M without boundary, and any $k \in \mathbb{Z}$, the bilinear map $H^k_{\mathrm{dR}}(M) \times H^{m-k}_{\mathrm{dR}}(M) \to \mathbb{R}$ defined by*

$$(\,[\omega], [\lambda]\,) \mapsto \mathcal{I}(\,[\omega]\cdot[\lambda]\,)$$

is nonsingular.

Two remarks: $\mathcal{I}(\,[\omega]\cdot[\lambda]\,)$ means $\int_M \omega \wedge \lambda$, and the word *nonsingular* does not require a "left" or "right" specifier because of corollary 11.4.2.

Even if we were only interested in compact manifolds without boundary, we would obviously want to see a proof. It must be admitted that such a proof could involve something like an induction on the number of charts needed to make an atlas for M. Since the charts have the form $V \to M$ where V is open in a euclidean space, and typically noncompact, it is hard to avoid noncompact manifolds altogether. Furthermore, if we have to deal with noncompact manifolds, then we also have to be prepared to deal with infinite dimensional vector spaces. For example, it is easy to produce an open set $U \subset \mathbb{R}$ such that $H^0_{\mathrm{dR}}(U)$ is infinite dimensional, or an open set $V \subset \mathbb{R}^2$ such that $H^1_{\mathrm{dR}}(V)$ is infinite dimensional.

Suppose therefore that M is any smooth oriented m-dimensional manifold M without boundary. Let $\Omega^*_c(M) \subset \Omega^*(M)$ be the cochain subcomplex consisting of the differential forms with compact support (i.e., those which are zero outside some compact subset of M). Write

$$H^k_{\mathrm{dR},c}(M) = \frac{\ker(d : \Omega^k_c(M) \to \Omega^{k+1}_c(M))}{\mathrm{im}(d : \Omega^{k-1}_c(M) \to \Omega^k_c(M))}$$

for the k-th cohomology group of $\Omega^*_c(M)$. The wedge product of an arbitrary differential k-form on M with a compactly supported differential ℓ-form on M is a compactly supported differential $(k+\ell)$-form on M. Therefore we have a product

$$H^k_{\mathrm{dR}}(M) \times H^\ell_{\mathrm{dR},c}(M) \longrightarrow H^{k+\ell}_{\mathrm{dR},c}(M).$$

This has the usual/expected associativity and bilinearity properties. Also, the element $1 \in H^0_{\mathrm{dR}}(M)$ is a "neutral element" for the multiplication. Note that there is no such thing as $1 \in H^0_{\mathrm{dR},c}(M)$ if M is noncompact, because then the support of the function 1 on M is not compact, being all of M.

From Stokes' theorem we know that there is a well defined linear map

$$\mathcal{I} : H^m_{\mathrm{dR},c}(M) \to \mathbb{R} \; ; \quad [\omega] \mapsto \int_M \omega \; .$$

Now we have the vocabulary to formulate a Poincaré duality statement in the noncompact case.

Theorem 13.2.3 *For a smooth oriented m-dimensional manifold M without boundary, and any $k \in \mathbb{Z}$, the bilinear map $H^k_{\mathrm{dR}}(M) \times H^{m-k}_{\mathrm{dR},c}(M) \to \mathbb{R}$ defined by*

$$(\,[\omega],[\lambda]\,) \mapsto \mathcal{I}(\,[\omega]\cdot[\lambda]\,)$$

is nonsingular on the left.

The rest of this chapter is all about the proof of theorem 13.2.3.

13.3 Poincaré duality: formal aspects of the proof

The proof of theorem 13.2.3 uses a form of induction which has an induction beginning and two types of induction steps. Let's set up the induction mechanism here and leave the details for the next subsection.

For the formulation of the "induction beginning", recall from chapter 8 that an open "rectangle" in $\mathbb{R}^m$ is a product $\prod_{i=1}^m U_i$ where each U_i is an open interval in $\mathbb{R}$.

Lemma 13.3.1 *The Poincaré duality statement 13.2.3 holds for any open rectangle M in $\mathbb{R}^m$.*

As promised, we postpone the proof.— Next, there is a fairly obvious induction step:

Lemma 13.3.2 *Suppose that M is a smooth oriented m-dimensional manifold without boundary, and that $M = V \cup W$ where V and W are open in M. If the Poincaré duality statement 13.2.3 holds for V, W and $V \cap W$, then it holds for $M = V \cup W$.*

We postpone the proof.— Then there is another little induction step, designed specifically to deal with infinite unions. The agreeable surprise here is that we only need to deal with infinite *disjoint* unions.

Lemma 13.3.3 *Suppose that M is a smooth oriented m-dimensional manifold without boundary, and that M is a disjoint union $\coprod_{i=1}^\infty M_i$. If the Poincaré duality statement 13.2.3 holds for each M_i , then it holds for M.*

We postpone the proof.

Proof of the Poincaré duality theorem 13.2.3 modulo lemmas 13.3.1, 13.3.2 and 13.3.3:
Step 1. The Poincaré duality theorem holds for an open rectangle M in $\mathbb{R}^m$ by lemma 13.3.1.
Step 2. The Poincaré duality theorem holds for any open subset M of $\mathbb{R}^m$ which is a union of finitely many open rectangles. This follows from step 1 and lemma 13.3.2 by induction. (For the details of the induction, take the proof of theorem 11.4.1 as a model.)
Step 3. The Poincaré duality theorem holds for any open subset $M \subset \mathbb{R}^m$. For the proof, note that it is always possible to write $M = M_1 \cup M_2$, where

- M_1 is a *disjoint* union of open sets $M_{1,\alpha}$ (possibly infinitely many), where each $M_{1,\alpha}$ is a finite union of open rectangles ;
- M_2 is also a *disjoint* union of open sets $M_{2,\alpha}$ (possibly infinitely many), where each $M_{2,\alpha}$ is a finite union of open rectangles.

By step 2, we know that the Poincaré duality theorem holds for each $M_{1,\alpha}$ and for each $M_{2,\beta}$ and also for $M_{1,\alpha} \cap M_{2,\beta}$. By lemma 13.3.3, the Poincaré duality theorem will then also hold for M_1 and M_2 and $M_1 \cap M_2$. Therefore, by lemma 13.3.2, it holds for $M = M_1 \cup M_2$.

Step 4. The Poincaré duality theorem holds for any smooth oriented manifold M which admits an atlas with only finitely many charts. This follows from step 3 and lemma 13.3.2 by induction on the number of charts.

Step 5. Any smooth oriented manifold M can be written in the form $M = M_1 \cup M_2$ where

- M_1 is a *disjoint* union of open sets $M_{1,\alpha}$ (possibly infinitely many), where each $M_{1,\alpha}$ admits an atlas with only finitely many charts ;
- M_2 is also a *disjoint* union of open sets $M_{2,\alpha}$ (possibly infinitely many), where each $M_{2,\alpha}$ admits an atlas with only finitely many charts.

By step 4, we know that the Poincaré duality theorem holds for each $M_{1,\alpha}$ and for each $M_{2,\beta}$ and also for $M_{1,\alpha} \cap M_{2,\beta}$. By lemma 13.3.3, the Poincaré duality theorem will then also hold for M_1 and M_2 and $M_1 \cap M_2$. Therefore, by lemma 13.3.2, it holds for $M = M_1 \cup M_2$. □

13.4 Mayer-Vietoris sequence in the compact support setting

The following theorem is important in its own right, but it is also an ingredient in the proof of lemma 13.3.2.

Theorem 13.4.1 *Let M be a smooth compact manifold, $M = V \cup W$ where V and W are open in M. Write $i : V \to M$, $j : W \to M$ and $e : V \cap W \to V$, $f : V \cap W \to W$ for the various inclusions. There is a long exact "Mayer-Vietoris" sequence in compactly supported deRham cohomology*

$$\cdots \longleftarrow H^k_{\mathrm{dR},c}(M) \xleftarrow{a} H^k_{\mathrm{dR},c}(V) \oplus H^k_{\mathrm{dR},c}(W) \xleftarrow{b} H^k_{\mathrm{dR},c}(V \cap W) \xleftarrow{\partial} H^{k-1}_{\mathrm{dR},c}(M) \xleftarrow{a} \cdots$$

where $a = \begin{pmatrix} i_* & j_* \end{pmatrix}$ *and* $b = \begin{pmatrix} e_* \\ -f_* \end{pmatrix}$.

Proof The construction is in some formal ways very similar to that of the Mayer-Vietoris sequence of example 11.1.6. The main difference is that here the arrows point in the oppposite direction to what we see in example 11.1.6. To explain that, let us look at the inclusion $i : V \to M$ for example, and what it does to differential forms. Restriction of differential forms determines a map

$$i^* : \Omega^*(M) \to \Omega^*(V)$$

but there is no similar restriction map $\Omega^*_c(M) \to \Omega^*_c(V)$ because the restriction to V of a compactly supported differential form on M need not have compact support in V. To make up for that we have an "inclusion" map

$$i_* : \Omega^*_c(V) \longrightarrow \Omega^*_c(M)$$

given by extending compactly supported differential forms on V *trivially* to M. So if $\omega \in \Omega^k_c(V)$ has compact support $K \subset V$, then $i_*\omega \in \Omega^k_c(M)$ still has the same support K, and that is still compact.

We now proceed by setting up a diagram of cochain complexes

$$\Omega^*_c(M) \xleftarrow{a} \Omega^*_c(V) \oplus \Omega^*_c(W) \xleftarrow{b} \Omega^*_c(V \cap W)$$

where

$$a = \begin{pmatrix} i_* & j_* \end{pmatrix} \ , \qquad b = \begin{pmatrix} e_* \\ -f_* \end{pmatrix} \ .$$

Then we only have to show that the diagram is short exact, and use theorem 11.1.5. It is clear that b is injective and that $\ker(a) = \mathrm{im}(b)$. Surjectivity of a follows from a partition-of-unity argument as in example 11.1.2. □

Remark 13.4.2 The operator

$$\partial : H^{k-1}_{\mathrm{dR},c}(M) \longrightarrow H^k_{\mathrm{dR},c}(V \cap W)$$

is given by $[\lambda] \mapsto [\omega]$ where $\omega = d(\varphi_V \lambda) = -d(\varphi_W \lambda)$, for a partition of unity $\{\varphi_V, \varphi_W\}$ subordinate to the open cover $\{V, W\}$ of M. *Prove it.*

13.5 Poincaré duality: details of the proof

The proofs of the three lemmas 13.3.1, 13.3.2 and 13.3.3 are all quite illuminating and each contains a new message.

Proof of lemma 13.3.1:
An open rectangle in $\mathbb{R}^m$ is either empty or diffeomorphic to $\mathbb{R}^m$, so we may suppose $M = \mathbb{R}^m$. Fix $P \in S^m$. There is a short exact sequence of cochain complexes

$$\Omega^*_c(\mathbb{R}^m) \to \Omega^*(S^m) \to \Omega^*_{\text{germ}}(S^m) \ .$$

The cochain map on the left is obtained by identifying $\mathbb{R}^m$ with $S^m \smallsetminus P$, and extending compactly supported differential forms on $S^m \smallsetminus P$ to all of S^m in the only possible way (choosing the value 0 at P for the extension). The *germ* subscript in the cochain complex on the right means that we work with equivalence classes of differential forms on S^m, two being equivalent if they agree on a neighbourhood of P. (The equivalence classes are called *germs*.) From the short exact sequence of chain complexes, we get a long exact sequence of cohomology groups. From the long exact sequence of cohomology groups, we see that it might be a good idea to show that the k-th cohomology of $\Omega^*_{\text{germ}}(S^m)$ is

$$\cong \begin{cases} \mathbb{R} & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k \neq 0 \ . \end{cases}$$

The case $k = 0$ is easy. For $k > 0$, suppose given $\omega \in \Omega^k(S^m)$ with $d\omega = 0$ *as a germ*. Then $d\omega|U$ is actually zero for a small open ball U about P. So $\omega|U = d\psi$ for some $\psi \in \Omega^{k-1}(U)$, because $H^k_{\text{dR}}(U) = 0$. Using e.g. partitions of unity, construct $\varphi \in \Omega^{k-1}(S^m)$ which agrees with ψ *as a germ*. Then $\omega = d\varphi$ *as germs*, showing that the cohomology class of the germ of ω is 0.
We have now shown that the cohomology of $\Omega^k_{\text{germ}}(S^m)$ is as claimed, linearly isomorphic to $\mathbb{R}$ if $k = 0$ and 0 for all other k. It follows from the long exact sequence that

$$H^k_{\text{dR},c}(\mathbb{R}^m) \cong \begin{cases} \mathbb{R} & \text{if} \quad k = m \\ 0 & \text{if} \quad k \neq m \ . \end{cases}$$

In fact it follows from the long exact sequence that the inclusion-induced map $H^m_{\text{dR},c}(\mathbb{R}^m) \to H^m(S^m)$ is an isomorphism. Since that map respects integration, we find that

$$\mathcal{I} : H^m_{\text{dR},c}(\mathbb{R}^m) \longrightarrow \mathbb{R}$$

is also an isomorphism. Together with the statement $H^k_{\text{dR},c}(\mathbb{R}^m) = 0$ for $k \neq m$ and what we know about $H^*_{\text{dR}}(\mathbb{R}^m) \cong H^*_{\text{dR}}(\text{point})$, this amounts to a proof of Poincaré duality for the manifold $\mathbb{R}^m$. ☐

Lemma 13.5.1 (The "five" lemma)*: Suppose that, in a commutative diagram of abelian groups and homomorphisms*

$$\begin{tikzcd}
\cdots \arrow[r] & A^k \arrow[r] \arrow[d, "f^k"] & B^k \arrow[r] \arrow[d, "g^k"] & C^k \arrow[r] \arrow[d, "h^k"] & A^{k+1} \arrow[r] \arrow[d, "f^{k+1}"] & B^{k+1} \arrow[r] \arrow[d, "h^{k+1}"] & \cdots \\
\cdots \arrow[r] & D^k \arrow[r] & E^k \arrow[r] & F^k \arrow[r] & D^{k+1} \arrow[r] & E^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

the rows are exact (kernel of each homomorphism equals image of preceding homomorphism) and the vertical arrows g^k and h^k are isomorphisms for all $k \in \mathbb{Z}$. Then the arrows f^k are isomorphisms for all $k \in \mathbb{Z}$.

Proof Exercise. ☐

Proof of lemma 13.3.2.
We introduce the abbreviations

$$A^k = H^k_{\text{dR}}(M) \ , \quad A_{m-k} = H^k_{\text{dR},c}(M) \ ,$$

$$B^k = H^k_{\text{dR}}(V) \oplus H^k_{\text{dR}}(W) \ , \quad B_{m-k} = H^k_{\text{dR},c}(V) \oplus H^k_{\text{dR},c}(W) \ ,$$

$$C^k = H^k_{\text{dR}}(V \cap W) \ , \quad C_{m-k} = H^k_{\text{dR}}(V \cap W).$$

Let's also use a "bullet" (rather than a star) for dual spaces, for example $A^\bullet_{m-k} = \hom_{\mathbb{R}}(A_{m-k}, \mathbb{R})$. The Poincaré duality statement that we want to prove for M can be reformulated as follows: a certain linear map $A^k \to A^\bullet_k$ given by

$$[\omega] \mapsto \left([\lambda] \mapsto \int_M \omega \wedge \lambda \right),$$

for $[\omega] \in A^k$ and $[\lambda] \in A_k$, is an isomorphism. We are *assuming* that the analogous maps $B^k \to B^\bullet_k$ and $C^k \to C^\bullet_k$ are isomorphisms.
These linear maps $A^k \to A^\bullet_k$ etc. can be arranged in a diagram of vector spaces and linear maps

$$\begin{tikzcd}
\cdots \arrow[r] & A^k \arrow[r] \arrow[d, "f^k"] & B^k \arrow[r] \arrow[d, "g^k"] & C^k \arrow[r] \arrow[d, "h^k"] & A^{k+1} \arrow[r] \arrow[d, "f^{k+1}"] & B^{k+1} \arrow[r] \arrow[d, "h^{k+1}"] & \cdots \\
\cdots \arrow[r] & A^\bullet_k \arrow[r] & B^\bullet_k \arrow[r] & C^\bullet_k \arrow[r] & A^\bullet_{k+1} \arrow[r] & B^\bullet_{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

The top row is the long exact sequence from example 11.1.6. The bottom row is what we get from the long exact sequence of theorem 13.4.1 by inflicting $\hom_{\mathbb{R}}(—, \mathbb{R})$. The diagram has exact rows and all arrows g^k and h^k are isomorphisms. Let us now show that the diagram is *almost* commutative, that is to say, commutative except for a sign deviation in some places. Then we can use the five lemma, lemma 13.5.1, to conclude that all arrows f^k are isomorphisms. (The sign deviation does not make the five lemma less applicable.)
Commutativity of the diagram is obvious where the squares involving f^k and g^k and the squares involving g^k and h^k are concerned. For the squares involving h^k and f^{k+1} , the almost-commutativity amounts to the statement

$$\mathcal{I}(\,\partial[\omega]\cdot[\lambda]\,) = \pm\mathcal{I}(\,[\omega]\cdot\partial[\lambda]\,)$$

where $[\omega] \in C^k = H^k_{\mathrm{dR}}(V \cap W)$ and $[\lambda] \in A_{k+1} = H^{m-k-1}_{\mathrm{dR},c}(M)$ and the sign $\pm$ should only depend on m and k. The definition of ∂ involves, in both cases, a partition of unity $\{\varphi_V, \varphi_W\}$ subordinate to the open cover $\{V, W\}$ of M. Using remark 13.4.2 and example 11.1.4, we get

$$\mathcal{I}(\,\partial[\omega]\cdot[\lambda]\,) \;=\; \int_M d(\varphi_V \cdot \omega) \wedge \lambda\,,$$

$$\mathcal{I}(\,[\omega]\cdot\partial[\lambda]\,) \;=\; \int_{V\cap W} \omega \wedge d(\varphi_V \cdot \lambda)\,.$$

We have $d(\varphi_V \cdot \omega) \wedge \lambda = d\varphi_V \wedge \omega \wedge \lambda = (-1)^k \omega \wedge d\varphi_V \wedge \lambda = \pm\omega \wedge d(\varphi_V \cdot \lambda)$ because $d\omega = 0$ and $d\lambda = 0$. Therefore

$$\mathcal{I}(\,\partial[\omega]\cdot[\lambda]\,) = (-1)^k \mathcal{I}(\,[\omega]\cdot\partial[\lambda]\,).$$

□

Lemma 13.5.2 *For $i = 1, 2, 3, \ldots,$ suppose given real vector spaces A_i and B_i and a bilinear map $g_i : A_i \times B_i \to \mathbb{R}$. If g_i is nonsingular on the left for $i = 1, 2, 3, \ldots,$ then the bilinear map*

$$g : \left(\prod_{i=1}^{\infty} A_i \right) \times \left(\bigoplus_{i=1}^{\infty} B_i \right) \longrightarrow \mathbb{R}$$

defined by $g((a_i)_{i\ge1}, (b_i)_{i\ge1}) = \sum_{i=1}^{\infty} g_i(a_i, b_i)$ is also nonsingular on the left.

Proof Exercise. □

Proof of lemma 13.3.3.
A differential k-form ω on $M = \coprod_i M_i$ is determined by its restrictions $\omega^{(i)} = \omega|M_i$, which can be prescribed arbitrarily. A compactly supported differential $(m-k)$-form λ on M is determined by its restrictions $\lambda^{(i)} = \lambda|M_i$, which can be prescribed almost arbitrarily, subject only to the conditions that each $\lambda^{(i)}$ has compact support and that $\lambda^{(i)}$ is zero for all but finitely many i. Therefore

$$H^k_{\mathrm{dR}}(M) \cong \prod_{i=1}^{\infty} H^k_{\mathrm{dR}}(M_i)\,, \quad H^{m-k}_{\mathrm{dR},c}(M) \cong \bigoplus_{i=1}^{\infty} H^{m-k}_{\mathrm{dR},c}(M_i)\,.$$

Also, for $\omega \in \Omega^k(M)$ and $\lambda \in \Omega_c^{m-k}(M)$ we have

$$\mathcal{I}(\,[\omega]\cdot[\lambda]\,) = \int_M \omega \wedge \lambda \;=\; \sum_{i=1}^{\infty} \int_{M_i} \omega^{(i)} \wedge \lambda^{(i)} = \sum_{i=1}^{\infty} \mathcal{I}(\,[\omega^{(i)}]\cdot[\lambda^{(i)}]\,).$$

So we are in a position to apply lemma 13.5.2, taking $H^k_{\mathrm{dR}}(M_i)$ for A_i and $H^{m-k}_{\mathrm{dR},c}(M_i)$ for B_i. $\square$