SPLITTING OFF THE REAL LINE AND PLANE

LINUS KRAMER

Abstract: We show that $S \times \mathbb{R}^n \cong \mathbb{R}^{m+k}$ implies $S \cong \mathbb{R}^k$ for $k \leq 2$.

AMS Classification: 57P05, 57N05. Keywords: Generalized manifolds, topology of $E^2$.

J. Rätz [4] and J. Tabor [5] prove that $S \times \mathbb{R}^n \cong \mathbb{R} \times \mathbb{R}^m$ implies $S \cong \mathbb{R}$, and mention that this was posed by M.C. Zdun as an open problem. However, a more general result follows easily from old theorems of A. Borel [2] and G. Young [6]. Since the result is of some interest in compact transformation groups and topological geometry, we give a direct proof of the more general statement.

Lemma Let $R$ be a principal ideal domain and let $X$ be a connected, separable and metrizable $n$-cm$_R$ (cohomology $n$-manifold over $R$, see Bredon [3] V.16.7). If $X$ factors as $X \cong S \times T$, and if $\dim_R(S) = k \leq 2$ (equivalently, if $k = \dim_R(X) - \dim_R(T) \leq 2$), then $S$ is a topological $k$-manifold.

Proof. The factors $S$ and $T$ are $k$- and $(n-k)$-cm$_R$'s respectively, see [3] V.16.11. A connected $k$-cm$_R$ is a $k$-hm$_R$ (homology $k$-manifold over $R$) [3] V.16.8, and a separable metrizable $k$-hm$_R$ is a topological manifold [3] V.16.32, provided that $k \leq 2$.

Corollary Let $X \cong S \times T$ be as above. Suppose that $X$ is 1-connected. If $k = 1$, then $S \cong \mathbb{R}$; if $k = 2$, then $S \cong \mathbb{R}^2$ or $S \cong S^2$. In particular, if $S \times \mathbb{R}^m \cong \mathbb{R}^{m+k}$, for $k \leq 2$, then $S \cong \mathbb{R}^k$.

Proof. By the Lemma, $S$ is a 1-connected $k$-manifold. It is well-known that every 1-connected 1-manifold is homeomorphic to the real line. Similarly, it follows from the classification of surfaces that a 1-connected surface is either homeomorphic to $\mathbb{R}^2$ or to the sphere $S^2$.

The result does not carry over to higher dimensions: there is a 3-cm$_R$ $E$ such that $E \times \mathbb{R} \cong \mathbb{R}^4$, but $E \not\cong \mathbb{R}^3$ [1].

References

Mathematisches Institut der Universität Würzburg, Am Hubland, D–97074 Würzburg, Germany
E-mail: kramer@mathematik.uni-wuerzburg.de

Eingegangen am 14. Januar 1999