# TAUTOLOGICAL CLASSES AND HIGHER SIGNATURES 

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#### Abstract

For a bundle of oriented closed smooth $n$-manifolds $\pi: E \rightarrow X$, the tautological class $\kappa_{\mathcal{L}_{k}}(E) \in H^{4 k-n}(X ; \mathbb{Q})$ is defined by fibre integration of the Hirzebruch class $\mathcal{L}_{k}\left(T_{v} E\right)$ of the vertical tangent bundle. More generally, given a discrete group $G$, a class $u \in H^{p}(B G ; \mathbb{Q})$ and a map $f: E \rightarrow B G$, one has tautological classes $\kappa_{\mathcal{L}_{k}, u}(E, f) \in H^{4 k+p-n}(X ; \mathbb{Q})$ associated to the Novikov higher signatures.

For odd $n$, it is well-known that $\kappa_{\mathcal{L}_{k}}(E)=0$ for all bundles with $n$ dimensional fibres. The aim of this note is to show that the question whether more generally $\kappa_{\mathcal{L}_{k}, u}(E, f)=0$ (for odd $n$ ) depends sensitively on the group $G$ and the class $u$.

For example, given a nonzero cohomology class $u \in H^{2}\left(B \pi_{1}\left(\Sigma_{g}\right) ; \mathbb{Q}\right)$ of a surface group, we show that always $\kappa_{\mathcal{L}_{k}, u}(E, f)=0$ if $g \geq 2$, whereas sometimes $\kappa_{\mathcal{L}_{k}, u}(E, f) \neq 0$ if $g=1$.

The vanishing theorem is obtained by a generalization of the index-theoretic proof that $\kappa_{\mathcal{L}_{k}}(E)=0$, while the nontriviality theorem follows with little effort from the work of Galatius and Randal-Williams on diffeomorphism groups of even-dimensional manifolds.


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## 1. Introduction

For a bundle $\pi: E \rightarrow X$ of smooth oriented closed $n$-manifolds and a characteristic class $c \in H^{k}(B S O(n) ; \mathbb{Q})$, the tautological classes are defined as

$$
\kappa_{c}(E):=\pi_{!}\left(c\left(T_{v} E\right)\right) \in H^{k-n}(X ; \mathbb{Q})
$$

where $T_{v} E \rightarrow E$ is the vertical tangent bundle and $\pi_{!}$the Gysin homomorphism in cohomology. If $M$ is the fibre of $\pi$, this construction gives in the universal case a

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class in

$$
H^{k-n}\left(B \mathrm{Diff}^{+}(M) ; \mathbb{Q}\right)
$$

More generally, we can consider a further space $B$ and oriented closed manifold bundles $\pi: E \rightarrow X$ together with maps $f: E \rightarrow B$; given $u \in H^{p}(B ; \mathbb{Q})$, we put

$$
\kappa_{c, u}(E, f):=\pi!\left(c\left(T_{v} E\right) \cup f^{*} u\right) \in H^{k+p-n}(X ; \mathbb{Q}) .
$$

The universal example of such a bundle has $E \operatorname{Diff}^{+}(M) \times_{\operatorname{Diff}^{+}(M)} \operatorname{map}(M ; B)$ as base space.

We are interested in the special case of the above where $B=B G$ is the classifying space of a discrete group and $c=\mathcal{L}_{k} \in H^{4 k}(B O ; \mathbb{Q})$ is the $k$ th component of the Hirzebruch $L$-class, or rather Atiyah-Singer's variant [3] thereof:

$$
\mathcal{L} \in \hat{H}^{4 *}(B O ; \mathbb{Q}):=\prod_{k \geq 0} H^{4 k}(B O ; \mathbb{Q})
$$

is the multiplicative characteristic class associated to the power series $\frac{x / 2}{\tanh (x / 2)} \in$ $\mathbb{Q}\left[\left[x^{2}\right]\right]$ (Hirzebruch's original class $L$ is associated to $\frac{x}{\tanh (x)}$ and differs in each degree from $\mathcal{L}$ by a power of 2 ).

In that case, the class $\kappa_{\mathcal{L}_{k}, u}(E, f) \in H^{4 k+p-n}(X ; \mathbb{Q})$ is a family version of the Novikov higher signature in the sense that when $4 k+p=n$, the class $\kappa_{\mathcal{L}_{k}, u}(E, f) \in$ $H^{0}(X ; \mathbb{Q})$ evaluated at $x \in X$ is Novikov's higher signature

$$
\operatorname{sign}_{u}\left(E_{x},\left.f\right|_{E_{x}}\right):=\left\langle\mathcal{L}_{k}\left(T E_{x}\right) \cup\left(\left.f\right|_{E_{x}}\right)^{*} u,\left[E_{x}\right]\right\rangle \in \mathbb{Q}
$$

of the fibre $E_{x}$ over $x$.
It is well-known that $\kappa_{\mathcal{L}_{k}}(E)=0$ for bundle with odd fibre dimension; the author has given an index-theoretic proof of this fact in [7]. One might ask whether for $u \in H^{*}(B G ; \mathbb{Q})$ and odd fibre dimension, one still has $\kappa_{L_{k}, u}(E, f)=0$ for all bundles with odd fibre dimension and maps $f$ to $B G$. The aim of this note is to disclose that the answer depends sensitively on $G$ and $u$; in a way that seems to be surprising and puzzling to the author. More specifically, we shall prove the following result; we believe it to be of interest because of the contrast between statements (1) and (2) on one side and (3) on the other.

Theorem 1.1. Let $\Gamma_{g}:=\pi_{1}\left(\Sigma_{g}\right)$ be the fundamental group of a surface of genus $g \geq 1$ and let $w \in H^{1}\left(B \Gamma_{g} ; \mathbb{Q}\right)$ and $u \in H^{2}\left(B \Gamma_{g} ; \mathbb{Q}\right)$ be nonzero.
(1) If $g \geq 1$, there is for each $m \geq 1$ and each $k$ with $4 k \geq 2 m$, some bundle of closed oriented $(2 m+1)$-manifolds $E \rightarrow X$ and a map $f: E \rightarrow B \Gamma_{g}$ such that $\kappa_{\mathcal{L}_{k}, w}(E, f) \neq 0 \in H^{4 k-2 m}(X ; \mathbb{Q})$.
(2) If $g=1$, there is, for each $m \geq 1$ and each $k$ with $4 k \geq 2 m$, some bundle of closed oriented $(2 m+1)$-manifolds $E \rightarrow X$ and a map $f: E \rightarrow B \Gamma_{1} \simeq T^{2}$ such that $\kappa_{\mathcal{L}_{k}, u}(E, f) \neq 0 \in H^{4 k+1-2 m}(X ; \mathbb{Q})$.
(3) If $g \geq 2$ and $n$ is odd, then $\kappa_{\mathcal{L}_{k}, u}(E, f)=0 \in H^{4 k+2-n}(X ; \mathbb{Q})$ for each bundle of closed oriented n-dimensional manifolds $E \rightarrow X$ and each $f$ : $E \rightarrow B \Gamma_{g}$.
The three statements are special cases of more general results that we now state. The general version of Theorem 1.1(1) and (2) reads as follows.

Theorem 1.2. Let $G$ and $H$ be groups, $v \in H^{q}(B G ; Q), w \in H^{p}(B H ; Q)$ be nonzero classes. Let numbers $m_{0}, m_{1}, k_{0}, k_{1} \in \mathbb{N}_{0}$ be given with

$$
m_{0} \geq 1,4 k_{0}+p-2 m_{0} \geq 0,2 m_{1}+1=4 k_{1}+q
$$

and let $m=m_{0}+m_{1}$ and $k=k_{0}+k_{1}$. Then there is a bundle of closed oriented $(2 m+1)$-manifolds $E \rightarrow X$ and a map $h: E \rightarrow B(G \times H)$ such that

$$
\kappa_{\mathcal{L}_{k}, u \times v}(E, h) \neq 0 \in H^{4 k+p+q-(2 m+1)}(X ; \mathbb{Q})
$$

Theorem 1.2 can be deduced without much effort from Galatius-Randal-Williams' work [9, 10] on diffeomorphism groups of even-dimensional manifolds; we give the short proof in §4. Their work implies a nontriviality result for the $\kappa_{\mathcal{L}_{k}, u}$-classes for bundles with even fibre dimension (see Lemma 4.4 below); Theorem 1.2 follows by taking products with suitable odd-dimensional manifolds.

To obtain Theorem $1.1(2)$, put $G=H=\mathbb{Z}$, let $v=w \in H^{1}(B \mathbb{Z} ; \mathbb{Q})$ be a generator, and let $m_{0}=m, k_{0}=k, m_{1}=m_{1}=0$. To obtain Theorem1.1(1), put $G=1, H=\Gamma_{g}, v=1$ and $w=w$, as well as $m_{0}=m, k_{0}=k, k_{1}=m_{1}=0$.

To formulate the general statement behind Theorem 1.1 (3), let $U(p, q)$ be the unitary group with signature $(p, q)$. The Lie group $U(p, q)$ contains $U(p) \times U(q)$ as a maximal compact subgroup, and so we have a map $\left(B U(p, q)^{\delta}\right.$ denotes the classifying space of the abstract group $U(p, q)$ )

$$
\begin{equation*}
\varphi: B U(p, q)^{\delta} \rightarrow B U(p, q) \simeq B U(p) \times B U(q) \xrightarrow{\Delta} B U \tag{1.3}
\end{equation*}
$$

where the last map takes the difference of the two universal bundles. We can pull back the components $\mathrm{ch}_{m}$ of the Chern character along $\varphi$ and obtain a class (super Chern character)

$$
\begin{equation*}
\operatorname{sch}_{m}:=\varphi^{*} \operatorname{ch}_{m} \in H^{2 m}\left(B U(p, q)^{\delta} ; \mathbb{Q}\right) \tag{1.4}
\end{equation*}
$$

and

$$
\operatorname{sch} \in \hat{H}^{2 *}\left(B U(p, q)^{\delta} ; \mathbb{Q}\right)
$$

Theorem 1.5. Let $\pi: E \rightarrow X$ be a bundle of closed smooth oriented manifolds of odd dimension $n$, and let $f: E \rightarrow B U(p, q)^{\delta}$ be a map. Then

$$
\kappa_{\mathcal{L}, \varphi^{*} \operatorname{sch}}(E, f)=0 \in H^{*-n}(X, \mathbb{Q})
$$

The deduction of Theorem 1.1 (3) from Theorem 1.5 is not entirely straightforward and is done in $\S 3$ below; the key ingredient here is [15].

The proof of Theorem 1.5 is an elaboration of the index-theoretic proof that $\kappa_{\mathcal{L}_{k}}=0$ given in 7]. We replace the odd signature operator introduced in [2] and used in [7] by a twisted version; this operator was introduced (on even-dimensional manifolds) by Lusztig [14] in relation to the Novikov conjecture (surely it is already implicit in [1]); we give the construction in odd dimensions in $\$ 2.2$. The necessary cohomological index formula for the odd twisted signature operator is shown in $\$ 2.3$ Here our procedure is somewhat different from the one in [7: the index theorem in the odd case is deduced from the even case, which is treated in adequate detail in [3], 5] and [13. This trick enforces to use Clifford algebras systematically.

## 2. Proof of the vanishing theorem

2.1. The family index of elliptic families. This section is devoted to the proof of Theorem 1.5. As we already said, the proof is index-theoretic, and we begin by reviewing the family index of a family of elliptic operators on a bundle of closed manifolds. Let $\pi: E \rightarrow X$ be a bundle of closed smooth manifolds and $W \rightarrow E$ is a fibrewise smooth complex vector bundle with a fibrewise smooth bundle metric. We assume that the vertical tangent bundle $T_{v} E \rightarrow E$ is equipped with a fibrewise smooth Riemannian metric; then each fibre $E_{x}:=\pi^{-1}(x)$ is a closed Riemannian
manifold. Let $D$ be a family of formally selfadjoint elliptic operators of order 1 acting on fibrewise smooth sections of $E$. We can restrict $D$ to each fibre and obtain a formally selfadjoint elliptic operator $\left.D\right|_{x}$ acting on smooth sections of the bundle $\left.W\right|_{x}:=\left.W\right|_{E_{x}} \rightarrow E_{x}$.

To define a family index of $D$, we shall firstly assume that $X$ is compact which is not a severe hypothesis for our purposes. Secondly, some linear algebraic symmetries on $D$ need to be present.

Definition 2.1. A $\mathrm{Cl}^{k, 0}$-structure on a metric vector bundle $W \rightarrow E$ consists of a $\mathbb{Z} / 2$-grading $\iota$ on $W$ and a graded $*$-algebra homomorphism $\alpha: \mathrm{Cl}^{k, 0} \rightarrow \operatorname{End}(T W)$. In some more detail, $\mathrm{Cl}^{k, 0}$ denotes the Clifford algebra (with anticommuting generators $e_{1}, \ldots, e_{k}$ such that $e_{i}^{2}=-1$ ) as a $\mathbb{Z} / 2$-graded algebra. The map $\alpha$ is determined by anticommuting skewadjoint operators $\alpha_{j}:=\alpha\left(e_{j}\right)$ which all anticommute with $\iota$ and have square -1 .

A Cl ${ }^{k, 0}$-graded elliptic family over a manifold bundle $E \rightarrow X$ consists of a family $D$ of formally selfadjoint elliptic operators of order 1 , acting on the sections of some metric vector bundle $W \rightarrow E$, and a $\mathrm{Cl}^{k, 0}$-structure $(\iota, \alpha)$ on $E$ such that the relations

$$
D \iota+\iota D=0=\alpha\left(e_{j}\right) D+D \alpha\left(e_{j}\right)
$$

hold (for all $j=1, \ldots, k$ ).
A Cl ${ }^{k, 0}$-graded elliptic family ( $D, \iota, \alpha$ ) over some bundle of closed smooth manifolds $\pi: E \rightarrow X$ over a finite CW base (this level of generality suffices for Theorem 1.5) has an index

$$
\operatorname{ind}_{k}(D, \iota, \alpha) \in K^{k}(X)
$$

The classical construction is essentially due to Atiyah-Singer and explained, with references, in [7, §3] in the case $k=0,1$ (to get it for arbitrary $k$, one invokes the general variant of the main result of [4]). A more modern and less cumbersome approach using Hilbert module techniques is described in [8].

A Cl ${ }^{0,0}$-graded elliptic family is essentially the same as a family of elliptic operators (no formal selfadjointness or further algebraic structure required); one splits $W=W_{+} \oplus W_{-}$into the eigenspaces of $\iota$ and looks at the restriction of $D$ to a $W_{+}$ which maps to $W_{-}$. The index $\operatorname{ind}_{0}(D, \iota)$ is just the ordinary family index of $\left.D\right|_{W_{+}}$ from sections of $W_{+}$to those of $W_{-}$.

A $\mathrm{Cl}^{1,0}$-graded elliptic family is essentially the same as a family of formally selfadjoint elliptic operators without further algebraic symmetries; one looks at the restriction of $\alpha_{1} D$ to the eigenbundle $W_{+}$(this connects the present setup with the one used in [7]).

The relation to the Bott periodicity isomorphism bott : $K^{0}(X) \cong K^{2}(X)$ is as follows. Let $(D, \iota, \alpha)$ be some $\mathrm{Cl}^{2,0}$-graded elliptic family, acting on some vector bundle $W \rightarrow E$ over some fibre bundle $E \rightarrow X$. Now $\varepsilon:=i \alpha\left(e_{1} e_{2}\right)$ is a selfadjoint involution on $W$, which commutes with $\iota$ and $D$. Hence we can restrict $D$ and $\iota$ to the eigenbundle $W_{+}=\operatorname{Eig}\left(i \alpha\left(e_{1} e_{2}\right),+1\right)$ and obtain a $\mathrm{Cl}^{0,0}$ graded elliptic family $\left(\left.D\right|_{W_{+}},\left.\iota\right|_{W_{+}}\right)$. This process is easily seen to be invertible, and we have

$$
\begin{equation*}
\operatorname{ind}_{2}(D, \iota, \alpha)=\operatorname{bott}\left(\operatorname{ind}_{0}\left(\left.D\right|_{W_{+}},\left.\iota\right|_{W_{+}}\right)\right) \tag{2.2}
\end{equation*}
$$

Let us describe the behaviour of the index with respect to products. Given a $\mathrm{Cl}^{k, 0}$-graded elliptic family $(D, \iota, \alpha)$ on some vector bundle $W \rightarrow E$ over $E \rightarrow X$
and a $\mathrm{Cl}^{l, 0}$-graded elliptic family $(B, \eta, \beta)$ on some other vector bundle $W \rightarrow F$ over $F \rightarrow Y$, we form the exterior product of these elliptic families as follows.

Consider the bundle $V \boxtimes W \rightarrow E \times F$ with grading $\iota \otimes \eta$. Here we have the graded tensor product differential operator $D \hat{\otimes} 1+1 \hat{\otimes} B$ (note that we use the graded tensor product here, with conventions as in [6] §14.4]), which is elliptic again with square $D^{2} \hat{\otimes} 1+1 \hat{\otimes} B^{2}$ and anticommutes with $\iota \otimes \eta$. We get a $\mathrm{Cl}^{k+l, 0}$-structure $\alpha \sharp \beta$ by sending $e_{i}$ with $i \leq k$ to $\alpha\left(e_{i}\right) \hat{\otimes} 1$ and $e_{i}$ with $i>k$ to $1 \hat{\otimes} \beta\left(e_{i-k}\right)$. Using the Koszul rule for the graded tensor product, it is easily checked that $D \hat{\otimes} 1+1 \hat{\otimes} B$ is $\mathrm{Cl}^{k+l, 0}$-graded. The external product in $K$-theory is so that

$$
\begin{equation*}
\operatorname{ind}_{k+l}(D \hat{\otimes} 1+1 \hat{\otimes} B, \iota \otimes \eta, \alpha \sharp \beta)=\operatorname{ind}_{k}(D, \iota, \alpha) \times \operatorname{ind}_{l}(B, \eta, \beta) . \tag{2.3}
\end{equation*}
$$

The following result, discovered in [12, Proposition 4.1] (and rediscovered in [7, Theorem 4.1]) is the analytical basis for our vanishing theorem.
Theorem 2.4. Assume that $\left(D, \iota, \alpha_{1}\right)$ is a $\mathrm{Cl}^{1,0}$-graded elliptic family and assume that the function $X \rightarrow \mathbb{N}_{0}, x \mapsto \operatorname{dim}\left(\operatorname{ker}\left(\left.D\right|_{x}\right)\right)$, is locally constant. Then

$$
\operatorname{ind}_{1}\left(D, \iota, \alpha_{1}\right)=0 \in K^{1}(X)
$$

One can give a more modern proof using Hilbert module theory; we refrain from giving it here.
2.2. The twisted signature operator. Let us first recapitulate the ordinary signature operator, assuming complex coefficients throughout. If $M$ is a closed Riemannian manifold of dimension $n$, we get an inner product on $\Lambda^{*} T^{*} M \otimes \mathbb{C}$, and the exterior derivative $d: \mathcal{A}^{p}(M) \rightarrow \mathcal{A}^{p+1}(M)$ has a formal adjoint $d^{*}$. The operator $D=d+d^{*}: \mathcal{A}^{*}(M) \rightarrow \mathcal{A}^{*}(M)$ is elliptic and formally selfadjoint. By the general Hodge decomposition theorem, the kernel of $D: \mathcal{A}^{p}(M) \rightarrow \mathcal{A}^{*}(M)$ can be identified with the de Rham cohomology $H_{d R}^{p}(M) \cong H^{p}(M ; \mathbb{C})$. Let $\iota$ be the even/odd grading on $\Lambda^{*} T^{*} M$; then clearly $D \iota+\iota D=0$, so that $(D, \iota)$ is $\mathrm{Cl}^{0,0}$-graded and $\operatorname{ind}(D, \iota)=\chi(M) \in \mathbb{Z}=K^{0}(*)$ is the Euler number.

If $M$ is oriented, we have the Hodge star operator $\star: \Lambda^{p} T^{*} M \rightarrow \Lambda^{n-p} T^{*} M$; with the help of $\star$, we can express the $L^{2}$-inner product on forms by

$$
\langle\alpha, \beta\rangle_{L^{2}}=\int_{M} \alpha \wedge \star \beta
$$

Note that $\star(1)$ is the volume form, and recall that on $p$-forms,

$$
\begin{equation*}
\star^{2}=(-1)^{p(n-p)} . \tag{2.5}
\end{equation*}
$$

It follows from Stokes' theorem and (2.5) that the adjoint $d^{*}: \mathcal{A}^{p}(M) \rightarrow \mathcal{A}^{p-1}(M)$ is given by the formula

$$
\begin{equation*}
d^{*}=(-1)^{n p-n+1} \star d \star \tag{2.6}
\end{equation*}
$$

We modify the Hodge star and define on $p$-forms

$$
\begin{equation*}
\tau:=i^{\frac{n(n+1)}{2}+2 n p+p(p-1)} \star \text {. } \tag{2.7}
\end{equation*}
$$

Equations (2.5), (2.6) and (2.7) imply
Lemma 2.8. The following relations hold.
(1) $\tau^{2}=1$,
(2) $\iota \tau=(-1)^{n} \tau \iota$,
(3) $d^{*}=(-1)^{n+1} \tau d \tau, d \tau=(-1)^{n+1} \tau d^{*}, \tau d=(-1)^{n+1} d^{*} \tau$,
(4) $D \tau=(-1)^{n+1} \tau D$.

Definition 2.9. For each $n$, we call the $\mathrm{Cl}^{0,0}$-graded elliptic operator $(D, \iota)$ the Euler characteristic operator. For even $n$, we call the $\mathrm{Cl}^{0,0}$-graded elliptic operator $(D, \tau)$ the signature operator. For odd $n$, we use the grading $\iota$ and define a Clifford structure $\alpha: \mathrm{Cl}^{1,0} \rightarrow \operatorname{End}\left(\Lambda^{*} T^{*} M\right)$ by $\alpha\left(e_{1}\right):=\iota \tau$ and call the resulting $\mathrm{Cl}^{1,0}{ }_{-}$ graded elliptic operator $(D, \iota, \alpha)$ the odd signature operator.

Some of these calculations can be made more transparent using Clifford algebras (though the connection to cohomology requires to use differential forms). This works as follows.

Let $T$ be a euclidean vector space of dimension $n$, and consider the usual Clifford action $c: \operatorname{Cl}(T) \rightarrow \operatorname{End}\left(\Lambda^{*} T\right)$, which sends $e \in T$ to $\operatorname{ext}_{e}-\operatorname{ext}_{e}^{*}$ ( $\operatorname{ext}_{e}$ is exterior multiplication with $e$, and its adjoint is the insertion operator). Let $\iota$ denote the even/odd grading on $\Lambda^{*} T$.

Assuming that $T$ is oriented, pick an oriented orthonormal basis $\left(e_{1}, \ldots, e_{n}\right)$, and let $\operatorname{vol}_{T} \in \Lambda^{n} T$ be the volume form. The Clifford volume element is defined as $\omega_{T}:=e_{1} \cdots e_{n} \in \operatorname{Cl}(V)$.

## Lemma 2.10.

$$
\begin{equation*}
\omega_{T}^{2}=(-1)^{\frac{n(n+1)}{2}} \tag{1}
\end{equation*}
$$

(2) The relation to the Hodge star operator is given by the identity

$$
c\left(\omega_{T}\right)=(-1)^{\frac{p(p-1)}{2}+n p} \star: \Lambda^{p}(T) \rightarrow \Lambda^{n-p}(T) .
$$

(3) $\tau=i^{\frac{n(n+1)}{2}} c\left(\omega_{T}\right)$.

Proof. (1) is straightforward, (2) is shown in [13, p.129] and (3) is trivial.
We continue to assume that $M$ is a closed oriented $n$-dimensional Riemannian manifold. In addition, let us suppose that $(V, \eta) \rightarrow M$ is a flat hermitian vector bundle; in other words $V$ is a complex vector bundle, together with a nondegenerate (possibly indefinite) hermitian form $\eta: V \times V \rightarrow \mathbb{C}$ and a flat connection $\nabla$ with respect to which $\eta$ is parallel. Alternatively, the structure group of $V$ is reduced to $U(p, q)^{\delta}$.

We can extend $\nabla$ to $d_{\nabla}: \mathcal{A}^{*}(M ; V) \rightarrow \mathcal{A}^{*+1}(M ; V)$ on $V$-valued differential forms, with $d_{\nabla}^{2}=0$ (since $\nabla$ is flat). So $0 \rightarrow \mathcal{A}^{0}(M ; V) \xrightarrow{d_{\nabla}} \ldots \stackrel{d}{\longrightarrow} \mathcal{A}^{n}(M ; V) \rightarrow 0$ is an elliptic complex, and a twisted version of de Rham's theorem states that $H^{*}\left(\mathcal{A}^{*}(M ; V), d_{\nabla}\right) \cong H^{*}(M ; V)$; here the right hand side is singular cohomology with coefficient system defined by the flat bundle $V$. So far, we have not used $\eta$.
Lemma 2.11. There exist pairs $(h, \sigma)$, consisting of a bundle metric $h$ on $V$ and an $h$-isometry $\sigma$ with $\sigma^{2}=1$ and

$$
h\left(\tau_{-},-\right)=\eta(-,-) .
$$

We can choose $(h, \sigma)$ to depend continuously on a bundle metric $h_{0}$. Such a pair is called a compatible pair.
Proof. Pick a bundle metric $h_{0}$ at random, which determines uniquely a bundle endomorphism $S$ such that $h_{0}\left(S_{-},-\right)=\eta(-,-)$, which is selfadjoint and invertible. Let

$$
h(-,-):=h_{0}\left(|S|_{-, ~}\right), \sigma:=S|S|^{-1}=|S|^{-1} S ;
$$

one checks that $(h, \sigma)$ is a compatible pair.

We fix a compatible pair $(h, \sigma)$. The bundle metric $h$, together with the original Riemannian metric on $M$, determines a bundle metric on $\Lambda^{*} T^{*} M \otimes V$. With respect to this metric, we form the formal adjoint $d_{\nabla}^{*}$ of $d_{\nabla}$, and let

$$
\begin{equation*}
D_{V}:=d_{\nabla}+d_{\nabla}^{*} \tag{2.12}
\end{equation*}
$$

The principal symbol of $D_{V}$ is given by the formula $\operatorname{smb}\left(D_{V}\right)=\operatorname{smb}(D) \otimes 1_{V}$, so $D_{V}$ is elliptic; and the kernel of $D_{V}$ is identified (by the Hodge theorem and the twisted de Rham theorem) with $H^{*}\left(\mathcal{A}^{*}(M ; V), d_{\nabla}\right) \cong H^{*}(M ; V)$ (flatness of $\nabla$ is needed only for this cohomological interpretation of the kernel of $D_{V}$ ).

A $\mathbb{Z} / 2$-grading on $\Lambda^{*} T^{*} M \otimes V$ is given by $\iota_{V}:=\iota \otimes 1_{V}$, and we have (obviously; look at the degrees of the involved forms!)

$$
D_{V} \iota_{V}+\iota_{V} D_{V}=0
$$

Assuming that $M$ is oriented, the inner product on $V$-valued forms is given by

$$
\begin{equation*}
\langle\alpha, \beta\rangle=\int_{M} h\left(\alpha \wedge\left(\star \otimes 1_{V}\right) \beta\right) \tag{2.13}
\end{equation*}
$$

here the wedge $\alpha \wedge \star \beta$ is a form with values in $V \otimes_{\mathbb{R}} V$ to which we can apply $h$ and get a $\mathbb{C}$-valued form that can be integrated.

Using that $(h, \sigma)$ is compatible $\eta$ on $V$, formula (2.13) becomes

$$
\begin{equation*}
\langle\alpha, \beta\rangle=\int_{M} \eta(\alpha \wedge(\star \otimes \sigma) \beta) . \tag{2.14}
\end{equation*}
$$

Using that $\eta$ is parallel, the proof of (2.6) carries over and yields the following.
Lemma 2.15. The formal adjoint

$$
d_{\nabla}^{*}: \mathcal{A}^{p}(M ; V) \rightarrow \mathcal{A}^{p-1}(M ; V)
$$

is given by the formula

$$
d_{\nabla}^{*}=(-1)^{n p-n+1}(\star \otimes \sigma) d_{\nabla}(\star \otimes \sigma)
$$

The proof of Lemma 2.8 was a formal consequence of (2.6). Hence we obtain the following generalization of Lemma 2.8, with the same proof. We introduce the involution $\tau_{V}:=\tau \otimes \sigma$ of $\Lambda^{*} T^{*} M \otimes V$.
Proposition 2.16. Let $M^{n}$ be an oriented Riemannian manifold and let $V \rightarrow M$ be a complex vector bundle with a flat connection $\nabla$ and a parallel nondegenerate hermitian form. Pick a compatible pair $(h, \sigma)$. Let $D_{V}:=d_{\nabla}+d_{\nabla}^{*}$. Then
(1) The operator $D_{V}$ is elliptic and its principal symbol is $\operatorname{smb}\left(D_{V}\right)=\operatorname{smb}_{D} \otimes$ $1_{V}$.
(2) If $M$ is closed, $\operatorname{ker}\left(D_{V}\right) \cong \bigoplus_{p=0}^{n} H^{p}(M ; V)$.
(3) $\iota_{V}^{2}=1, \tau_{V}^{2}=1$ and

$$
\iota_{V} \tau_{V}=(-1)^{n} \tau_{V} \iota_{V} .
$$

$$
\begin{gather*}
D_{V} \iota_{V}+\iota_{V} D_{V}=0  \tag{4}\\
D_{V} \tau_{V}+(-1)^{n} \tau_{V} D_{V}=0 \tag{5}
\end{gather*}
$$

Definition 2.17. In the situation of Proposition 2.16, we make the following definitions.
(1) The $\mathbb{Z} / 2$-graded elliptic operator $\left(D_{V}, \iota_{V}\right)$ is the twisted Euler characteristic operator.
(2) If $n$ is even, the $\mathbb{Z} / 2$-graded elliptic operator $\left(D_{V}, \tau_{V}\right)$ is the even twisted signature operator.
(3) If $n$ is odd, we define $\alpha: \mathrm{Cl}^{1,0} \rightarrow \operatorname{End}\left(\Lambda^{*} T^{*} M \otimes V\right)$ by $\alpha\left(e_{1}\right)=\iota_{V} \tau_{V}$. The $\mathrm{Cl}^{1,0}$-elliptic operator $\left(D_{V}, \iota_{V}, \alpha\right)$ is called the odd twisted signature operator.

The index of the twisted Euler characteristic operator is

$$
\operatorname{ind}\left(D_{V}, \iota_{V}\right)=\sum_{p=0}^{n}(-1)^{p} \operatorname{dim} H^{p}(M ; V)=\chi(M) \operatorname{rank}(V)
$$

and when $n=2 m$, the index of the even twisted signature operator is the signature of the following hermitian form on $H^{m}(M ; V)$ :

$$
(\alpha, \beta) \mapsto\left\{\begin{array}{lll}
\int_{M} \omega(\alpha \wedge \beta) & m \equiv 0 & (\bmod 2) \\
-i \int_{M} \omega(\alpha \wedge \beta) & m \equiv 1 & (\bmod 2)
\end{array}\right.
$$

This is shown by the same argument that identifies the index of the signature operator with the signature of $M$.

The whole discussion carries over to families. Let $\pi: E \rightarrow X$ be a bundle of smooth closed oriented Riemannian $n$-manifolds. We say that a fibrewise flat hermitian bundle $(V, \eta) \rightarrow E$ is a complex vector bundle with a hermitian form $\eta$ and a family of connections $\nabla$ on $V$ such that each $\left.\nabla\right|_{x}$ is flat and such that $\left.h\right|_{E_{x}}$ is $\left.\nabla\right|_{x}$-parallel for each $x$. This means that the structure group of each $\left.V\right|_{E_{x}}$ can be reduced to $U(p, q)^{\delta}$; and we say that $(V, \eta)$ is globally flat if the structure group of $V$ can be reduced to $U(p, q)^{\delta}$. Note that being globally flat is a much stronger condition than being fibrewise flat.

If $(V, \eta)$ is fibrewise flat, we can pick a compatible pair $(h, \sigma)$. In that case, we can define the twisted operator family $D_{V}$; the restriction to each fibre is $\left.D_{V}\right|_{E_{x}}$. We also have the two involutions $\iota_{V}$ and $\tau_{V}$; and all the identities of Proposition 2.16 continue to hold. We define family indices

$$
\chi(E \rightarrow X ; V):=\operatorname{ind}_{0}\left(D_{V}, \iota_{V}\right) \in K^{0}(X)
$$

and for $n$ even

$$
\operatorname{sign}_{\text {even }}(E \rightarrow X ; V):=\operatorname{ind}_{0}\left(D_{V}, \tau_{V}\right) \in K^{0}(X)
$$

For $n$ odd, we define

$$
\operatorname{sign}_{\text {odd }}(E \rightarrow X ; V):=\operatorname{ind}_{1}\left(D_{V}, \iota_{V}, \alpha\left(e_{1}\right):=\iota_{V} \tau_{V}\right) \in K^{1}(X)
$$

Corollary 2.18. Let $E \rightarrow X$ be a bundle of smooth closed oriented odd-dimensional manifolds and let $(V, \eta) \rightarrow E$ be globally flat hermitian vector bundle. Then

$$
\operatorname{sign}_{\text {odd }}(E \rightarrow X ; V)=0 \in K^{1}(X)
$$

Proof. The dimension of the kernel of $\left.\left(D_{V}\right)\right|_{x}$ is equal to $\sum_{k=0}^{n} \operatorname{dim} H^{k}\left(E_{x}, V_{x}\right)$. Since $V$ is globally flat, this is a locally constant function of $x$. Hence Theorem 2.4 applies and gives the desired conclusion.
2.3. The index formula for the twisted signature operator. In this subsection, we derive the cohomological formula for the index of the odd twisted signature operator, which reads as follows.
Theorem 2.19 (Index theorem for the odd signature operator). Let $\pi: E \rightarrow X$ be a bundle of closed oriented $2 m+1$-dimensional manifolds over a finite $C W$ complex. Let $(V, \eta) \rightarrow E$ be a fibrewise flat hermitian vector bundle. Then

$$
\operatorname{ch}\left(\operatorname{sign}_{\mathrm{odd}}(E \xrightarrow{\pi} X ; V)\right)= \pm 2^{m} \pi_{!}\left(\mathcal{L}\left(T_{v} E\right) \operatorname{sch}(V)\right) \in H^{2 *+1}(X ; \mathbb{Q})
$$

Together with Corollary 2.18, Theorem 2.19 immediately shows Theorem 1.5 Since it does not matter for our overall purpose, we can afford to leave the sign in Theorem 2.19 undetermined, and the author is too lazy to figure out the correct sign (which in any case only depends on $m$ ). We have written the proof so that the sign can in principle not be tracked; figuring out the sign will start with giving a different proof of Lemma 2.33 below.

The first step in the proof is the cohomological index formula for the twisted signature operator in even dimensions.
Theorem 2.20. If $n=2 m$, we have

$$
\operatorname{ch}\left(\operatorname{sign}_{\text {even }}(E \xrightarrow[\rightarrow]{\rightarrow} X ; V)\right)=(-1)^{m} 2^{m} \pi_{!}\left(\mathcal{L}\left(T_{v} E\right) \operatorname{sch}(V)\right) \in H^{2 *}(X ; \mathbb{Q})
$$

This is straightforward from the usual family index theorem, see [3, p.577] for the derivation of the cohomological formula. The $\operatorname{sign}(-1)^{m}$ is not present in loc.cit; here is the reason why it appears here. For $n=2 m$, the definition of the grading involution $\tau 2.7$ is $\tau=(-1)^{m} i^{m+p(p-1) \star}$ and differs from the involution with the same name in [3, p. 575] by the $\operatorname{sign}(-1)^{m}$.

As next step, we need two facts about the twisted Euler characteristic operator.
Proposition 2.21.
(1) For two bundles of closed manifolds $\pi_{i}: E_{i} \rightarrow X_{i}$ and fibrewise flat hermitian vector bundles $\left(V_{i}, \eta_{i}\right) \rightarrow E_{i}$, we have
$\chi\left(E_{0} \times E_{1} \rightarrow X_{0} \times X_{1} ; V_{0} \boxtimes V_{1}\right)=\chi\left(E_{0} \rightarrow X_{0}, V_{0}\right) \times \chi\left(E_{1} \rightarrow X_{1}, V_{1}\right) \in K^{0}\left(X_{0} \times X_{1}\right)$.
(2) For a bundle $\pi: E \rightarrow X$ of closed, oriented and odd-dimensional manifolds and a fibrewise flat hermitian vector bundle $(V, \eta) \rightarrow E$, we have

$$
\chi(E \rightarrow X ; V)=0 \in K^{0}(X)
$$

Proof. (1): Let us adopt the convention that we use the $\boxtimes$-symbol and the graded variant $\hat{\boxtimes}$ when we tensor operators on different manifolds (bundles), and reserve $\otimes$ and $\hat{\otimes}$ for the tensor product of bundle endomorphisms on an individual manifold (bundle). We only spell out the computation when $E_{i}=M_{i} \rightarrow *$, the general case is only notationally more involved.

The operator $D_{V_{0}} \hat{\otimes} 1+1 \hat{\otimes} D_{V_{1}}$ acts on the vector bundle $\Lambda^{*} T^{*}\left(M_{0} \times M_{1}\right) \otimes V_{0} \boxtimes$ $V_{1}$ ); here we use the graded isomorphism

$$
\begin{equation*}
\Lambda^{*} T^{*} M_{0} \hat{\otimes} \Lambda^{*} T^{*} M_{1} \cong \Lambda^{*} T^{*}\left(M_{0} \times M_{1}\right) \tag{2.22}
\end{equation*}
$$

As the grading of each $D_{V_{i}}$ is given the even-odd grading on forms, $D_{V_{0}} \hat{\otimes} 1+1 \hat{\otimes} D_{V_{1}}$ agrees as an operator with $D_{V_{0} \boxtimes V_{1}}$ (using the tensor product connection, metric and hermitian form on $V_{0} \boxtimes V_{1}$ ). The grading is the even/odd grading. Hence the claim is a consequence of the product formula (2.3).
(2): the operator $i_{V} \tau_{V}$ is odd, anticommutes with $D_{V}$ and has square +1 . This means (by the homotopy invariance of the index) that

$$
\chi(E \rightarrow X ; V)=\operatorname{ind}_{0}\left(D_{V}+i \iota_{V} \tau_{V}, \iota_{V}\right)
$$

but $\left(D_{V}+i \iota_{V} \tau_{V}\right)^{2}=D_{V}^{2}+1$ is invertible. Hence $\chi(E \rightarrow X ; V)=0 \in K^{0}(X)$.
Proposition 2.23. Let $\pi_{i}: E_{i} \rightarrow X_{i}, i=0,1$, be two bundles of closed oriented manifolds, and let $\left(V_{i}, \eta_{i}\right) \rightarrow E_{i}$ be two fibrewise flat hermitian vector bundles. If the fibre dimensions $n_{i}=2 m_{i}+1$ are both odd, we have the identity

$$
\begin{gathered}
\operatorname{sign}_{\text {even }}\left(E_{0} \times E_{1} \rightarrow X_{0} \times X_{1} ; V_{0} \boxtimes V_{1}\right)= \\
=2(-1)^{m_{0}+m_{1}} \operatorname{bott}^{-1}\left(\operatorname{sign}_{\text {odd }}\left(E_{0} \rightarrow X_{0} ; V_{0}\right) \times \operatorname{sign}_{\text {odd }}\left(E_{1} \rightarrow X_{1} ; V_{1}\right)\right)
\end{gathered}
$$

in the group $K^{0}\left(X_{0} \times X_{1}\right)\left[\frac{1}{2}\right]$.
In [17, Lemma 6], analogous formulas for the $K$-homology classes of the signature operators are established. These do not quite imply Proposition 2.23 (but a family version of [17, Lemma 6] which should not be too hard would do it). In loc.cit., these formulas are stated without inverting 2. However, the author has been unable to understand the details of the proof given in [17, so he prefers to give an independent treatment.

Proof of Proposition 2.23. We first express $\operatorname{sign}_{\text {odd }}\left(E_{0} \rightarrow X_{0} ; V_{0}\right) \times \operatorname{sign}_{\text {odd }}\left(E_{1} \rightarrow\right.$ $\left.X_{1} ; V_{1}\right) \in K^{2}\left(X_{0} \times X_{1}\right)$ as the $\mathrm{Cl}^{2,0}$-index of another operator family on $E_{0} \times E_{1} \rightarrow$ $X_{0} \times X_{1}$. Computing exactly as in the proof of Proposition2.21(1) (and using that the grading of the odd twisted signature operator is the even/odd grading, as it is the case for the twisted Euler characteristic operator), we find that $\operatorname{sign}_{\text {odd }}\left(E_{0} \rightarrow\right.$ $\left.X_{0} ; V_{0}\right) \times \operatorname{sign}_{\text {odd }}\left(E_{1} \rightarrow X_{1} ; V_{1}\right)$ is the $\mathrm{Cl}^{2,0}$-index of the operator $D_{V_{0}} \hat{\otimes} 1+1 \hat{\otimes} D_{V_{1}}$ acting on the vector bundle $\Lambda^{*} T^{*}\left(M_{0} \times M_{1}\right) \otimes V_{0} \boxtimes V_{1}$ with the grading $\iota_{V_{0} \boxtimes V_{1}}$. The $\mathrm{Cl}^{2,0}$-structure is given by the two generators $\alpha_{0} \hat{\otimes} 1,1 \hat{\boxtimes} \alpha_{1}$, where

$$
\alpha_{i}=\iota_{V_{i}} \tau_{V_{i}} .
$$

Next we calculate $\operatorname{bott}^{-1}\left(\operatorname{sign}_{\text {odd }}\left(E_{0} \rightarrow X_{0} ; V_{0}\right) \times \operatorname{sign}_{\text {odd }}\left(E_{1} \rightarrow X_{1} ; V_{1}\right)\right) \in$ $K^{0}\left(X_{0} \times X_{1}\right)$ using the recipe (2.2). Using the notation introduced before that formula, the involution $\varepsilon$ is given by the formula

$$
\varepsilon=i\left(\alpha_{0} \hat{\otimes} 1\right)\left(1 \hat{\otimes} \alpha_{1}\right)=i \iota_{V_{0}} \tau_{V_{0}} \hat{\otimes} \iota_{V_{1}} \tau_{V_{1}} .
$$

Keeping in mind the Koszul rule, the definition of $\tau_{V_{i}}$ and that $\tau_{V_{0}}$ is even, we rewrite this as

$$
\begin{equation*}
\varepsilon=i\left(\iota_{0} \hat{\boxtimes} \iota_{1}\right)\left(\tau_{0} \hat{\boxtimes} \tau_{1}\right)\left(\sigma_{0} \hat{\boxtimes} \sigma_{1}\right), \tag{2.24}
\end{equation*}
$$

with the following interpretation understood: $\iota_{j}$ is the even/odd-grading on forms on $M_{j}$, tensored with the identity on $V_{j}, \tau_{j}$ is the $\tau$-operator on forms on $M_{j}$, tensored with the identity on $V_{j}$, and $\sigma_{j}$ is the compatible involution on $V_{j}$, tensored with the identity on the exterior algebra of $M_{j}$.

Under the isomorphism (2.22), $\iota_{0} \hat{\boxtimes} \iota_{1}$ corresponds to the even/odd grading $\iota$ on $\Lambda^{*} T^{*}\left(M_{0} \times M_{1}\right)$, tensored with the identity on $V_{0} \boxtimes V_{1}$, and $\sigma_{0} \hat{\boxtimes} \sigma_{1}$ is the compatible involution on $V_{0} \boxtimes V_{1}$, tensored with the identity on the exterior algebra.

Next recall Lemma 2.10. If $\omega_{i}$ is the volume element in the Clifford algebra of $T M_{i}$, we see that

$$
\begin{equation*}
\left(\tau_{0} \hat{\boxtimes} \tau_{1}\right)=i^{\frac{\left(2 m_{0}+1\right)\left(2 m_{0}+2\right)}{2}+\frac{\left(2 m_{1}+1\right)\left(2 m_{1}+2\right)}{2}} c\left(\omega_{0}\right) \hat{\boxtimes} c\left(\omega_{1}\right) . \tag{2.25}
\end{equation*}
$$

The term $c\left(\omega_{0}\right) \hat{\boxtimes} c\left(\omega_{1}\right)$ corresponds under (2.22) to $c(\omega)$, with $\omega$ the volume element in the Clifford algebra of $T\left(M_{0} \times M_{1}\right)$ (using the standard orientation conventions for products). Using Lemma 2.10 (3), we get

$$
\begin{equation*}
\left(\tau_{0} \hat{\otimes} \tau_{1}\right)=i^{\frac{\left(2 m_{0}+1\right)\left(2 m_{0}+2\right)}{2}+\frac{\left(2 m_{1}+1\right)\left(2 m_{1}+2\right)}{2}} c(\omega)=i(-1)^{m_{0}+m_{1}+1} \tau \tag{2.26}
\end{equation*}
$$

Putting (2.24), (2.25) and (2.26) together, we obtain

$$
\begin{equation*}
\varepsilon=(-1)^{m_{0}+m_{1}} \iota \tau \sigma=(-1)^{m_{0}+m_{1}} \iota_{V_{0} \boxtimes V_{1}} \tau_{V_{0} \boxtimes V_{1}} . \tag{2.27}
\end{equation*}
$$

Formula (2.2) hence shows that $\operatorname{bott}^{-1}\left(\operatorname{sign}_{\text {odd }}\left(E_{0} \rightarrow X_{0} ; V_{0}\right) \times \operatorname{sign}_{\text {odd }}\left(E_{1} \rightarrow\right.\right.$ $\left.X_{1} ; V_{1}\right)$ ) is the $K^{0}$-index of $D_{V_{0} \boxtimes V_{1}}$, restricted to the +1 -eigenspace of $\epsilon$ as computed in (2.27), with the even/odd grading.

To relate this to the index of the twisted signature operator on the product, we now contemplate on the following construction. For even n, the twisted Euler characteristic and twisted signature operator admit a further decomposition as follows.

Regardless of any product structure, the operator $\iota_{V} \tau_{V}$ is an involution which commutes with $\iota_{V}, \tau_{V}$ and $D_{V}$. We thus can decompose $\Lambda^{*} T_{v}^{*} E \otimes V=W_{+} \oplus W_{-}$ into eigenbundles of $\iota_{V} \tau_{V}$; the three operators $\iota_{V}, \tau_{V}$ and $D_{V}$ preserve this decomposition. We can therefore write the $K^{0}$-index of the twisted Euler characteristic operator as

$$
\chi(E \rightarrow X ; V)=\operatorname{ind}_{0}\left(\left.D_{V}\right|_{W_{+}}, \iota_{V}\right)+\operatorname{ind}_{0}\left(\left.D_{V}\right|_{W_{-}}, \iota_{V}\right)
$$

and that of the twisted signature operator as

$$
\operatorname{sign}_{\text {even }}(E \rightarrow X ; V)=\operatorname{ind}_{0}\left(\left.D_{V}\right|_{W_{+}}, \tau_{V}\right)+\operatorname{ind}_{0}\left(\left.D_{V}\right|_{W_{-}}, \tau_{V}\right)
$$

On $W_{ \pm}$, we have $\tau_{V}= \pm \iota_{V}$. This, together with the general relation $\operatorname{ind}_{0}(D,-\iota)=$ $-\operatorname{ind}_{0}(D, \iota)$ for $\mathrm{Cl}^{0,0}$-graded elliptic families, shows that

$$
\operatorname{sign}_{\text {even }}(E \rightarrow X ; V)=\operatorname{ind}_{0}\left(\left.D_{V}\right|_{W_{+}}, \iota_{V}\right)-\operatorname{ind}_{0}\left(\left.D_{V}\right|_{W_{-}}, \iota_{V}\right)
$$

Using the notation

$$
\mu_{ \pm}(E \rightarrow X ; V):=\operatorname{ind}_{0}\left(\left.D_{V}\right|_{W_{ \pm}}, \iota_{V}\right)
$$

we hence have

$$
\begin{equation*}
\chi(E \rightarrow X ; V)=\mu_{+}(E \rightarrow X ; V)+\mu_{-}(E \rightarrow X ; V) \tag{2.28}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{sign}_{\text {even }}(E \rightarrow X ; V)=\mu_{+}(E \rightarrow X ; V)-\mu_{-}(E \rightarrow X ; V) \tag{2.29}
\end{equation*}
$$

Our previous calculation of $\operatorname{bott}^{-1}\left(\operatorname{sign}_{\text {odd }}\left(E_{0} \rightarrow X_{0} ; V_{0}\right) \times \operatorname{sign}_{\text {odd }}\left(E_{1} \rightarrow X_{1} ; V_{1}\right)\right) \in$ $K^{0}\left(X_{0} \times X_{1}\right)$ can therefore be restated by writing $\operatorname{bott}^{-1}\left(\operatorname{sign}_{\text {odd }}\left(E_{0} \rightarrow X_{0} ; V_{0}\right) \times \operatorname{sign}_{\text {odd }}\left(E_{1} \rightarrow X_{1} ; V_{1}\right)\right)=\mu_{(-1)^{m_{0}+m_{1}}}\left(E_{0} \times E_{1} \rightarrow X_{0} \times X_{1} ; V_{0} \boxtimes V_{1}\right)$.

Inverting 2, we can solve (2.28) and (2.29) for $\mu_{ \pm}$and obtain

$$
\begin{equation*}
\mu_{ \pm}(E \rightarrow X ; V)=\frac{1}{2}(\chi(E \rightarrow X ; V) \pm \operatorname{sign}(E \rightarrow X ; V)) \tag{2.31}
\end{equation*}
$$

Inserting (2.31) into (2.30) therefore leads to the formula

$$
\begin{array}{r}
2 \operatorname{bott}^{-1}\left(\operatorname{sign}\left(E_{0} \rightarrow X_{0} ; V_{0}\right) \times \operatorname{sign}\left(E_{1} \rightarrow X_{1} ; V_{1}\right)\right)= \\
=\chi(E \rightarrow X ; V)+(-1)^{m_{0}+m_{1}} \operatorname{sign}(E \rightarrow X ; V) \in K^{0}\left(X_{0} \times X_{1}\right)\left[\frac{1}{2}\right] . \tag{2.32}
\end{array}
$$

Proposition 2.21 shows that $\chi(E \rightarrow X ; V)=0$, and the proof is complete.
To proceed with the proof of Theorem [2.19, we consider an explicit example, which is due to Lusztig [14, §4]. Consider the trivial fibre bundle $\operatorname{pr}_{1}: S^{1} \times S^{1} \rightarrow S^{1}$. In loc.cit, it is shown that the complex line bundle $L \rightarrow S^{1} \times S^{1}$ whose first Chern class is the generator $u \times u \in H^{2}\left(S^{1} \times S^{1} ; \mathbb{Z}\right)\left(u \in H^{1}\left(S^{1}\right)\right.$ is an arbitrary generator) has a positive definite hermitian metric and a hermitian connection $\nabla$ whose restriction to each fibre $\mathrm{pr}_{2}^{-1}(z)$ is flat (one should think of $\left.L\right|_{\{z\} \times S^{1}}$ as the flat line bundle with monodromy $z$ ).

Lemma 2.33. The analytical index $\operatorname{sign}\left(S^{1} \times S^{1} \xrightarrow{\mathrm{pr}_{1}} S^{1} ; L\right) \in K^{1}\left(S^{1}\right) \cong \mathbb{Z}$ is a generator, and $\left\langle\left(\operatorname{pr}_{1}\right)!\left(\mathcal{L}\left(T_{v} S^{1} \times S^{1}\right) \operatorname{ch}(L)\right),\left[S^{1}\right]\right\rangle= \pm 1 \in \mathbb{Q}$.

The proof we shall give will not be able to detect the sign of the analytical index. One can compute the analytical index also by a spectral flow argument, which in principle allows to find the sign. This still depends on keeping track of many sign conventions. As the sign does not matter for our ultimate goal, we refrain from doing so. Hence, we can afford to be deliberately sloppy about signs in the proof.

Proof. We have $c_{1}(L)=u \times u$. The vertical tangent bundle of the projection $\operatorname{pr}_{1}$ is trivial; hence

$$
\left(\operatorname{pr}_{1}\right)!\left(\mathcal{L}\left(T_{v} S^{1} \times S^{1}\right) \operatorname{ch}(L)\right)=\left(\operatorname{pr}_{1}\right)!(1+u \times u)= \pm u \in H^{1}\left(S^{1} ; \mathbb{Z}\right)
$$

is a generator. We compute the analytical index (up to a sign) with the help of Proposition 2.23. By that formula (and since bott is an isomorphism), it is sufficient to check that

$$
\begin{equation*}
\operatorname{sign}\left(S^{1} \times S^{1} \times S^{1} \times S^{1} \xrightarrow{\mathrm{pr}_{1} \times \mathrm{pr}_{3}} S^{1} \times S^{1} ; L \boxtimes L\right) \in K^{0}\left(S^{1} \times S^{1}\right) \tag{2.34}
\end{equation*}
$$

lies in the reduced group $\tilde{K}^{0}\left(S^{1} \times S^{1}\right) \cong \mathbb{Z}$ and is twice a generator. The Chern character on $S^{1} \times S^{1}$ takes values in integral cohomology and is an isomorphism $K^{0}\left(S^{1} \times S^{1}\right) \rightarrow H^{2 *}\left(S^{1} \times S^{1} ; \mathbb{Z}\right)$. So we must show that the Chern character of (2.34) is twice a generator of $H^{2}\left(S^{1} \times S^{1} ; \mathbb{Z}\right)$. This can be done with the help of Theorem 2.20

$$
\begin{aligned}
\operatorname{ch}\left(\operatorname { s i g n } \left(S^{1} \times\right.\right. & \left.\left.S^{1} \times S^{1} \times S^{1} \xrightarrow{\mathrm{pr}_{1} \times \mathrm{pr}_{3}} S^{1} \times S^{1} ; L \boxtimes L\right)\right)= \\
& -2\left(\operatorname{pr}_{1} \times \operatorname{pr}_{3}\right)!(\operatorname{sch}(L \boxtimes L))
\end{aligned}
$$

using that the vertical tangent bundle is trivial. Now $L$ and $L \boxtimes L$ have trivial grading, so

$$
\begin{gathered}
\operatorname{sch}(L \boxtimes L)=\operatorname{ch}(L \boxtimes L)=\operatorname{ch}(L) \times \operatorname{ch}(L)=(1+u \times u) \times(1+u \times u)= \\
=1+(u \times u \times 1 \times 1)+(1 \times 1 \times u \times u)+(u \times u \times u \times u)
\end{gathered}
$$

Applying $-2\left(\operatorname{pr}_{1} \times \mathrm{pr}_{3}\right)$ ! yields as degree 0 component 0 , and as degree 2 component

$$
-2\left(\operatorname{pr}_{1} \times \operatorname{pr}_{3}\right)!(u \times u \times u \times u)= \pm 2 u \times u
$$

as claimed.
Proof of Theorem 2.19, Let $\pi: E \rightarrow X$ be a bundle with $2 m$-1-dimensional fibres and let $V \rightarrow E$ be a fibrewise flat hermitian bundle. We compute in $H^{*}\left(X \times S^{1} ; \mathbb{Q}\right)$
$\pm \operatorname{ch}(\operatorname{sign}(E \xrightarrow{\pi} X ; V)) \times u \stackrel{(2.33)}{=} \operatorname{ch}(\operatorname{sign}(E \xrightarrow{\pi} X ; V)) \times \operatorname{ch}\left(\operatorname{sign}\left(S^{1} \times S^{1} \xrightarrow{\mathrm{pr}_{1}} S^{1} ; L\right)\right)=$
(multiplicativity of the Chern character)

$$
\begin{gathered}
\operatorname{ch}\left(\operatorname{sign}(E \xrightarrow{\pi} X ; V) \times \operatorname{sign}\left(S^{1} \times S^{1} \xrightarrow{\mathrm{pr}_{1}} S^{1} ; L\right)\right) \stackrel{(2.23)}{=} \\
\frac{(-1)^{m-1}}{2} \operatorname{ch}\left(\operatorname{bott}\left(\operatorname{sign}\left(E \times S^{1} \times S^{1} \xrightarrow{\pi \times \mathrm{pr}_{1}} X \times S^{1} ; V \boxtimes L\right)\right)\right)=
\end{gathered}
$$

(using that the Chern character commutes with the forma Bott map)

$$
\begin{gathered}
\frac{(-1)^{m-1}}{2} \operatorname{ch}\left(\operatorname{sign}\left(E \times S^{1} \times S^{1} \xrightarrow{\pi \times \mathrm{pr}_{1}} X \times S^{1} ; V \boxtimes L\right)\right) \stackrel{\stackrel{(2.20}{=}}{-2^{m-1}\left(\pi \times \operatorname{pr}_{1}\right)!\left(\left(\mathcal{L}\left(T_{v} E\right) \times 1\right)(\operatorname{sch}(V) \times(1 \pm u \times u))\right)=} \\
\left.-2^{m-1}\left(\pi \times \operatorname{pr}_{1}\right)!\left(\mathcal{L}\left(T_{v} E\right) \operatorname{sch}(V) \times(1 \pm u \times u)\right)\right)= \\
\pm 2^{m-1} \pi_{!}\left(\mathcal{L}\left(T_{v} E\right) \operatorname{sch}(V)\right) \times u
\end{gathered}
$$

As the cross product with $u$ is injective, the claim follows.

## 3. The case of surface groups

To deduce Theorem 1.1 from Theorem [1.5, it remains to show the following.
Lemma 3.1. For $g \geq 2$, there is a flat hermitian vector bundle $V \rightarrow \Sigma_{g}$ of signature $(1,1)$ with $\left\langle\operatorname{sch}_{1}(V),\left[\Sigma_{g}\right]\right\rangle=2-2 g$.

Proof. Recall that $\mathbb{P S U}(1,1)=\mathrm{SU}(1,1) / \pm 1$ is the orientation-preserving isometry group of the Poincare disc model $\mathbb{E}$ of hyperbolic space; the action is given by Möbius transformations. The isotropy group of $\operatorname{SU}(1,1)$ at the origin is $U(1)$, given by $z \mapsto \operatorname{diag}\left(z, z^{-1}\right)$.

The Riemann uniformization theorem provides us with a homomorphism $\gamma$ : $\Gamma_{g} \rightarrow \mathbb{P S U}(1,1)$ such that $\mathbb{E} / \gamma\left(\Gamma_{g}\right) \cong \Sigma_{g}$. The composition

$$
\Sigma_{g} \xrightarrow{B \gamma} B \mathbb{P S U}(1,1)^{\delta} \rightarrow B \mathbb{P} U(1,1) \simeq B(U(1) / \pm 1) \simeq B S O(2)
$$

is a classifying map for the tangent bundle $T \Sigma_{g}$; this is nicely explained in [16, p.312f]. Since $\Sigma_{g}$ is spin, the map $\Sigma_{g} \rightarrow B \mathbb{P} U(1,1)$ can thus lifted to a map $\Sigma_{g} \rightarrow B U(1,1)$. The diagram

is homotopy cartesian; hence we can lift $B \gamma: \Sigma_{g} \rightarrow B \operatorname{PSU}(1,1)^{\delta}$ to a map $\Sigma_{g} \rightarrow$ $B \mathrm{SU}(1,1)^{\delta}$, which is $B \rho$ for some lift $\rho: \Gamma_{g} \rightarrow \mathrm{SU}(1,1)$ of $\gamma$. By construction, $\Sigma_{g} \rightarrow B \mathrm{SU}(1,1)^{\delta} \rightarrow B \mathrm{SU}(1,1) \simeq B U(1)$ is a complex line bundle of Chern number $1-g$. Hence the hermitian vector bundle $V$ given by $B \rho$ has $\left\langle\operatorname{sch}_{1}(V),\left[\Sigma_{g}\right]\right\rangle=2-2 g$. To see where the factor of 2 originates, note the commutative diagram

whose horizontal maps are homotopy equivalences.

Remark 3.2. In $\S 8 \frac{2}{7}$ of [11, Gromov gives a beautiful geometric computation of the index of the even twisted signature operator on $V \rightarrow \Sigma_{g}$; together with the index theorem, this also yields a computation of $\operatorname{sch}_{1}(V)$.

## 4. Nontriviality result

In this section, we give the proof of Theorem 1.2. The idea is to borrow from the work of Galatius and Randal-Williams bundles of even-dimensional manifolds, especially [10], and then to take products with fixed manifolds. Let us first record a simple product formula.
Lemma 4.1. Let $\pi_{j}: E_{j} \rightarrow X_{j}, f_{j}: E_{j} \rightarrow B G_{j}, j=0,1$, be two bundles of closed oriented $n_{j}$-manifolds with maps to $B G_{j}$. Then for $u_{j} \in H^{*}\left(B G_{j} ; \mathbb{Q}\right)$, we have

$$
\kappa_{\mathcal{L}, u_{0} \times u_{1}}\left(E_{0} \times E_{1}, f_{0} \times f_{1}\right)=(-1)^{n_{1}\left|u_{0}\right|} \kappa_{\mathcal{L}, u_{0}}\left(E_{0}, f_{0}\right) \times \kappa_{\mathcal{L}, u_{1}}\left(E_{1}, f_{1}\right) .
$$

Proof. From the formulas $p_{!}\left(x \cup p^{*} y\right)=p_{!}(x) \cup y$ and $(p \circ q)_{!}=p_{!} \circ q!$ for the Gysin maps, one deduces the formula $\left(\pi_{0} \times \pi_{1}\right)!\left(x_{0} \times x_{1}\right)=(-1)^{n_{1}\left|x_{0}\right|}\left(\pi_{0}\right)!\left(x_{0}\right) \times\left(\pi_{1}\right)_{!}\left(x_{1}\right)$. Hence, using the multiplicativity of the $\mathcal{L}$-class,

$$
\begin{array}{r}
\kappa_{\mathcal{L}, u_{0} \times u_{1}}\left(E_{0} \times E_{1}, f_{0} \times f_{1}\right) \\
=\left(\pi_{0} \times \pi_{1}\right)!\left(\mathcal{L}\left(T_{v} E_{0}\right) \times \mathcal{L}\left(T_{v} E_{1}\right) \cup\left(f_{0}^{*}\left(u_{0}\right)\right) \times f_{1} *\left(u_{1}\right)\right) \\
=\left(\pi_{0} \times \pi_{1}\right)!\left(\left(\mathcal{L}\left(T_{v} E_{0}\right) \cup f_{0}^{*}\left(u_{0}\right)\right) \times\left(\mathcal{L}\left(T_{v} E_{1}\right) \cup f_{1}^{*}\left(u_{1}\right)\right)\right)  \tag{4.2}\\
=(-1)^{n_{1}\left|u_{0}\right|} \kappa_{\mathcal{L}, u_{0}}\left(E_{0}, f_{0}\right) \times \kappa_{\mathcal{L}, u_{1}}\left(E_{1}, f_{1}\right) .
\end{array}
$$

When we split the $\mathcal{L}$-class into its components, we obtain

$$
\kappa_{\mathcal{L}_{m}, u_{0} \times u_{1}}\left(E_{0} \times E_{1}, f_{0} \times f_{1}\right)=\sum_{k+l=m}(-1)^{n_{1}\left|u_{0}\right|} \kappa_{\mathcal{L}_{k}, u_{0}}\left(E_{0}, f_{0}\right) \times \kappa_{\mathcal{L}_{l}, u_{1}}\left(E_{1}, f_{1}\right)
$$

We can apply this product formula when $\pi_{1}: E_{1} \rightarrow X_{1}$ is the map $c_{N}: N \rightarrow *$ is a single closed manifold of dimension $n_{1}$. If $4 l+\left|u_{1}\right|=n_{1}$, the previous formula collapses to

$$
\begin{equation*}
\kappa_{\mathcal{L}_{m}, u_{0} \times u_{1}}\left(E_{0} \times N, f_{0} \times f_{1}\right)=(-1)^{n_{1}\left|u_{0}\right|} \kappa_{\mathcal{L}_{k}-l, u_{0}}\left(E_{0}, f_{0}\right) \times \operatorname{sign}_{u_{1}}\left(N, f_{1}\right) . \tag{4.3}
\end{equation*}
$$

From the work of Galatius and Randal-Williams, we extract the following result.
Lemma 4.4. Let $n \geq 1$, let $G$ be a discrete group, $v \neq 0 \in H^{q}(B G ; \mathbb{Q})$ and $k \in \mathbb{N}_{0}$ such that $4 k+q \geq 2 n$. Then there is a bundle $\pi: E \rightarrow X$ of closed oriented 2n-manifolds, together with a map $h: E \rightarrow B G$ such that

$$
\kappa_{\mathcal{L}_{k}, v}(E, h) \neq 0 \in H^{4 k+q-2 n}(X ; \mathbb{Q})
$$

Proof. In the case $k+q=2 n$, the lemma just asserts the existence of a closed $2 n$-manifold $N$ and a map $h: N \rightarrow B G$ such that $\operatorname{sign}_{v}(N, f) \neq 0$; this is straightforward from Thom's classical result $\Omega_{*}^{S O}(B G) \otimes \mathbb{Q} \cong H_{*}(B S O \times B G ; \mathbb{Q})$; the obvious analogue for odd-dimensional manifolds holds by the same argument.

So let us suppose $k+q>2 n$. There is a finite CW complex $Y$ and a map $g: Y \rightarrow$ $B G$ such that $g^{*} v \neq 0 \in H^{q}(Y ; \mathbb{Q})$, by the universal coefficient theorem for rational cohomology. We consider the fibration $\theta: B S O(2 n) \times Y \rightarrow B S O(2 n) \rightarrow B O(2 n)$, with associated Madsen-Tillmann spectrum $\operatorname{MT} \theta(2 n) \simeq \operatorname{MTSO}(2 n) \wedge Y_{+}$. The idea is to apply [10, Theorem 7.3] to $\theta$. Surgery below the middle dimension produces a closed $2 n$-manifold $N$ with a $\theta$-structure whose underlying map $N \rightarrow B S O(2 n) \times Y$
is $n$-connected. Let $W$ be $N$ with a disc removed; note that this gives a $\theta$-structure to $S^{2 n-1}=\partial W$, which can be extended to a $\theta$-structure on $D^{2 n}$.

Let $\mathscr{N}_{n}^{\theta}\left(S^{2 n-1}\right)$ denote the space of all $\theta$-nullbordisms of $S^{2 n-1}$ with $n$-connected structure map to $B S O(2 n) \times Y$. Theorem 7.3 of [10] states that there is a homology isomorphism

$$
\alpha: \operatorname{hocolim}_{g \rightarrow \infty}\left(\mathscr{N}_{n}^{\theta}\left(S^{2 n-1}\right) \rightarrow \mathscr{N}_{n}^{\theta}\left(S^{2 n-1}\right) \rightarrow \ldots\right) \rightarrow \Omega^{\infty}\left(\operatorname{MTSO}(2 n) \wedge Y_{+}\right),
$$

where the stabilization map is by adding $\left(S^{2 n-1} \times[0,1]\right) \sharp\left(S^{n} \times S^{n}\right)$.
Under the Thom isomorphism $H^{*+2 n}(B S O(2 n) \times Y ; \mathbb{Q}) \cong H^{*}(\operatorname{MTSO}(2 n) \wedge$ $\left.Y_{+} ; \mathbb{Q}\right)$, the class $\mathcal{L}_{k} \times v$ corresponds to a class in $H^{4 k+q-2 n}\left(\operatorname{MTSO}(2 n) \wedge Y_{+} ; \mathbb{Q}\right)$, which produces a class in $H^{4 k+q-2 n}\left(\Omega^{\infty} \operatorname{MTSO}(2 n) ; \mathbb{Q}\right)$ which is nontrivial as $v \neq 0$, $\mathcal{L}_{k} \neq 0$ and $4 k+q-2 n \geq 0$. Hence this class becomes nonzero when pulled back to $\mathscr{N}_{n}^{\theta}\left(S^{2 n-1}\right)$. On the other hand, the latter space classifies oriented manifold bundles with maps to $B S O(2 n)$ on the total space, and the class corresponds to $\kappa_{\mathcal{L}_{k}, v}$. Now we close the boundary by adding a fixed copy of $D^{2 n}$; this finishes the proof.

The proof of Theorem [1.2, is now straightforward: Lemma 4.4 provides a bundle $\pi: E \rightarrow X$ with oriented $2 n_{0}$-dimensional fibres and $f: E \rightarrow B G$ such that $\kappa_{\mathcal{L}_{m_{0}}, v}(E, f) \neq 0$, and there is a closed $2 n_{1}+1$-dimensional $N$ and $g: N \rightarrow B H$ with $\operatorname{sign}_{w}(N, g) \neq 0$. The bundle $E \times N \rightarrow X$ with the map $f \times g: E \times N \rightarrow B(G \times H)$ does the job by Lemma 4.1 .

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