Exercises for Index theory II

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To be discussed: 26.06.14 - ??

The purpose of this set of exercises is to discuss the Spin groups in terms of Clifford algebras. You might want to consult references: the classic paper on the material is Atiyah, Bott, Shapiro: "Clifford modules"; a must-read. A textbook reference is Lawson-Michelsohn: "Spin geometry".

Recall that $\mathrm{Cl}^{p,q}$ is the (unital, associative) $\mathbb{R}$-algebra generated by elements $e_1, \ldots, e_p, \epsilon_1, \ldots, \epsilon_q$, subject to the relations

$$e_i e_j + e_j e_i = -2\delta_{ij};\ \epsilon_i\epsilon_j + \epsilon_j\epsilon_i = 2\delta_{ij};\ e_i\epsilon_j + \epsilon_j e_i = 0.$$

We consider $\mathbb{R}^{p+q} = \mathrm{span}\{e_1, \ldots, e_p, \epsilon_1, \ldots, \epsilon_q\}$ as a subspace of $\mathrm{Cl}^{p,q}$. The grading involution on $\mathrm{Cl}^{p,q}$ is the unique automorphism α of $\mathrm{Cl}^{p,q}$ such that $\alpha(v) = -v$. Let $\mathrm{Cl}^{p,q}_i \subset \mathrm{Cl}^{p,q}$ be the eigenspace of α to the eigenvalue $(-1)^i$. Moreover, we let $* : \mathrm{Cl}^{p,q} \to \mathrm{Cl}^{p,q}$ be the unique antiautomorphism with $e_i \mapsto -e_i$ and $\epsilon_i \mapsto \epsilon_i$.

We identify $\mathbb{R}^n$ with its dual space using the standard inner product. Let $(v_1, \ldots, v_n)$ be the standard basis. Consider the exterior algebra $\Lambda^*\mathbb{R}^n$ and the operators

$$a_i(\omega) := v_i \wedge \omega;\ b_i(\omega) = \iota_{v_i}\omega$$

on $\Lambda^*\mathbb{R}^n$ given by the wedge product and the insertion. The exterior algebra has the standard even/odd grading, given by the involution ι which is $(-1)^p$ on $\Lambda^p\mathbb{R}^n$.

Exercise 1. Prove that the operators $e_i := a_i - b_i$ and $\epsilon_i := a_i + b_i$ define an algebra homomorphism $\gamma : \mathrm{Cl}^{n,n} \to \mathrm{End}(\Lambda^*\mathbb{R}^n)$, which is moreover graded. Moreover, prove that $\gamma(x^*) = \gamma(x)^*$ (the latter is the adjoint with respect to the standard scalar product on the exterior algebra) for all $x \in \mathrm{Cl}^{n,n}$. Thus γ is a $*$-homomorphism. We call the resulting Clifford module by $\mathbb{S}_{n,n}$ and call it the *spinor representation*.

Exercise 2. Prove that the map

$$c : \mathrm{Cl}^{n,0} \subset \mathrm{Cl}^{n,n} \overset{x \mapsto \gamma(x)1}{\to} \Lambda^*\mathbb{R}^n$$

is an isomorphism of vector spaces (not of algebras). Hint: for dimension reasons, it is enough to prove surjectivity.

Exercise 3. Prove that $\mathbb{S}_{n,n} \otimes \mathbb{S}_{m,m} \cong \mathbb{S}_{m+n,m+n}$ (here we use the exterior tensor product of graded Clifford modules). Prove by induction on n that $\gamma : \mathrm{Cl}^{n,n} \to \mathrm{End}(\Lambda^*\mathbb{R}^n)$ is an isomorphism of algebras.

Exercise 4. Let $(\mathrm{Cl}^{n,0})^\times$ be the group of units in the Clifford algebra. We define two subgroups: $\Delta_n \subset (\mathrm{Cl}^{n,0})^\times$ is the group of all units x such that $\gamma(x) \in \mathrm{End}(\Lambda^*\mathbb{R}^n)$ is orthogonal (equivalently $\gamma(x)^*\gamma(x) = \gamma(x^*x) = 1$ or $x^*x = 1$), and $\Gamma_n \subset (\mathrm{Cl}^{n,0})^\times$ is the group of all units x such that $\alpha(x)yx^{-1} \in \mathbb{R}^n$ for all $y \in \mathbb{R}^n$. We define $\mathrm{Pin}(n) := \Delta_n \cap \Gamma_n$. In a similar way, consider the complexification $\mathbb{S}_{n,n} \otimes \mathbb{C}$, with induced homomorphism $\gamma^c : \mathrm{Cl}^{n,0} \otimes \mathbb{C} \to \mathrm{End}(\mathbb{S}_{n,n} \otimes \mathbb{C})$. Let $\Delta_n^c \subset (\mathrm{Cl}^{n,0} \otimes \mathbb{C})^\times$ be those elements x with $\gamma(x)$ unitary and let Γ_n^c be the group of all $x \in (\mathrm{Cl}^{n,0} \otimes \mathbb{C})^\times$ with $\alpha(x)yx^{-1} \in \mathbb{R}^n$ for all $y \in \mathbb{R}^n$. We let $\mathrm{Pin}^c(n) = \Gamma_n^c \cap \Delta_n^c$.
Prove that $\mathrm{Pin}(n)$ and $\mathrm{Pin}^c(n)$ are compact Lie groups. Hint: use the nontrivial result from Lie theory that a closed subgroup of $\mathrm{GL}_k(\mathbb{R})$ is a Lie group.

The groups Γ_n and Γ_n^c come with homomorphism $\rho : \Gamma_n \to \mathrm{GL}_n(\mathbb{R})$ and $\rho^c : \Gamma_n^c \to \mathrm{GL}_n(\mathbb{R})$: $x \mapsto (y \mapsto \alpha(x)yx^{-1})$.

Exercise 5. Prove that the kernel of ρ^c consists of all $z1$, $z \in \mathbb{C}^\times$. Hint: here you have to work a bit. Pick x in the kernel and write x as a linear combination of the elements $e_{j_1} \cdots e_{j_k}$. Hence

$$\ker(\rho^c : \mathrm{Pin}^c(n) \to O(n)) \cong S^1; \ \ker(\rho : \mathrm{Pin}(n) \to O(n)) \cong \pm 1.$$

Exercise 6. Prove the inclusions (hence equalities)

$$O(n) \subset \mathrm{Im}(\rho) \subset \mathrm{Im}(\rho^c) \subset O(n).$$

Hint: the second inclusion is clear. For the first one, let $x \in \mathbb{R}^n \subset \mathrm{Cl}^{n,0}$ be a unit vector. Prove that $x \in \mathrm{Pin}(n)$ and that $\rho(x) \in \mathrm{GL}_n(\mathbb{R})$ is the reflection at the hyperplane $x^\perp$. Use that the reflections generate the orthogonal group; this classical result is known as the *Cartan–Dieudonné theorem*. For the third one, use that $\mathrm{Pin}^c(n)$ is compact and that $O(n) \subset \mathrm{GL}_n(\mathbb{R})$ is a maximal compact subgroup. This latter statement can be proven nicely using invariant integration: let K be compact, $O(n) \subset K \subset \mathrm{GL}_n(\mathbb{R})$. By invariant integration, K leaves an inner product on $\mathbb{R}^n$ invariant. Since this inner product is also invariant under $O(n)$, it must be a multiple of the standard scalar product. Hence $K \subset O(n)$.

Altogether, the above exercises prove that there are short exact sequences

$$1 \to \pm 1 \to \mathrm{Pin}(n) \to O(n) \to 1; \ 1 \to S^1 \to \mathrm{Pin}^c(n) \to O(n) \to 1.$$

Exercise 7. Show that $\mathrm{Pin}(n) \cap \mathrm{Cl}_0^{n,0} = \rho^{-1}(SO(n))$. This group is called $\mathrm{Spin}(n)$, the *Spin group*. Similar, $\mathrm{Spin}^c(n) = \mathrm{Pin}^c(n) \cap \mathrm{Cl}_0^{n,0} \otimes \mathbb{C} = (\rho^c)^{-1}(SO(n))$.

Exercise 8. Show that Spin(n) and Spin^c(n) are connected, if $n \geq 2$. Hint: why is it enough to study Spin(n)? Show that $x(t) = \cos(t) + \sin(t)e_1e_2$ is a path that connects the two elements in the kernel of ρ. Conclude that for $n \geq 3$, the group Spin(n) is simply-connected (since $\pi_1(SO(n)) = \mathbb{Z}/2$).

Exercise 9. Let $\gamma : \mathbb{R}^n \to \mathrm{End}(\mathbb{S}_{n,n})$ be the Clifford multiplication. Prove that γ is Spin(n)-equivariant in the following sense. On the source of γ, Spin(n) acts through the homomorphism ρ. As Spin(n) is a subgroup of the units in $\mathrm{Cl}^{n,0}$, it acts on $\mathbb{S}_{n,n}$ through γ and thus on $\mathrm{End}(\mathbb{S}_{n,n})$ (how?).

Exercise 10. Let $V \to X$ be an n-dimensional Riemannian vector bundle. We define a Spin-structure on V to be a pair (P, η), where $P \to X$ is a Spin(n)-principal bundle and $\eta : P \times_{\mathrm{Spin}(n)} \mathbb{R}^n \to V$ an isometry of Riemannian vector bundles. In a similar way, one defines a Spin^c-structure, replacing Spin(n) by Spin^c(n).
Assume that a Spin-structure on V is given. Show (using the last exercise) that $P \times_{\mathrm{Spin}(n)} \mathbb{S}_{n,n} \otimes \mathbb{C}$ is a graded $\mathrm{Cl}(V \oplus \mathbb{R}^{0,n})$-module bundle. Now let n be even. Under the algebraic Bott periodicity, we obtain a graded $\mathrm{Cl}(V)$-module bundle $\$_V \to X$, the complex spinor bundle.

Now let M be a Riemannian manifold and (P, η) be a spin structure on TM. Let $\$_M$ be the spinor bundle constructed in the last exercise. It has a graded Dirac operator $\not{D}$, the *Atiyah-Singer-Dirac operator*. We wish to compute the index of this operator. This is done in the following steps. Recall that

$$\mathrm{ind}(\not{D}) = \int_M \lambda(\$_M)\mathrm{Td}(TM \otimes \mathbb{C}).$$

This leaves the computation of $\lambda(\$_M)$, which was defined in the following way. If $V \to X$ is a rank $2n$ vector bundle with a spin structure, let $\lambda(\$_V) := \mathrm{th}^{-1}(\mathrm{ch}(\mathrm{abs}(E))) \in H^*(X; \mathbb{R})$, using the Atiyah-Bott-Shapiro map, the Chern character and the Thom isomorphism. This is a characteristic class for Spin($2n$)-principal bundles.
There is a map $\mathrm{Spin}(2)^n \to \mathrm{Spin}(2n)$, which is *not* injective, but it is a covering of maximal tori. Since the map $I(\mathrm{Spin}(2n)) \to I(\mathrm{Spin}(2)^n)$ is injective, it is enough to compute $\lambda(\$_V)$ for bundles with structural group $\mathrm{Spin}(2)^n$. Use the multiplicative structure of all data at hand to reduce to the case $n = 1$.
For the case $n = 1$, give a direct calculation. Hint: for the result to be proven, you might consult the literature.