Exercises for Index theory II

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To be discussed: 08.05.14

Exercise 1. (K-theory of a surface) Last term, we proved that a connected oriented closed surface has $K^0(X) \cong \mathbb{Z}^2$. The main ingredient was the theory of the first Chern class and the theorem that homotopy classes $[X, S^2]$ are given by the mapping degree. Give a proof that uses Bott periodicity instead. Hint: use the classification of surfaces; X is a 2-dimensional CW complex with $X^{(1)} = \bigvee^{2g} S^1$ and $X/X^{(1)} \cong S^2$. You need to know $K^0(S^2)$ and a tiny bit of the first Chern class.

Exercise 2. (The Gysin sequence) Let $\pi : V \to X$ be a hermitian vector bundle over a compact space, with Thom class $\mathbf{t}_V \in K^0(DV, DV)$ and zero section $\iota : V \to DV$. We let $\mathbf{e}(V) := \iota^* \mathbf{t}_V \in K^0(X)$ be the K-theoretic Euler class. Use the long exact sequence of the pair (DV, SV) and the Thom isomorphism to derive a long exact sequence

$$K^0(X) \stackrel{\mathbf{e}\cdot}{\to} K^0(X) \stackrel{\pi^*}{\to} K^0(SV) \to K^{-1}(X) \ldots,$$

the *Gysin sequence*

The following exercises will give the proof of the following result

Theorem. Let $\mathbf{x} := 1 - [H] \in K^0(\mathbb{CP}^n)$. There is an isomorphism $\mathbb{Z}[\mathbf{x}]/(\mathbf{x}^{n+1}) \cong K^0(\mathbb{CP}^n)$; and $K^{-1}(\mathbb{CP}^n) = 0$.

Exercise 3. Prove that $K^{-1}(\mathbb{CP}^n) = 0$ and that $K^0(\mathbb{CP}^n)$ is free abelian of rank $n+1$; by induction on n. Use the pair $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$.

Exercise 4. Show that $\mathbf{x}^{n+1} = 0 \in K^0(\mathbb{CP}^n)$. Hint: Use the fact that for each codimension 1 subspace $V \subset \mathbb{C}^{n+1}$, there is a section s of H such that $s^{-1}(0) = \mathbb{P}V$. Therefore, $\mathbf{x}$ lies in the image of $K^0(\mathbb{CP}^n, \mathbb{CP}^n - \mathbb{P}V) \to K^0(\mathbb{CP}^n)$. Use that if $V_0, \ldots, V_n \subset \mathbb{C}^{n+1}$ are in general position, then $\cap_{i=0}^n \mathbb{P}V_i = \emptyset$.

Exercise 5. Use the Gysin sequence and the previous two exercises to reduce the proof of the theorem to the following purely algebraic lemm:

Lemma. Let R be a commutative ring with unit, and assume that $R \cong \mathbb{Z}^{n+1}$ as abelian group. Let $I \subset R$ be an ideal, such that $I \oplus \mathbb{Z} \to R$, $(x, a) \mapsto x + a \cdot 1$ is an isomorphism of abelian groups. Let $\mathbf{x} \in I \subset R$ be an element with $\mathbf{x}^{n+1} = 0$ and such that $\mu_{\mathbf{x}} : R \to I$, $y \mapsto \mathbf{x}y$ is surjective. Show that $\mathbb{Z}[\mathbf{x}]/(\mathbf{x}^{n+1}) \to R$ is an isomorphism of rings.

Exercise 6. Prove the lemma!