Exercises for Index theory II

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To be discussed: 24.04.14

Exercise 1. (Working with the Bott element)

Recall that the *Bott element* $\mathbf{b}$ is the class in $K^0(D^2, S^1)$ represented by the cycle $(\underline{\mathbb{C}}, \underline{\mathbb{C}}, \alpha)$, where $\alpha(x, z) = (x, \bar{x}z)$. Let $D_\infty \subset \mathbb{CP}^1$ be the disc $\{[z : 1] | |z| \geq 1\}$. Prove that under the isomorphisms (which ones?)

$$K^0(D^2, S^1) \cong K^0(\mathbb{CP}^1, D_\infty) \cong \tilde{K}^0(\mathbb{CP}^1),$$

the Bott class corresponds to $[H] - 1$, where $H \to \mathbb{CP}^1$ is the dual tautological line bundle. Moreover, let $\bar{\mathbf{b}}$ be the element represented by $(\underline{\mathbb{C}}, \underline{\mathbb{C}}, \bar{\alpha})$. Show that $\mathbf{b} + \bar{\mathbf{b}} = 0 \in K^0(D^2, S^1)$.

Exercise 2. (The low-dimensional shadow of Bott periodicity)

[This requires some basic homotopy theory] Let $e : U(n) \to S^{2n-1}$ be the map $A \mapsto Ae_n$. This is an $U(n-1)$-principal bundle. Use the long exact sequence of homotopy groups to prove that the map $\pi_i(U(n-1)) \to \pi_i(U(n))$ is surjective if $i < 2n-1$ and injective if $i < 2n-2$. Let $U = \operatorname{colim}_n U(n)$. Prove that $\pi_i(U) = \pi_i(U(n))$, for $i < 2n$. Use this to compute $\pi_i(U)$ for $i \leq 3$.

Exercise 3. (Working with K-cycles)

Let (X, A) be a space pair. We consider the self-map T of $X \times [-1, 1]$ given by $T(x, t) = (x, -t)$. Prove that T induces $-\mathrm{id}$ on $K^0((X, A) \times ([-1, 1], \{\pm 1\})$. Hint: any element in $K^0((X, A) \times ([-1, 1], \{\pm 1\})$ can be represented by a cycle of the form $(E \times [-1, 1], E \times [-1, 1], \alpha_t)$, where α_t, $t \in [-1, 1]$, is a family of endomorphisms of E. Needless to say, $\alpha_{\pm 1}$ is invertible.

Let $\pi : V \to X$ be a complex vector bundle of rank n equipped with a hermitian metric. Let DV be the disc bundle and SV be the sphere bundle. The *Thom element* or *Thom class* $\tau \in K^0(DV, SV)$ is the element represented by the chain complex

$$0 \to \pi^* \Lambda^0 V \to \pi^* \Lambda^1 V^* \to \ldots \to \pi^* \Lambda^n V^* \to 0;$$

the map $\pi^* \Lambda^p V^* \to \pi^* \Lambda^{p+1} V^*$ is given over $v \in V$ by taking the wedge product with v. Using the Thom class, we get a homomorphism

$$\mathrm{th} = \mathrm{th}_V : K^0(X, A) \to K^0(DV, SV \cup D(V|_A));\ \mathbf{x} \mapsto \pi^* \mathbf{x} \cdot \tau_V,$$

the *Thom isomorphism.* It is of course a theorem and not a triviality that th_V is an isomorphism. This theorem will be easy once we know Bott periodicity, and the following exercise does the reduction to the Bott theorem.

Exercise 4. (the Thom isomorphism theorem) Formulate and prove that for two vector bundles V and W, the exterior product $\tau_V \times \tau_W$ is the Thom element of $V \times W$, in an appropriate sense. Now consider $\pi : V \to X$ and $W \to X$. Show that $\mathrm{th}_{\pi^* W} \circ \mathrm{th}_V = \mathrm{th}_{V \oplus}$. Prove that for the trivial line bundle $V = X \times \mathbb{C}$, the Thom homomorphism is the Bott isomorphism. Use the fact that each complex vector bundle has a complement to derive that th_V is an isomorphism.