# An Introduction to Hochschild and Cyclic Homology

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**Abstract:** We define Hochschild and cyclic (co)homology for simplicial and cyclic modules. The theory for an algebra $A$ is then obtained from the canonical simplicial/cyclic module $C_*(A)$ associated to this algebra. The definitions are purely algebraic. In the last sections we connect (co)homology theories and differential calculi of an algebra. Cyclic homology will be seen to be a natural generalization of de Rham cohomology to non-commutative algebras. As a justification for developing the machinery of cyclic homology, we give some applications using the generalized Chern character.

**Notations:** We consider modules and algebras over a commutative ring $k$ with identity. To simplify statements we assume that $\mathbb{Q} \subset k$. Throughout this paper, $A, A'$ denote unital $k$-algebras, $V$ a smooth compact manifold (without boundary), $C^\infty(V)$ the smooth $\mathbb{C}$-valued functions on $V$.

**Main References:** A very readable introduction to the topic is [Lo,92]: "Cyclic Homology" by Jean-Louis Loday. Other good references are [Co,94]: "Noncommutative Geometry" by Alain Connes (which is available online for free at www.noncommutativegeometry.net) and [FVB,00]: "Elements of Noncommutative Geometry" by Joseph C. Varilly, Hector Figueroa and Jose M. Gracia-Bondia.

## Contents

# 0 Preliminaries

We recall some concepts from homological algebra, such as the notion of a bicomplex, which is used to define cyclic homology.

(0.1) **Remark:** For a $k$-algebra $A$, $A^o$ denotes the opposite algebra (where $a \cdot b = ba$), $A^e = A \otimes A^o$ is the enveloping algebra of $A$. One sets $A^* = \mathrm{Hom}_k(A, k)$, $\tilde{A} = A \oplus k$ and if $A$ is unital, $\bar{A} = A/k$. By $\otimes$ is always meant tensoring over $k$, i.e. $\otimes = \otimes_k$. Every $A$-bimodule (especially $A$ itself) is in a natural way a $A^e$-module.

(0.2) **Definition:** A **bicomplex** $C_{**}$ (or $C$ for short) is a collection of modules $C_{p,q}$, $p, q \in \mathbb{Z}$ with two differentials $d^h : C_{p,q} \to C_{p-1,q}$ and $d^v : C_{p,q} \to C_{p,q-1}$, called **horizontal** and **vertical** differential, s.t. $d^h d^h = d^v d^v = d^h d^v + d^v d^h = 0$. The **total complex** associated to $C$ is $(\mathrm{Tot}\ C, d)$ where $(\mathrm{Tot}\ C)_n = \prod_{p+q=n} C_{p,q}$ and $d = d^h + d^v$ (note that $\prod_{p+q=n} C_{p,q} \cong \bigoplus_{p+q=n} C_{p,q}$ if $C_{p,q} \neq 0$ for only finitely many $p + q = n$). The homology groups $H_*(C)$ of the bicomplex $C$ are defined to be $H_n(C) := H_n(\mathrm{Tot}\ C)$.

$$\begin{array}{ccccccc}
\ddots & & \vdots & & \vdots & & \ddots \\
 & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \\
\cdots & \xleftarrow[d^h]{} & C_{p,q+1} & \xleftarrow[d^h]{} & C_{p+1,q+1} & \xleftarrow[d^h]{} & \cdots \\
 & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \\
\cdots & \xleftarrow[d^h]{} & C_{p,q} & \xleftarrow[d^h]{} & C_{p+1,q} & \xleftarrow[d^h]{} & \cdots \\
 & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \\
\ddots & & \vdots & & \vdots & & \ddots
\end{array}$$

(0.3) **Definition:** A complex $(C, d)$ is called **contractible** with **contracting homotopy** $h : C_n \to C_{n+1}$ if $dh + hd = \mathrm{id}$ (i.e. id and 0 are chain homotopic via $h$). Every contractible complex is **acyclic** (i.e. $H_*(C) = 0$)

(0.4) **Proposition:** [Lo,92] **Killing contractible subcomplexes**
Let $(A_* \oplus A'_*, d)$ be a complex with $d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : A_n \oplus A'_n \to A_{n-1} \oplus A'_{n-1}$, s.t. $(A'_*, \delta)$ is a contractible complex with contracting homotopy $h : A'_n \to A'_{n+1}$. Then the following inclusion of complexes is a quasi-isomorphism (i.e. it induces the identity on homology):

$$(\mathrm{id}\ , -h\gamma) : (A_*, \alpha - \beta h \gamma) \to (A_* \oplus A'_*, d)$$

(0.5) **Definition: Bar Complex**
Let $A$ be an algebra. The **bar complex** $C^{\mathrm{bar}}_*(A)$ of $A$ is the complex of $A^e$-modules $C^{\mathrm{bar}}_n(A) := A^{\otimes n+2}, n \geq 0$ with boundary map $b' : C^{\mathrm{bar}}_n \to C^{\mathrm{bar}}_{n-1}$, $b'(a_0, \ldots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1})$.

(0.6) **Proposition:** If $A$ is unital, then $C^{\mathrm{bar}}_*(A)$ is a resolution of $A$ as $A^e$-module. The augmentation $\mu : C^{\mathrm{bar}}_*(A) \to A$ is simply multiplication.

$$C^{\mathrm{bar}}_*(A) \xrightarrow{\mu} A : \qquad \cdots \xrightarrow{b'} A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2} \xrightarrow{\mu} A \longrightarrow 0$$

A contracting homotopy is given by $s : C^{\mathrm{bar}}_n(A) \to C^{\mathrm{bar}}_{n+1}(A), s(a_0, \cdots, a_{n+1}) = (1, a_0, \cdots, a_{n+1})$. (i.e. $sb' + b's = \mathrm{id}$ ). The map $s$ is also called the extra degeneracy map.

# 1 Some Categories

In this section we recall the pre-simplicial and simplicial categories $\Delta^{\text{pre}}$ and $\Delta$. A contravariant functor from $\Delta$ into the category of $k$-modules is called a simplicial $k$-module and is all we need to define Hochschild (co)homology (which is done in sections 2 and 3). Then, we define the cyclic category $\Lambda$, which was introduced by Alain Connes. Contravariant functors from $\Lambda$ into the category of $k$-modules are called cyclic $k$-modules and are used to define cyclic (co)homology (see sections 4 and 5).

## 1.1 The Pre-Simplicial Category $\Delta^{\text{pre}}$

(1.1) **Definition:** The **pre-simplicial category** $\Delta^{\text{pre}}$ is the small category of finite, totally ordered sets and increasing functions. That means:

- The objects are $[n] = \{\ 0 < 1 < \ldots < n\ \} \quad n = 0, 1, \ldots$
- $\text{Mor}_{\Delta^{\text{pre}}}([m],[n]) = \{\ f : [m] \to [n] \mid f(0) < f(1) < \ldots < f(m)\ \}$

In $\text{Mor}_{\Delta^{\text{pre}}}([n],[n+1])$ one has the special elements $\delta_i$ $(i = 0, \ldots, n+1)$, called **face maps**, These maps are characterized by "skipping" the i-th element, i.e. $i \notin \text{im}(\delta_i)$:

- $\delta_i : [n] \to [n+1] \quad \delta_i(x) = \begin{cases} x & \text{, if } x < i \\ x+1 & \text{, if } x \geq i \end{cases}$

(1.2) **Proposition:** The face maps fulfill the following relation:

(1.2.1) $\quad \delta_j \delta_i = \delta_i \delta_{j-1} \quad \text{if } i < j$

Further, if $m > n$, then $\text{Mor}_{\Delta^{\text{pre}}}([m],[n]) = \emptyset$. If $m \leq n$ and $f \in \text{Mor}_{\Delta^{\text{pre}}}([m],[n])$, then there are unique $i_1 \leq \ldots \leq i_r$ s.t. $m = n + r$ and $f = \delta_{i_1} \ldots \delta_{i_r}$ (if $r = 0$, this is understood to be $\text{id}_{[n]}$). This means that the $\delta_i$ give a presentation of $\Delta^{\text{pre}}$.

## 1.2 The Simplicial Category $\Delta$

(1.3) **Definition:** The **simplicial category** $\Delta$ is the small category of finite, totally ordered sets and non-decreasing functions. That means:

- The objects are $[n] = \{\ 0 < 1 < \ldots < n\ \} \quad n = 0, 1, \ldots$
- $\text{Mor}_{\Delta}([m],[n]) = \{\ f : [m] \to [n] \mid f(0) \leq f(1) \leq \ldots \leq f(m)\ \}$

Besides the face maps $\delta_i$ from (1.1) one has in $\text{Mor}_{\Delta}([n],[n-1])$ the special elements $\sigma_j$ $(j = 0, \ldots, n-1)$, called **degeneracy maps**. These maps are characterized by "repeating" (only) the j-th element once, i.e. $\sigma_j(j) = \sigma_j(j+1)$ :

- $\sigma_j : [n] \to [n-1] \quad \sigma_j(x) = \begin{cases} x & \text{, if } x \leq j \\ x-1 & \text{, if } x > j \end{cases}$

(1.4) **Proposition:** The face and degeneracy maps fulfill the following relations:

(1.4.1) $\quad \delta_j \delta_i = \delta_i \delta_{j-1} \quad \text{if } i < j$

(1.4.2) $\quad \sigma_j \sigma_i = \sigma_i \sigma_{j+1} \quad \text{if } i \leq j$

(1.4.3) $\quad \sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{, if } i < j \\ \text{id}_{[n]} & \text{, if } i = j \text{ or } i = j+1 \\ \delta_{i-1} \sigma_j & \text{, if } i > j+1 \end{cases}$

If $f \in \text{Mor}_{\Delta}([m],[n])$, then there are unique $i_1 \leq \ldots \leq i_r, j_1 < \ldots < j_s$ s.t. $m = n + r - s$ and $f = \delta_{i_1} \ldots \delta_{i_r} \sigma_{j_1} \ldots \sigma_{j_s}$. This means that the $\delta_i, \sigma_j$ give a presentation of $\Delta$.

## 1.3 The Cyclic Category $\Lambda$

(1.5) **Definition:** The **cyclic category** $\Lambda$ is a small category with objects the finite, totally ordered sets $[n]$. The maps are defined as follows [Co,94]: Identify $\mathbb{Z}_{n+1}$ with the $(n+1)$-th roots of unity in $S^1$. We fix an orientation on $S^1$. This gives the notion of a non-decreasing function $f : S^1 \to S^1$. Maps from $[m]$ to $[n]$ are defined as homotopy classes of continuous non-decreasing functions $f : S^1 \to S^1$ of degree 1 with $f(\mathbb{Z}_{m+1}) \subset \mathbb{Z}_{n+1}$.

One might need some time to get acquainted with this definition. There are helpful pictures in section $III.A.\beta$ of [Connes,94]. When we interpret the elements of $\mathrm{Mor}_\Lambda([m],[n])$ as maps $[m] \to [n]$, then $\Delta$ is a subcategory of $\Lambda$. But besides the face and degeneracy maps from (1.1) and (1.3) one has in $\mathrm{Mor}_\Lambda([n],[n])$ the special elements $\tau_n$, called **cyclic maps** :

- $\tau_n : [n] \to [n] \quad \tau_n(x) = \begin{cases} x-1 & \text{, if } x \geq 1 \\ n & \text{, if } x = 0 \end{cases}$

(1.6) **Proposition:** The face, degeneracy and cyclic maps fulfill the following relations:

$$
\begin{array}{llll}
(1.6.1) & \delta_j\delta_i = \delta_i\delta_{j-1} & \text{if } i < j & \\
(1.6.2) & \sigma_j\sigma_i = \sigma_i\sigma_{j+1} & \text{if } i \leq j & \\
(1.6.3) & \sigma_j\delta_i = \begin{cases} \delta_i\sigma_{j-1} & \text{, if } i < j \\ \mathrm{id}_{[n]} & \text{, if } i = j \text{ or } i = j+1 \\ \delta_{i-1}\sigma_j & \text{, if } i > j+1 \end{cases} & & \\
(1.6.4) & \tau_n\delta_i = \delta_{i-1}\tau_{n-1} & \text{if } 1 \leq i \leq n & \tau_n\delta_0 = \delta_n \\
(1.6.5) & \tau_n\sigma_i = \sigma_{i-1}\tau_{n+1} & \text{if } 1 \leq i \leq n & \tau_n\sigma_0 = \sigma_n\tau_{n+1}^2 \\
(1.6.6) & \tau_n^{n+1} = \mathrm{id}_{[n]} & &
\end{array}
$$

If $f \in \mathrm{Mor}_\Lambda([m],[n])$, then there are unique $\tilde{f} \in \mathrm{Mor}_\Delta([m],[n])$ and $e \in \mathbb{Z}_{m+1}$ s.t. $f = \tilde{f}\tau_m^e$. This means that $\Lambda$ can be "decomposed" into $\Delta$ and a collection of cyclic groups $\mathbb{Z}_{n+1}$ being the automorphisms (on each $[n]$). One writes $\Lambda = \Delta C$ to indicate this decomposition.

(1.7) **Remark:** One can generalize this idea and form other categories $\Delta G$, where $G$ stands for a sequence of groups $G_n$, each being the automorphism group of $[n]$. It is required that each morphism in $\Delta G$ can be written uniquely as an element of $\Delta$ and an element of $G_n$. One calls $G$ a crossed simplicial group, and defines (co)homology theories for it. There is a complete classification of crossed simplicial groups, among them are the families of trivial, cyclic, dihedral and quaternionic groups, giving rise to Hochschild, cyclic, dihedral and quaternionic homology. See [FLo,91] and section 6.3 of [Lo,92].

## 1.4 Simplicial and Cyclic Modules

(1.8) **Definition:** Let $\mathcal{C}$ be a category. A **pre-simplicial (simplicial, cyclic) object** in $\mathcal{C}$ is a contravariant functor $X : \Delta^{\mathrm{pre}} \to \mathcal{C}$ ($X : \Delta \to \mathcal{C}$, $X : \Lambda \to \mathcal{C}$). A **pre-simplicial (simplicial, cyclic) set (space)** is such an object in the category of sets (spaces).

(1.9) **Definition:** A **simplicial (cyclic) $k$-module** is a family of $k$-modules $C_n, n \geq 0$ together with $k$-linear face maps $d_i : C_n \to C_{n-1}, i = 0, \cdots, n$, degeneracy maps $s_j : C_n \to C_{n+1}, j = 0, \cdots, n$, and for cyclic modules also cyclic maps $t_n : C_n \to C_n$ that fulfill the relations (i)-(iii) for a simplicial module and (i)-(vi) for a cyclic module:

$$
\begin{array}{llll}
(1.9.\mathrm{i}) & d_id_j = d_{j-1}d_i & \text{if } i < j & \\
(1.9.\mathrm{ii}) & s_is_j = s_{j+1}s_i & \text{if } i \leq j & \\
(1.9.\mathrm{iii}) & d_is_j = \begin{cases} s_{j-1}d_i & \text{, if } i < j \\ \mathrm{id}_{C_n} & \text{, if } i = j \text{ or } i = j+1 \\ s_jd_{i-1} & \text{, if } i > j+1 \end{cases} & & \\
(1.9.\mathrm{iv}) & d_it_n = -t_{n-1}d_{i-1} & \text{if } 1 \leq i \leq n & d_0t_n = (-1)^n d_n \\
(1.9.\mathrm{v}) & s_it_n = -t_{n+1}s_{i-1} & \text{if } 1 \leq i \leq n & s_0t_n = (-1)^n t_{n+1}^2 s_n \\
(1.9.\mathrm{vi}) & t_n^{n+1} = \mathrm{id}_{C_n} & &
\end{array}
$$

(1.10) **Remark:** Notice that a simplicial module is equivalent to a contravariant functor from $\Delta$ into $k$-modules. The formulas (1.9.i)-(1.9.iii) are just a restatement of (1.4.1)-(1.4.3) for $d_i = (\delta_i)^*, s_j = (\sigma_j)^*$. But a cyclic module is not equivalent to a contravariant functor from $\Lambda$ into $k$-modules, since the cyclic maps have signs. In fact, the formulas in (1.9) can be obtained from the formulas in (1.6) by setting $t_n = (-1)^n(\tau_n)^*$.

## 2 Hochschild Homology

**Assumptions:** Throughout this section $M$ denotes an $A$-bimodule, $C$ a simplicial $k$-module.

We first define Hochschild homology $H_*(C)$ for a simplicial $k$-module $C$. Given an algebra $A$ and a $A$-bimodule $M$, we then define the simplicial $k$-module $C_*(A, M)$, whose homology $H_*(A, M)$ is the Hochschild homology of $A$ with coefficients in $M$.

(2.1) **Definition:** Let $C$ be a simplicial module. Define $b : C_n \to C_{n-1}$ as:

- $b = \sum_{i=0}^{n}(-1)^i d_i$

This defines a boundary map on $C$ (i.e. $b^2 = 0$). The complex $(C_*, b)$ is called the **Hochschild complex** (associated to $C$). Its homology $H_n(C_*, b)$ is called the **Hochschild homology** of $C$.

(2.2) **Definition:** For an algebra $A$ (algebras are assumed to be unital throughout the paper) and an $A$-bimodule $M$ we define $C_n(A, M) := M \otimes A^{\otimes n}$ and let:

- $d_0(m, a_1, \ldots, a_n) = (ma_1, a_2, \ldots, a_n)$
- $d_i(m, a_1, \ldots, a_n) = (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \qquad ,i = 1, \ldots, n-1$
- $d_n(m, a_1, \ldots, a_n) = (a_n m, a_1, \ldots, a_{n-1})$
- $s_j(m, a_1, \ldots, a_n) = (m, a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n) \quad ,j = 0, \ldots, n$

This makes $C_*(A, M)$ into a simplicial module (i.e. the formulas (1.8.i) to (1.8.iii) are fulfilled), called **the simplical module associated to a** $A$. The **Hochschild homology** $H_*(A, M)$ of $A$ with coefficients in $M$ is now defined to be the homology of $C_*(A, M)$. We let $HH_*(A) := H_*(A, A)$.

(2.3) **Remark:** One can define Hochschild homology using the tensored bar complex $M \otimes_{A^e} C_*^{\text{bar}}(A)$. The boundary map of $M \otimes_{A^e} C_*^{\text{bar}}(A)$ is understood as $1_M \otimes b'$. Since the tensoring over $A^e$ kills the leftmost and rightmost occurrence of $A$, we have $M \otimes_{A^e} C_n^{\text{bar}}(A) \cong C_n(A, M)$, and under this identification $b$ of $C_*(A, M)$ corresponds to $1 \otimes b'$ of $M \otimes_{A^e} C_n^{\text{bar}}(A)$. So $H_n(A, M) \cong H_n(M \otimes_{A^e} C_*^{\text{bar}}(A))$.

(2.4) **Proposition:** Let us quickly give some basic properties of Hochschild homology:

1. The Hochschild homology of the underlying ring $k$ can be computed as:

(2.4.1) $\quad HH_n(k) \cong \begin{cases} k & ,n = 0 \\ 0 & ,n \geq 1 \end{cases}$

2. The 0-st Hochschild homology group is the symmetrization of $M$, i.e. the biggest quotient of $M$ that is a symmetric bimodule:

(2.4.2) $\quad H_0(A, M) \cong M/\langle am - ma | a \in A, m \in M\rangle$

Thus, if $M$ is symmetric, then $H_0(A, M) = M$. We also get $HH_0(A) = A/[A, A] = A_{ab}$.

3. If $A$ is commutative, then there is a canonical isomorphism

(2.4.3) $\quad H_1(A, M) \cong M \otimes_A \Omega^1_{ab}(A)$

where $\Omega^1_{ab}(A)$ is the module of 1-forms, defined in (6.14). See also (6.18) for a generalization.

4. If $A$ is projective, then $C_*^{\text{bar}}(A)$ is a projective resolution of $A$, and remark (2.3) shows:

(2.4.4) $\quad H_n(A, M) \cong \text{Tor}_n^{A^e}(M, A)$

5. For the product of two unital algebras $A, A'$ we get:

(2.4.5) $\quad HH_*(A \times A') \cong HH_*(A) \oplus HH_*(A')$

6. Hochschild homology is functorial, i.e. $H_*(A, M)$ is a covariant functor in $M$, and $HH_*(A)$ is a covariant functor in $A$.

One important property of Hochschild and cyclic (co)homology is their Morita invariance, i.e. they cannot distinguish two Morita equivalent algebras. The precise statement is as follows:

(2.5) **Theorem:** Let $A, A'$ be unital $k$-algebras that are Morita equivalent via the $(A, A')$-bimodule $P$ and the $(A', A)$-bimodule $Q$, and let $M$ be an $A$-bimodule. Then:

(2.5.1) $H_*(A, M) \cong H_*(A', Q \otimes_A M \otimes_A P)$

For the case of matrices we get: Let $\mathcal{M}_r(A)$ (resp. $\mathcal{M}_r(M)$) denote the $(r, r)$-matrices over $A$ (resp. $M$) ($r = 1, 2, \ldots, \infty$). Then $A$ and $\mathcal{M}_r(A)$ are Morita equivalent, so

(2.5.2) $H_*(A, M) \cong H_*(\mathcal{M}_r(A), \mathcal{M}_r(M)) \quad r = 1, 2, \ldots, \infty$

This isomorphism is induced by the trace and inclusion map (they induce maps inverse to each other). See also section 1.2 of [Lo,92].

# 3 Hochschild Cohomology

**Assumptions:** Throughout this section $M, M'$ denote $A$-bimodules, $C$ a simplicial $k$-module.

The definition of Hochschild cohomology is now straightforward. We define it first for simplicial modules, and then for algebras with coefficients in a bimodule.

(3.1) **Definition:** The **Hochschild cohomology** $H^*(C)$ of a simplicial module $C$ is defined to be the cohomology of $\mathrm{Hom}_k(C_*, k)$ where $(C_*, b)$ is the Hochschild complex associated to $C$.

(3.2) **Definition:** The Hochschild cohomology of $A$ with coefficients in $M$ is *not* defined as cohomology of $\mathrm{Hom}(C_*(A, M), k)$ but as follows: Define $C^n(A, M) := \mathrm{Hom}_k(A^{\otimes n}, M)$ and let: (for $f : A^{\otimes n} \to M$)

- $$(\beta f)(a_1, \ldots, a_{n+1}) = a_1 f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^n f(a_1, \ldots, a_n) a_{n+1}$$

The **Hochschild cohomology** of $A$ with coefficients in $M$ is defined to be $H^n(C^*(A, M), \beta)$. We let $C^n(A) := C^n(A, A^*) \cong \mathrm{Hom}_k(A^{\otimes n+1}, k)$ and $HH^n(A) := H^n(A, A^*)$.

(3.3) **Remark:** As in (2.3), one can define Hochschild cohomology using the bar complex of $A$. An $A^e$-linear map $\phi : C_n^{\mathrm{bar}}(A) = A^{\otimes n+2} \to M$ can be identified with a $k$-linear map $f : A^{\otimes n} \to M$ via $\phi(a_0, \ldots, a_{n+1}) = a_0 f(a_1, \ldots, a_n) a_{n+1}$. So $C^*(A, M) \cong \mathrm{Hom}_{A^e}(C_*^{\mathrm{bar}}(A), M)$. The coboundary maps correspond and we have $H^n(A, M) \cong H^n(\mathrm{Hom}_{A^e}(C_*^{\mathrm{bar}}(A), M))$.

(3.4) **Proposition:** Let us collect basic properties of Hochschild cohomology:

1. The Hochschild cohomology of the underlying ring $k$ is:

(3.4.1) $HH^n(k) \cong \begin{cases} k & , n = 0 \\ 0 & , n \geq 1 \end{cases}$

2. The 0-st Hochschild cohomology group is :

(3.4.2) $H^0(A, M) \cong \langle am - ma | a \in A, m \in M \rangle$

3. The first Hochschild cohomology group is exactly the group of outer derivations of $A$ in $M$ (see (6.2)):

(3.4.3) $H^1(A, M) \cong \mathrm{Der}(A, M)/\mathrm{Der}'(A, M)$

4. If $A$ is projective, then $C_*^{\mathrm{bar}}(A)$ is a projective resolution of $A$, and remark (3.3) shows:

(3.4.4) $H^n(A, M) \cong \mathrm{Ext}^n_{A^e}(M, A)$

6. Hochschild cohomology is functorial, i.e. $H^*(A, M)$ is a covariant functor in $M$, and $HH^*(A)$ is a covariant functor in $A$.

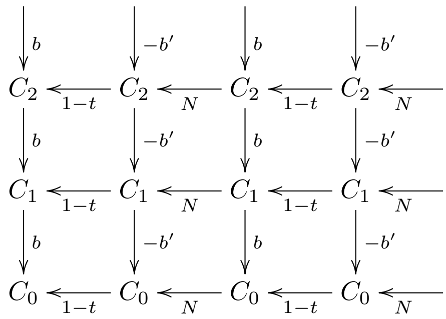
7. Hochschild cohomology is Morita invariant.

# 4 Cyclic Homology

**Assumptions:** Throughout this section $C$ denotes a cyclic $k$-module.

As for Hochschild homology, we first work in a rather abstract setting (namely for a cyclic $k$-module $C$). We define three complexes $CC_{**}$, $\mathcal{B}C_{**}$ and $C_*^\lambda$, that all give the same homology, called cyclic homology. Then, we associate to an algebra $A$ a cyclic $k$-module $A^\natural$, whose homology is the cyclic homology $HC_*(A)$ of $A$.

(4.1) **Definition:** Let $C$ be a cyclic module. The associated **cyclic bicomplex** $CC$ is:



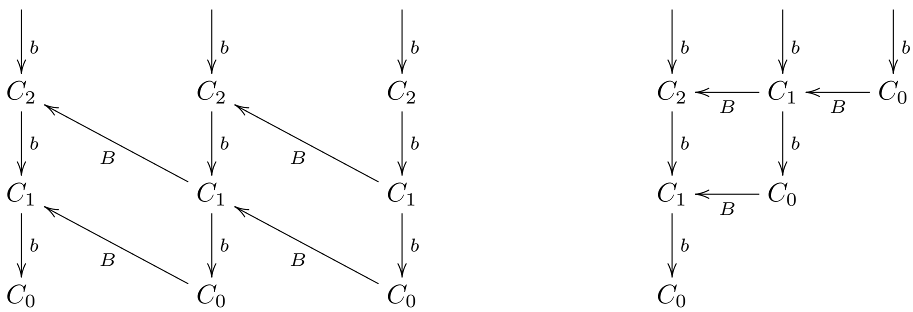

where $b : C_n \to C_{n-1}, b = \sum_{i=0}^{n}(-1)^i d_i, b' : C_n \to C_{n-1}, b' = \sum_{i=0}^{n-1}(-1)^i d_i, N : C_n \to C_n, N = \sum_{i=0}^{n} t_n^i$. Its homology $H_*(CC) := H_*(\text{Tot } CC)$ is called the **cyclic homology** of $C$ and denoted $HC_*(C)$. That $CC$ is a bicomplex means that $N(1-t) = (1-t)N = 0$ (which is obvious) and also $b^2 = b'^2 = 0, (1-t)b' = b(1-t)$ (which needs some formal calculations).

(4.2) **Definition:** The complex $(C_*^\lambda, b)$ where $C_n^\lambda := C_n/(1-t_n)$ is called **Connes's complex**. The boundary map is well-defined since $(1-t_{n-1})b' = b(1-t_n)$. The homology of this complex is denoted $H_*^\lambda(C)$.

$$C_*^\lambda : \qquad \cdots \xrightarrow{b} C_n^\lambda \xrightarrow{b} C_{n-1}^\lambda \xrightarrow{b} \cdots \xrightarrow{b} C_0^\lambda$$

(4.3) **Proposition:** Let $CC^{\{1\}}$ denote the first column of the bicomplex $CC$. Let further $p : \text{Tot } CC \to CC^{\{1\}}/(1-t) \cong C_*^\lambda$ be the map which is zero on all columns except the first one, where it is the quotient map $CC^{\{1\}} \to CC^{\{1\}}/(1-t)$. Then $p$ is a quasi-isomorphism, i.e. $H_*^\lambda(C) \cong HC_*(C)$ canonically.

(4.4) **The bicomplex $\mathcal{B}C$:** In $CC$, the columns with vertical differential $b'$ are contractible. The contracting homotopy is $s = s_{n+1} : C_n \to C_{n+1}$, $s = (-1)^n t_n s_n$. (i.e. $sb' + b's = \text{id}$ ) This map is called extra degeneracy. Applying proposition (0.4) to each of these columns we get the bicomplex on the left, with $B = (1-t)sN$. After rearranging, we get the bicomplex $\mathcal{B}C$ on the right side:



$B$ is called **Connes' boundary map** and fulfills $bB + Bb = 0$ (so $\mathcal{B}C$ really is a bicomplex). From (0.4) we get that $CC$ and $\mathcal{B}C$ have the same homology:

(4.4.1) $H_n(\mathcal{B}C) = H_n(\text{Tot } \mathcal{B}C_{**}) \cong HC_n(C) = H_n(\text{Tot } CC_{**})$

Let $CC^{\{2\}}$ denote the bicomplex consisting of the first two columns of $CC$, and $CC[2,0]$ the shifted bicomplex with $(CC[2,0])_{p,q} = CC_{p+2,q}$. Then we get a short exact sequence of bicomplexes:

$$0 \longrightarrow CC^{\{2\}} \xrightarrow{I} CC \xrightarrow{S} CC[2,0] \longrightarrow 0$$

Since the second column is contractible, we have $H_*(CC^{\{2\}}) \cong HH_*(C)$. This shows:

(4.5) **Theorem:** There are natural long exact sequences, called **Connes' Periodicity Exact Sequences**:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & HC_{n-1}(C) & \xrightarrow{B} & HH_n(C) & \xrightarrow{I} & HC_n(C) & \xrightarrow{S} & HC_{n-2}(C) & \xrightarrow{B} & HH_{n-1}(C) & \longrightarrow \cdots \\
& \cong \downarrow p & & & & \cong \downarrow p & & \cong \downarrow p & & & \\
\cdots \longrightarrow & H^\lambda_{n-1}(C) & \xrightarrow{B} & HH_n(C) & \xrightarrow{I} & H^\lambda_n(C) & \xrightarrow{S} & H^\lambda_{n-2}(C) & \xrightarrow{B} & HH_{n-1}(C) & \longrightarrow \cdots
\end{array}$$

$I$ is induced by inclusion, $B$ is Connes' boundary map, $S$ is called **periodicity map**.

(4.6) **Proposition:** There is a formula for the periodicity map $S : C^\lambda_n \to C^\lambda_{n-2}$. First, let $b^{[2]} : C_n \to C_{n-2}$ be given by:

- $b^{[2]} := \sum_{0 \le i < j \le n} (-1)^{i+j} d_i d_j$

Then $S : H^\lambda_n(C) \to H^\lambda_{n-2}(C)$ is induced by:

(4.6.1) $C_n \ni x \mapsto \frac{-1}{n(n-1)} b^{[2]}(x) \in C_{n-2}$

(4.7) **Corollary:** Let $C, C'$ be two cyclic modules. From (4.5) and the Five-Lemma we get:

(4.7.1) $HH_*(C) \cong HH_*(C') \Leftrightarrow HC_*(C) \cong HC_*(C')$

(4.8) **Definition:** The simplicial module $C_*(A)$ can be equipped with cyclic maps $t_n$ defined as follows:

- $t_n(a_0, \dots, a_n) := (-1)^n (a_n, a_0, \dots, a_{n-1})$

This makes $C_*(A)$ into a cyclic module, called **the cyclic module associated to** $A$, which is denoted by $A^\natural$. The **cyclic homology** $HC_*(A)$ of the algebra $A$ is defined to be the cyclic homology of the cyclic module $A^\natural$.

Let $C^\lambda_n(A) := A^{\otimes n+1}/(1-t)$. The Hochschild boundary map $b : C_n(A) \to C_{n-1}(A)$ drops to $b : C^\lambda_n(A) \to C^\lambda_{n-1}(A)$. In (4.3) we have shown that $H^\lambda_n(A) := H_n(C^\lambda_*(A), b)$ is isomorphic to $HC_n(A)$. Defining cyclic homology this way is sometimes called Connes' approach.

(4.9) **Explicit Formulas:** The map $I : HH_n(A) \to H^\lambda_n(A)$ is simply induced by the projection $A^{\otimes n+1} \to A^{\otimes n+1}/(1-t)$. From the definition $B = (1-t)sN$ we compute $B : A^{\otimes n+1} \to A^{\otimes n+2}$ as:

(4.9.1) $B(a_0, \dots, a_n) = \sum_{i=0}^{n-1} ((-1)^{ni}(1, a_i, \dots, a_n, a_0, \dots, a_{i-1}) - -(-1)^{n(i-1)}(a_{i-1}, 1, a_i, \dots, a_n, a_0, \dots, a_{i-2}))$

This induces maps $HH_n(A) \to HH_{n+1}(A)$, as well as $H^\lambda_n(A) \to HH_{n+1}(A)$ (all denoted by $B$). To define $S : H^\lambda_n(A) \to H^\lambda_{n-2}(A)$ use formula (4.6.1).

(4.10) **Proposition:** The basic properties of cyclic homology are:

1. The cyclic homology of the underlying ring $k$ can be computed as:

   (4.10.1) $HC_n(k) \cong \begin{cases} k & , n \text{ even} \\ 0 & , n \text{ odd} \end{cases}$

2. The 0-st cyclic homology group equals the 0-st Hochschild homology group:

   (4.10.2) $HC_0(A) \cong HH_0(A) \cong A/[A,A] = A_{ab}$

3. If $A$ is commutative, then there is a canonical isomorphism

(4.10.3) $\quad HC_1(A) \cong \Omega^1_{ab}(A)/(dA)$

where $\Omega^1_{ab}(A)$ is the module of 1-forms, defined in (6.14). See also (6.19) for a generalization.

4. Cyclic homology is Morita invariant.

5. For the product of two unital algebras $A, A'$ we get:

(4.10.4) $\quad HC_*(A \times A') \cong HC_*(A) \oplus HC_*(A')$

# 5 Cyclic Cohomology

**Assumptions:** Throughout this section $C$ denotes a cyclic $k$-module.

The definition of cyclic cohomology is as expected. We define it first for simplicial modules, and then for algebras.

(5.1) **Definition:** Let $C$ be a cyclic module. We dualize the cyclic bicomplex $CC_{**}$ to get a bicomplex of cochains $CC^{**}$ with $CC^{p,q} := \mathrm{Hom}_k(CC_{p,q}, k) = \mathrm{Hom}_k(C_q, k)$. The differentials are $b^*, b'^*, (1-t)^*$ and $N^*$. The cohomology $HC^n(C) := H^n(\mathrm{Tot}\ CC^{**})$ of this bicomplex is called the **cyclic cohomology** of $C$.

(5.2) **The bicomplex $\mathcal{B}^{**}C$:** We dualize $\mathcal{B}C$ to get the bicomplex $\mathcal{B}^{**}C$ with differentials $b^*$ and $B^*$. As in (4.4) we get:

(5.2.1) $\quad H^n(\mathcal{B}^{**}C) = H^n(\mathrm{Tot}\ \mathcal{B}^{**}C) \cong HC^n(C) = H^n(\mathrm{Tot}\ CC^{**})$

$$CC^{**}: \quad \begin{array}{ccccccc} \uparrow b^* & & \uparrow -b'^* & & \uparrow b^* & & \\ C^1 & \xrightarrow{(1-t)^*} & C^1 & \xrightarrow{N^*} & C^1 & \xrightarrow{(1-t)^*} & \\ \uparrow b^* & & \uparrow -b'^* & & \uparrow b^* & & \\ C^0 & \xrightarrow{(1-t)^*} & C^0 & \xrightarrow{N^*} & C^0 & \xrightarrow{(1-t)^*} & \end{array} \qquad \mathcal{B}^{**}C: \quad \begin{array}{ccccc} C^2 & \xrightarrow{B^*} & C^1 & \xrightarrow{B^*} & C^0 \\ \uparrow b^* & & \uparrow b^* & & \\ C^1 & \xrightarrow{B^*} & C^0 & & \\ \uparrow b^* & & & & \\ C^0 & & & & \end{array}$$

(5.3) **Definition:** The **cyclic cohomology** $HC^*(A)$ of $A$ is defined to be the cohomology of $A^\natural$. We also construct a complex $C^*_\lambda(A)$ (a subcomplex of $C^*(A)$) as follows: A cochain $f \in C^n(A) = \mathrm{Hom}_k(A^{\otimes n+1}, k)$ is said to be **cyclic** if: $(\forall\ a_0, \ldots, a_n \in A)$

(5.3.i) $\quad f(a_0, \ldots, a_n) = (-1)^n f(a_n, a_0, \ldots, a_{n-1})$

Then $\beta = b^*$ is a coboundary on $C^*_\lambda(A)$ and its cohomology is denoted $H^\lambda_*(A)$. We get $H^\lambda_*(A) \cong HC^*(A)$ canonically induced by inclusion.

(5.4) **Theorem: Connes' Periodicity Exact Sequence**

As in (4.5) we get natural long exact sequences (the dualized maps are still denoted $I, B, S$) :

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & HC^n(A) & \xrightarrow{I} & HH^n(A) & \xrightarrow{B} & HC^{n-1}(A) & \xrightarrow{S} & HC^{n+1}(A) & \xrightarrow{I} & HH^{n+1}(A) & \longrightarrow \cdots \\ & \cong \big| p & & & & \cong \big| p & & \cong \big| p & & & \\ \cdots \longrightarrow & H^n_\lambda(A) & \xrightarrow{I} & HH^n(A) & \xrightarrow{B} & H^{n-1}_\lambda(A) & \xrightarrow{S} & H^{n+1}_\lambda(A) & \xrightarrow{I} & HH^{n+1}(A) & \longrightarrow \cdots \end{array}$$

(5.5) **Proposition: Basic Properties of Cyclic Cohomology**

(5.5.1) $\quad HC^{2n}(k) \cong k, HC^{2n+1}(k) = 0$

(5.5.2) $\quad HC^0(A) = \{f : A \to k \mid f(ab) = f(ba)\} =$ traces on $A$

(5.5.3) $\quad$ Cyclic Cohomology is Morita invariant

# 6 Differential Calculus

We begin the discussion of differential calculus with the notion of a derivation. Every derivation gives rise to a differential calculus. In particular, the universal derivation $\Omega^1(A)$ of an algebra $A$ is used to construct the universal differential calculus $\Omega^*(A)$ of $A$. The non-commutative de Rham complex is the abelianization of $\Omega^*(A)$, and non-commutative de Rham homology $HDR_*(A)$ is the homology of that complex. We show the connection to the other homology theories of $A$. If $A$ is smooth (which includes being commutative), we have a canonical isomorphism $HC_n(A) \cong \Omega^n_{dR}(A)/(d\Omega^{n-1}_{dR}(A)) \oplus H^{n-2}_{dR}(A) \oplus H^{n-4}_{dR}(A) \oplus \ldots \oplus H^l_{dR}(A)$ where $l = 0$ or 1. This is the reason, why cyclic homology is considered to be the generalization of de Rham cohomology to non-commutative algebras.

## 6.1 Derivations

**Assumptions:** Throughout this section $M$ denotes an $A$-bimodule.

(6.1) **Definition:** A **derivation** from $A$ into $M$ is a $k$-linear map $D : A \to M$ s.t.: $(\forall\ a, b \in A)$

(6.1.i) $D(ab) = a(Db) + (Da)b$

We denote by $\mathrm{Der}(A, M)$ the set of all derivations from $A$ into $M$, and let $\mathrm{Der}(A) := \mathrm{Der}(A, A)$. For each $m \in M$ we define the derivation $\mathrm{ad}(m) : A \to M$ by $\mathrm{ad}(m)a = ma - am$. These are called **inner derivations**. We denote by $\mathrm{Der}'(A, M)$ the set of all inner derivations from $A$ into $M$.

(6.2) **Remark:** $\mathrm{Der}(A, M)$ is a $k$-module, $\mathrm{Der}'(A, M)$ a submodule. $\mathrm{Der}(A)$ has a natural Lie-algebra and $A$-module structure. $M$ is called symmetric iff $\mathrm{Der}'(A, M) = 0$. (i.e. if $am = ma\ \forall\ a \in A, m \in M$). If $A$ is commutative, $\mathrm{Der}(A, M)$ and $\mathrm{Der}'(A, M)$ have natural $A$-bimodule structures.

(6.3) **Definition:** A **universal derivation** (for $A$) is an $A$-bimodule $\Omega^1(A)$ together with a derivation $d : A \to \Omega^1(A)$ s.t. every derivation of $A$ factors uniquely through $\Omega^1(A)$, i.e. for every derivation $D : A \to M$ there exists a unique $A$-bimodule map $i_D : \Omega^1(A) \to M$ s.t. $D = i_D \circ d$.

(6.4) **Proposition:** The universal derivation of $A$ is unique (up to isomorphism) and is denoted by $\Omega^1(A)$ (or $\Omega^1_A$). It can be constructed as follows: (recall $\bar{A} = A/k$)

(1) $\Omega^1(A) \cong A \otimes \bar{A}$ with $A$-bimodule structure given by
$x(a \otimes \bar{b}) = xa \otimes \bar{b}$, $(a \otimes \bar{B})y = a \otimes \bar{by} - ab \otimes \bar{y}$
$d : a \mapsto 1 \otimes a$, or:

(2) $\Omega^1(A) \cong I := \ker(m : A \otimes A \to A) < A \otimes A$, where $m$ is just multiplication and the $A$-bimodule structure is the natural one; $d : a \mapsto 1 \otimes a - a \otimes 1$

One often writes $a_0 da_1$ for the element $a_0 \otimes a_1 \in A \otimes \bar{A}$ (resp. $a \otimes b - ab \otimes 1 \in I$).

## 6.2 The Differential Envelope and non-commutative de Rham Homology

We now turn to the notion of a differential calculus over an algebra $A$. For the case $A = C^\infty(V)$ ($V$ a smooth compact manifold) there is the well-known differential calculus of p-forms $\mathcal{A}^p(V)$ on $V$ with exterior derivative $d$. We generalize that to:

(6.5) **Definition:** A **differential graded algebra** (also **DG-algebra**) $(R^*, \delta)$ is a graded algebra $R^* = \bigoplus_{k\geq 0} R^k$ with graded product and graded differential $\delta$ of degree $+1$, i.e.

(6.5.i) $R^k R^l \subset R^{k+l}$

(6.5.ii) $\delta : R^k \to R^{k+1}, \delta^2 = 0$

(6.5.iii) $\delta(\omega_k \eta) = (\delta\omega_k)\eta + (-1)^k \omega_k \delta\eta \quad (\omega_k \in R^k, \eta \in R)$

A **differential calculus** over $A$ is a DG-algebra over $A$, i.e. a DG-algebra $R^*$ together with a homomorphism $\psi : A \to R^0$.

(6.6) **Remark:** Every derivation $D: A \to M$ gives rise to a differential calculus $\Omega_M^*(A)$ over $A$ as follows: Let $\bar{\Omega}_M^1(A) := \text{im } D$ (a k-submodule of $M$) and:

- $\Omega_M^0(A) := A$
- $\Omega_M^1(A) := \langle \bar{\Omega}_M^1(A) \rangle$ (the $A$-subbimodule of $M$ generated by $\bar{\Omega}_M^1(A)$)
- $\Omega_M^n(A) := \Omega_M^1(A) \otimes_k \bar{\Omega}_M^1(A) \otimes_k \ldots \otimes_k \bar{\Omega}_M^1(A)$ ((n-1)-times $\bar{\Omega}_M^1(A)$) $,n \geq 2$

From (6.1.i) we see that $\Omega_M^n(A) \cong \Omega_M^1(A) \otimes_A \ldots \otimes_A \Omega_M^1(A)$ (n-times). To shorten notation write $a_0 \otimes Da_1 \otimes \ldots \otimes Da_n$ as $a_0 Da_1 \ldots Da_n$. We make $\Omega_M^*(A)$ into a DG-algebra by defining:

- $D(a_0 Da_1 \ldots Da_n) = 1Da_0 Da_1 \ldots Da_n$
- $x(a_0 Da_1 \ldots Da_n) = (xa_0) Da_1 \ldots Da_n$
- $(a_0 Da_1 \ldots Da_n)y = a_0 Da_1 \ldots D(a_n y) + \sum_{i=1}^{n-1} (-1)^{n-i} a_0 Da_1 \ldots D(a_i a_{i+1}) \ldots Da_n +$ $+(-1)^n a_0 a_1 Da_2 \ldots Da_n$
- $(a_0 Da_1 \ldots Da_n)(b_0 Db_1 \ldots Db_n) = ((a_0 Da_1 \ldots Da_n) b_0) Db_1 \ldots Db_n$

(6.7) **Definition:** A **universal DG-algebra** over $A$ is a DG-algebra $(\Omega^*(A), d)$ over $A$ with $\Omega^0(A) = A$, s.t. every DG-algebra $(R^*, \delta)$ over $A$ gives rise to a unique map $\psi : \Omega^*(A) \to R^*$ of degree 0 intertwining $d$ and $\delta$ that extends the given $\psi : A = \Omega^0(A) \to R^0$.

(6.8) **Proposition:** The universal DG-algebra over $A$ is unique (up to isomorphism). It is called the **differential envelope** of $A$ and is denoted by $\Omega^*(A)$ (or $\Omega(A)$, $\Omega_A^*$). It can be constructed by applying the method in (6.9) to the universal derivation $\Omega^1(A)$. We get $\Omega^n(A) := \Omega^1(A) \otimes_A \ldots \otimes_A \Omega^1(A) \cong A \otimes \bar{A} \otimes \ldots \otimes \bar{A}$. We write an element of $\Omega^n(A)$ as $a_0 da_1 \ldots da_n$. It is universal because for another DG-algebra $(R^*, \delta)$ over $A$ we can set $\psi(a_0 da_1 \ldots da_n) := \psi(a_0)\delta(\psi(a_1)) \ldots \delta(\psi(a_n))$. The elements of $\Omega^n(A)$ are called non-commutative n-forms on $A$.

(6.9) **Proposition:** If $A$ can be decomposed as $A = \bar{A} \oplus k$ (for example if $k$ is a field), then $h : \Omega^n(A) \cong A \otimes \bar{A}^{\otimes n} \to \Omega^{n-1}(A) \cong A \otimes \bar{A}^{\otimes n-1}$ $,(a_0 + \lambda) \otimes a_1 \otimes \ldots \otimes a_n \mapsto \lambda a_1 \otimes a_2 \otimes \ldots \otimes a_n$ is a contracting homotopy for $d$ and the cohomology of the differential envelope is:

(6.9.1) $$H^i(\Omega^*(A), d) = \begin{cases} k & ,i = 0 \\ 0 & ,i \geq 1 \end{cases}$$

Thus, the following is an exact sequence:

$$0 \longrightarrow k \xrightarrow{\epsilon} A = \Omega^0(A) \xrightarrow{d} \Omega^1(A) \longrightarrow \cdots \xrightarrow{d} \Omega^n(A) \xrightarrow{d} \Omega^{n+1}(A) \longrightarrow \cdots$$

where $\epsilon : k \to A$ is given by $c \mapsto c1_A$.

We want to define the non-commutative de Rham homology of $A$. The previous proposition shows that the complex $(\Omega^*(A), d)$ is unsuitable. We use the abelianized differential envelope instead:

(6.10) **Definition:** Let $\Omega_{ab}^*(A) := \Omega^*(A)/[\Omega^*(A), \Omega^*(A)]$ be the quotient of $\Omega^*(A)$ by all graded commutators $[w_k, w_l] = w_k w_l + (-1)^{kl} w_l w_k$. Since $d$ is a graded differential, it is well-defined on $\Omega_{ab}^*(A)$. The **non-commutative de Rham homology** of $A$ is $HDR_n(A) := H^n(\Omega_{ab}^*(A), d)$.

(6.11) **Remark:** There is a nice connection between $HDR_*(A), HH_*(A)$ and $HC_*(A)$. The non-commutative de Rham homology $HDR_n(A)$ is essentially the kernel of the map $B : HC_n(A) \to HH_{n+1}(A)$ from (4.5) (or equivalently the image of $S : HC_{n+2}(A) \to HC_n(A)$). The precise statements need the notion of reduced homology theories and can be found in [Lo,92], theorem 2.6.7 and [Ro,94] theorem 6.1.40.

(6.12) **Remark:** In the literature one often finds another definition of the universal differential calculus over an algebra. One defines $\tilde{\Omega}^n(A) := \tilde{A} \otimes A^{\otimes n}$, where $\tilde{A} = A \oplus k$ is obtained from $A$ by adjoining a unit (even if $A$ is already unital). The advantage of this construction is that it works for non-unital algebras. The difference in the unital case is that $d1_A = 0$ in $\Omega^*(A)$ while $d1_A \neq 0$ in $\tilde{\Omega}^*(A)$. This comes down to the question of wether $1_A$ has to act as identity on $A$-(bi)modules. In $\tilde{A}^*(A)$ we do not assume that, so that in some sense this differential calculus is more universal than $\Omega^*(A)$.

## 6.3 The Commutative Case

**Assumptions:** Throughout this section, $A$ denotes a commutative $k$-algebra, $M$ an $A$-module.

We now turn to the case that $A$ is commutative. The notion of derivation and differential calculus is essentially the same as in the non-commutative case. But we consider $A$-modules instead of $A$-bimodules. Therefore the universal differential calculus is smaller than in the non-commutative setting. We define de Rham cohomology, which is not the same as non-commutative de Rham homology. For the class of smooth algebras we state some nice results that connect differential calculus and homology theories.

(6.13) **Definition:** If $A$ is commutative, then any $A$-module is in a natural way an $A$-bimodule. Thus, we can define derivations from $A$ into $M$ as in (6.1). They are $k$-linear maps $D : A \to M$ s.t. $(\forall\ a, b \in A)$

(6.13.i) $\quad D(ab) = a(Db) + b(Da)$

Every $A$-bimodule $N$ is also a module over $A$. In that case we have two different notions of a derivation from $A$ into $M$, namely in the sense of (6.1) or (6.13). If confusion is possible, one calls a derivation as in (6.13) also a commutative derivation.

(6.14) **Definition:** We define a **universal derivation** in the commutative case as in (6.3): as an $A$-module $\Omega^1_{ab}(A)$ together with a (commutative) derivation $d : A \to \Omega^1_{ab}(A)$ s.t. every (commutative) derivation of $A$ factors uniquely through $\Omega^1_{ab}(A)$ (via a unique $A$-module map).

(6.15) **Proposition:** The universal derivation $\Omega^1_{ab}(A)$ of $A$ is unique (up to isomorphism). It is universal for fewer objects (namely only for $A$-modules, instead of $A$-bimodules) and therefore smaller than $\Omega^1(A)$. We can construct $\Omega^1_{ab}(A)$ from $\Omega^1(A)$ as $\Omega^1_{ab}(A) = \Omega^1(A)/\langle am - ma | a \in A, m \in \Omega^1(A)\rangle$ which can be shown to equal $\Omega^1(A)/(\Omega^1(A))^2$. The elements of $\Omega^1_{ab}(A)$ are called **Kähler differentials**. Specifically we get:

(1) $\quad \Omega^1_{ab}(A) \cong I/I^2$, where $I = \ker(m : A \otimes A \to A) < A \otimes A$, and $m$ is just multiplication
$d : a \mapsto 1 \otimes a - a \otimes 1 + I^2$

Note that the $\Omega^1_{ab}(A)$ is the same as in (6.10), so the notation is not in conflict.

(6.16) **Definition:** Let $\Omega^0_{dR}(A) := A$, $\Omega^n_{dR}(A) := \Omega^1_{ab}(A) \wedge_A \ldots \wedge_A \Omega^1_{ab}(A)$ (n-times exterior product) for $n \geq 1$. We get a complex $(\Omega^*_{dR}(A), d)$, called **de Rham complex** of $A$:

$$\Omega^*_{dR}(A) : \qquad A = \Omega^0_{dR}(A) \xrightarrow{d} \Omega^1_{dR}(A) \longrightarrow \cdots \longrightarrow \Omega^n_{dR}(A) \xrightarrow{d} \Omega^{n+1}_{dR}(A) \longrightarrow \cdots$$

Its homology $H^n_{dR}(A) := H^n(\Omega^*_{dR}(A))$ is called **de Rham cohomology** of $A$. The elements of $\Omega^n_{dR}(A)$ are called n-forms on $A$.

For the case of a smooth, compact manifold $V$, the de Rham complex $\Omega^*_{dR}(C^\infty(V))$ can be identified with the well-known de Rham complex $\mathcal{A}^*(V)$ of differential forms on $V$. For a commutative $A$, the de Rham cohomology $H^*_{dR}(A)$ is not isomorphic to the non-commutative de Rham homology $HDR_*(A)$, as defined in (6.10). To understand the connection between the two we need first to define the notion of a smooth algebra:

(6.17) **Definition:** [Cu, 00] A commutative algebra $A$ is **smooth** if any homomorphism $\alpha : A \to B/N$ where $B$ is a commutative algebra and $N$ an ideal in $B$ with $N^2 = 0$ can be lifted to a homomorphism $\hat{\alpha} : A \to B$ s.t. $\pi \circ \hat{\alpha} = \alpha$ for the quotient map $\pi : B \to B/N$.

There are several other equivalent definitions of smoothness, see [Lo,92] proposition 3.4.2. For our purposes it is enough to know that the algebra $C^\infty(V)$ of smooth functions on a compact manifold is smooth.

We can now state the Hochschild-Kostant-Rosenberg theorem. It expresses the Hochschild homology of a smooth algebra as its de Rham complex.

(6.18) **Theorem: Hochschild-Kostant-Rosenberg (HKR) Theorem**:
For a smooth algebra $A$ we get a canonical isomorphism of graded algebras:
(6.18.1) $HH_*(A) \cong \Omega^*_{dR}(A)$

(6.19) **Theorem:** For a smooth algebra we get isomorphisms:
(6.19.1) $HC_{2n}(A) \cong \Omega^{2n}_{dR}(A)/(d\Omega^{2n-1}_{dR}(A)) \oplus H^{2n-2}_{dR}(A) \oplus H^{2n-4}_{dR}(A) \oplus \cdots \oplus H^0_{dR}(A)$
(6.19.2) $HC_{2n+1}(A) \cong \Omega^{2n+1}_{dR}(A)/(d\Omega^{2n}_{dR}(A)) \oplus H^{2n-1}_{dR}(A) \oplus H^{2n-3}_{dR}(A) \oplus \cdots \oplus H^1_{dR}(A)$
(6.19.3) $HDR_{2n}(A) \cong H^{2n}_{dR}(A) \oplus H^{2n-2}_{dR}(A) \oplus \cdots \oplus H^2_{dR}(A) \oplus (H^0_{dR}(A)/k)$
(6.19.4) $HDR_{2n+1}(A) \cong H^{2n+1}_{dR}(A) \oplus H^{2n-1}_{dR}(A) \oplus \cdots \oplus H^3_{dR}(A) \oplus H^1_{dR}(A)$
For a proof of this theorem and the HKR-theorem see [Lo,92] section 3.4. Alain Connes proved in [Co,82] the HKR-theorem for the case $A = C^\infty(V)$.

# 7 A generalized Chern character

In this section we give some applications using the generalized Chern character. We do not make that explicit or precise. It is just meant as a justification for developing cyclic (co)homology.

Let $X$ be a compact space. Then, to each vector bundle over $X$ one can assign elements (called Chern classes) in the de Rham homology of $X$. This extends to a ring homomorphism $ch : K^0(X) = K_0(C(X)) \to H^{ev}_{dR}(C(X)) := \prod_{n\geq 0} H^{2n}_{dR}(C(X))$, called the classical Chern character. One can generalize this to the case of general commutative, unital algebras, and gets $ch : K_0(A) \to H^{ev}_{dR}(A)$.

One can generalize this further. Assume $A$ is non-commutative. The domain $K_0(A)$ of $ch$ is still defined. The target $H^{ev}_{dR}(A)$ is not defined anymore, but we see from (6.19) that we should substitute $\prod_{n\geq 2} HC_{2n}(A)$ for it. And in fact one can define (non-trivial) maps:

- $ch_n : K_0(A) \to HC_{2n}(A)$

that have the nice property:
(7.0.5) $S \circ ch_n = ch_{n-1}$
where $S : HC_{2n}(A) \to HC_{2n-2}(A)$ is as in (4.5).

Using the generalized Chern character, one can prove a deep statement about idempotents in the reduced group $C^*$-algebra of free groups:

(7.1) **Theorem:** Let $F_n$ denote the free group with $n$ generators ($n = 1, 2, \ldots, \infty$), then $C^*_r(F_n)$ has no idempotents other than 0 and 1.

This was an unsolved problem for a long time. It was first solved by Pimsner and Voiculescu, but the proof using Chern characters is much easier compared to their proof. (see [Co,82])

Another application can be found in [Ri,87]. There, Marc Rieffel characterizes and constructs (finitely generated) projective modules over non-rational non-commutative tori $A_\theta$ using the generalized Chern character.

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