An Introduction to Hochschild and Cyclic Homology

Hannes Thiel


Abstract: We define Hochschild and cyclic (co)homology for simplicial and cyclic modules. The theory for an algebra $A$ is then obtained from the canonical simplicial/cyclic module $C_*(A)$ associated to this algebra. The definitions are purely algebraic. In the last sections we connect (co)homology theories and differential calculi of an algebra. Cyclic homology will be seen to be a natural generalization of de Rham cohomology to non-commutative algebras. As a justification for developing the machinery of cyclic homology, we give some applications using the generalized Chern character.

Notations: We consider modules and algebras over a commutative ring $k$ with identity. To simplify statements we assume that $k \subset k$. Throughout this paper, $A, A'$ denote unital $k$-algebras, $V$ a smooth compact manifold (without boundary), $\mathcal{C}^\infty(V)$ the smooth $\mathbb{C}$-valued functions on $V$.

Main References: A very readable introduction to the topic is [Lo,92]: "Cyclic Homology" by Jean-Louis Loday. Other good references are [Co,94]: "Noncommutative Geometry" by Alain Connes (which is available online for free at www.noncommutativegeometry.net) and [FVB,00]: "Elements of Noncommutative Geometry" by Joseph C. Varilly, Hector Figueroa and Jose M. Gracia-Bondia.

Contents

0 Preliminaries ii

1 Some Categories 1
1.1 The Pre-Simplicial Category $\Delta^{pre}$ 1
1.2 The Simplicial Category $\Delta$ 1
1.3 The Cyclic Category $\Lambda$ 2
1.4 Simplicial and Cyclic Modules 2

2 Hochschild Homology 3

3 Hochschild Cohomology 4

4 Cyclic Homology 5

5 Cyclic Cohomology 7

6 Differential Calculus 8
6.1 Derivations 8
6.2 The Differential Envelope and non-commutative de Rham Homology 8
6.3 The Commutative Case 10

7 A generalized Chern character 11

A References 12
0 Preliminaries

We recall some concepts from homological algebra, such as the notion of a bicomplex, which is used to define cyclic homology.

(0.1) Remark: For a $k$-algebra $A$, $A^o$ denotes the opposite algebra (where $a \cdot b = ba$), $A^e = A \otimes A^o$ is the enveloping algebra of $A$. One sets $A^e = \Hom_k(A,k)$, $\bar{A} = A \oplus k$ and if $A$ is unital, $\bar{A} = A/k$. By $\otimes$ is always meant tensoring over $k$, i.e. $\otimes = \otimes_k$. Every $A$-bimodule (especially $\bar{A}$ itself) is in a natural way a $A^e$-module.

(0.2) Definition: A bicomplex $C_{\bullet \bullet}$ (or $C$ for short) is a collection of modules $C_{p,q}, p, q \in \mathbb{Z}$ with two differentials $d^h : C_{p,q} \to C_{p-1,q}$ and $d^v : C_{p,q} \to C_{p,q-1}$, called horizontal and vertical differential, s.t. $d^h d^h = d^v d^v = d^h d^v + d^v d^h = 0$. The total complex associated to $C$ is $(\text{Tot} \, C)_n = \prod_{p+q=n} C_{p,q}$ and $d = d^h + d^v$ (note that $\prod_{p+q=n} C_{p,q} \cong \bigoplus_{p+q=n} C_{p,q}$ if $C_{p,q} \neq 0$ for only finitely many $p + q = n$). The homology groups $H_\bullet(C)$ of the bicomplex $C$ are defined to be $H_n(C) := H_n(\text{Tot} \, C)$.

\[
\begin{array}{cccccc}
\cdots & C_{p,q} & C_{p,q+1} & \cdots \\
\downarrow & d^v & d^h & \downarrow \\
\cdots & C_{p+1,q} & C_{p+1,q+1} & \cdots \\
\end{array}
\]

(0.3) Definition: A complex $(C, d)$ is called contractible with contracting homotopy $h : C_n \to C_{n+1}$ if $dh + hd = \text{id}$ (i.e. id and 0 are chain homotopic via $h$). Every contractible complex is acyclic (i.e. $H_\bullet(C) = 0$)

(0.4) Proposition: \cite{Lo,92} Killing contractible subcomplexes

Let $(A_\bullet \oplus A'_\bullet, d)$ be a complex with $d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : A_n \oplus A'_n \to A_{n-1} \oplus A'_{n-1}$, s.t. $(A'_\bullet, \delta)$ is a contractible complex with contracting homotopy $h : A'_n \to A'_{n+1}$. Then the following inclusion of complexes is a quasi-isomorphism (i.e. it induces the identity on homology):

$$(\text{id} - h\gamma) : (A_\bullet, \alpha - \beta h\gamma) \to (A_\bullet \oplus A'_\bullet, d)$$

(0.5) Definition: Bar Complex

Let $A$ be an algebra. The bar complex $C^{\text{bar}}(A)$ of $A$ is the complex of $A^e$-modules $C^{\text{bar}}_n(A) := A^{\otimes n+2}, n \geq 0$ with boundary map $b' : C^{\text{bar}}_n \to C^{\text{bar}}_{n-1}, b'(a_0, \ldots, a_{n+1}) = \sum_{i=0}^n (-1)^i (a_0, \ldots, a_1 a_{i+1}, \ldots, a_{n+1})$.

(0.6) Proposition: If $A$ is unital, then $C^{\text{bar}}(A)$ is a resolution of $A$ as $A^e$-module. The augmentation $\mu : C^{\text{bar}}_\bullet(A) \to A$ is simply multiplication.

\[
C^{\text{bar}}_n(A) \xrightarrow{\mu} A : \quad \cdots \xrightarrow{b'} A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2} \xrightarrow{\mu} A \longrightarrow 0
\]

A contracting homotopy is given by $s : C^{\text{bar}}_n(A) \to C^{\text{bar}}_{n+1}(A), s(a_0, \ldots, a_{n+1}) = (1, a_0, \ldots, a_{n+1})$. (i.e. $sb' + b's = \text{id}$). The map $s$ is also called the extra degeneracy map.
1 Some Categories

In this section we recall the pre-simplicial and simplicial categories $\Delta^{\text{pre}}$ and $\Delta$. A contravariant functor from $\Delta$ into the category of $k$-modules is called a simplicial $k$-module and is all we need to define Hochschild (co)homology (which is done in sections 2 and 3). Then, we define the cyclic category $\Lambda$, which was introduced by Alain Connes. Contravariant functors from $\Lambda$ into the category of $k$-modules are called cyclic $k$-modules and are used to define cyclic (co)homology (see sections 4 and 5).

1.1 The Pre-Simplicial Category $\Delta^{\text{pre}}$

(1.1) Definition: The pre-simplicial category $\Delta^{\text{pre}}$ is the small category of finite, totally ordered sets and increasing functions. That means:

- The objects are $[n] = \{0 < 1 \ldots < n\}$, $n = 0, 1, \ldots$
- $\text{Mor}_{\Delta^{\text{pre}}}(\mathbb{1}, [n]) = \{ f : \mathbb{1} \rightarrow [n] \mid f(0) < f(1) < \ldots < f(m) \}$

In $\text{Mor}_{\Delta^{\text{pre}}}(\mathbb{1}, [n])$ one has the special elements $\delta_i$ ($i = 0, \ldots, n + 1$), called face maps. These maps are characterized by “skipping” the $i$-th element, i.e. $i \notin \text{im}(\delta_i)$:

- $\delta_i : [n] \rightarrow [n + 1]$ \quad $\delta_i(x) = \begin{cases} x & \text{if } x < i \\ x + 1 & \text{if } x \geq i \end{cases}$

(1.2) Proposition: The face maps fulfill the following relation:

$(1.2.1)$ \quad $\delta_i \delta_j = \delta_j \delta_{i-1}$ if $i < j$

Further, if $m > n$, then $\text{Mor}_{\Delta^{\text{pre}}}(\mathbb{1}, [n]) = \emptyset$. If $m \leq n$ and $f \in \text{Mor}_{\Delta^{\text{pre}}}(\mathbb{1}, [n])$, then there are unique $i_1 \leq \ldots \leq i_r$ s.t. $m = n + r$ and $f = \delta_{i_1} \ldots \delta_{i_r}$ (if $r = 0$, this is understood to be $\text{id}_{[n]}$). This means that the $\delta_i$ give a presentation of $\Delta^{\text{pre}}$.

1.2 The Simplicial Category $\Delta$

(1.3) Definition: The simplicial category $\Delta$ is the small category of finite, totally ordered sets and non-decreasing functions. That means:

- The objects are $[n] = \{0 < 1 \ldots < n\}$, $n = 0, 1, \ldots$
- $\text{Mor}_{\Delta}(\mathbb{1}, [n]) = \{ f : \mathbb{1} \rightarrow [n] \mid f(0) \leq f(1) \leq \ldots \leq f(m) \}$

Besides the face maps $\delta_i$ from (1.1) one has in $\text{Mor}_{\Delta}(\mathbb{1}, [n])$ the special elements $\sigma_j$ ($j = 0, \ldots, n - 1$), called degeneracy maps. These maps are characterized by “repeating” (only) the $j$-th element once, i.e. $\sigma_j(j) = \sigma_{j+1}(j)$:

- $\sigma_j : [n] \rightarrow [n - 1]$ \quad $\sigma_j(x) = \begin{cases} x & \text{if } x \leq j \\ x + 1 & \text{if } x > j \end{cases}$

(1.4) Proposition: The face and degeneracy maps fulfill the following relations:

$(1.4.1)$ \quad $\delta_i \delta_j = \delta_j \delta_{i-1}$ if $i < j$

$(1.4.2)$ \quad $\sigma_i \sigma_j = \sigma_{i+1} \sigma_j$ if $i \leq j$

$(1.4.3)$ \quad $\sigma_i \delta_j = \begin{cases} \delta_{i-1} \sigma_j & \text{if } i > j + 1 \\ \text{id}_{[n]} & \text{if } i = j \text{ or } i = j + 1 \\ \delta_{i-1} \sigma_j & \text{if } i < j \end{cases}$

If $f \in \text{Mor}_{\Delta}(\mathbb{1}, [n])$, then there are unique $i_1 \leq \ldots \leq i_r, j_1 \leq \ldots < j_s$ s.t. $m = n + r - s$ and $f = \delta_{i_1} \ldots \delta_{i_r} \sigma_{j_1} \ldots \sigma_{j_s}$. This means that the $\delta_i, \sigma$ give a presentation of $\Delta$. 


1.3 The Cyclic Category \( \Lambda \)

(1.5) **Definition:** The cyclic category \( \Lambda \) is a small category with objects the finite, totally ordered sets \([n]\). The maps are defined as follows [Co.94]: Identify \( \mathbb{Z}_{n+1} \) with the \((n+1)\)-th roots of unity in \( S^1 \). We fix an orientation on \( S^1 \). This gives the notion of a non-decreasing function \( f : S^1 \to S^1 \). Maps from \([m]\) to \([n]\) are defined as homotopy classes of continuous non-decreasing functions \( f : S^1 \to S^1 \) of degree 1 with \( f(\mathbb{Z}_{m+1}) \subset \mathbb{Z}_{n+1} \).

One might need some time to get acquainted with this definition. There are helpful pictures in section III.A.3 of [Connes,94]. When we interpret the elements of \( \text{Mor}_\Lambda([m],[n]) \) as maps \([m] \to [n] \), then \( \Delta \) is a subcategory of \( \Lambda \). But besides the face and degeneracy maps from (1.1) and (1.3) one has in \( \text{Mor}_\Lambda([n],[n]) \) the special elements \( \tau_n \), called cyclic maps:

- \( \tau_n : [n] \to [n] \) if \( x \geq 1 \)
- \( \tau_n(x) = \begin{cases} x-1 & \text{if } x \geq 1 \\ n & \text{if } x = 0 \end{cases} \)

(1.6) **Proposition:** The face, degeneracy and cyclic maps fulfill the following relations:

\[
\begin{align*}
(1.6.1) \quad & \delta_j \delta_i = \delta_i \delta_{j-1} & \text{if } i < j \\
(1.6.2) \quad & \sigma_j \sigma_i = \sigma_i \sigma_{j+1} & \text{if } i \leq j \\
(1.6.3) \quad & \delta_j \delta_i = \begin{cases} \text{id}_{[n]} & \text{if } i = j \text{ or } i = j + 1 \\
\delta_{j-1} \sigma_j & \text{if } i > j + 1 \\
\end{cases} \\
(1.6.4) \quad & \tau_n \delta_i = \delta_{i-1} \tau_{n-1} & \text{if } 1 \leq i \leq n \\
(1.6.5) \quad & \tau_n \sigma_i = \sigma_{i-1} \tau_{n+1} & \text{if } 1 \leq i \leq n \\
(1.6.6) \quad & \tau_1^{n+1} = \text{id}_{[n]} \\
\end{align*}
\]

If \( f \in \text{Mor}_\Lambda([n],[n]) \), then there are unique \( \tilde{f} \in \text{Mor}_\Lambda([m],[n]) \) and \( \epsilon \in \mathbb{Z}_{m+1} \) s.t. \( f = \tilde{f} \tau_m^n \). This means that \( \Lambda \) can be "decomposed" into \( \Delta \) and a collection of cyclic groups \( \mathbb{Z}_{m+1} \) being the automorphisms (on each \([n]\)). One writes \( \Lambda = \Delta \mathbb{C} \) to indicate this decomposition.

(1.7) **Remark:** One can generalize this idea and form other categories \( \Delta \mathbb{G} \), where \( \mathbb{G} \) stands for a sequence of groups \( G_n \), each being the automorphism group of \([n]\). It is required that each morphism in \( \Delta \mathbb{G} \) can be written uniquely as an element of \( \Delta \) and an element of \( G_n \). One calls \( G \) a crossed simplicial group, and defines (co)homology theories for it. There is a complete classification of crossed simplicial groups, among them are the families of trivial, cyclic, dihedral and quaternionic groups, giving rise to Hochschild, cyclic, dihedral and quaternionic homology. See [FLo,91] and section 6.3 of [Lo,92].

1.4 Simplicial and Cyclic Modules

(1.8) **Definition:** Let \( \mathcal{C} \) be a category. A pre-simplicial (simplicial, cyclic) object in \( \mathcal{C} \) is a contravariant functor \( X : \Delta^{\text{pre}} \to \mathcal{C} \) \((X : \Delta \to \mathcal{C}, X : \Lambda \to \mathcal{C})\). A pre-simplicial (simplicial, cyclic) set (space) is such an object in the category of sets (spaces).

(1.9) **Definition:** A simplicial (cyclic) \( k \)-module is a family of \( k \)-modules \( C_n, n \geq 0 \) together with \( k \)-linear face maps \( d_i : C_n \to C_{n-1}, i = 0, \cdots, n \), degeneracy maps \( s_j : C_n \to C_{n+1}, j = 0, \cdots, n \), and for cyclic modules also cyclic maps \( t_n : C_n \to C_n \) that fulfill the relations (i)-(iii) for a simplicial module and (i)-(vi) for a cyclic module:

\[
\begin{align*}
(1.9.i) \quad & d_id_j = d_{j-1}d_i & \text{if } i < j \\
(1.9.ii) \quad & s_is_j = s_{j+1}s_i & \text{if } i < j \\
(1.9.iii) \quad & d_is_j = \begin{cases} \text{id}_{C_n} & \text{if } i = j \text{ or } i = j + 1 \\
\sigma_j \tau_i & \text{if } i > j + 1 \\
\end{cases} \\
(1.9.iv) \quad & d_it_n = -t_{n-1}d_i & \text{if } 1 \leq i \leq n \\
(1.9.v) \quad & s_it_n = -t_{n+1}s_i & \text{if } 1 \leq i \leq n \\
(1.9.vi) \quad & t_n^{n+1} = \text{id}_{C_n} \\
\end{align*}
\]
Definition: Let $A$ be a simplicial module. Define $b : C_n \to C_{n-1}$ as:

- $b = \sum_{i=0}^{n} (-1)^i d_i$

This defines a boundary map on $C$ (i.e. $b^2 = 0$). The complex $(C_*, b)$ is called the Hochschild complex (associated to $C$). Its homology $H_n(C_*, b)$ is called the Hochschild homology of $C$.

Proposition: Let us quickly give some basic properties of Hochschild homology:

1. The Hochschild homology of the underlying ring $k$ can be computed as:

\[
HH_n(k) \cong \begin{cases} k & n = 0 \\ 0 & n \geq 1 \end{cases}
\]

2. The 0-st Hochschild homology group is the symmetrization of $M$, i.e. the biggest quotient of $M$ that is a symmetric bimodule:

\[
H_0(A, M) \cong M/\langle am - ma : a \in A, m \in M \rangle
\]

Thus, if $M$ is symmetric, then $H_0(A, M) = M$. We also get $HH_0(A) = A/[A, A] = A_{ab}$.

3. If $A$ is commutative, then there is a canonical isomorphism

\[
H_1(A, M) \cong M \otimes_A \Omega^1_{ab}(A)
\]

where $\Omega^1_{ab}(A)$ is the module of 1-forms, defined in (6.14). See also (6.18) for a generalization.

4. If $A$ is projective, then $C_n^{\text{bar}}(A)$ is a projective resolution of $A$, and remark (2.3) shows:

\[
H_n(A, M) \cong \text{Tor}_n^A(M, A)
\]

5. For the product of two unital algebras $A, A'$ we get:

\[
HH_*(A \times A') \cong HH_*(A) \otimes HH_*(A')
\]

6. Hochschild homology is functorial, i.e. $H_*(A, M)$ is a covariant functor in $M$, and $HH_*(A)$ is a covariant functor in $A$. 

2 Hochschild Homology

Assumptions: Throughout this section $M$ denotes an $A$-bimodule, $C$ a simplicial $k$-module.

We first define Hochschild homology $H_*(C)$ for a simplicial $k$-module $C$. Given an algebra $A$ and a $A$-bimodule $M$, we then define the simplicial $k$-module $C_*(A, M)$, whose homology $H_*(A, M)$ is the Hochschild homology of $M$ with coefficients in $M$.

(2.1) Definition: Let $C$ be a simplicial module. Define $b : C_n \to C_{n-1}$ as:

- $b = \sum_{i=0}^{n} (-1)^i d_i$

This defines a boundary map on $C$ (i.e. $b^2 = 0$). The complex $(C_*, b)$ is called the Hochschild complex (associated to $C$). Its homology $H_n(C_*, b)$ is called the Hochschild homology of $C$.

(2.2) Definition: For an algebra $A$ (algebras are assumed to be unital throughout the paper) and an $A$-bimodule $M$ we define $C_n(A, M) := M \otimes A^{\otimes n}$ and let:

- $d_0(m, a_1, \ldots, a_n) = (ma_1, a_2, \ldots, a_n)$
- $d_i(m, a_1, \ldots, a_n) = (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \quad i = 1, \ldots, n-1$
- $d_n(m, a_1, \ldots, a_n) = (a_n m, a_1, \ldots, a_{n-1})$
- $s_j(m, a_1, \ldots, a_n) = (m, a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n) \quad j = 0, \ldots, n$

This makes $C_*(A, M)$ into a simplicial module (i.e. the formulas (1.8.i) to (1.8.iii) are fulfilled), called the simplicial module associated to $A$. The Hochschild homology $H_*(A, M)$ of $A$ with coefficients in $M$ is now defined to be the homology of $C_*(A, M)$. We let $HH_*(A) := H_*(A, A)$.

(2.3) Remark: One can define Hochschild homology using the tensored bar complex $M \otimes_{A^e} C_0^{\text{bar}}(A)$. The boundary map of $M \otimes_{A^e} C_0^{\text{bar}}(A)$ is understood as $1_M \otimes b'$. Since the tensoring over $A^e$ kills the leftmost and rightmost occurrence of $A$, we have $M \otimes_{A^e} C_0^{\text{bar}}(A) \cong C_0(A, M)$, and under this identification $b$ of $C_*(A, M)$ corresponds to $1 \otimes b'$ of $M \otimes_{A^e} C_0^{\text{bar}}(A)$. So $H_n(A, M) \cong H_n(M \otimes_{A^e} C_0^{\text{bar}}(A))$.

(2.4) Proposition: Let us quickly give some basic properties of Hochschild homology:

1. The Hochschild homology of the underlying ring $k$ can be computed as:

\[
HH_n(k) \cong \begin{cases} k & n = 0 \\ 0 & n \geq 1 \end{cases}
\]

2. The 0-st Hochschild homology group is the symmetrization of $M$, i.e. the biggest quotient of $M$ that is a symmetric bimodule:

\[
H_0(A, M) \cong M/\langle am - ma : a \in A, m \in M \rangle
\]

Thus, if $M$ is symmetric, then $H_0(A, M) = M$. We also get $HH_0(A) = A/[A, A] = A_{ab}$.

3. If $A$ is commutative, then there is a canonical isomorphism

\[
H_1(A, M) \cong M \otimes_A \Omega^1_{ab}(A)
\]

where $\Omega^1_{ab}(A)$ is the module of 1-forms, defined in (6.14). See also (6.18) for a generalization.

4. If $A$ is projective, then $C_n^{\text{bar}}(A)$ is a projective resolution of $A$, and remark (2.3) shows:

\[
H_n(A, M) \cong \text{Tor}_n^A(M, A)
\]

5. For the product of two unital algebras $A, A'$ we get:

\[
HH_*(A \times A') \cong HH_*(A) \otimes HH_*(A')
\]

6. Hochschild homology is functorial, i.e. $H_*(A, M)$ is a covariant functor in $M$, and $HH_*(A)$ is a covariant functor in $A$. 

One important property of Hochschild and cyclic (co)homology is their Morita invariance, i.e. they cannot distinguish two Morita equivalent algebras. The precise statement is as follows:

(2.5) **Theorem:** Let \( A, A' \) be unital \( k \)-algebras that are Morita equivalent via the \((A, A')\)-bimodule \( P \) and the \((A', A)\)-bimodule \( Q \), and let \( M \) be an \( A \)-bimodule. Then:

\[
H_*(A, M) \cong H_*(A', Q \otimes_A M \otimes_A P)
\]

For the case of matrices we get: Let \( M_r(A) \) (resp. \( M_r(M) \)) denote the \((r, r)\)-matrices over \( A \) (resp. \( M \)) \((r = 1, 2, \ldots, \infty)\). Then \( A \) and \( M_r(A) \) are Morita equivalent, so

\[
H_* (A, M) \cong H_* (M_r(A), M_r(M)) \quad r = 1, 2, \ldots, \infty
\]

This isomorphism is induced by the trace and inclusion map (they induce maps inverse to each other). See also section 1.2 of [Lo92].

### 3 Hochschild Cohomology

**Assumptions:** Throughout this section \( M, M' \) denote \( A \)-bimodules, \( C \) a simplicial \( k \)-module.

The definition of Hochschild cohomology is now straightforward. We define it first for simplicial modules, and then for algebras with coefficients in a bimodule.

(3.1) **Definition:** The **Hochschild cohomology** \( H^*(C) \) of a simplicial module \( C \) is defined to be the cohomology of \( \text{Hom}_k(C_*, k) \) where \((C_*, b)\) is the Hochschild complex associated to \( C \).

(3.2) **Definition:** The Hochschild cohomology of \( A \) with coefficients in \( M \) is **not** defined as cohomology of \( \text{Hom}(C_*(A, M), k) \) but as follows: Define \( C^n(A, M) := \text{Hom}_k(A^\otimes n, M) \) and let: (for \( f : A^\otimes n \to M \))

- \((\beta f)(a_1, \ldots, a_{n+1}) = a_1f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n}(-1)^i f(a_1, \ldots, a_ia_{i+1}, \ldots, a_{n+1}) + (-1)^n f(a_1, \ldots, a_n)a_{n+1}\)

The **Hochschild cohomology** of \( A \) with coefficients in \( M \) is defined to be \( H^n(C^*(A, M), \beta) \). We let \( C^n(A) := C^n(A, A^*) \cong \text{Hom}_k(A^\otimes n+1, k) \) and \( HH^n(A) := H^n(A, A^*) \).

(3.3) **Remark:** As in (2.3), one can define Hochschild cohomology using the bar complex of \( A \). An \( A^e \)-linear map \( \phi : C^n_{\text{bar}}(A) = A^\otimes n+2 \to M \) can be identified with a \( k \)-linear map \( f : A^\otimes n \to M \) via \( \phi(a_0, \ldots, a_{n+1}) = a_0f(a_1, \ldots, a_n)a_{n+1} \). So \( C^n(A, M) \cong \text{Hom}_{A^e}(C^n_{\text{bar}}(A), M) \). The coboundary maps correspond and we have \( H^n(A, M) \cong H^n(\text{Hom}_{A^e}(C^n_{\text{bar}}(A), M)) \).

(3.4) **Proposition:** Let us collect basic properties of Hochschild cohomology:

1. The Hochschild cohomology of the underlying ring \( k \) is:

   \[
   HH^n(k) \cong \begin{cases} k & n = 0 \\ 0 & n \geq 1 \end{cases}
   \]

2. The 0-st Hochschild cohomology group is:

   \[
   H^0(A, M) \cong \langle am - ma | a \in A, m \in M \rangle
   \]

3. The first Hochschild cohomology group is exactly the group of outer derivations of \( A \) in \( M \) (see (6.2)):

   \[
   H^1(A, M) \cong \text{Der}(A, M)/\text{Der}^e(A, M)
   \]

4. If \( A \) is projective, then \( C^n_{\text{bar}}(A) \) is a projective resolution of \( A \), and remark (3.3) shows:

   \[
   H^n(A, M) \cong \text{Ext}_{A^e}^n(M, A)
   \]

5. Hochschild cohomology is functorial, i.e. \( H^*(A, M) \) is a covariant functor in \( M \), and \( HH^*(A) \) is a covariant functor in \( A \).

6. Hochschild cohomology is Morita invariant.
4 Cyclic Homology

Assumptions: Throughout this section $C$ denotes a cyclic $k$-module.

As for Hochschild homology, we first work in a rather abstract setting (namely for a cyclic $k$-module $C$). We define three complexes $CC_\ast$, $BC_\ast$ and $C^\lambda_\ast$, that all give the same homology, called cyclic homology. Then, we associate to an algebra $A$ a cyclic $k$-module $A^\natural$, whose homology is the cyclic homology $HC_\ast(A)$ of $A$.

(4.1) Definition: Let $C$ be a cyclic module. The associated cyclic bicomplex $CC$ is:

\[
\begin{array}{cccc}
C_n & C_{n-1} & \ldots & C_0 \\
\downarrow b & \downarrow -b' & \downarrow b & \downarrow -b' \\
C_n & C_{n-1} & \ldots & C_0 \\
\downarrow b & \downarrow -b' & \downarrow b & \downarrow -b' \\
C_n & C_{n-1} & \ldots & C_0 \\
\end{array}
\]

where $b : C_n \to C_{n-1}, b = \sum_{i=0}^n (-1)^i d_i$, $b' : C_n \to C_{n-1}, b' = \sum_{i=0}^{n-1} (-1)^i d_i$. Its homology $H_\ast(CC) := H_\ast(\text{Tot} CC)$ is called the cyclic homology of $C$ and denoted $HC_\ast(C)$. That $CC$ is a bicomplex means that $N(1-t) = (1-t)N = 0$ (which is obvious) and also $b^2 = b'^2 = 0, (1-t)b' = b(1-t)$ (which needs some formal calculations).

(4.2) Definition: The complex $(C^\lambda_\ast, b)$ where $C^\lambda_n := C_n / (1 - t_n)$ is called Connes’s complex. The boundary map is well-defined since $(1 - t_{n-1})b' = b(1 - t_n)$. The homology of this complex is denoted $H^\lambda_\ast(C)$.

\[
\begin{array}{cccc}
C^\lambda_n & C^\lambda_{n-1} & \ldots & C^\lambda_0 \\
\downarrow b & \downarrow b & \downarrow b \\
C^\lambda_n & C^\lambda_{n-1} & \ldots & C^\lambda_0 \\
\downarrow b & \downarrow b & \downarrow b \\
C^\lambda_n & C^\lambda_{n-1} & \ldots & C^\lambda_0 \\
\end{array}
\]

(4.3) Proposition: Let $CC^{(1)}$ denote the first column of the bicomplex $CC$. Let further $p : \text{Tot} CC \to CC^{(1)}/(1-t) \cong C^\lambda_1$ be the map which is zero on all columns except the first one, where it is the quotient map $CC^{(1)} \to CC^{(1)}/(1-t)$. Then $p$ is a quasi-isomorphism, i.e. $H^\lambda_\ast(C) \cong HC_\ast(C)$ canonically.

(4.4) The bicomplex $BC$: In $CC$, the columns with vertical differential $b'$ are contractible. The contracting homotopy is $s = s_{n+1} : C_n \to C_{n+1}, s = (-1)^n t_n s_n$. (i.e. $sb' + b's = \text{id}$) This map is called extra degeneracy. Applying proposition (4.4) to each of these columns we get the bicomplex on the left, with $B = (1-t)sN$. After rearranging, we get the bicomplex $BC$ on the right side:
B is called Connes’ boundary map and fulfills $bB + Bb = 0$ (so $BC$ really is a bicomplex). From (0.4) we get that $CC$ and $BC$ have the same homology:

\[(4.1) \quad H_n(BC) = H_n(Tot BC_*) \cong HC_n(C) = H_n(Tot CC_*)\]

Let $CC^{[2]}$ denote the bicomplex consisting of the first two columns of $CC$, and $CC^{[2,0]}$ the shifted bicomplex with $(CC^{[2,0]})_{p,q} = CC_{p+2,q}$. Then we get a short exact sequence of bicomplexes:

\[0 \rightarrow CC^{(2)} \xrightarrow{I} CC \xrightarrow{S} CC^{[2,0]} \rightarrow 0\]

Since the second column is contractible, we have $H_*(CC^{[2]}) \cong H_*(C)$. This shows:

\[\textbf{Theorem:} \quad \text{There are natural long exact sequences, called Connes’ Periodicity Exact Sequences:}\]

\[\cdots \rightarrow HC_{n-1}(C) \xrightarrow{B} HH_{n-1}(C) \xrightarrow{I} HC_n(C) \xrightarrow{S} HC_{n-2}(C) \xrightarrow{B} HH_{n-1}(C) \rightarrow \cdots\]

\[\xrightarrow{\cong p} \quad \xrightarrow{\cong p} \quad \xrightarrow{\cong p}\]

$I$ is induced by inclusion, $B$ is Connes’ boundary map, $S$ is called periodicity map.

\[\textbf{Proposition:} \quad \text{There is a formula for the periodicity map } S : C_n^\lambda \rightarrow C_{n-2}^\lambda. \text{ First, let } b^{[2]} : C_n \rightarrow C_{n-2} \text{ be given by:}\]

\[b^{[2]} := \sum_{0 \leq i < j \leq n} (-1)^{i+j}d_id_j\]

Then $S : H_n^\lambda(C) \rightarrow H_{n-2}^\lambda(C)$ is induced by:

\[(4.6.1) \quad C_n \ni x \mapsto \frac{-1}{n+1} b^{[2]}(x) \in C_{n-2}\]

\[\textbf{Corollary:} \quad \text{Let } C, C’ \text{ be two cyclic modules. From (4.5) and the Five-Lemma we get:}\]

\[(4.7.1) \quad HH_*(C) \cong HH_*(C’) \Leftrightarrow HC_*(C) \cong HC_*(C’)\]

\[\textbf{Definition:} \quad \text{The simplicial module } C_*(A) \text{ can be equipped with cyclic maps } t_n \text{ defined as follows:}\]

\[t_n(a_0, \ldots, a_n) := (-1)^n (a_n, a_0, \ldots, a_{n-1})\]

This makes $C_*(A)$ into a cyclic module, called the cyclic module associated to $A$, which is denoted by $A^\lambda$. The cyclic homology $HC_*(A)$ of the algebra $A$ is defined to be the cyclic homology of the cyclic module $A^\lambda$.

Let $C_n^\lambda(A) := A^\otimes n+1/(1 - t)$. The Hochschild boundary map $b : C_n(A) \rightarrow C_{n-1}(A)$ drops to $b : C_n^\lambda(A) \rightarrow C_{n-1}^\lambda(A)$. In (4.3) we have shown that $H_n^\lambda(A) := H_n(C_n^\lambda(A), b)$ is isomorphic to $HC_n(A)$. Defining cyclic homology this way is sometimes called Connes’ approach.

\[\textbf{Explicit Formulas:} \quad \text{The map } I : HH_n(A) \rightarrow H_n^\lambda(A) \text{ is simply induced by the projection } A^\otimes n+1 \rightarrow A^\otimes n+1/(1 - t). \text{ From the definition } B = (1 - t)sN \text{ we compute } B : A^\otimes n+1 \rightarrow A^\otimes n+2 \text{ as:}\]

\[(4.9.1) \quad B(a_0, \ldots, a_n) = \sum_{i=0}^{n-1} ((-1)^n (1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1}) - (-1)^{n+i}(a_n, a_0, \ldots, a_i, \ldots, a_{n-1}))\]

This induces maps $HH_n(A) \rightarrow HH_{n+1}(A)$, as well as $H_n^\lambda(A) \rightarrow HH_{n+1}(A)$ (all denoted by $B$). To define $S : H_n^\lambda(A) \rightarrow H_{n-2}^\lambda(A)$ use formula (4.6.1).

\[\textbf{Proposition:} \quad \text{The basic properties of cyclic homology are:}\]

1. The cyclic homology of the underlying ring $k$ can be computed as:

\[(4.10.1) \quad HC_n(k) = \begin{cases} k & \text{n even} \\ 0 & \text{n odd} \end{cases}\]

2. The 0-st cyclic homology group equals the 0-st Hochschild homology group:

\[(4.10.2) \quad HC_0(A) \cong HH_0(A) \cong A/[A, A] = A_{ab}\]
3. If $A$ is commutative, then there is a canonical isomorphism
\[(4.10.3) \quad HC_1(A) \cong \Omega^1_{ab}(A)/(dA)\]
where $\Omega^1_{ab}(A)$ is the module of 1-forms, defined in (6.14). See also (6.19) for a generalization.

4. Cyclic homology is Morita invariant.

5. For the product of two unital algebras $A, A'$ we get:
\[(4.10.4) \quad HC_\ast(A \times A') \cong HC_\ast(A) \otimes HC_\ast(A')\]

## 5 Cyclic Cohomology

**Assumptions:** Throughout this section $C$ denotes a cyclic $k$-module.

The definition of cyclic cohomology is as expected. We define it first for simplicial modules, and then for algebras.

**Definition:** Let $C$ be a cyclic module. We dualize the cyclic bicomplex $CC_{\ast\ast}$ to get a bicomplex of cochains $CC_{\ast\ast}$ with $CC^{p,q} := \text{Hom}_k(CC_{p,q}, k) = \text{Hom}_k(C_q, k)$. The differentials are $b^\ast, b'^\ast, (1-t)^\ast$ and $N^\ast$. The cohomology $HC^n(C) := H^n(Tot CC_{\ast\ast})$ of this bicomplex is called the cyclic cohomology of $C$.

**The bicomplex $B^{\ast\ast}C$:** We dualize $BC$ to get the bicomplex $B^{\ast\ast}C$ with differentials $b^\ast$ and $B'^\ast$. As in (4.4) we get:
\[(5.2.1) \quad H^n(B^{\ast\ast}C) = H^n(Tot B^{\ast\ast}C) \cong HC^n(C) = H^n(Tot CC_{\ast\ast})\]

**Definition:** The cyclic cohomology $HC_\ast(A)$ of $A$ is defined to be the cohomology of $A^2$. We also construct a complex $C^\ast_\lambda(A)$ (a subcomplex of $C^\ast(A)$) as follows: A cochain $f \in C^n(A) = \text{Hom}_k(A^\otimes n+1, k)$ is said to be cyclic if: $(\forall a_0, \ldots, a_n \in A)$
\[(5.3.1) \quad f(a_0, \ldots, a_n) = (-1)^n f(a_n, a_0, \ldots, a_{n-1})\]

Then $\beta = b^\ast$ is a coboundary on $C^\ast_\lambda(A)$ and its cohomology is denoted $H^\lambda_\ast(A)$. We get $H^\lambda_\ast(A) \cong HC^\ast(A)$ canonically induced by inclusion.

**Theorem: Connes' Periodicity Exact Sequence**
As in (4.5) we get natural long exact sequences (the dualized maps are still denoted $I, B, S$):
\[
\begin{array}{cccccccc}
\cdots & H^{n}(A) & \xrightarrow{I} & H^{n+1}(A) & \xrightarrow{B} & H^{n+1}(A) & \xrightarrow{S} & H^{n+1}(A) & \xrightarrow{I} & H^{n+1}(A) & \cdots \\
\cong & \p & \cong & \p & \cong & \p & \\
\cdots & H^\lambda_n(A) & \xrightarrow{I} & H^\lambda_{n+1}(A) & \xrightarrow{B} & H^\lambda_{n+1}(A) & \xrightarrow{S} & H^\lambda_{n+1}(A) & \xrightarrow{I} & H^\lambda_{n+1}(A) & \cdots \\
\end{array}
\]

**Proposition: Basic Properties of Cyclic Cohomology**
\[(5.5.1) \quad HC^{2n}(k) \cong k, HC^{2n+1}(k) = 0\]
\[(5.5.2) \quad HC^0(A) = \{ f : A \to k \mid f(ab) = f(ba) \} = \text{traces on } A\]
\[(5.5.3) \quad \text{Cyclic Cohomology is Morita invariant}\]
6 Differential Calculus

We begin the discussion of differential calculus with the notion of a derivation. Every derivation gives rise to a differential calculus. In particular, the universal derivation $Ω_1(A)$ of an algebra $A$ is used to construct the universal differential calculus $Ω^*(A)$ of $A$. The non-commutative de Rham complex is the abelianization of $Ω^*(A)$, and non-commutative de Rham homology $HDR_n(A)$ is the homology of that complex. We show the connection to the other homology theories of $A$. If $A$ is smooth (which includes being commutative), we have a canonical isomorphism $HC_n(A) ∼= Ω^n_{DR}(A) / (dΩ^{n-1}_{DR}(A) ⊕ H^{n-2}_{DR}(A) ⊕ H^{n-3}_{DR}(A) ⊕ \ldots ⊕ H^1_{DR}(A))$ where $l = 0$ or $1$. This is the reason, why cyclic homology is considered to be the generalization of de Rham cohomology to non-commutative algebras.

6.1 Derivations

Assumptions: Throughout this section $M$ denotes an $A$-bimodule.

(6.1) **Definition:** A derivation from $A$ into $M$ is a $k$-linear map $D : A → M$ s.t.: $(\forall a, b ∈ A)$

\[ D(ab) = a(Db) + (Da)b \]

We denote by Der$(A,M)$ the set of all derivations from $A$ into $M$, and let Der$(A) :=$ Der$(A,A)$. For each $m ∈ M$ we define the derivation ad$(m) : A → M$ by ad$(m)a = ma - am$. These are called inner derivations. We denote by Der'$A(M)$ the set of all inner derivations from $A$ into $M$.

(6.2) **Remark:** Der$(A,M)$ is a $k$-module, Der'$A(M)$ a submodule. Der$(A)$ has a natural Lie-algebra and $A$-module structure. $M$ is called symmetric iff Der'$A(M) = 0$. (i.e. if $am = ma \forall a ∈ A, m ∈ M$). If $A$ is commutative, Der$(A,M)$ and Der'$A(M)$ have natural $A$-bimodule structures.

(6.3) **Definition:** A universal derivation (for $A$) is an $A$-bimodule $Ω^1(A)$ together with a derivation $d : A → Ω^1(A)$ s.t. every derivation of $A$ factors uniquely through $Ω^1(A)$, i.e. for every derivation $D : A → M$ there exists a unique $A$-bimodule map $i_D : Ω^1(A) → M$ s.t. $D = i_D ∘ d$.

(6.4) **Proposition:** The universal derivation of $A$ is unique (up to isomorphism) and is denoted by $Ω^1(A)$ (or $Ω^1_A$). It can be constructed as follows: (recall $\bar{A} = A/k$)

\[ (1) \quad Ω^1(A) ∼= A ⊗ \bar{A} \text{ with } A\text{-bimodule structure given by} \]
\[ x(a ⊗ b) = xa ⊗ b, (a ⊗ \bar{b})y = a ⊗ by - ab ⊗ \bar{y} \]
\[ d : a → 1 ⊗ a, \text{ or:} \]
\[ (2) \quad Ω^1(A) ∼= I := \ker(m : A ⊗ A → A) ⊆ A ⊗ A, \text{ where } m \text{ is just multiplication and the } A\text{-bimodule structure is the natural one; } d : a → 1 ⊗ a - a ⊗ 1 \]

One often writes $a_0a_1d$ for the element $a_0 ⊗ a_1 ∈ A ⊗ A$ (resp. $a ⊗ b - ab ⊗ 1 ∈ I$).

6.2 The Differential Envelope and non-commutative de Rham Homology

We now turn to the notion of a differential calculus over an algebra $A$. For the case $A = C^∞(V)$ ($V$ a smooth compact manifold) there is the well-known differential calculus of $p$-forms $\mathcal{A}^p(V)$ on $V$ with exterior derivative $d$. We generalize that to:

(6.5) **Definition:** A differential graded algebra (also DG-algebra) $(R^*, δ)$ is a graded algebra $R^* = \bigoplus_{k≥0} R^k$ with graded product and graded differential $δ$ of degree $+1$, i.e.

\[ (6.5.i) \quad R^k R^l ⊆ R^{k+l} \]
\[ (6.5.ii) \quad δ : R^k → R^{k+1}, δ^2 = 0 \]
\[ (6.5.iii) \quad δ(ωδη) = δωδη + (-1)^kωδη (ω, η ∈ R^k, δη) \]

A differential calculus over $A$ is a DG-algebra over $A$, i.e. a DG-algebra $R^*$ together with a homomorphism $ψ : A → R^0$. 

8
Remark: Every derivation \( D : A \to M \) gives rise to a differential calculus \( \Omega^\ast_M(A) \) over \( A \) as follows: Let \( \Omega^1_M(A) := \text{im } D \) (a \( k \)-submodule of \( M \)) and:

- \( \Omega^0_M(A) := A \)
- \( \Omega^1_M(A) := \Omega^1_M(A) \) (the \( A \)-subbimodule of \( M \) generated by \( \Omega^1_M(A) \))
- \( \Omega^n_M(A) := \Omega^n_M(A) \otimes_k \Omega^{n-1}_M(A) \otimes_k \cdots \otimes_k \Omega^1_M(A) \) (\( n \)-times) for \( n \geq 2 \).

From (6.1) we see that \( \Omega^n_M(A) \cong \Omega^n_M(A) \otimes_A \cdots \otimes_A \Omega^1_M(A) \) (\( n \)-times). To shorten notation write \( a_0 \otimes Da_1 \otimes \cdots \otimes Da_n \) as \( a_0 Da_1 \cdots Da_n \). We make \( \Omega^n_M(A) \) into a DG-algebra by defining:

- \( D(a_0 Da_1 \cdots Da_n) = 1 Da_0 Da_1 \cdots Da_n \)
- \( x(a_0 Da_1 \cdots Da_n) = (xa_0) Da_1 \cdots Da_n \)
- \( (a_0 Da_1 \cdots Da_n)y = a_0 Da_1 \cdots D(a_n y) + \sum_{i=1}^{n-1} (-1)^{n-i} a_0 Da_1 \cdots D(a_ia_{i+1}) \cdots Da_n + (-1)^n a_0 a_1 Da_2 \cdots Da_n \)
- \( (a_0 Da_1 \cdots Da_n)(b_0 Db_1 \cdots Db_n) = ((a_0 Da_1 \cdots Da_n)b_0) Db_1 \cdots Db_n \)

Definition: A universal DG-algebra over \( A \) is a DG-algebra \( (\Omega^\ast(A), d) \) over \( A \) with \( \Omega^0(A) = A \), such that in some sense this differential calculus is more universal than \( \Omega^*_{\ast M}(A) \).

Proposition: The universal DG-algebra over \( A \) is unique (up to isomorphism). It is called the differential envelope of \( A \) and is denoted by \( \Omega^\ast(A) \) (or \( \Omega(A), \Omega^\ast_M \)). It can be constructed by applying the method in (6.9) to the universal derivation \( \bar{\Omega}^\ast(A) \). We get \( \Omega^\ast(A) := \Omega^1(A) \otimes_A \cdots \otimes_A \Omega^1(A) \cong A \otimes \cdots \otimes A \). We write an element of \( \Omega^\ast(A) \) as \( a_0 Da_1 \cdots Da_n \). It is universal because for another DG-algebra \( (\Omega^\ast, \delta) \) over \( A \) we can set \( \psi(a_0 Da_1 \cdots Da_n) := \psi(a_0) \delta(\psi(a_1)) \cdots \delta(\psi(a_n)) \). The elements of \( \Omega^\ast(A) \) are called non-commutative n-forms on \( A \).

Proposition: If \( A \) can be decomposed as \( A = \tilde{A} \oplus k \) (for example if \( k \) is a field), then \( h : \Omega^n(A) \cong A \otimes A^\otimes n \to \Omega^{n+1}(A) \cong \Omega(A) \otimes A^\otimes n-1 \).\( (a_0 + \lambda) \otimes a_1 \cdots \otimes a_n \mapsto \lambda a_1 \otimes a_2 \otimes \cdots \otimes a_n \) is a contracting homotopy for \( d \) and the cohomology of the differential envelope is:

\[
H^i(\Omega^\ast(A), d) = \begin{cases} k, & i = 0 \\ 0, & i \geq 1 \end{cases}
\]

Thus, the following is an exact sequence:

\[
0 \to k \xrightarrow{\epsilon} A \xrightarrow{d} \Omega^1(A) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(A) \xrightarrow{d} \Omega^{n+1}(A) \to \cdots
\]

where \( \epsilon : k \to A \) is given by \( c \mapsto cl_A \).

We want to define the non-commutative de Rham homology of \( A \). The previous proposition shows that the complex \( (\Omega^\ast(A), d) \) is unsuitable. We use the abelianized differential envelope instead:

Definition: Let \( \Omega^\ast_{ab}(A) := \Omega^\ast(A)/[\Omega^\ast(A), \Omega^\ast(A)] \) be the quotient of \( \Omega^\ast(A) \) by all graded commutators \( [w_k, w_l] = w_k w_l + (-1)^{kl} w_l w_k \). Since \( d \) is a graded differential, it is well-defined on \( \Omega^\ast_{ab}(A) \). The non-commutative de Rham homology of \( A \) is \( HDR_n(A) := H^n(\Omega^\ast_{ab}(A), d) \).

Remark: There is a nice connection between \( HDR_n(A), HH_n(A) \) and \( HC_n(A) \). The non-commutative de Rham homology \( HDR_n(A) \) is essentially the kernel of the map \( B : HC_n(A) \to HH_{n+1}(A) \) from (4.5) (or equivalently the image of \( \bar{B} : HC_{n+2}(A) \to HC_n(A) \)). The precise statements need the notion of reduced homology theories and can be found in [Lo,92], theorem 2.6.7 and [Ro,94] theorem 6.1.40.

Remark: In the literature one often finds another definition of the universal differential calculus over an algebra. One defines \( \hat{\Omega}^\ast(A) := \hat{A} \otimes A^\otimes n \), where \( \hat{A} = A \oplus k \) is obtained from \( A \) by adjoining a unit (even if \( A \) is already unital). The advantage of this construction is that it works for non-unital algebras. The difference in the unital case is that \( d1_A = 0 \) in \( \Omega^\ast(A) \) while \( d1_A \neq 0 \) in \( \hat{\Omega}^\ast(A) \). This comes down to the question of whether \( 1_A \) has to act as identity on \( A \)-(bi)modules. In \( \hat{A}^\ast(A) \) we do not assume that, so that in some sense this differential calculus is more universal than \( \Omega^\ast(A) \).
6.3 The Commutative Case

**Assumptions:** Throughout this section, $A$ denotes a commutative $k$-algebra, $M$ an $A$-module.

We now turn to the case that $A$ is commutative. The notion of derivation and differential calculus is essentially the same as in the non-commutative case. But we consider $A$-modules instead of $A$-bimodules. Therefore the universal differential calculus is smaller than in the non-commutative setting. We define de Rham cohomology, which is not the same as non-commutative de Rham homology. For the class of smooth algebras we state some nice results that connect differential calculus and homology theories.

(6.13) **Definition:** If $A$ is commutative, then any $A$-module is in a natural way an $A$-bimodule. Thus, we can define derivations from $A$ into $M$ as in (6.1). They are $k$-linear maps $D : A \to M$ s.t. $(\forall a, b \in A)$

(6.13.i) $D(ab) = a(Db) + b(Da)$

Every $A$-bimodule $N$ is also a module over $A$. In that case we have two different notions of a derivation from $A$ into $M$, namely in the sense of (6.1) or (6.13). If confusion is possible, one calls a derivation as in (6.13) also a commutative derivation.

(6.14) **Definition:** We define a universal derivation in the commutative case as in (6.3): as an $A$-module $\Omega^1_{ab}(A)$ together with a (commutative) derivation $d : A \to \Omega^1_{ab}(A)$ s.t. every (commutative) derivation of $A$ factors uniquely through $\Omega^1_{ab}(A)$ (via a unique $A$-module map).

(6.15) **Proposition:** The universal derivation $\Omega^1_{ab}(A)$ of $A$ is unique (up to isomorphism). It is universal for fewer objects (namely only for $A$-modules, instead of $A$-bimodules) and therefore smaller than $\Omega^1_1(A)$. We can construct $\Omega^1_{ab}(A)$ from $\Omega^1(A)$ as $\Omega^1_{ab}(A) = \Omega^1(A)/(am - ma | a \in A, m \in \Omega^1(A))$ which can be shown to equal $\Omega^1(A)/(\Omega^1(A))^2$. The elements of $\Omega^1_{ab}(A)$ are called Kähler differentials. Specifically we get:

(1) $\Omega^1_{ab}(A) \cong I/I^2$, where $I = \ker(m : A \otimes A \to A) < A \otimes A$, and $m$ is just multiplication $d : a \mapsto 1 \otimes a - a \otimes 1 + I^2$.

Note that the $\Omega^1_{ab}(A)$ is the same as in (6.10), so the notation is not in conflict.

(6.16) **Definition:** Let $\Omega^*_{dR}(A) := A$, $\Omega^1_{dR}(A) := \Omega^1_{ab}(A) \wedge \ldots \wedge A \Omega^1_{ab}(A)$ (n-times exterior product) for $n \geq 1$. We get a complex $(\Omega^*_{dR}(A), d)$, called de Rham complex of $A$:

$$\Omega^*_{dR}(A) : \quad A = \Omega^0_{dR}(A) \xrightarrow{d} \Omega^1_{dR}(A) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^m_{dR}(A) \xrightarrow{d} \Omega^{m+1}_{dR}(A) \xrightarrow{d} \ldots$$

Its homology $H^*_{dR}(A) := H^*(\Omega^*_{dR}(A))$ is called de Rham cohomology of $A$. The elements of $\Omega^n_{dR}(A)$ are called n-forms on $A$.

For the case of a smooth, compact manifold $V$, the de Rham complex $\Omega^*_dR(C^\infty(V))$ can be identified with the well-known de Rham complex $\mathcal{A}^*(V)$ of differential forms on $V$. For a commutative $A$, the de Rham cohomology $H^*_dR(A)$ is not isomorphic to the non-commutative de Rham homology $HDR_*(A)$, as defined in (6.10). To understand the connection between the two we need first to define the notion of a smooth algebra:

(6.17) **Definition:** [Cu, 00] A commutative algebra $A$ is smooth if any homomorphism $\alpha : A \to B/N$ where $B$ is a commutative algebra and $N$ an ideal in $B$ with $N^2 = 0$ can be lifted to a homomorphism $\hat{\alpha} : A \to B$ s.t. $\pi \circ \hat{\alpha} = \alpha$ for the quotient map $\pi : B \to B/N$.

There are several other equivalent definitions of smoothness, see [Lo,92] proposition 3.4.2. For our purposes it is enough to know that the algebra $C^\infty(V)$ of smooth functions on a compact manifold is smooth.

We can now state the Hochschild-Kostant-Rosenberg theorem. It expresses the Hochschild homology of a smooth algebra as its de Rham complex.
(6.18) **Theorem:** Hochschild-Kostant-Rosenberg (HKR) Theorem:

For a smooth algebra $A$ we get a canonical isomorphism of graded algebras:

$$HH_*(A) \cong \Omega^*_{dR}(A)$$

(6.19) **Theorem:** For a smooth algebra we get isomorphisms:

1. $H_{2n}(A) \cong \Omega^*_{dR}(A)/(d\Omega^*_{dR}(A)) \oplus H^*_{dR}(A)$
2. $H_{2n+1}(A) \cong \Omega^*_{dR}(A)/d\Omega^*_{dR}(A)$
3. $H_{2n+1}(A) \cong H^*_{dR}(A) \oplus H^*_{dR}(A)$
4. $H_{2n+1}(A) \cong H^*_{dR}(A)$

For a proof of this theorem and the HKR-theorem see [Lo,92] section 3.4. Alain Connes proved in [Co,82] the HKR-theorem for the case $A = C^\infty(V)$.

## 7 A generalized Chern character

In this section we give some applications using the generalized Chern character. We do not make that explicit or precise. It is just meant as a justification for developing cyclic (co)homology.

Let $X$ be a compact space. Then, to each vector bundle over $X$ one can assign elements (called Chern classes) in the de Rham homology of $X$. This extends to a ring homomorphism $ch : K_0(X) = K_0(C(X)) \to H^*_{dR}(C(X)) := \prod_{n \geq 0} H^*_{dR}(C(X))$, called the classical Chern character. One can generalize this to the case of general comutative, unital algebras, and gets $ch : K_0(A) \to H^*_{dR}(A)$.

One can generalize this further. Assume $A$ is non-commutative. The domain $K_0(A)$ of $ch$ is still defined. The target $H^*_{dR}(A)$ is not defined anymore, but we see from (6.19) that we should substitute $\prod_{n \geq 2} H^*_{dR}(A)$ for it. And in fact one can define (non-trivial) maps:

- $ch_n : K_0(A) \to H_{2n}(A)$

that have the nice property:

$$(7.0.5) \quad S \circ ch_n = ch_{n-1}$$

where $S : H_{2n}(A) \to H_{2n-2}(A)$ is as in (4.5).

Using the generalized Chern character, one can prove a deep statement about idempotents in the reduced group $C^*$-algebras of free groups:

(7.1) **Theorem:** Let $F_n$ denote the free group with $n$ generators ($n = 1, 2, \ldots, \infty$), then $C^*_r(F_n)$ has no idempotents other than 0 and 1.

This was an unsolved problem for a long time. It was first solved by Pimsner and Voiculescu, but the proof using Chern characters is much easier compared to their proof. (see [Co,82])

Another application can be found in [Ri,87]. There, Marc Rieffel characterizes and constructs (finitely generated) projective modules over non-rational non-commutative tori $A_\theta$ using the generalized Chern character.
A References


[Cu,00] J. Cuntz, Cyclic Theory and the Bivariant Chern-Connes Character, preprint, *SFB 478, Münster*


