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Definable Henselian Valuations and Absolute Galois Groups

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Abstract

This thesis investigates the connections between henselian valuations and absolute Galois groups. There are fundamental links between these: On one hand, the absolute Galois group of a field often encodes information about (henselian) valuations on that field. On the other, in many cases a henselian valuation imposes a certain structure on an absolute Galois group which makes it easier to study.

We are particularly interested in the question of when a field admits a non-trivial parameter-free definable henselian valuation. By a result of Prestel and Ziegler, this does not hold for every henselian valued field. However, improving a result by Koenigsmann, we show that there is a non-trivial parameter-free definable valuation on every henselian valued field. This allows us to give a range of conditions under which a henselian field does indeed admit a non-trivial parameter-free definable *henselian* valuation. Most of these conditions are in fact of a Galois-theoretic nature. Since the existence of a parameter-free definable henselian valuation on a field ensures that henselianity is elementary in $\mathcal{L}_{ring}$, we also study henselianity as an $\mathcal{L}_{ring}$-property.

Throughout the thesis, we discuss a number of applications of our results. These include fields elementarily characterized by their absolute Galois group, model complete henselian fields and henselian NIP fields of positive characteristic, as well as PAC and hilbertian fields.

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Introduction

Valued fields play a fundamental role in several different mathematical disciplines, most notably in number theory and algebraic geometry. This has led to considerable model-theoretic interest in valuations. A valuation assigns a certain size or multiplicity to elements of a field, and is a generalization of the notion of an absolute value.

At the end of the 19th century, Kurt Hensel was the first to introduce and study the p-adic numbers, and hence implicitly formed the idea of a valuation ([Hen97]). The first formal definition of a valuation (with values in the additive group of the real numbers) came from Jószef Kürschák ([Kür13]). Wolfgang Krull later developed the notion of a valuation as we know it today ([Kru30]). From the model-theoretic point of view, allowing arbitrary ordered abelian value groups is vital – clearly, being a subgroup of the additive group of the real numbers is not preserved under elementary equivalence in the language of ordered groups.

On the model-theoretic side, valued fields were already studied by Abraham Robinson. He proved that the theory of algebraically closed valued fields (ACVF) is model complete in the language of valued fields, i.e. the language of fields together with a symbol for the valuation ring ([Rob56]). In fact, it follows from his proof that ACVF also eliminates quantifiers in this language. Since then, a lot of work has been done on the model theory of ACVF; in particular, using the advanced model-theoretic technique of stable domination ([HHM08]). There is plenty of ongoing research in this area, not least because model-theoretic results on ACVF give information about quantifier-free definable sets in any valued field.

A major breakthrough in a different area of the model theory of valued fields came in the 1960's with the work of James Ax and Simon Kochen ([AK65]), as well as independently obtained results by Yuri Eršov ([Erš65]). Motivated by the p-adics, they studied the elementary theory of henselian valued fields. Being henselian is a natural notion of completeness for valuations and roughly speaking requires polynomials which have a point 'close enough to zero' to actually have a zero in the field. This is a key property of the p-adic numbers. They found an axiomatisation of the p-adic numbers and proved that their theory is decidable. One can deduce from their work – and Eršov also proves this explicitely – that, in the case that the residue field has characteristic 0, the theory of a henselian valued field in the language of valued fields only depends on the elementary theory of the residue field (as a pure field) and the value group (as an ordered abelian group). As a consequence, decidability of the valued field depends only on the decidability of those simpler structures.

Since these revolutionary results, much more work has been done on the model theory of henselian valued fields. Mostly, they were studied in the language of valued fields or even with additional structure. In the language of valued fields, to be henselian turns out to be an elementary property of a valuation, although it can not be axiomatized by finitely many sentences. Inspired by the work of Ax, Kochen and Eršov, Angus Macintyre considered the p-adics in a richer language, namely the language of valued fields together with predicates for n^{th} powers for all natural numbers n. He proved that the p-adic numbers have quantifier elimination with respect to this language ([Mac76]). Another frequently used language is the Denef-Pas language. It consists of sorts for the field, the residue field and the value group (in their natural languages) together with a valuation map from the field to the value group and an angular component map from the field to the residue field. Henselian valued fields with value group $\mathbb{Z}$ and residue characteristic 0 admit relative quantifier elimination in this language ([Pas89]). The Cluckers-Loeser approach to motivic integration considers integrals over Denef-Pas definable sets ([CL08]) in henselian valued fields of characteristic 0. In contrast, the Hrushovski-Kazdan approach to motivic integration uses the RV-language ([HK06]).

There are several areas of current research following the Ax-Kochen/Eršov theorem in some way or another. On the one hand, the Ax-Kochen/Eršov theorem has been generalized to wider settings, for example by Franz-Viktor Kuhlmann ([Kuh09]). On the other hand, François Delon showed that henselian valued fields of equicharacteristic 0 (in the language of valued fields) have relative quantifier elimination down to the residue field and the value group (in their respective languages). Delon also showed that such a field has the independence property if and only if the residue field does ([Del81]). This result has since been adapted to NTP_2 instead of NIP by Artem Chernikov ([Che12]), and to some henselian valued fields of positive characteristic by Itay Kaplan, Thomas Scanlon and Frank Wagner ([KSW11]).

Studying henselian valued fields in the language of rings started with a paper on topological fields by Alexander Prestel and Martin Ziegler ([PZ78]). They showed that a field is elementarily equivalent to a (non-trivially) henselian valued field in the language of rings if and only if it admits a field topology with a definable basis which 'looks like' the topology coming from a henselian valuation. Furthermore, they gave an example of a (pure) field elementarily equivalent to a henselian valued field which does itself not admit a non-trivial henselian valuation.

This leads naturally to the question of whether and when (henselian) valuations are definable in a field using only the language of rings. By the results of Prestel and Ziegler, there are henselian fields which admit no parameter-free definable non-trivial henselian valuations. Jochen Koenigsmann showed that every henselian valued field admits a definable valuation which induces the same topology as the topology coming from any non-trivial henselian valuation on the field ([Koe94]). There is also considerable recent interest in the definability of henselian valuations. In a yet unpublished joint article by Raf Cluckers, Jamshid Derakhshan, Eva Leenknegt and Angus Macintyre, they study uniform definitions of the p-adic integers in the p-adic numbers ([CDLM12]). Jizhan Hong has shown in a preprint that under certain restrictions on the value group, a henselian valuation on a field is definable in the language of rings ([Hon12]).

Applications of definable valuations include a result by Robert Rumely, showing that any two global fields are elementarily equivalent if and only if they are isomorphic ([Rum80]).

The key step in the proof of this theorem is that the ring of integers is uniformly definable in any global field. Pop's conjecture states that in fact any two finitely generated fields are elementarily equivalent if and only if they are isomorphic, and Florian Pop has recently shown that this holds at least for finitely generated fields of Kronecker dimension 1 or 2 ([Pop11]). An interesting application of definable henselian valuations is Jochen Koenigsmann's proof of the birational section conjecture for the p-adic numbers ([Koe05a]), for which no number-theoretic proof is known yet. Definable henselian valuations also play an important role in Koenigsmann's classification of fields elementarily characterized by their absolute Galois group ([Koe04]).

The Present Work

In the first chapter, we introduce some prerequisites from infinite Galois theory and valuation theory and describe their interactions. Only towards the end, we cover less well-known ground: We define the canonical henselian valuation and state a theorem from [Koe03] which gives a purely Galois-theoretic condition for a field to carry a non-trivial henselian valuation.

In the second chapter, we study different notions of henselianity. We start by considering p-henselian valuations, specifically canonical p-henselian valuations and their definability. We give a generalization of a theorem by Koenigsmann by showing the following

Corollary 2.1.6. *Let $p > 2$ and consider the class of fields*

$$\mathcal{K}_p := \{\, K \mid K \neq K(p) \text{ and if } \operatorname{char}(K) \neq p \text{ then } \zeta_p \in K \,\}.$$

Then, there is a parameter-free $\mathcal{L}_{ring}$-formula ϕ_p such that $\phi_p(K)$ defines the valuation ring of the canonical p-henselian valuation in any p-henselian $K \in \mathcal{K}_p$.

The notation is explained in section 2.1. We deduce that for a fixed prime $p > 2$, the canonical p-henselian valuation is uniformly definable. Note that a similar but weaker result holds for $p = 2$ (Theorem 2.1.5).

The next section introduces t-henselianity, a topological condition first studied in [PZ78] which holds exactly in all fields elementarily equivalent to henselian fields. Note that separably and real closed fields play a special role – every real closed field and every algebraically closed field of positive characteristic is elementarily equivalent to both a henselian and a non-henselian field. All other separably closed fields are henselian. Most importantly, neither separably nor real closed fields admit any non-trivial definable valuations. Thus, we work under the general assumption that our fields are neither separably nor real closed. However, Prestel and Ziegler give an example of a t-henselian field which is not henselian and not separably or real closed, so henselianity is in general not an $\mathcal{L}_{ring}$-elementary property. Using Corollary 2.1.6, we show:

Theorem 2.2.2. *Any t-henselian field K carries a $\emptyset$-definable valuation which induces the (unique) henselian topology on K.*

Next, we quote some known conditions under which henselianity becomes elementary and give several new ones, all under purely Galois-theoretic assumptions. We show that if K is t-henselian, and G_K is pro-p or pro-nilpotent, then K is henselian (Proposition 2.2.4 and Proposition 2.2.5). In order to generalize these, we use a definition from profinite group theory.

Definition. *Let G be a profinite group. We say that G is* universal *if every finite group occurs as a continuous subquotient of G.*

Combining this with a theorem by Koenigsmann, we then obtain the following

Corollary 2.2.9. *Let G_K be non-universal, and assume that K is t-henselian. Then K is henselian.*

In the last part of the chapter, we show that neither PAC nor hilbertian fields can be non-trivially p-henselian. Finally, we give an explicit sentence which separates the theory of a non p-closed PAC field from any such henselian field (Theorem 2.3.9).

The third chapter contains results on definable henselian valuations. If a field carries a non-trivial parameter-free definable henselian valuation, then henselianity becomes an elementary property. Thus, not every henselian field admits a non-trivial parameter-free definable henselian valuation. Apart from separably or real closed fields, there are no known examples of henselian fields which – allowing parameters – do not admit a definable non-trivial henselian valuation. We use our definitions of p-henselian valuations to define henselian valuations. First, we give conditions on the residue field Kv of a non-trivially henselian valued field (K, v) to ensure that K admits a non-trivial parameter-free definable valuation. We show that this is the case if the value group vK is not divisible and Kv is separably or real closed, as well as whenever Kv admits no abelian extensions (Theorem 3.1.3, Corollary 3.1.4 and Lemma 3.1.5). Furthermore, v is definable if Kv is sufficiently non-henselian:

Theorem 3.1.8. *Let (K, v) be a non-trivially henselian valued field with $p \mid \#G_{Kv}$, and if $p = 2$ assume that Kv is not euclidean. If Kv is not virtually p-henselian then v is $\emptyset$-definable on K.*

As a consequence, any henselian valuation with non-separably closed PAC or hilbertian residue field is parameter-free definable. For hilbertian residue fields, we can even show that there is a uniform definition (Corollary 3.1.7).

Next, we discuss Galois-theoretic conditions to ensure the existence of a definable henselian valuation on a henselian field. Similar to the previous chapter, it is quite straightforward to show that if K is henselian and G_K is pro-p or pro-nilpotent, then K admits a parameter-free definable non-trivial henselian valuation (Proposition 3.1.2 and Proposition 3.2.1). The centrepiece of this chapter is our proof of the same statement for non-universal absolute Galois group:

Theorem 3.2.3. *Let G_K be non-universal, and assume that K is neither separably nor real closed. If K is t-henselian, then K is henselian with respect to a non-trivial $\emptyset$-definable valuation.*

In fact, if (K, v) is henselian, it is equivalent to ask for G_K or G_{Kv} to be non-universal (Observation 3.2.6). Thus, once more we get a condition on the residue field. Note that by a result of Kaplan, Scanlon and Wagner, the absolute Galois group of every NIP field of positive characteristic is non-universal. This allows us to draw some conclusions about the existence of definable henselian valuations on henselian NIP fields (Proposition 3.2.10).

The fourth section proves that if a field with a small absolute Galois group carries a definable henselian valuation, then often no parameters are needed (Theorem 3.3.1). We end by using our results about definable henselian valuations to draw some conclusions about model complete henselian fields (Theorem 3.4.9).

The last chapter is still work in progress. The idea is to improve the main theorem in [Koe04] which states that there are exactly five classes of fields which are elementarily characterized by their absolute Galois group. Two of these classes are conjectured to be empty. In the first section, we give the relevant definitions and quote the theorem. The next two sections use definable henselian valuations to give results towards a new characterization. In particular, we prove the following

Proposition 4.2.2. *Let K be elementarily characterized by G_K, and assume K is not real closed. Then either K is henselian with respect to a non-trivial $\emptyset$-definable valuation, or for some saturated $L \equiv K$ the field Lv_L is t-henselian but not henselian.*

The notation used in the statement of the proposition is introduced in section 1.3. All fields elementarily characterized by their absolute Galois group have small absolute Galois group. Therefore, the main open question in this context is whether there are t-henselian fields with small absolute Galois group which are not henselian (Question 4.2.3). Using our knowledge of the existence of parameter-free definable henselian valuations on henselian fields with non-universal absolute Galois group, we also give a slightly different characterization to Koenigsmann's (Corollary 4.2.10).

In the last section, we make a few remarks on a completely different approach to the same problem. Here, we discuss how one can see the classes we encountered earlier as abstract elementary classes. In particular, we show:

Proposition 4.3.1. *Let G be a profinite group. We define*

$$\mathcal{K}_G = \{\, K \mid K \text{ is a field, } G_K \cong G \,\} .$$

Furthermore, we define a notion of embedding $\prec_G$ on $\mathcal{K}_G$ given by $K \prec_G L$ just in case the extension $K \subseteq L$ is regular and the canonical projection $\mathrm{pr}_{L/K} : G_L \to G_K$ is an isomorphism. Then, the class $(\mathcal{K}_G, \prec_G)$ is an abstract elementary class.

Finally, we investigate some of the basic properties of this and other related abstract elementary classes (e.g. Proposition 4.3.4).

Chapter 1

Galois Theory and Valued Fields

1.1 Galois Theory

1.1.1 The Absolute Galois Group

The first chapter of [FJ08] provides a good introduction to Galois theory; almost everything contained in this section can be found there. Our main aim is to clarify notation and give the most important ideas. Throughout this chapter, let K be a field. First, we recall the definition of a Galois group.

Definition. *1. Let L/K be an algebraic field extension, such that $K \subseteq L$ is Galois, i.e. normal and separable. The* Galois group *of L/K is defined as*

$$\mathrm{Gal}(L/K) = \mathrm{Aut}(L/K),$$

with composition as group operation.

2. The absolute Galois group *of K is defined to be*

$$G_K = \mathrm{Gal}(K^{sep}/K).$$

We start with some well-known

Examples. *1. K is separably closed iff $G_K = \{e\}$.*

2. $G_{\mathbb{R}} \cong \mathbb{Z}/2\mathbb{Z}$.

3. If K is finite, then $G_K \cong \hat{\mathbb{Z}}$.

Galois theory has always been motivated by the fact that it provides a means to study fields, using tools developed for groups. The rationale for absolute Galois theory is that the absolute Galois group contains all the data encoded by Galois groups of finite extensions simultaneously. In order to see this, we want to consider G_K as a profinite group.

Definition. *Let I be a directed set. An* inverse system of finite groups *over I is a family $(G_i)_{i\in I}$ of finite groups together with connecting homomorphisms $\pi_{ij} : G_i \to G_j$, for all $i, j \in I$ with $j \leq i$, such that $\pi_{ii} = id_{G_i}$ and $\pi_{ik} = \pi_{ij} \circ \pi_{jk}$, for all $i, j, k \in I$ with $k \leq j \leq i$, hold. The* inverse limit *of this family is the set of all compatible sequences, i.e.*

$$\varprojlim G_i := \left\{ (g_i)_{i\in I} \in \prod_{i\in I} G_i \;\middle|\; \pi_{ij}(g_i) = g_j \text{ for all } j \leq i \right\}.$$

The inverse limit is a subgroup of the product group $\prod_{i\in I} G_i$. If one equips all the finite groups with the discrete topology and accordingly $\prod_{i\in I} G_i$ with the product topology, then the induced topology on G turns out to be compact, Hausdorff, and totally disconnected. In fact all compact, Hausdorff, totally disconnected topological groups can be seen as inverse limits of finite groups and, indeed, can be realized as the Galois group of some Galois extension of fields (Corollary 1.3.4 in [FJ08]). We call such a group a *profinite group.*

Note that because of compactness, open normal subgroups of profinite groups are precisely the closed normal subgroups of finite index. Well-known examples of profinite groups are $\hat{\mathbb{Z}}$ and $\mathbb{Z}_p$ (for any prime p). To consider G_K as a profinite group, we make the following

Observation 1.1.1. *Let $K \subseteq N$ be a Galois extension. Then, there is a canonical isomor-*

phism

$$\mathrm{Gal}(N/K) \cong \varprojlim \mathrm{Gal}(L/K),$$

where L ranges over all finite Galois extensions of K in N, and for two such fields L and L' with $L \subseteq L'$ the connecting homomorphism is given by the restriction map

$$\mathrm{res} : \mathrm{Gal}(L'/K) \longrightarrow \mathrm{Gal}(L/K).$$

This isomorphism gives G_K the structure of a topological group and hence defines the *Krull topology* on G_K. In this topology the open normal subgroups are, by Galois correspondence, just the absolute Galois groups of intermediate fields L, where L/K is a finite Galois extension. From now on, when we refer to subgroups, quotients or homomorphisms of profinite group, each shall respectively refer to closed subgroups, continuous quotients and continuous homomorphisms.

1.1.2 Sylow Subgroups of Profinite Groups

Profinite groups inherit many of the nice properties of finite groups. In particular, a profinite group has p-Sylow subgroups.

Definition. *Let G be a profinite group and p a prime. G is called a* pro-p group *if all finite quotients of G are p-groups, i.e. groups of p-power order.*

Analogously to the finite case, we define the notion of a p-Sylow group:

Definition. *Let G be a profinite group. A maximal pro-p subgroup of G is called a* p-Sylow subgroup. *We write* $\mathrm{Syl}_p(G)$ *for the set of p-Sylow subgroups of G.*

The p-Sylow subgroups satisfy the same Sylow theorem as in the finite case.

Theorem 1.1.2 (Sylow)**.** *Let G be any profinite group and p a prime. Then*

1. *G contains a p-Sylow subgroup.*

2. Any p-subgroup of G is contained in a p-Sylow subgroup.

3. Any two p-Sylow subgroups of G are conjugate.

Proof: [RZ00], Corollary 2.3.6.

1.1.3 Profinite Groups as Elementary Structures

The elementary theory of K in the language of rings 'sees' all finite Galois extensions of K as they are finite-dimensional K-vector spaces, which are made into fields with the multiplication described by the (monic) minimal polynomial of a primitive element. The multiplication can be defined uniformly in the coefficients of said polynomials. In particular, all finite Galois extensions of K are interpretable in K, using parameters for the coefficients of the minimal polynomial of a primitive element.

If there are only finitely many Galois extensions of degree n for each $n \in \mathbb{N}$, then the absolute Galois group is pro-interpretable in the field (see [CvdDM80], §1). Some easy applications of this fact are the following

Examples. *1.* $G_K \cong \mathbb{Z}/2\mathbb{Z}$ *if* $K \equiv \mathbb{R}$ *(this is in fact an equivalence),*

2. $G_K \cong \hat{\mathbb{Z}}$ *for any pseudofinite field* K*.*

In order to study the theory of an absolute Galois group, we need to find the right language in which to consider it in. Generally, the group language sees 'too little' of the structure of profinite groups; there are profinite groups which are isomorphic to each other as abstract groups but not as topological groups ([Cha84], Appendix I). Thus, we often want to consider profinite groups in the language of inverse systems which was introduced in [CvdDM80]. To do this, we associate with a profinite group G the set

$$S(G) := \{ gN \mid g \in G, N \lhd_o G \},$$

namely the set of all cosets of open normal subgroups of G. We put an ω-sorted structure on

$S(G)$, where we assign sorts as follows:

$$gN \text{ has sort } n \Longleftrightarrow [G : N] \leq n.$$

Then we define a relational structure on $S(G)$ which encodes the product on the quotient groups G/N as well as the projection and inclusion maps. With the right axiomatization of structures in this language, we get a theory T_{PG} and a one-to-one correspondence between the models of said theory and the complete systems of profinite groups (see Chapter 3 in [Cha84]).

As always with multi-sorted structures, not only finite but sortwise finite structures are the unique models of their theories. In the case of absolute Galois groups, this corresponds to the field having only finitely many Galois extensions of degree n for each $n \in \mathbb{N}_{>0}$. We call such groups *small*. By the considerations above, the theory of the field determines whether the absolute Galois group of the field is small, and in such case the theory also determines its isomorphism type as a profinite group.

1.1.4 Projective Profinite Groups

In this subsection, we will define what it means for a profinite group to be projective and state some properties of projective groups which we will use later.

Definition. *A profinite group G is called* projective *if for all finite groups A, B and for every epimorphism $\alpha : B \to A$ and every epimorphism $\phi : G \to A$ there exists a homomorphism $\gamma : G \to B$ such that $\alpha \circ \gamma = \phi$.*

Example. *The groups $\mathbb{Z}_p$ and $\hat{\mathbb{Z}}$ are both projective.*

This implies that every epimorphism onto a projective group splits:

Observation 1.1.3. *Suppose G is projective and that $\pi : H \to G$ is an epimorphism of profinite groups. Then there exists an embedding $\pi' : G \to H$ such that $\pi \circ \pi'$ is the identity map.*

Proof: [FJ08], Remark 22.4.2.

There is also a cohomological description of which groups are projective. See [Ser97] for an introduction to Galois cohomology, and in particular chapter I.3 thereof for the definition of (p)-cohomological dimension, i.e. $\mathrm{cd}_p(G)$ and $\mathrm{cd}(G)$.

Theorem 1.1.4. *A profinite group G is projective iff* $\mathrm{cd}(G) \leq 1$.

Proof: [FJ08], Corollary 22.4.3.

What makes projective groups important for us is that projective groups are exactly those groups which occur as absolute Galois groups of PAC fields.

Definition. *A field K is called* pseudo algebraically closed (PAC) *if every absolutely irreducible variety V defined over K has a K-rational point.*

Theorem 1.1.5 (Ax)**.** *If K is PAC, then G_K is projective.*

Proof: [FJ08], Theorem 11.6.2.

A converse to this was proven by Lubotzki and van den Dries.

Theorem 1.1.6 (Lubotzki-van den Dries)**.** *Given a projective group G and a field K, there is an extension F of K which is perfect and PAC with $G_F \cong G$.*

Proof: [FJ08], Corollary 23.1.2.

1.2 Valued Fields

This section is an overview of the notation and tools we will be using in this thesis. For a detailed introduction to valued fields, see [EP05].

1.2.1 Definitions and Notation

Let K be a field and Γ_v an ordered abelian group. Recall that a *valuation* on K is a map $v : K \twoheadrightarrow \Gamma_v \cup \{\infty\}$ such that, for all $x, y \in K$,

$$v(x) = \infty \Longleftrightarrow x = 0, \tag{1.1}$$

$$v(xy) = v(x) + v(y), \tag{1.2}$$

$$v(x + y) \geq \min(v(x), v(y)). \tag{1.3}$$

We sometimes denote Γ_v by vK. Here, ∞ is a symbol and by definition $\Gamma_v \cup \{\infty\}$ is ordered by putting ∞ above all elements of the group. Furthermore, ∞ satisfies the obvious rules for addition.

Recall further that the ring $\mathcal{O}_v = \{ x \in K \mid v(x) \geq 0 \}$ is a *valuation ring* of K, i.e. for all $x \in K$ we have $x \in \mathcal{O}_v$ or $x^{-1} \in \mathcal{O}_v$. We say that v is non-trivial if $v|_{K^\times} \neq 0$ or equivalently $\mathcal{O}_v \neq K$.

Examples. *1. Any finite field allows only the trivial valuation.*

2. For any prime p, we have a p-adic valuation on $\mathbb{Q}$ given by $v_p\left(p^\nu \frac{a}{b}\right) = \nu$ for $p \nmid a, b$. The p-adic valuations are in fact the only valuations on $\mathbb{Q}$.

Any valuation ring has a unique maximal ideal $\mathfrak{m}_v = \{ x \in K \mid v(x) > 0 \}$. We call the quotient $\mathcal{O}_v/\mathfrak{m}_v$ the *residue field* of (K, v) and denote it by Kv. The characteristic of the residue field can be equal to the characteristic of K or, when $\mathrm{char}(K) = 0$, some prime p. We say (K, v) has characteristic $(\mathrm{char}(K), \mathrm{char}(Kv))$ when we want to distinguish between these cases.

For an element $x \in \mathcal{O}_v$, we write $\overline{x}$ for its image in the residue field. Similarly, for $f \in \mathcal{O}_v[X]$, we write $\overline{f}$ for its image under the corresponding map from $\mathcal{O}_v[X]$ to $Kv[X]$.

Note that a valuation can be described through the valuation map, the valuation ring or the residue field map. Up to isomorphism, these all contain the same information. We will always consider two valuations to be the same if they induce the same valuation ring.

Let $\mathcal{O}$ be a valuation ring with value group Γ and corresponding valuation v. Since the convex subgroups of Γ are linearly ordered by inclusion, we can define the *rank of* Γ, $\mathrm{rk}(\Gamma)$, as the order type of the collection of all convex subgroups.

As we want to study extensions of fields, we need a notion of extensions of valued fields.

Definition. *Let $K \subseteq L$ be fields, each respectively equipped with a valuation v and w. We say that (L, w) is an* extension *of (K, v) and that w is a* prolongation *of v if $\mathcal{O}_w \cap K = \mathcal{O}_v$ holds. In this case, we call*

$$e = e(w/v) = [\Gamma_w : \Gamma_v]$$

the ramification index *and*

$$f = f(w/v) = [Lw : Kv]$$

the inertia degree *of the extension. If $e = f = 1$, we say that the extension is* immediate*.*

Note that the definitons of inertia degree and ramification index make sense as Γ_v is a subgroup of Γ_w and Kv embeds into Lv. Due to Chevalley's Extension Theorem, any valuation v on K has a prolongation to any field extension L of K. Furthermore, the degree of the field extension bounds e and f.

Theorem 1.2.1. *If $(K, v) \subseteq (L, w)$ is a finite extension of valued fields, then $ef \leq [L : K]$.*

Proof: [EP05], Corollary 3.2.3.

There may of course be several valuations on a field. We say that two valuation rings of a field are *incomparable* if neither is contained in the other. Note that for a valuation ring $\mathcal{O}_v$ of a field K there is a natural correspondence

$$\{ \text{ overrings of } \mathcal{O}_v \} \stackrel{1:1}{\longleftrightarrow} \{ \text{ convex subgroups of } \Gamma_v \}$$

via

$$v(\mathcal{O}_w^\times) < \Gamma_v$$

for $\mathcal{O}_w$ an overring of $\mathcal{O}_v$ and

$$K^\times \xrightarrow{v} \Gamma_v \xrightarrow{\text{res}} \Gamma_v/\Delta$$

for $\Delta < \Gamma_v$ a convex subgroup. In this case, Δ is just the value group of the valuation $\overline{v}$ on Kw which is induced by v. Hence if Γ_v has rank 1, then v has no proper coarsenings.

1.2.2 Hilbert Theory

There are several reasons why one studies valued fields in the context of Galois theory. One is that valued fields have a very well-structured absolute Galois group.

Definition. *Let $(L, w)/(K, v)$ be a Galois extension of valued fields, i.e. L/K Galois and $(L, w)/(K, v)$ an extension of valued fields. Let us denote $G = \mathrm{Gal}(L/K)$. We define subgroups of G by*

$$\begin{aligned} D &= \{\, \sigma \in G \mid \sigma(\mathcal{O}_w) = \mathcal{O}_w \,\} = \{\, \sigma \in G \mid \forall x \in \mathcal{O}_w\ \sigma(x) - x \in \mathcal{O}_w \,\}, \\ I &= \{\, \sigma \in G \mid \forall x \in \mathcal{O}_w\ \sigma(x) - x \in \mathfrak{m}_w \,\}, \\ R &= \{\, \sigma \in G \mid \forall x \in \mathcal{O}_w\ \sigma(x) - x \in x \cdot \mathfrak{m}_w \,\} = \{\, \sigma \in G \mid \forall x \in K\ \sigma(x) - x \in x \cdot \mathfrak{m}_w \,\} \end{aligned}$$

and respectively call them the decomposition, inertia *and* ramification subgroup of G *(with respect to w). The corresponding fixed fields*

$$L_D \coloneqq \mathrm{Fix}(D),\ L_I \coloneqq \mathrm{Fix}(I),\ L_R \coloneqq \mathrm{Fix}(R)$$

are called the decomposition, inertia *and* ramification subfields of L *(with respect to w).*

It is clear that these groups are subgroups of each other, i.e.

$$R \leq I \leq D \leq G.$$

Furthermore, this composition behaves well in towers of fields, so it is sufficient to study the

Galois groups of finite Galois extensions. The most important properties of these groups are described in the following three propositions which we will refer to several times later. Proofs of these can be found in [EP05], sections 5.2 and 5.3. For the remainder of this section, let $(L, w)/(K, v)$ be a Galois extension of valued fields.

Proposition 1.2.2 (D). *Let $w_D := w|_{L_D}$ be the restriction of w to L_D. Then*

1. *w is the unique prolongation of w_D to L,*
2. *v has $[G : D] = [L_D : K]$ many prolongations to L_D,*
3. *the residue fields and the value groups of all prolongations of v to L are isomorphic,*
4. *(L_D, w_D) is an immediate extension of (K, v).*

The next proposition states the most important properties of the inertia group.

Proposition 1.2.3 (I). *The map*

$$\begin{aligned} \varphi : D &\longrightarrow \mathrm{Aut}(Lw/Kv) \\ \sigma &\mapsto [\overline{\sigma} : Lw \to Lw,\ \overline{x} \mapsto \overline{\sigma x}] \end{aligned}$$

is a well-defined epimorphism with kernel I, so in particular $I \lhd D$. Furthermore, we have $L_I w_I = Lw \cap Kv^{sep}$ and the extension $L_D \subseteq L_I$ is inert ($n = f$).

Lastly, we come to the ramification group.

Proposition 1.2.4 (R). *1. The homomorphism*

$$\begin{aligned} \psi_0 : I &\longrightarrow \mathrm{Hom}(L^\times, Lw^\times) \\ \sigma &\mapsto [L^\times \to Lw^\times,\ x \mapsto \overline{\left(\frac{\sigma x}{x}\right)}] \end{aligned}$$

induces an epimorphism

$$\psi : I \longrightarrow \mathrm{Hom}(\Gamma_w/\Gamma_v, \mu(Lw))$$

with kernel R, where $\mu(Lw)$ denotes the group of roots of unity in Lw. In particular $R \lhd I$ and I/R is abelian.

2. *If $\mathrm{char}(Kv) = p$, then R is the unique Sylow-p subgroup of I (hence $R \lhd D$). If $\mathrm{char}(Kv) = 0$, then R is trivial. In any case, L_R/L_I is purely ramified ($n = e$) and tame ($p \nmid e$).*

Combining the three propositions, we obtain Figure 1.1.

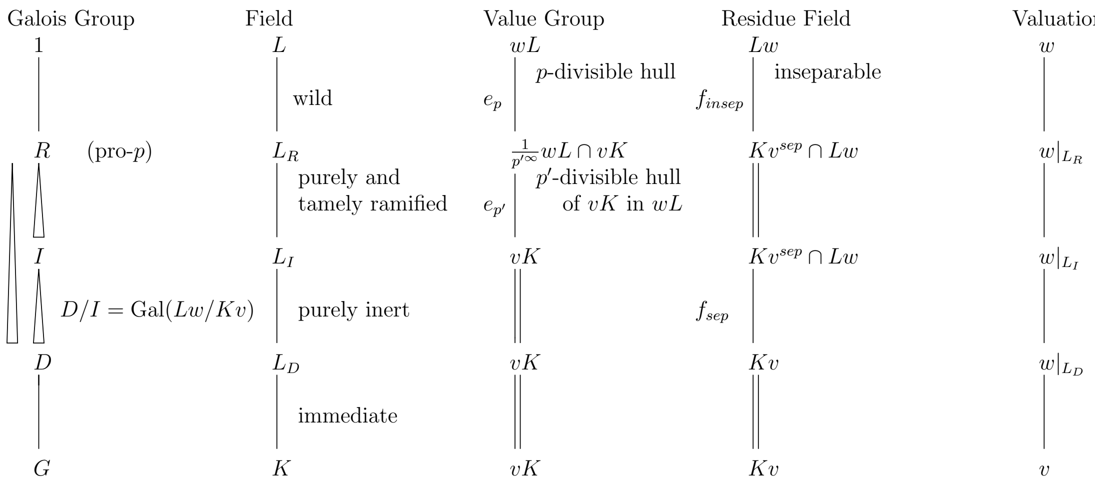


Figure 1.1: Galois Theory of Valued Fields

In particular, all extensions of a valuation to a Galois extension have the same inertia degree and ramification index.

Corollary 1.2.5. *If $[L : K] = n$, then $n = rdef$, where r is the number of prolongations of v to L, $d = 1$ when the residue characteristic is 0 and some power of p when the residue characteristic is p, and $e = e(w/v)$ and $f = f(w/v)$ as before.*

In the notation of the above corollary, $d = d(w/v)$ is called the *defect* of the extension $(K, v) \subseteq (L, w)$. If $d = 1$, the extension is called *defectless.*

1.3 Henselian Valued Fields

1.3.1 Henselian Valuations

Definition. *A* henselian valuation ring $\mathcal{O} \subseteq K$ *is a valuation ring which extends uniquely to every algebraic extension of* K*. A field* K *is called* henselian *if some non-trivial valuation ring of* K *is henselian.*

We have already encountered one non-trivial

Example. *For any non-trivially valued field* (K, v) *and any extension* w *of* v *to* K^{sep}*, consider the fixed field* $\mathrm{Fix}(D_w)$ *of the decomposition group with respect to* w*. Then* $w|_{\mathrm{Fix}(D_w)}$ *extends uniquely from* $\mathrm{Fix}(D_w)$ *to* K^{sep} *(cf. Proposition* 1.2.2*). As valuations always extend uniquely to purely inseparable extensions, this is an example of a henselian field.*

Thus every valued field embeds into a henselian valued field, namely $\mathrm{Fix}(D_w)$, using the notation from the previous section. Since the decomposition groups of all prolongations to the separable closure are conjugate in G_K (see again [EP05], Lemma 5.2.1), their fixed fields are isomorphic. Note that (K, v) is henselian if and only if $G_K = D_w$ for any (all) prolongation(s) w of v to K^{sep}. Also, $\mathrm{Fix}(D_w)$ is the 'smallest' henselian field containing K, i.e. it embeds as a valued field into any henselian valued extension of (K, v).

Definition. *Let* (K, w) *be a valued field and* w *a prolongation to* K^{sep}*. The (up to isomorphism unique) fixed field of the decomposition group* D_w *is called the* henselization *of* K *and denoted by* K^h*.*

The following theorem, which can be found as Theorem 4.1.3 in [EP05], gives conditions equivalent to henselianity.

Theorem 1.3.1 (Hensel's Lemma)**.** *For a valued field* (K, v)*, the following are equivalent.*

1. $\mathcal{O}_v$ *is henselian.*

2. *For each* $f \in \mathcal{O}_v[X]$ *and* $a \in \mathcal{O}_v$ *with* $\overline{f}(\overline{a}) = 0$ *and* $\overline{f}'(\overline{a}) \neq 0$ *there exists* $\alpha \in \mathcal{O}_v$ *with*

$f(\alpha) = 0$ and $\overline{\alpha} = \overline{a}$.

3. *For all $f \in \mathcal{O}_v[X]$ and for all $a \in \mathcal{O}_v$ with $v(f(a)) > 2v(f'(a))$ there exists $b \in \mathcal{O}_v$ with $f(b) = 0$ and $v(b - a) > v(f'(a))$.*

Note that the original version of Hensel's Lemma states that complete valued fields, which include $\mathbb{Q}_p$ with the p-adic valuation and $K((t))$ with the t-adic valuation, satisfy these conditions; for a proof see [EP05], Theorem 1.3.1.

The main theorem about henselian fields from the model-theoretic viewpoint is the famous theorem of Ax-Kochen and Eršov, which proves that the theory of a henselian valued field of residue characteristic 0 is determined by the theory of residue field and the theory of the value group. For the formal setup, we take valued fields either in the language of fields together with a unary predicate for the valuation ring, or as a 3-sorted structures with valuation and angular component map. Note that the preceeding theorem shows that henselianity is expressible in these languages.

Theorem 1.3.2 (Ax-Kochen/Eršov)**.** *Let (K_1, v_1) and (K_2, v_2) be two henselian valued fields of characteristic $(0, 0)$. Then*

$$(K_1, v_1) \equiv (K_2, v_2) \iff K_1v_1 \equiv K_2v_2 \text{ and } \Gamma_1 \equiv \Gamma_2.$$

Proof: [Pre86], Theorem 4.26.

1.3.2 The Canonical Henselian Valuation

Since a field can carry many valuations, it can be henselian with respect to several valuations. However, these valuations cannot in fact be too different if the field is not separably closed. Every valuation induces a topology on the field since the sets

$$\mathcal{U}_\gamma(a) = \{\, x \in K \mid v(x - a) > \gamma \,\}$$

for each $a \in K$ and with γ ranging over Γ_v, form a basis of open neighbourhoods of a.

Definition. *Two valuations on a field K are said to be* independent *if they induce different topologies on K.*

This notion can be used to prove the following

Theorem 1.3.3 (à la F.K. Schmidt)**.** *If K has two independent non-trivial henselian valuations, then K is separably closed.*

Proof: [EP05], Theorem 4.4.1.

This means we can order the non-trivial henselian valuations on a field in a certain way. Divide the class of henselian valuations on K into two subclasses, namely

$$H_1(K) = \{ \, v \text{ henselian on } K \mid Kv \neq Kv^{sep} \, \}$$

and

$$H_2(K) = \{ \, v \text{ henselian on } K \mid Kv = Kv^{sep} \, \} \, .$$

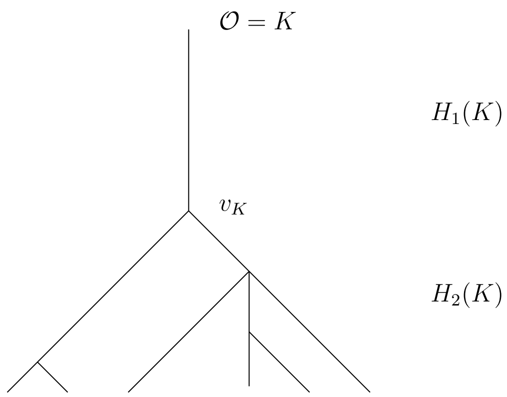


Figure 1.2: The canonical henselian valuation

A corollary of the above theorem is that any valuation $v_2 \in H_2(K)$ is *finer* than any $v_1 \in H_1(K)$, i.e. $\mathcal{O}_{v_2} \subsetneq \mathcal{O}_{v_1}$, and that any two valuations in $H_1(K)$ are comparable. Furthermore, if $H_2(K)$ is non-empty, then there exists a unique coarsest v_K in $H_2(K)$; otherwise there exists a unique finest $v_K \in H_1(K)$. In either case, v_K is called the *canonical henselian valuation*.

This is illustrated by Figure 1.2.

If (K, v) is a henselian valued field and L is an algebraic extension of K, then the unique extension of v to L is again henselian. This clearly need not be true when we go down to a subfield of K.

Example. *Consider $\mathbb{Q}_p \cap \mathbb{Q}^{alg}$, the algebraic part of the p-adics. Like the p-adics, this field carries a henselian valuation v with residue field $\mathbb{F}_p$ and value group $\mathbb{Z}$. The restriction of that valuation to $\mathbb{Q}$ is not henselian as $\mathbb{Q}$ does not admit any non-trivial henselian valuation.*

The next theorem gives conditions under which henselianity is inherited by subfields:

Theorem 1.3.4. *Let (L, w) be a valued field, and assume that L is not separably closed and that w is a (not necessarily proper) coarsening of v_L. If $K \subseteq L$ is an algebraic subfield, then $v = w|_K$ is a coarsening of v_K in all of the following cases:*

1. *L/K normal.*
2. *L/K finite.*
3. *$G_L \in \mathrm{Syl}_p(G_K)$ and if $p = 2$ and Lw is real closed, then no proper coarsening of w has real closed residue field.*

Proof: [EP05], Theorems 4.4.3, 4.4.4 and [Koe03], Proposition 3.1.

1.3.3 Finding Henselian Valuations via Galois Groups

The discussion in subsection 1.2.2 might suggest that the absolute Galois group of a henselian field is always has certain group-theoretic properties. However, this is not the case, as any absolute Galois group occurs over some henselian field:

Observation 1.3.5. *Let G_K be the absolute Galois group of some field K. Then there is a henselian field L such that $G_K \cong G_L$.*

Proof: We may assume that K is perfect as $G_K \cong G_{K^{\mathrm{perf}}}$. A standard construction, see

Corollary 2.22 in [Kuh09], gives a defectless henselian valued field (L, v) with $\mathrm{char}(L) = 0$, $Lv = K$ and $G_L \cong G_K$. □

In many cases, the absolute Galois group nonetheless encodes information about a henselian valuation on the field. The theorem cited below is one of the key ingredients for chapter 4 and was first proven in [Koe01]. The version stated here can be found as Theorem 2.15 in [Koe03]. It provides a criterion for a field to be henselian that depends only on its absolute Galois group.

Definition. *Let (K, v) be a valued field and p a prime. We call v* tamely branching *at p if* $\mathrm{char}(Kv) \neq p$, *Γ_v is not p-divisible and if $[\Gamma_v : p\Gamma_v] = p$, then Kv has a finite separable field extension of degree divisible by p^2.*

This technical definition gives us the tools to recover a henselian valuation from the absolute Galois group.

Theorem 1.3.6 (Koenigsmann)**.** *A field K is henselian with respect to a valuation v which is tamely branching at p iff there exists a non-trivial, abelian $N \lhd P \in \mathrm{Syl}_p(G_K)$ with $P \not\cong \mathbb{Z}_p$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}_2 \rtimes \mathbb{Z}/2\mathbb{Z}$.*

Chapter 2

Notions of Henselianity

2.1 p-Henselianity

2.1.1 The Canonical p-Henselian Valuation

In the previous chapter, we studied valuations which extend uniquely to the separable closure of a field, i.e. in particular to all Galois extensions. In this section, we again ask for a unique extension of a valuation, but only to all Galois extensions of p-power degree for some prime p.

Definition. *Let K be a field. We define $K(p)$ to be the compositum of all Galois extensions of K of p-power degree. A valuation v on K is called* p-henselian *if v extends uniquely to $K(p)$. If $K \neq K(p)$ and K admits a non-trivial p-henselian valuation, then K is called* p-henselian.

Clearly, this definition only imposes a condition on v if K admits Galois extensions of p-power degree, that is if $K \neq K(p)$.

Proposition 2.1.1 (p-Hensel's Lemma)**.** *For a valued field (K, v), the following are equivalent:*

1. *v is p-henselian,*

2. *v extends uniquely to every Galois extension of K of p-power degree,*

3. *v extends uniquely to every Galois extension of K of degree p,*

4. *for every polynomial $f \in \mathcal{O}_v$ which splits in $K(p)$ and for every $a \in \mathcal{O}_v$ with $\overline{f}(\overline{a}) = 0$ and $\overline{f}'(\overline{a}) \neq 0$ there exists $\alpha \in \mathcal{O}_v$ with $f(\alpha) = 0$ and $\overline{\alpha} = \overline{a}$,*

5. *for every polynomial $f \in \mathcal{O}_v$ which splits in $K(p)$ and for every $a \in \mathcal{O}_v$ with $v(f(a)) > 2v(f'(a))$ there exists a unique $b \in K$ with $f(b) = 0$ and $v(b - a) > v(f'(a))$.*

Proof: [Koe95], Propositions 1.2 and 1.3.

Just like for fields carrying a henselian valuation, there is again a canonical p-henselian valuation. This is due to the following analogue of Theorem 1.3.3:

Theorem 2.1.2. *If K carries two independent non-trivial p-henselian valuations, then $K = K(p)$.*

Proof: [Brö76], Corollary 1.5.

Assume that $K \neq K(p)$. We again divide the class of p-henselian valuations on K into two subclasses,

$$H_1^p(K) = \{\, v \text{ } p\text{-henselian on } K \mid Kv \neq Kv(p) \,\}$$

and

$$H_2^p(K) = \{\, v \text{ } p\text{-henselian on } K \mid Kv = Kv(p) \,\}.$$

As before, one can deduce from the above theorem that any valuation $v_2 \in H_2^p(K)$ is *finer* than any $v_1 \in H_1^p(K)$, i.e. $\mathcal{O}_{v_2} \subsetneq \mathcal{O}_{v_1}$, and that any two valuations in $H_1^p(K)$ are comparable. Furthermore, if $H_2^p(K)$ is non-empty, then there exists a unique coarsest v_K^p in $H_2^p(K)$; otherwise there exists a unique finest $v_K^p \in H_1^p(K)$. In either case, v_K^p is called the *canonical p-henselian valuation.* Note that if K is p-henselian, then v_K^p is non-trivial. The p-henselian valuations on a field K are illustrated in Figure 2.1.

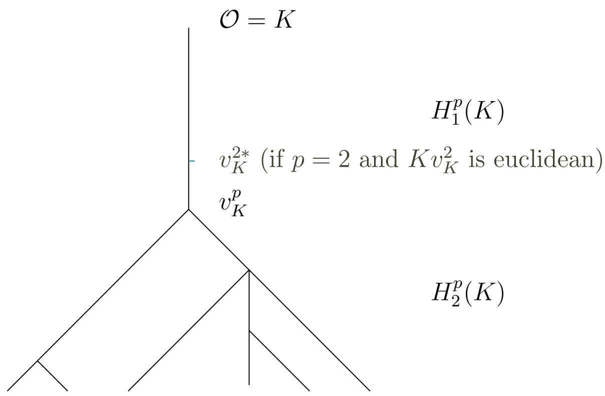


Figure 2.1: The canonical p-henselian valuation on a field K with $K \neq K(p)$

2.1.2 Definitions of p-Henselian Valuations

As we will discuss in the next section, admitting a non-trivial henselian valuation is not necessarily an elementary property of a field in $\mathcal{L}_{ring}$. This is different for p-henselianity, at least in the presence of a primitive pth root of unity ζ_p.

When it comes to henselian valued fields, real closed fields always play a special role. No real closed field admits a definable henselian valuation, and there are real closed fields which admit no henselian valuations (like $\mathbb{R}$, see for example [EP05], Corollary 2.2.6) whereas others do (like $\mathbb{R}((\mathbb{Q}))$). These difficulties are reflected by 2-henselian valuations on euclidean fields.

Definition. *A field K is called* euclidean *if* $[K(2) : K] = 2$.

Any euclidean field is uniquely ordered, the positive elements being exactly the squares. Real closed fields are in particular euclidean. Conversely, if a euclidean field has no odd-degree extensions, then it is real closed. In particular, there is an $\mathcal{L}_{ring}$-sentence ρ such that ρ holds in any field K iff K is non-euclidean. Note that euclidean fields are the only fields for which $K(p)$ is a finite non-trivial extension of K.

Proposition 2.1.3. *For $p > 2$, consider the class of fields*

$$\mathcal{K}_p := \{\ K \mid K \neq K(p) \ \textit{and if}\ \mathrm{char}(K) \neq p \ \textit{then}\ \zeta_p \in K\ \},$$

and define furthermore

$$\mathcal{K}_2 := \{\, K \mid K \neq K(2) \text{ and } K \text{ is not euclidean} \,\}.$$

Then for any prime p there is an $\mathcal{L}_{ring}$-sentence φ_p such that for any $K \in \mathcal{K}_p$, φ_p holds in K iff K admits a non-trivial p-henselian valuation.

Proof: This follows from the proof of [Koe95], Corollary 2.2. The sentence φ_p expresses

- if $\mathrm{char}(K) \neq p$, the sets
$$\big(a(K^\times)^p + b\big) \cap \big(c(K^\times)^p + d\big)$$
for $a, c \neq 0$ form the base of a V-topology and
- if $\mathrm{char}(K) = p$, the sets
$$\left\{ \frac{ax+b}{cx+d} \,\middle|\, x \in K^{(p)}, x \neq -\frac{d}{c} \right\}$$
with $K^{(p)} = \{\, y^p - y \mid y \in K \,\}$ and $ad - bc \neq 0$ form the base of a V-topology.

A V-topology is a non-discrete Hausdorff field topology on K, such that for any neighbourhood base τ of 0

$$\forall W \in \tau \, \exists U \in \tau \, \forall x, y \in K \, (xy \in U \longrightarrow x \in W \vee y \in W)$$

holds (see section 2.2 and [EP05], Appendix B). □

We call a valuation on a field K *definable* if there is some $\mathcal{L}_{ring}$-formula φ defining the valuation ring. Koenigsmann uses the above proposition in [Koe95] to show that the canonical p-henselian valuation is $\emptyset$-definable for any p-henselian $K \in \mathcal{K}_p$ for $p > 2$. For $p = 2$, he gives a parameter-free definition of v_K^2 for any 2-henselian $K \in \mathcal{K}_2$ with Kv_K^2 not euclidean. For our applications, we need to show that the definitions of p-henselian valuations in [Koe95] can be done uniformly, as this allows us to deal with parameters. First, we define a 2-henselian valuation in case the residue field is euclidean, as this is not treated in [Koe95].

Theorem 2.1.4. *Let $K \neq K(2)$, and assume that K is not euclidean and that Kv_K^2 is* *euclidean. Then the coarsest* 2*-henselian valuation* v_K^{2*} *on* K *which has euclidean residue field is* $\emptyset$*-definable.*

Proof: We follow the proof of Theorem 3.2 in [Koe95]. As Kv_K^2 is euclidean, all 2-henselian valuations are comparable and furthermore coarsenings of v_K^2. Since K is not euclidean, there must be a coarsest 2-henselian valuation v_K^{2*} such that Kv_K^{2*} is euclidean.

We use Beth's Definability Theorem to show that v_K^{2*} is definable. If we add a symbol for $\mathcal{O}_v$ to the ring language, then we claim that $\mathcal{O}_v = \mathcal{O}_{v_K^{2*}}$ is axiomatized by the properties

(i) v is 2-henselian,

(ii) Kv is euclidean,

(iii) vK is not 2-divisible and no non-trivial convex subgroup of vK is 2-divisible:

$$\forall \alpha > 0 \in vK \, \exists \gamma \in vK \ \text{ such that } 0 < \gamma \leq \alpha \wedge 2 \nmid \gamma.$$

Clearly, v_K^{2*} satisfies the first two of these axioms. Furthermore, since Kv_K^{2*} is euclidean but K is not, $v_K^{2*}K$ is not 2-divisible. As every coarsening of v_K^{2*} has non-euclidean residue field, $v_K^{2*}K$ has no non-trivial convex 2-divisible subgroups. Since all 2-henselian valuations are comparable, v_K^{2*} is the only 2-henselian valuation with euclidean residue field and value group having no non-trivial 2-divisible convex subgroup, hence it is indeed characterized by these properties.

Note that the same characterization defines v_K^{2*} in any field $L \equiv K$ and no parameters were needed. Therefore v_K^{2*} is $\emptyset$-definable by Beth's Definability Theorem (see [Hod97], Theorem 5.5.4). □

2.1.3 A Uniform Definition

Using Beth's Definability Theorem once more, we can show that there is a uniform definition of the canonical p-henselian valuation.

Theorem 2.1.5. *For a fixed characteristic q, consider the class of fields*

$$\mathcal{K}_{p,q} := \{ K \in \mathcal{K}_p \mid \mathrm{char}(K) = q \textit{ and } K \textit{ is } p\textit{-henselian} \} .$$

For $(p,q) \neq (2,0)$, there is a parameter-free $\mathcal{L}_{ring}$-formula $\phi_{p,q}(x)$ such that $\phi_{p,q}(K)$ defines the canonical p-henselian valuation ring $\mathcal{O}_{v_K^p}$ in any $K \in \mathcal{K}_{p,q}$. In case $(p,q) = (2,0)$, there is a parameter-free $\mathcal{L}_{ring}$-formula $\phi_{2,0}(x)$ such that $\phi_{2,0}(K)$ defines a non-trivial coarsening of the canonical 2-henselian valuation ring $\mathcal{O}_{v_K^2}$ in any $K \in \mathcal{K}_{2,0}$.

Proof: We show that the definition in [Koe95], Theorem 3.2, can be found uniformly. In order to do that, we use Beth's Definability Theorem to get a definition modulo the (elementary) class $\mathcal{K}_{p,q}$. We start with $(p,q) \neq (2,0)$. In this case, $\mathcal{O}_v = \mathcal{O}_{v_K^p}$ is axiomatized by

(i) $\mathcal{O}_v$ is p-henselian,

(ii) if $Kv \neq Kv(p)$ then Kv is not p-henselian,

(iii) if $Kv = Kv(p)$ and $\mathrm{char}(K) \neq p$, then vK has no non-trivial p-divisible convex subgroup,

(iv) if $Kv = Kv(p)$ and $\mathrm{char}(K) = p$, then either vK contains no non-trivial p-divisible convex subgroup or $v(\{ x^p - x \mid x \in K \})$ contains no non-trivial convex subgroup.

Note that there can't be more than one p-henselian valuation satisfying one of the clauses (ii)-(iv) for any field in $\mathcal{K}_{p,q}$. Furthermore, all statements are elementary by the proof of Theorem 3.2 in [Koe95]. Hence, Beth's Definability Theorem gives a uniform, parameter-free definition of v_K^p modulo $\mathcal{K}_{p,q}$ as claimed.

In case $p = 2$, we need to include one further case in order to deal with the special case of a euclidean residue field. We get a characterization of some 2-henselian non-trivial coarsening of v_K^2 (namely depending on K either v_K^2 or v_K^{2*}) by replacing (ii) by

(ii)' if $Kv \neq Kv(2)$ then either Kv is not euclidean and not 2-henselian, or Kv is euclidean and vK is not 2-divisible and no convex subgroup of vK is 2-divisible.

As no euclidean field admits a 2-henselian valuation with 2-closed residue field, the cases are again mutually exclusive and we get a parameter-free definition, defining v_K^2 on fields $K \in \mathcal{K}_{2,0}$ in case Kv_K^2 is not euclidean and v_K^{2*} on all others. □

Since we only needed to differentiate between characteristic p and characteristic non-p in the above proof, we can even give a uniform definition regardless of characteristic.

Corollary 2.1.6. *Let $p > 2$. There is a parameter-free $\mathcal{L}_{ring}$-formula ϕ_p such that $\phi_p(K)$ defines the valuation ring of the canonical p-henselian valuation in any p-henselian $K \in \mathcal{K}_p$.*

2.1.4 p-Henselianity and Subfields

Note that unlike henselianity, being p-henselian does not go up arbitrary algebraic extensions, as a superfield might have far more extensions of p-power degree. Nevertheless, as with henselianity, sometimes p-henselianity goes down:

Proposition 2.1.7. *Let K be a field, $K \neq K(p)$. Assume that L is a normal algebraic extension of K, where L is p-henselian and $L \neq L(p)$. If*

1. *$K \subseteq L \subsetneq K(p)$ or*
2. *L/K is finite*

then K is p-henselian.

Proof: 1.: See [Koe03], Proposition 2.10.
2.: Assume K is not p-henselian, and let v be a valuation on K. By the first part of the proposition, v has infinitely many extensions to $K(p)$: If there were only n extensions of v to $K(p)$, then there would be some finite algebraic extension L' of K, with $L' \subsetneq K(p)$, such that v had n extensions to L'. The normal hull of L' and thus K would be p-henselian.
Now assume $L = K(a_1, \ldots, a_m)$ finite and normal, then $K(p)(a_1, \ldots, a_m) \subseteq L(p)$. If w is a valuation on L, then $v = w|_K$ has infinitely many prolongations to $K(p)$. As v has only finitely many prolongations to L, and all these are conjugate, w must have infinitely many prolongations to $K(p)(a_1, \ldots, a_m)$ and hence to $L(p)$. □

2.2 t-Henselianity

2.2.1 t-Henselian Fields

This section deals with the question to what extent henselianity is an elementary property in $\mathcal{L}_{ring}$.

Definition. *Let K be a field and τ a filter of neighbourhoods of 0 on K. Then (K, τ) is called* t-henselian *if the following axioms hold, where U and V range over elements of τ and x, y range over elements of K:*

(T1) $\forall U\ \{0\} \subsetneq U,\ \forall x \neq 0\, \exists V\ x \notin V$

(T2) $\forall U\, \exists V\ V - V \subseteq U$

(T3) $\forall U\, \exists V\ V \cdot V \subseteq U$

(T4) $\forall U\, \forall x\, \exists V\ xV \subseteq U$

(T5) $\forall U\, \exists V\, \forall x, y\ (x \cdot y \in V \longrightarrow (x \in U\ \vee\ y \in U))$

(T6) (for every $n \in \mathbb{N}$) $\exists U\, \forall f \in X^{n+1} + X^n + U[X]^{n-1}\, \exists x\ f(x) = 0$

Here, $U[X]^m$ denotes the set of polynomials with coefficients in U and degree at most m.

Note that the first four axioms ensure that τ consists of the neighbourhoods of 0 of a non-discrete Hausdorff ring topology of K. The fifth axiom implies that the topology is a V-topology and – together with (T1)–(T4) – that it is in fact a field topology. The final axiom scheme can be seen as a non-uniform version of henselianity.

Being t-henselian is an elementary property (in $\mathcal{L}_{ring}$): If K is not separably closed, then K admits only one t-henselian topology and this topology is first-order definable in the language of rings. Fix any irreducible, separable polynomial $f \in K[X]$ with $\deg(f) > 1$ and $a \in K$ satisfying $f'(a) \neq 0$. We define

$$U_{f,a} := \left\{ f(x)^{-1} - f(a)^{-1} \;\middle|\; x \in K \right\}.$$

Then the sets $c \cdot U_{f,a}$ for $c \in K^\times$ form a basis of open neighbourhoods of 0 of the (unique) t-henselian topology on K (see [Pre91], p. 203).

In particular, we get the following

Remark ([PZ78], Remark 7.11)**.** *If K is not separably closed and admits a t-henselian topology, then every field elementarily equivalent to K carries a t-henselian topology.*

Note that henselian fields are of course t-henselian with the topology being the valuation topology. In the axiom scheme, we can choose $U = \mathfrak{m}$ for any $n \in \mathbb{N}$. If we take a t-henselian field, any sufficiently saturated elementarily equivalent field will carry a henselian valuation:

Theorem 2.2.1. *Let K be a non separably closed field. Then K is t-henselian iff K is elementarily equivalent to some field admitting a non-trivial henselian valuation.*

Proof: [PZ78], Theorem 7.2.

Thus t-henselianity – like henselianity – goes up finite extensions.

We would like to know for which (elementary) classes of fields t-henselianity is in fact equivalent to henselianity. As we already know, this does not hold for real closed fields. Unfortunately, these are not the only exception:

Example ([PZ78], p. 338)**.** *There are t-henselian fields which are neither henselian nor real closed.*

For the remainder of this section, let K be neither separably nor real closed.

2.2.2 Finding a Parameter-Free Definable Valuation

The question of how t-henselianity relates to henselianity is closely connected to the question whether a henselian field carries a definable henselian valuation. If a field admits a parameter-free definable non-trivial henselian valuation, then the same formula defines a non-trivial henselian valuation in any field elementarily equivalent to K. In case the definition requires parameters, there could still be a field $L \equiv K$ which is not henselian, since any given parameter from L might only ensure that Hensel's lemma holds for polynomials up to some degree n.

We can use the uniform definition theorem for canonical p-henselian valuations from the last section to improve one of the main theorems in [Koe94], which shows that every t-henselian field admits a definable valuation inducing the henselian topology.

Theorem 2.2.2. *Any t-henselian field K carries a $\emptyset$-definable valuation which induces the (unique) t-henselian topology on K.*

Proof: Note that without loss of generality, we may assume that K is henselian. By Theorem 2.2.1, any sufficiently saturated elementary extension K' of K is henselian. A base of the t-henselian topology can be defined using the same formulas on K' as on K (see the remark in section 2.2.1). Thus, it suffices to give a parameter-free definition of a valuation on K' inducing the t-henselian topology. The same formula will then define such a valuation on K.

First we assume that there is some p with $K \neq K(p)$ (and $p \neq 2$ if K is euclidean). In case $\mathrm{char}(K) \neq p$ and K does not contain a primitive pth root of unity, consider $K(\zeta_p)$. Since $K(\zeta_p)$ is a finite Galois extension of K and the coefficients of the minimal polynomial of this extension are all in $\mathrm{dcl}_K(\emptyset)$, $K(\zeta_p)$ is interpretable without parameters in K. Hence, it suffices to define a valuation on $K(\zeta_p)$ without parameters which induces the same topology on $K(\zeta_p)$ as $v_{K(\zeta_p)}$. The restriction of such a valuation to K is then again $\emptyset$-definable and induces the henselian topology on K by Theorem 1.3.4. But by Theorem 2.1.5, some non-trivial coarsening of $v^p_{K(\zeta_p)}$ is $\emptyset$-definable on $K(\zeta_p)$. As $v_{K(\zeta_p)}$ is in particular p-henselian, these valuations are comparable and thus induce the same topology.

Otherwise, we have that $K = K(p)$ for all primes p with $p \mid \#G_K$ (except for $p = 2$ if K is euclidean). We may assume that K is not euclidean, since – as above – it suffices to define a suitable valuation without parameters on $K(i)$. Furthermore, there must be at least two primes p_1 and p_2 with $p_1, p_2 \mid \#G_K$, else K would be separably or real closed. Say $\mathrm{char}(Kv_K) \neq p_1$. If $\mathrm{char}(K) \neq p_1$, we may again assume $\zeta_{p_1} \in K$.

Claim: For any finite Galois extension L of K with $L \neq L(p_1)$, we have $Lv_L \neq Lv_L(p_1)$.

Proof of Claim: As $K = K(p_1)$, we get $Kv_K = Kv_K(p_1)$. Thus $v_K K$ is p_1-divisible. Let now L be a finite Galois extension of K with $L \neq L(p_1)$. Note that v_L is the unique prolongation of v_K to L, so $v_L L$ is also p_1-divisible. Since $\mathrm{char}(Lv_L) = \mathrm{char}(Kv_K) \neq p_1$, we conclude $Lv_L \neq Lv_L(p_1)$.

Let n be an integer such that there exists a Galois extension L of K, $[L : K] = n$, with $L \neq L(p_1)$. By the claim, we get that v_L is a coarsening of $v_L^{p_1}$ for any such L. Hence, $v_K = v_L|_K$ is a coarsening of $v_L^{p_1}|_K$. Consider the valuation ring

$$\bigcup \Big(\mathcal{O}_{v_L^{p_1}} \cap K \;\Big|\; K \subseteq L \text{ Galois, } [L : K] = n,\, L \neq L(p_1) \Big)$$

on K. By Theorem 2.1.5 and the remarks above, this is a $\emptyset$-definable refinement of v_K. Thus, it gives a non-trivial $\emptyset$-definable valuation on K inducing the same topology as v_K. □

We will come back to definitions (with and without parameters) of henselian valuations later, and focus on when t-henselianity implies henselianity for now.

2.2.3 Henselianity as an Elementary Property

Note that Theorem 1.3.6 gives a criterion to test the existence of a henselian valuation:

Corollary 2.2.3. *Let K be a field, and assume that G_K is small. If some $L \equiv K$ admits a henselian valuation which is tamely branching at some prime p, then so does K.*

Proof: Assume that $L \equiv K$ admits a henselian valuation tamely branching at p. Then $G_L \equiv G_K$, and – as G_K is small – we get $G_L \cong G_K$. Thus, if G_L has a non pro-cyclic p-Sylow subgroup $P \ncong \mathbb{Z}_2 \rtimes \mathbb{Z}/2\mathbb{Z}$ with a non-trivial abelian normal subgroup, then so has G_K. □

As the corollary depends on the absolute Galois group of K, there is no single elementary sentence ensuring the existence of a tamely branching henselian valuation on a field – it is rather an elementary property in the sense that it is inherited by elementarily equivalent fields.

¿From the previous section, we get the following

Proposition 2.2.4. *Let K be a field such that G_K is pro-p for some prime p. If K is t-henselian, then K is henselian.*

Proof: By our general assumption, K is not real closed and hence in particular not euclidean. Note that if $\mathrm{char}(K) \neq p$, then K contains a primitive pth root of unity since otherwise ζ_p would generate a Galois extension of degree prime to p. As K is not separably closed and G_K is pro-p, we get $K \neq K(p) = K^{sep}$.

As K is t-henselian, $K \equiv L$ for a henselian valued field L. Since $L \neq L(p)$, L is in particular p-henselian. By Proposition 2.1.3, this is an elementary property. Thus, K is p-henselian and hence, as G_K is pro-p, henselian. □

As we have seen before, henselianity goes down from fixed fields of Sylow subgroups. This implies the following criterion:

Proposition 2.2.5. *Let K be a t-henselian field such that G_K is pro-nilpotent. Then K is henselian.*

Proof: First we remark that we may assume that K is not euclidean: If K is euclidean, then $G_{K(i)}$ is also pro-nilpotent. By Theorem 1.3.4, it suffices to show that $K(i)$ is henselian. As K was not real closed, $K(i)$ is also not separably closed.

If G_K is pro-nilpotent, then by Proposition 2.3.8 in [RZ00], G_K has a unique p-Sylow $\mathrm{Syl}_p(G_K)$ for each prime p and

$$G_K \cong \prod_p \mathrm{Syl}_p(G_K).$$

If G_K is pro-p, then $K(p) = K^{sep}$ and hence the definable p-henselian valuation on K is a $\emptyset$-definable henselian valuation. Thus, we may assume that there are at least two primes $p_1, p_2 \mid \#G_K$. As in the previous proofs, we may further assume that K contains a primitive p_1th and p_2th root of unity.

Claim 1: There is a prime p and a p-henselian valuation v on K such that the (unique) prolongation of v to $K(p) \neq K$ is defectless.

Proof of Claim 1: We start with the case $\mathrm{char}(K) > 0$. Take p such that $p \neq \mathrm{char}(K)$ and $p \mid \#G_K$. Then the extension of v_K^p to $K(p)$ is defectless.

Now assume $\mathrm{char}(K) = 0$. We need to find a p-henselian valuation v such that $\mathrm{char}(Kv) \neq p$. Take $p_1, p_2 \mid \#G_K$. If $v_K^{p_1}$ and $v_K^{p_2}$ have characteristic p_1 and p_2 respectively, we consider the finest common coarsening v of $v_K^{p_1}$ and $v_K^{p_2}$. As K is t-henselian, $v_K^{p_1}$ and $v_K^{p_2}$ induce the same topology and are thus dependent. Hence, v is a non-trivial p_1-henselian valuation on K. Since v is coarser than both $v_K^{p_1}$ and $v_K^{p_2}$, we obtain $\mathrm{char}(Kv) = 0$.

Fix p and v as given by the claim. Now $P := \mathrm{Syl}_p(G_K)$ is the group-theoretic complement of $Q := \prod_{q \neq p} \mathrm{Syl}_q(G_K)$ in G_K and $K(p) \cong \mathrm{Fix}(Q)$.

Claim 2: $\mathrm{Fix}(P)$ is henselian.

Proof of Claim 2: Let w be some prolongation of v to $\mathrm{Fix}(P)$ and consider a finite Galois extension L' of $\mathrm{Fix}(P)$. As $G_P \cong \mathrm{Syl}_p(G_K)$ is pro-p, we get $[L' : \mathrm{Fix}(P)] = p^n$ for some $n \in \mathbb{N}$. Since Q is a complement to $\mathrm{Syl}_p(G_K)$, L' corresponds exactly to a finite Galois extension $K \subseteq K' \subsetneq K(p)$. As v extends uniquely and without defect to $K(p)$, w extends uniquely to K'.

Hence by Theorem 1.3.4, v is henselian. □

The next generalization of Proposition 2.2.5 is very useful. Recall our general assumption that K **is neither separably nor real closed.**

Proposition 2.2.6. *Let K be a t-henselian field. If G_K is pro-soluble or if K is of finite transcendence degree over its prime field, then K is henselian.*

Proof: [Koe04], Lemma 3.5.

To conclude this section, we give a stronger version of Proposition 2.2.6. The following group-theoretic definition is taken from [NS07], p. 174.

Definition. *Let G be a profinite group. We say that G is* universal *if every finite group occurs as a continuous subquotient of G.*

Note that for a field K, G_K is non-universal iff there is some $n \in \mathbb{N}$ such that the symmetric group S_n does not occur as a Galois group over any finite Galois extension of K.

Example. *All of the following profinite groups are non-universal:*

1. *pro-abelian groups,*
2. *pro-nilpotent groups,*
3. *pro-soluble groups,*
4. *any group G such that $p \nmid \#G$ for some prime p.*

Non-abelian free profinite groups are of course universal, and so are absolute Galois groups of hilbertian fields.

We will make use of the following statement:

Theorem 2.2.7. *Let K be a field and L and M algebraic extensions of K which both carry non-trivial henselian valuations. Assume further that G_L is non-trivial pro-p and G_M non-trivial pro-q for primes $p < q$. Let v and w be (not necessarily proper) coarsenings of the canonical henselian valuations on L and M respectively, and, if $p = 2$ and Lv is real closed, assume v to be the coarsest henselian valuation on L with real closed residue field. Then either G_K is universal or $v|_K$ and $w|_K$ are comparable and the coarser valuation is henselian on K.*

Proof: [Koe05b], Theorem 3.1.

Now we can use Theorem 2.2.7 to deduce henselianity from p- and q-henselianity:

Proposition 2.2.8. *Suppose G_K is non-universal, and $K(p) \neq K \neq K(q)$ for two primes $p < q$. In case $p = 2$, assume further that K is not euclidean. If K is p- and q-henselian, then K is henselian.*

Proof: Consider the henselization L' (respectively M') of K with respect to the canonical p-henselian valuation v_K^p (the canonical q-henselian valuation v_K^q) on K. Then define L to be the fixed field of a p-Sylow of L', and M accordingly.

Claim: L is not separably closed.

Proof of Claim: We need to show that L' is not p-closed. But if $\alpha \in K(p)$ has degree p^n over K, then – as v_K^p is p-henselian – α is moved by some element of $D(K(p)/K)$. As decomposition groups behave well in towers, we get $\alpha \notin L$.

In case $p = 2$, the same argument shows that L is also not real closed. Since L is p-henselian and G_L is pro-p, L is also henselian, and likewise is M. Now we consider the canonical henselian valuations v_L on L and the canonical henselian valuation v_M on M. If $p = 2$ and Lv_L is real closed, we replace v_L by the coarsest henselian valuation on L with real closed residue field. As L is not real closed, this is again a non-trivial henselian valuation.

By Theorem 2.2.7, the restrictions $v_L|_K$ and $v_M|_K$ are comparable and the coarser one is henselian. As L and M are algebraic extensions of K, none of the restrictions is trivial. Hence, K is henselian. □

Corollary 2.2.9. *Let G_K be non-universal, and assume that K is t-henselian. Then K is henselian.*

Proof: We consider the following cases. Either G_K is pro-p for some prime p or there are two primes p and q with $p, q \mid \#G_K$. As before, we may assume that K is not euclidean since it suffices to show that $K(i)$ is henselian (see Theorem 1.3.4). Note that if G_K is non-universal, then so is $G_{K(i)}$.

If G_K is pro-p, then K is henselian by Proposition 2.2.4.

Otherwise, there are primes $p < q$ such that $p, q \mid \#G_K$. Let $K' \supset K$ be a finite Galois extension containing a primitive pth and qth root of unity such that $K'(p) \neq K' \neq K'(q)$. As t-henselianity goes up finite algebraic extensions, K' is again t-henselian. By Proposition 2.1.3, K' is in particular p- and q-henselian. Applying the proposition to K' yields a non-trivial henselian valuation v on K'. In particular, the canonical henselian valuation $v_{K'}$ is non-trivial. As the extension is Galois, the restriction $v_{K'}|_K$ gives a non-trivial henselian valuation on K

by Theorem 1.3.4. □

Remark. *We will give several parameter-free definitions of henselian valuations in the third chapter. This will give examples of more classes of fields for which henselianity is an* $\mathcal{L}_{ring}$*-elementary property.*

2.3 An Example: PAC and Hilbertian Fields

There are well-known examples of fields which are not henselian, such as number fields and PAC fields. For these one can show that they are not even p-henselian using the same methods as to show that they are not henselian.

2.3.1 Hilbertian Fields

Number fields, as well as function fields of curves over finite fields, are examples of hilbertian fields. In order to introduce them, we need the notion of a Hilbert set.

Definition. *Let* K *be a field and* $\overline{T}$ *and* $\overline{X}$ *two sets of variables. Consider polynomials* $f_1(\overline{T},\overline{X}),\ldots,f_n(\overline{T},\overline{X})$ *in* $\overline{X}$ *with coefficients in* $K(\overline{T})$*, and assume that these are irreducible in* $K(\overline{T})[\overline{X}]$*. For* $g \in K[\overline{T}] \setminus \{0\}$*, define the set*

$$H_K(f_1,\ldots,f_n;g) = \left\{ \overline{a} \in K \;\middle|\; g(\overline{a}) \neq 0 \textit{ and } f_1(\overline{a},\overline{X}),\ldots,f_n(\overline{a},\overline{X}) \textit{ are defined and irreducible in } K[\overline{X}] \right\}.$$

Then $H_K(f_1,\ldots,f_n;g)$ *is called a* Hilbert set *of* K*.*

Note that *Hilbert's Irreducibility Theorem* states that any Hilbert set of $\mathbb{Q}$ is nonempty. This motivates the following

Definition. *A field* K *is called* hilbertian *if all its Hilbert sets are nonempty.*

One consequence of Hilbert's irreducibilty theorem is that all finite symmetric groups S_n and alternating groups A_n are realizable as Galois groups over $\mathbb{Q}$. This holds in fact for any hilbertian field (see [FJ08], Proposition 16.6.4). Hence, absolute Galois groups of hilbertian fields are universal.

Example ([FJ08], Theorem 13.4.2)**.** *Global fields (i.e. number fields and function fields of curves over finite fields) are hilbertian.*

Proposition 2.3.1. *No hilbertian field is henselian.*

Proof: [FJ08], Lemma 15.5.4.

The proof of the above proposition can be adapted to prove the following

Lemma 2.3.2. *Let K is a hilbertian field. Then for any prime p we have $K \neq K(p)$ and K is not p-henselian.*

Proof: If K is hilbertian, then any finite abelian group is realizable as a Galois group over K by [FJ08], Corollary 16.3.6. In particular, $K \neq K(p)$ for any p. We first treat the case $\mathrm{char}(K) \neq p$. Then we may assume that K contains a primitive pth root of unity as $K(\zeta_p)$ is again hilbertian (see [FJ08], Theorem 13.9.1). If $K(\zeta_p)$ was p-henselian then so would be K by Proposition 2.1.7.

Let v be a non-trivial valuation on K. Choose $m \in \mathfrak{m}_v \setminus \{0\}$ and consider the irreducible polynomial $f(T, X) = X^p + mT - 1$ in $K(T)[X]$. If K is hilbertian, there exists an $a \in K^\times$ such that $f(a, X)$ is irreducible in $K[X]$. Furthermore, by exercise 13.4 in [FJ08], a may be chosen in $\mathcal{O}_v$. But now $f(a, X)$ splits in $K(p)$, and has a simple zero in Kv. Hence, by Proposition 2.1.1, v cannot be p-henselian.

In case $\mathrm{char}(K) = p$, the same argument as above applies to the polynomial $f(T, X) = X^p + X + mT - 2$. □

2.3.2 PAC Fields

We now want to show the corresponding result for PAC fields. Recall that a field K is PAC if every absolutely irreducible variety over K has a K-rational point. Note that this implies in particular that every absolutely irreducible variety over K has infinitely many K-rational points ([FJ08], Proposition 11.1.1). Even though the properties of hilbertianity and PAC are somewhat opposing, there are fields which are hilbertian and PAC (cf. [FJ08], Proposition 13.4.6).

The statement that a field is PAC is elementary in $\mathcal{L}_{ring}$. Furthermore, every algebraic extension of a PAC field is again PAC. This can be used to show the following

Theorem 2.3.3 (Frey-Prestel). *If K is PAC and henselian, then K is separably closed.*

Proof: [FJ08], Corollary 11.5.6.

Following the proof of the above theorem, we can prove that PAC fields which allow Galois extensions of degree p are in particular not p-henselian.

Lemma 2.3.4 (Kaplansky-Krasner for p-henselian valuations). *Assume that (K, v) is p-henselian and take $f \in K[X]$ separable, $\deg(f) > 1$, such that f splits in $K(p)$. Suppose for each $\gamma \in vK$ there exists some $x \in K$ such that $v(f(x)) > \gamma$. Then f has a zero in K.*

Proof: Without loss of generality we may assume that f is monic and that $\deg(f) = n > 0$. Write

$$f(X) = \prod_{i=1}^{n} (X - x_i)$$

for $x_i \in K(p)$. Take $\gamma > n \cdot \max\{\, v(x_i - x_j) \mid 1 \leq i < j \leq n \,\}$, and choose $x \in K$ such that

$$v(f(x)) = \sum_{i=1}^{n} v(x - x_i) > \gamma.$$

Hence, for some j with $1 \leq j \leq n$, we get $v(x - x_j) > \gamma/n$. If $x_j \notin K$, then there is some

$\sigma \in \mathrm{Gal}(K(p)/K)$ such that $\sigma(x_j) \neq x_j$. Thus, we get

$$v(x - \sigma(x_j)) = v(\sigma(x - x_j)) = v(x - x_j) > \frac{\gamma}{n},$$

where the last equality holds as v is p-henselian. Therefore,

$$v(x_j - \sigma(x_j)) \geq \min\{\, v(x_j - x), v(x - \sigma(x_j))\,\} > \frac{\gamma}{n}$$

which contradicts the choice of γ. Hence, we conclude $x_j \in K$, so f has a zero in K. □

Corollary 2.3.5. *Let K be a field with $K \neq K(p)$. If K is PAC, then K is not p-henselian.*

Proof: Assume that K is PAC and p-henselian. We show that then $K = K(p)$. Take $f \in K[X]$ a separable, irreducible polynomial with $\deg(f) > 1$ splitting in $K(p)$. It suffices to show that for all $c \in K^\times$ there exists an $x \in K$ such that $v(f(x)) \geq v(c)$, as then f has a zero in K.

Consider the curve $g(X,Y) = f(X)f(Y) - c^2$. Consider $g(X,Y)$ as a polynomial over $K^{sep}[Y]$. Eisenstein's criterion ([FJ08], 2.3.10(b)) applies over this ring to any linear factor of $f(Y)$, thus $g(X,Y)$ is absolutely irreducible. As K is PAC, there exist $x, y \in K$ such that $f(x)f(y) = c^2$. Thus, either $v(f(x)) \geq v(c)$ or $v(f(y)) \geq v(c)$. □

2.3.3 A Separating Sentence

In the next part of this section, we give an elementary statement which separates the theory of any PAC field K with $K \neq K(p)$ and $\mathrm{char}(K) \neq p$ from the theory of any such henselian field. As PAC is an elementary property, the theories of any given PAC field K_1 and henselian field K_2 are separated by the sentence that some absolutely irreducible curve C over K_2 has no rational point. But as the degree of C might vary, this sentence can not be chosen uniformly when we vary K_1 and K_2. We use the p-henselian topology to find a uniform statement. We already know that being p-henselian gives a separating sentence which holds in any henselian field which allows p-degree extensions but in no such PAC field, and it neither holds in any hilbertian field.

Conjecturally, any field K with $G_K \cong \mathbb{Z}_p$ is either henselian or PAC. We now give a simplified criterion to separate PAC and henselian fields which we hope might become useful for proving this dichotomy.

Theorem 2.3.6. *Let K be a p-henselian field with* $\mathrm{char}(K) \neq p$ *and* $\zeta_p \in K$*. The sets*

$$\left\{ ax^p - a \;\middle|\; x \in K^\times \right\} \cap \left\{ bx^p - b \;\middle|\; x \in K^\times \right\}$$

with $a, b \neq 0$ form a base of neighbourhoods of 0 for the p-henselian topology on K.

Proof: [Koe95], Theorem 2.1.

Let us consider the set $U = (K^\times)^p - 1$. We will now show that for a PAC field K with $K \neq K(p)$ and $\mathrm{char}(K) \neq p$, the axiom (T2) of V-topologies does not hold for the sets defined above. Define

$$V_a \coloneqq \left\{ ax^p - a \;\middle|\; x \in K^\times \right\}.$$

Then we show that there are no $a, b \in K^\times$ such that

$$(V_a \cap V_b) - (V_a \cap V_b) \subseteq U$$

holds.

Let us first consider the case $a = b$. Then

$$V_a - V_a = \left\{ a(x^p - y^p) \;\middle|\; x, y \in K^\times \right\}.$$

We claim that $V_a - V_a = K$. If G_K is small, this already follows from the following fact:

Theorem 2.3.7. *Let K be an infinite field and $G < K^\times$ a subgroup of finite index. Then $K = G - G$.*

Proof: [BS92], Theorem 1.7 and Remark 1.9.

For the general case, we consider the parameterized curve

$$W_{z,a} : a(X_1^p - X_2^p) = z$$

for any $z \in K$. Then $z \in V_a - V_a$ iff $W_{z,a}$ has a K-rational point. Consider the polynomial

$$f_{z,a}(X_2) = X_2^p - \frac{z}{a}.$$

Lemma 2.3.8. *Let $g_1, \ldots, g_m \in K[X]$ be irreducible polynomials, none of which is a multiple of the other. Assume that* $\mathrm{char}(K) \nmid n$*. Then the algebraic set W defined by the system*

$$g_i(X) - Y_i^n = 0 \textit{ for } i = 1, \ldots, m$$

is an absolutely irreducible variety.

Proof: [FJ08], Exercise 10.4.

Let $g_{1,z,a}, \ldots, g_{m,z,a}$ be the irreducible factors of $f_{z,a}$ in $K[x]$. As $\mathrm{char}(K) \neq p$, $f_{z,a}$ has only simple roots in K^{sep}, so no two of the $g_{i,z,a}$ are multiplicatively dependent. Since K is PAC, the variety defined by

$$g_{i,z,a} - Y_i^p = 0 \text{ for } i = 1, \ldots, m$$

has a K-rational point. Hence, $W_{z,a}$ has a K-rational point for all $z \in K$ and $a \in K^\times$. Thus $V_a - V_a = K$.

Now we assume $a \neq b$. For $z \in K$ and $a, b \in K^\times$ consider the curve

$$W_{z,a,b} : (aX_1{}^p - a) - (bX_2{}^p - b) = 0 \wedge (aY_1{}^p - a) - (bY_2{}^p - b) = 0 \wedge z = a(X_1{}^p - Y_1{}^p).$$

Clearly, if $W_{z,a,b}$ has a rational point, then $z \in V_a \cap V_b - V_a \cap V_b$. We can rewrite $W_{z,a,b}$ as

$$\begin{aligned} X_1^p &= f_{1,z,a,b}(Y_1) \coloneqq Y_1^p + \frac{z}{a} \\ X_2^p &= f_{2,z,a,b}(Y_1) \coloneqq \frac{1}{b}(z + aY_1^p - a + b) \\ Y_2^p &= f_{3,z,a,b}(Y_1) \coloneqq \frac{1}{b}(aY_1^p - a + b) \end{aligned}$$

Note that for given $a, b \in K^\times$, there are at most finitely many $z \in K$ such that two of the irreducible factors of the polynomials $f_{i,z,a,b}$ in $K[X]$ are multiples of each other. Hence, using Lemma 2.3.8 and the fact that K is PAC as before, we obtain that for almost all $z \in K$, $W_{z,a,b}$ has a K-rational point. In particular, $V_a \cap V_b - V_a \cap V_b$ is cofinite in K, thus

$$V_a \cap V_b - V_a \cap V_b \not\subset U.$$

We summarize the above as the following

Theorem 2.3.9. *Let K be a field containing a primitive pth root of unity such that $K \neq K(p)$ and* $\mathrm{char}(K) \neq p$. *Consider*

$$\varphi \equiv \exists a, b \neq 0\ \forall z\ \Big(\exists x_1, x_2, y_1, y_2\ ax_1^p - a = bx_2^p - b \wedge ay_1^p - a = by_2^p - b \\ \wedge z = a\left(x_1^p - y_1^p\right) \longrightarrow \exists w\ z = w^p - 1\Big)$$

If K is p-henselian, then $K \models \varphi$. If K is PAC, then $K \models \neg\varphi$.

Chapter 3

Definable Henselian Valuations

3.1 Conditions on the Residue Field

3.1.1 p-Henselian Valuations as Henselian Valuations

Let K be a henselian field and p a prime such that $K \neq K(p)$. We want to find out in which cases v_K^p is henselian, as this gives us a $\emptyset$-definable non-trivial henselian valuation on K. Then, we can use v_K^p to define henselian valuations on Galois subfields of finite index in K which do not admit any Galois extensions of p-power degree.

As any henselian valuation is in particular p-henselian and all p-henselian valuations are comparable to v_K^p, we have either $v_K^p \supseteq v_K$ or $v_K^p \subsetneq v_K$. In the first case, v_K^p is henselian. As we will make use of this fact many times later, we note here that this is in fact an equivalence:

Lemma 3.1.1. *Let K be a henselian field with $K \neq K(p)$ for some prime p. Then v_K^p is henselian iff v_K^p coarsens v_K.*

Proof: Obviously, coarsenings of henselian valuations – like v_K – are henselian. Conversely, assume that v_K^p is henselian and a proper refinement of v_K. Then, by the definition of v_K, we get $v_K^p \in H_2(K)$ and hence $v_K \in H_2(K)$. In this case, v_K^p has a proper coarsening with p-closed residue field, contradicting the definition of v_K^p. □

As there are fields which are t-henselian but not henselian, not every henselian field admits a parameter-free definable non-trivial henselian valuation. However, the only known examples of henselian valued fields which do not admit any non-trivial definable henselian valuation are separably closed and real closed fields. In fact, no separably or real closed field carries any non-trivial definable valuation (see [Koe94], p. 1/2). We return to our standing assumption from the previous chapter:

In this section, let K be neither separably nor real closed.

In the previous chapter (Proposition 2.2.4), we have already seen that p-henselian valuations are henselian when G_K is pro-p.

Proposition 3.1.2. *Let K be henselian and assume that G_K is pro-p. Then K admits a $\emptyset$-definable non-trivial henselian valuation.*

Proof: As G_K is pro-p, we get $K^{sep} = K(p)$ and $\zeta_p \in K$ as in Proposition 2.2.4. Hence, any p-henselian valuation is henselian on K. As K is not real closed, it is not euclidean and thus admits a $\emptyset$-definable non-trivial p-henselian valuation by either Theorem 2.1.5 or Theorem 2.1.4. □

3.1.2 Separably Closed, Real Closed, or p-Closed Residue Fields

We now give several conditions on the residue field of a henselian valued field (K, v) which imply the existence of some $\emptyset$-definable non-trivial henselian valuation on K. First, we consider henselian fields K on which some henselian valuation has a separably closed residue field, i.e. with $v_K \in H_2(K)$.

Theorem 3.1.3. *Assume that K is henselian with respect to a valuation with separably closed residue field. Then K admits a non-trivial $\emptyset$-definable henselian valuation.*

Proof: We show first that G_K is pro-soluble. If K is henselian with respect to a valuation with separably closed residue field, then v_K has also separably closed residue field. Let w be the prolongation of v_K to K^{sep}. Recall that there is an exact sequence

$$I_w \longrightarrow G_K \longrightarrow G_{Kv_K}$$

where I_w denotes the inertia group of w over K (Proposition 1.2.3). Hence, as I_w is pro-soluble (see Proposition 1.2.4), so is G_K.

Thus there is some prime p with $K \neq K(p)$. But now v_K^p is indeed a coarsening of v_K: Otherwise, the definition of v_K^p would imply $Kv_K \neq Kv_K(p)$. Note that as Kv_K is separably closed, K is not formally real. Thus, if K contains a primitive pth root of unity or $\mathrm{char}(K) = p$, then v_K^p is $\emptyset$-definable and henselian. Else, we consider the $\emptyset$-definable extension $K(\zeta_p)$. The canonical henselian valuation on $K(\zeta_p)$ still has separably closed residue field, therefore $v^p_{K(\zeta_p)}|_K$ gives a $\emptyset$-definable non-trivial henselian valuation on K. □

The next corollary generalizes this to what are called *almost real closed* fields in [DF96].

Corollary 3.1.4. *Let K be henselian with respect to a valuation with real closed residue field. Then K admits a non-trivial $\emptyset$-definable henselian valuation.*

Proof: By our general assumption, K is not real closed. As K is henselian with respect to a valuation v with real closed residue field, K is formally real. Now the unique prolongation of v to $K(i)$ has separably closed residue field. By Theorem 3.1.3, there is some prime p such that $v^p_{K(i)}$ is a non-trivial $\emptyset$-definable henselian valuation on $K(i)$. Its restriction to K is henselian by Lemma 3.1.1, and thus – interpreting $K(i)$ in K – we get a non-trivial $\emptyset$-definable henselian valuation on K. □

We already know that v_K^p is henselian if it coarsens v_K, e.g. in case $v_K \in H_2^p(K)$.

Lemma 3.1.5. *Let K be a henselian field with $K \neq K(p)$. In case* $\mathrm{char}(K) \neq p$*, assume further $\zeta_p \in K$. If $Kv_K = Kv_K(p)$, then K is henselian with respect to a non-trivial $\emptyset$-definable valuation.*

Proof: By assumption and Theorem 2.1.5, v_K^p is $\emptyset$-definable. As $Kv_K = Kv_K(p)$, v_K^p coarsens v_K and is thus henselian. □

Example. *Let K be a field which is not separably closed but admits no abelian extensions. If Γ is a non-divisible ordered abelian group, then $K((\Gamma))$ is henselian with respect to a non-trivial $\emptyset$-definable valuation by Lemma* 3.1.5.

3.1.3 Not p-Henselian Residue Fields

Next, we show that we can use the canonical p-henselian valuation to define a henselian valuation on henselian fields for which the residue fields are not p-henselian.

Proposition 3.1.6. *Let (K, v) be a non-trivially henselian valued field. Assume that the residue field Kv is not p-henselian and that $Kv \neq Kv(p)$. If $p = 2$, assume further that Kv is not euclidean. Then v is $\emptyset$-definable.*

Proof: Let p and (K, v) be as above. If $\mathrm{char}(K) \neq p$, we assume furthermore $\zeta_p \in K$ for now.

Note that $K \neq K(p)$ (see [EP05], Theorem 4.2.6). Thus K is p-henselian. We claim that $v_K^p = v$. As v is henselian, it is in particular p-henselian and hence comparable to v_K^p. Since Kv is not p-henselian, v_K^p is a coarsening of v, as otherwise v_K^p would induce a p-henselian valuation on Kv ([EP05], Corollary 4.2.7). Assume v_K^p is a proper coarsening of v. Then we get $v \in H_2^p(K)$ and hence $Kv = Kv(p)$, contradicting our assumption on Kv. This proves the claim.

For $p = 2$, we get from our assumption that $Kv_K^2 = Kv$ is not euclidean. Thus v_K^p is henselian and $\emptyset$-definable by Theorem 2.1.5.

In case $\mathrm{char}(K) \neq p$ and K does not contain a primitive pth root of unity, we consider $K' = K(\zeta_p)$. As this is a $\emptyset$-definable extension of K, it suffices to define the – by henselianity unique – prolongation v' of v to K'. Since $K'v'$ is a finite normal extension of Kv, it still satisfies $K'v' \neq K'v'(p)$ and is furthermore not p-henselian by Proposition 2.1.7. Now v' is $\emptyset$-definable as above, and thus also v. □

Combining this with our observations about hilbertian fields, we get the following

Corollary 3.1.7. *There is an $\mathcal{L}_{ring}$-formula ϕ which defines the valuation ring of the power* *series valuation v on $K((\Gamma))$, where K ranges over all hilbertian fields and Γ ranges over all non-trivial ordered abelian groups.*

Proof: If K is hilbertian, then $K \neq K(p)$ for all p by Lemma 2.3.2. Furthermore, K is not p-henselian for any p. Fix some prime $p > 2$, and let $L := K((\Gamma))$. In case $\mathrm{char}(L) = p$ or $\zeta_p \in L$, v_L^p is uniformly definable by Corollary 2.1.6, and by the proof of the above proposition defines the power series valuation on L. If $\mathrm{char}(L) \neq p$ and $\zeta_p \notin L$, the restriction of the power series valuation on $L(\zeta_p)$ to L is again uniformly definable and gives the power series valuation on L. By differentiating between these two cases, we get a uniform definition of the power series valuation on $K((\Gamma))$. □

3.1.4 Not Virtually p-Henselian Residue Fields

In order to generalize the proposition to a wider class of fields, we need the following

Definition. *Let K be a field. We call K* virtually p-henselian *if $p \mid \#G_K$ and there is some finite Galois extension L of K with $L \neq L(p)$ such that L is p-henselian.*

If $K \neq K(p)$, then K is virtually p-henselian iff it is p-henselian by Proposition 2.1.7.

Example. *A PAC field K is not virtually p-henselian for any prime p with $p \mid \#G_K$: Any algebraic extension of K is again PAC, so if L is a finite Galois extension of K and $L \neq L(p)$, then L is not p-henselian by Corollary* 2.3.5.

Theorem 3.1.8. *Let (K, v) be a non-trivially henselian valued field with $p \mid \#G_{Kv}$, and if $p = 2$ assume that Kv is not euclidean. If Kv is not virtually p-henselian, then v is $\emptyset$-definable on K.*

Proof: If Kv is not virtually p-henselian and $Kv \neq Kv(p)$, then v is $\emptyset$-definable by Proposition 3.1.6.

In case $Kv = Kv(p)$, we take some finite Galois extension L of Kv with $L \neq L(p)$. By assumption, no such L is p-henselian. As Kv is not euclidean, L is also not euclidean.

Using Proposition 2.1.7, we may assume that L contains a primitive pth root of unity in case $\mathrm{char}(Kv) \neq p$. Say $[L : Kv] = n$.

Consider any finite Galois extension M of K, with w the unique prolongation of v to M, such that $Mw = L$. As before, w is $\emptyset$-definable on M and hence, by interpreting M in K using parameters, so is its restriction v to K.

Thus it remains to show that a definition can be found without parameters. The interpretation of Galois extensions of a fixed degree of K can be done uniformly with respect to the parameters (namely the coefficients of a minimal polynomial generating the extension). By Theorem 2.1.5, the definition of the p-henselian valuations on these can also be done uniformly. To make sure that the residue field of the canonical p-henselian valuation of a finite Galois extension of K corresponds to a field L as described above, we just need to restrict to extensions M of K with $v_M^p \in H_1^p(M)$. Hence, we get the desired definition by

$$\bigcap \big(\mathcal{O}_{v_M^p} \cap K \;\big|\; K \subseteq M \text{ Galois, } [M : K] = n,\, M \neq M(p), \\ \zeta_p \in M \text{ if } \mathrm{char}(M) \neq p,\, v_M^p \in H_1^p(M)\big).$$

□

Corollary 3.1.9. *Let p be a prime and let K be a field such that $p \mid \#G_K$ and that K is not virtually p-henselian. If $p = 2$, assume that K is not euclidean. Then the power series valuation is $\emptyset$-definable on $K((\Gamma))$, for any ordered abelian group Γ.*

Example. *Let K be PAC and not separably closed. Then – as PAC fields are not euclidean – the power series valuation is definable on $K((\Gamma))$, for any ordered abelian group Γ.*

3.2 Galois-Theoretic Conditions

Next, we define henselian valuations on t-henselian fields with non-universal absolute Galois group. We return to our general assumption:

In this section, let K be neither separably nor real closed.

3.2.1 Pro-Nilpotent Absolute Galois Groups

First, we show that the henselian valuation we found in Proposition 2.2.5 is in fact definable:

Proposition 3.2.1. *Let K be a t-henselian field such that G_K is pro-nilpotent. Then K is henselian with respect to a non-trivial $\emptyset$-definable valuation.*

Proof: Recall that in the proof of Proposition 2.2.5, we constructed a $\emptyset$-definable p-henselian valuation v on K, such that any prolongation of v to $\mathrm{Fix}(\mathrm{Syl}_p)$ is henselian. By Theorem 1.3.4, it suffices to show that any prolongation w of v to $\mathrm{Fix}(\mathrm{Syl}_p)$ is a coarsening of the canonical henselian valuation on $\mathrm{Fix}(\mathrm{Syl}_p)$. As Syl_p is pro-p, the canonical henselian valuation coincides with the canonical p-henselian valuation on $\mathrm{Fix}(\mathrm{Syl}_p)$.

If $v \in H_1^p(K)$, we have by definition $Kv \neq Kv(p)$. As $p \nmid \#\mathrm{Gal}(\mathrm{Fix}(\mathrm{Syl}_p)/K)$, we also get $w \in H_1^p(\mathrm{Fix}(\mathrm{Syl}_p))$ for any prolongation w of v. Otherwise, we have $v = v_K^p$ and thus the same argument works for any proper coarsening of v. Hence in both cases, any proper coarsening of w is contained in $H_1^p(\mathrm{Fix}(\mathrm{Syl}_p))$, so w is a coarsening of the canonical p-henselian valuation on $\mathrm{Fix}(\mathrm{Syl}_p)$. □

3.2.2 Non-Universal Absolute Galois Groups

We want to use our results from the previous chapter to show that the same holds for any field with a non-universal absolute Galois group.

Proposition 3.2.2. *Let G_K be non-universal. Assume that there are two primes $p < q$ with $p, q \mid \#G_K$ and that $K(p) \neq K \neq K(q)$. If K is henselian, then K is henselian with respect to a non-trivial $\emptyset$-definable valuation.*

Proof: As long as we define a coarsening of v_K without parameters, we may assume that $\zeta_p, \zeta_q \in K$ if $\mathrm{char}(K) \neq p$ or q respectively. Thus in case $p = 2$, K is not formally real and so Kv_K^2 cannot be euclidean. All these extensions still have non-universal absolute Galois group.

As K is henselian, it is in particular p- and q-henselian. We consider the canonical p-henselian (q-henselian) valuation v_K^p (v_K^q respectively) on K. If v_K^p or v_K^q is henselian, then we have found a $\emptyset$-definable henselian valuation.

But this must always be the case: Assume that neither v_K^p nor v_K^q is henselian. Then v_K is a proper coarsening of v_K^p, and thus Kv_K is p-henselian and satisfies $Kv_K \neq Kv_K(p)$. Similarly, Kv_K is q-henselian and $Kv_K \neq Kv_K(q)$ holds. Therefore, by Proposition 2.2.8, Kv_K is henselian. This contradicts the definition of v_K. □

Theorem 3.2.3. *Let G_K be non-universal. If K is t-henselian, then K is henselian with respect to a non-trivial $\emptyset$-definable valuation.*

Proof. If G_K is pro-p, then $K(p) = K^{sep}$ and hence the $\emptyset$-definable p-henselian valuation is a $\emptyset$-definable henselian valuation. Thus we may assume that there are at least two primes p, q with $p, q \mid \#G_K$. Let L be some finite non-euclidean Galois extension of K with $L(p) \neq L \neq L(q)$, containing ζ_p and ζ_q just in case $\mathrm{char}(K) \neq p$ and q respectively. Then the the smallest common coarsening of v_L^p and v_L^q, namely $\mathcal{O}_{v_L^p} \cdot \mathcal{O}_{v_L^q}$, is non-trivial, $\emptyset$-definable and henselian on L. By Lemma 3.1.1, this is in particular a coarsening of v_L.

Now fix an integer n such that there is some Galois extension L of K of degree n as described above. Then the intersection

$$\bigcap \big((\mathcal{O}_{v_L^p} \cdot \mathcal{O}_{v_L^q}) \cap K \;\big|\; K \subseteq L \text{ Galois, } [L:K] = n,\ L(p) \neq L \neq L(q),$$
$$\zeta_p \in L \text{ if } \mathrm{char}(K) \neq p,\ \zeta_q \in L \text{ if } \mathrm{char}(K) \neq q\big)$$

gives a non-trivial $\emptyset$-definable henselian valuation on K. □

Note that in general, our construction only allows us to define a coarsening of v_K. This changes if v_K has finite rank.

Corollary 3.2.4. *Let K be henselian and G_K non-universal. If v_K has finite rank and no henselian valuation on K has real closed residue field, then v_K is definable.*

Proof: By Theorem 1.3.4 and the proof of Theorem 3.2.3, we can define a coarsening v_1 of v_K. Consider the residue field Kv_1. If v_1 is a proper coarsening of v_K, then Kv_1 is t-henselian and neither real nor separably closed. As the sequence

$$I_v \longrightarrow G_K \longrightarrow G_{Kv_1}$$

is exact, G_{Kv_1} is non-universal, so (by Theorem 3.2.3) Kv_1 is henselian with respect to a definable valuation v_2. Again, v_2 and $v_2 \circ v_1$ are coarsenings of v_{Kv_1} and v_K respectively. Since v_K has finite rank, finitely many iterations of this process allow us to define v_K. □

Similar to Theorem 3.1.8, we can also prove the following

Theorem 3.2.5. *Let (K, v) be a henselian valued field with G_K non-universal. Assume that Kv is not t-henselian, and neither separably nor real closed. Then v is $\emptyset$-definable.*

Proof: As Kv is not separably closed and not t-henselian, v is in fact the canonical henselian valuation on K. Furthermore, there is some prime p with $p \mid \#G_{Kv}$. Assume first that G_{Kv} is pro-p, then it follows that $Kv \neq Kv(p)$ and thus $K \neq K(p)$ ([EP05], Theorem 4.2.6). In particular, v must be a coarsening of v_K^p. But if v was a proper coarsening of v_K^p, then Kv would be p-henselian and hence t-henselian or real closed. Since by assumption Kv is neither real closed nor t-henselian, we get $v = v_K^p$. As in previous proofs (see for example the proof of Theorem 3.1.8), we may assume $\zeta_p \in K$ in case $\mathrm{char}(K) \neq p$, so v is $\emptyset$-definable.

Now consider the case that there are (at least) two primes $p < q$ with $p, q \mid \#G_{Kv}$. Thus also $p, q \mid G_K$. If $Kv(p) \neq Kv \neq Kv(q)$, then – using Theorem 4.2.6 in [EP05] once more – $K(p) \neq K \neq K(q)$. By the proof of Proposition 3.2.2, one of v_K^p and v_K^q is henselian. Say v_K^p is henselian, then we get $v \subset v_K^p$ by Lemma 3.1.1. But v is also a coarsening of v_K^p, as $Kv \neq Kv(p)$. Thus we conclude $v = v_K^p$, and hence v is again $\emptyset$-definable.

Finally, if there are two primes $p, q \mid G_{Kv}$, but $Kv = Kv(p)$ or $Kv = Kv(q)$, we want to consider finite Galois extensions L of Kv with $L(p) \neq L \neq L(q)$. Let M be a finite Galois extension of K, and let w be the unique prolongation of v to M. Note that G_M is again non-universal, and as above we can conclude $w = v_M$. If $Mw(p) \neq Mw \neq Mw(q)$, then w is $\emptyset$-definable on M by v_M^p or v_M^q as above. As Kv is not real closed, the proof of Proposition 2.2.8 shows that v_M^p and v_M^q are comparable and that the coarser one is henselian. In particular, the finest common coarsening of v_K^p and v_K^q is equal to the coarser one of the two and furthermore $\emptyset$-definable and henselian.

Now, fix n such that there is a Galois extension M of K (containing ζ_p and ζ_q if necessary) such that $Mw(p) \neq Mw \neq Mw(q)$. Just like in Theorem 3.1.8, we get a parameter-free definition of v by

$$\bigcap \big((\mathcal{O}_{v_M^p} \cdot \mathcal{O}_{v_M^q}) \cap K \;\big|\; K \subseteq M \text{ Galois, } [M:K] = n,\, M(p) \neq M \neq M(q),$$
$$\zeta_p \in M \text{ if } \mathrm{char}(M) \neq p,\, \zeta_q \in M \text{ if } \mathrm{char}(M) \neq q,\, v_M^p \in H_1^p(M),\, v_M^q \in H_1^q(M)\big).$$

□

3.2.3 Galois-Theoretic Conditions on the Residue Field

We want to use the above results to define power series valuations over fields with non-universal absolute Galois group.

Observation 3.2.6. *Let (K, v) be a henselian valued field. Then G_K is non-universal iff G_{Kv} is non-universal.*

Proof: Recall the exact sequence

$$I_v \longrightarrow G_K \longrightarrow G_{Kv}$$

from Proposition 1.2.3. If G_K is non-universal, then some finite group does not appear as a Galois group over any finite extension of K, and hence the same holds for Kv.

On the other hand, if G_{Kv} is non-universal, there is some $n_0 \in \mathbb{N}$ such that neither S_n nor A_n (for $n \geq n_0$) occur as a subquotients of G_{Kv}. As I_v is soluble, S_n (for $n \geq \max\{ 5, n_0 \}$) is not a subquotient of G_K, either. □

Corollary 3.2.7. *Let K be a field with G_K non-universal. Then for any non-trivial ordered abelian group Γ, there is a $\emptyset$-definable non-trivial henselian valuation on $K((\Gamma))$. If K is not t-henselian and neither separably nor real closed, then the power series valuation is $\emptyset$-definable.*

Proof: The first statement is immediate from the previous observation and Theorem 3.2.3. The second statement follows from Theorem 3.2.5. □

Remark. *Not all absolute Galois groups are non-universal: The main examples of universal absolute Galois groups are Galois groups of hilbertian fields and non-abelian free profinite groups. Since all free profinite groups are projective, any such group occurs in particular as the absolute Galois group of a PAC field. Note that if K is either PAC or hilbertian, then the power series valuation on $K((\Gamma))$ is already definable by Theorem* 3.1.8.

3.2.4 NIP Fields

NIP theories are a subject of considerable interest in pure model theory. Examples include stable and o-minimal theories. Let us first recall the definition.

Definition. *Let $\mathcal{T}$ be a complete theory in some language $\mathcal{L}$, and let $\mathbb{C}$ be a monster model for $\mathcal{T}$. An $\mathcal{L}$-formula $\phi(\bar{x}, \bar{y})$ has the* independence property (IP) *if there is some infinite $B \subset \mathbb{C}$ such that for every $C \subset B$, there is some $\bar{y}_C \in \mathbb{C}$ such that for any $\bar{x} \in B$*

$$\mathbb{C} \models \phi(\bar{x}, \bar{y}_C) \iff \bar{x} \in C$$

holds. We say that $\mathcal{T}$ is NIP *if no formula has the independence property. A field K is called* NIP *if* Th(K) *is NIP.*

Examples of NIP fields include stable fields, o-minimal fields and the p-adic numbers. By a conjecture of Assaf Hasson and Saharon Shelah, an NIP field K with $K^\times/(K^\times)^n$ finite for all $n \in \mathbb{N}$ is separably closed, real closed or admits a non-trivial definable valuation. Due to the next theorem, henselian valued field provide an important class of examples of NIP fields.

Theorem 3.2.8. *Let (K, v) be a henselian valued field of residue characteristic 0. Then (K, v) is NIP iff Kv is NIP.*

Proof: Follows immediately from Theorem 8 in [Del81] and Theorem 3.1 in [GS84].

In [KSW11], the authors show that if K is NIP and has positive characteristic, then K admits no Artin-Schreier extensions. Using this, they obtain the following

Corollary 3.2.9. *Let K be an infinite NIP field of characteristic $p > 0$, and consider a finite separable extension L of K. Then p does not divide $[L : K]$.*

Proof: [KSW11], Corollary 4.5.

Thus, any NIP field of positive characteristic has a non-universal absolute Galois group, and Theorem 3.2.3 applies:

Proposition 3.2.10. *Let (K, v) be a non-trivially henselian valued field. If*

- *K is NIP and* $\mathrm{char}(K) > 0$*, or*
- *Kv is NIP and* $\mathrm{char}(Kv) > 0$*,*

then K admits a non-trivial $\emptyset$-definable henselian valuation.

Proof: The first statement follows from Theorem 3.2.3. The second statement is now a consequence of Observation 3.2.6. □

3.3 Elimination of Parameters

For all the henselian valuations which we have defined so far, we have managed to avoid the need for parameters by exploiting the uniformity of the definitions of the canonical p-henselian valuations. In a general situation, it might however be possible that there is a non-trivial $\emptyset$-definable henselian valuation on some finite Galois extension L of K. By interpreting L in K using parameters for the minimal polynomial, this would yield a non-trivial definable henselian valuation on K. We now want to show that if there are only finitely many Galois extensions of the same degree as L over K (e.g. in case G_K is small, that is if K has only finitely many Galois extensions of degree n for each n), then we can also define a henselian valuation on K without parameters. So far, we do not know any applications of this.

As we discussed in the first chapter, the theory of a field lists all finite groups which occur as Galois groups over K. In particular, the fact that G_K is small or non-universal (as some S_m does not occur as a Galois group over any finite extension of K) is inherited by any $L \equiv K$.

Note that the result discussed in this section is not already covered by our previous work: Even though many small groups are non-universal, there are universal small profinite groups.

Example. *The free product $\hat{\mathbb{Z}} * \hat{\mathbb{Z}}$ is small and universal. By Theorem A in [Mel99], it also occurs as the absolute Galois group of some (henselian) field.*

Theorem 3.3.1. *Let K be a field such that K has only finitely many Galois extensions of degree n for some n. Assume that some finite Galois extension L of K with $[L : K] = n$ carries a non-trivial $\emptyset$-definable henselian valuation $v \supseteq v_L$. Then there is a non-trivial $\emptyset$-definable henselian valuation on K.*

Proof: Consider n and L as given by the assumption. Thus there are only finitely many (say m) Galois extensions of K of degree n. Take (monic) minimal polynomials corresponding to these extensions $L = L_1, \ldots, L_m$ and assume that they are given by tuples of coefficients $\bar{a}_i$ for $1 < i \leq m$. We call the minimal polynomial depending on the coefficients $f_{\bar{a}_i}$.

By our assumption, L_1 carries a $\emptyset$-definable non-trivial henselian valuation. Its restriction to K is definable using the tuple $\bar{a}_1$ as a parameter, say via the formula $\phi(x, \bar{a}_1)$. Note that since the interpretation of the L_i in K can be done uniformly in the coefficients of the minimal polynomial, we may choose ϕ in such a way that $\phi(x, \bar{a}_i) = \phi(x, \bar{b}_i)$ for any choice of coefficients $\bar{b}_i$ such that $f_{\bar{b}_i}$ generates the same extension over K as $f_{\bar{a}_i}$. Now consider the partial type

$$\Sigma(\bar{y}) = \big\{ f_{\bar{y}} \text{ generates a Galois extension } L_{\bar{y}} \text{ of degree } n, \\ \phi(x,\bar{y}) \text{ is a non-trivial henselian valuation ring} \big\}.$$

Claim: $\Sigma(\bar{y})$ is principal.

Proof of Claim: Take $K' \equiv K$. Then $G_{K'} \cong G_K$ and hence also small. The type $\Sigma(\bar{y})$ is finitely satisfiable, so for each $k \in \mathbb{N}$ there is some Galois extension L' of K' such that $[L' : K'] = n$, v is a non-trivial valuation on L', and Hensel's lemma holds for polynomials up to degree k. Since there are only finitely many choices for L' and because different choices of a generator for L' give rise to the same valuation, v must induce a henselian valuation on some such L'.

Assume

$$\varphi(\bar{y}) \vdash \Sigma(\bar{y}).$$

We show that the formula

$$\chi(x) \equiv \exists \bar{y}\, \varphi(\bar{y}) \wedge \phi_1(x,\bar{y})$$

defines a non-trivial henselian valuation on K. After possibly renumerating the $\bar{a}_i$, we get

$$\chi(K) = \bigcup_{1 \leq i \leq m'} \phi(K, \bar{a}_i)$$

for some $m' \leq m$. Since $\phi(K, \bar{a}_1)$ is a coarsening of the canonical henselian valuation on K, all the henselian valuations occuring in the union are comparable to $\phi(K, \bar{a}_1)$. As m' is finite, there is a $i_0 \leq m'$ such that $\phi(K, \bar{a}_{i_0})$ is coarser than any of the other valuation rings. In particular, $\phi(K, \bar{a}_{i_0})$ is henselian and non-trivial. This is the henselian valuation ring defined by $\chi(x)$. □

3.4 An Application: Model Complete Fields

We now use the definability of henselian valuations to make some comments on henselian fields which are model complete in $\mathcal{L}_{ring}$. Throughout the section, we write $K \equiv L$ if two fields L and K are elementarily equivalent in the language of rings, and $(K, v) \equiv (L, w)$ if two valued fields are elementarily equivalent in $\mathcal{L}_{ring} \cup \{\mathcal{O}_v\}$. We begin by recalling the definition of model completeness and some well-known examples of model complete fields.

Definition. *A field K is called* model complete *(in the language of rings) if for all fields $K_1, K_2 \equiv K$ the following holds:*

$$K_1 \subseteq K_2 \Longrightarrow K_1 \prec K_2.$$

Similarly, when we discuss model completeness of a valued field (K, v), we consider the theory of K as a structure in $\mathcal{L}_{ring} \cup \{\mathcal{O}_v\}$. Note that a theory is model complete iff every formula is equivalent to an existential formula (see [TZ12], Theorem 8.7).

Example. *Finite fields, algebraically closed fields, real closed fields and p-adically closed fields are model complete.*

One can loosen these criteria slightly to obtain fields which are similar enough to those above. We defined PAC fields in the last chapter, and analogously, there are PRC and PpC fields:

Definition. *A field K is called* pseudo real closed *(*pseudo p-adically closed*) if every absolutely irreducible variety defined over K has a K-rational point provided that it has a simple $\overline{K}$-rational point for every real (respective p-adic) closure $\overline{K}$ of K.*

In particular, algebraically closed fields are PAC, real closed fields are PRC and p-adically closed fields are PpC. Not all PAC, PRC, or PpC fields are model complete, but it is well-classified which ones are (see [Whe79], Theorem 6.3 and [Whe83], Theorem 5.2). There are non-separably closed model complete PAC fields of any characteristic. All known examples of model complete fields have a small absolute Galois group. This lead to the following

Question 3.4.1 (Macintyre, McKenna)**.** *Let K be a model complete field. Is its absolute Galois group G_K small?*

Furthermore, all previous examples of model complete fields behave quite rigidly when it comes to henselian valuations:

Theorem 3.4.2. *1. If K is PAC, then the henselian closure of each valuation on K is separably closed.*

2. If K is PRC, then the henselian closure of each valuation on K is either real closed or separably closed.

3. If K is PpC, then the henselian closure of each valuation on K is either p-adically closed or algebraically closed.

Proof: [FJ08], Corollary 11.5.6 and [GJ91], Theorems A and B.

Hence, we want to consider the following

Question 3.4.3. *Are there any henselian model complete fields which are neither p-adically nor separably nor real closed?*

We will not give an answer to this question, but we will discuss an example which indicates that the answer might be negative. Note that Theorem 1.3.2 gives a criterion for model completeness of valued fields:

Corollary 3.4.4. *Let (K, v) be a henselian valued field of characteristic $(0, 0)$. Then (K, v) is model complete (in the language of valued fields) iff both Kv and vK are model complete.*

Proof: Follows immediately from the version of Theorem 1.3.2 given on p. 183 in [KP84]. □

We want to find out how this is related to definable henselian valuations.

Proposition 3.4.5. *Assume that (K, v) is henselian and v is $\emptyset$-definable. Then K is model complete iff (K, v) is model complete and v is existentially and universally $\emptyset$-definable.*

Proof: If K is model complete, then any formula is equivalent to an existential and a universal formula modulo $\mathrm{Th}(K)$ (see Theorem 8.7 in [TZ12]). Thus v is existentially and universally definable. Let now $(K_1, v_1) \subset (K_2, v_2)$ be an extension of valued fields such that $(K_i, v_i) \equiv (K, v)$. Then $K_i \equiv K$ and thus $K_1 \prec K_2$ hold. As v is defined both existentially and universally, this implies $(K_1, v_1) \prec (K_2, v_2)$.
Conversely, assume that (K, v) is model complete and that v is given by an existential and a universal formula. Consider $K_1 \equiv K_2 \equiv K$ with $K_1 \subset K_2$. Let v_i denote the respective valuation defined by the same formula as v on each K_i. As there is an existential and a universal definition, this gives rise to an extension of valued fields $(K_1, v_1) \subset (K_2, v_2)$ with $(K_i, v_i) \equiv (K, v)$. Thus $K_1 \prec K_2$ follows from the model completeness of (K, v). □

We will now give an example of a model complete henselian valued field (K, v) which is not model complete as a pure field.

Observation 3.4.6. *If an ordered abelian group is model complete, then it is divisible. Furthermore,* $\mathrm{Th}(\mathbb{Q})$ *is model complete as an ordered abelian group.*

Proof: [Mar02], Corollary 3.1.17.

From the earlier sections in this chapter, we know quite well how to construct a field $K((\mathbb{Q}))$ such that the power series valuation is $\emptyset$-definable.

Proposition 3.4.7. *Let K be a model complete PAC field of characteristic 0, and assume that K is not separably closed. Then $(K((\mathbb{Q})), v)$ is model complete as a valued field, neither separably nor p-adically closed, and v is $\emptyset$-definable.*

Proof: By the previous observations, $(K((\mathbb{Q})), v)$ is model complete. By Theorem 3.1.8 and the fact that a PAC field K is not virtually p-henselian for any p with $p \mid \#G_K$ (see Corollary 2.3.5), the power series valuation on $K((\mathbb{Q}))$ is $\emptyset$-definable. □

However, the field which we have just constructed is not model complete in $\mathcal{L}_{ring}$:

Proposition 3.4.8. *Let K be a PAC field of characteristic 0 which is not separably closed. Then $K((\mathbb{Q}))$ is not model complete.*

Proof: Using Theorem 23.1.1 in [FJ08], we can find a PAC field $L \supset K$ with $K((\mathbb{Q})) \subseteq L$ and such that the canonical restriction map

$$\mathrm{res}_{L/K} : G_L \longrightarrow G_K$$

is an isomorphism. Then by Corollary 20.3.4 in [FJ08], we have $L \prec K$. Now consider the tower of fields

$$K \subset K((\mathbb{Q})) \subset L \subset L((\mathbb{Q})).$$

By Theorem 1.3.2, the fields $K((\mathbb{Q}))$ and $L((\mathbb{Q}))$ are elementarily equivalent. By Proposition 3.4.7, the power series valuation is $\emptyset$-definable on both of them. As there are no other henselian valuations on either of these fields, the power series valuation is in particular given by the same formula in both cases. However, the power series valuation on $L((\mathbb{Q}))$ is trivial when restricted to $K((\mathbb{Q}))$, as it is trivial when restricted to the intermediate field L. Hence the extension is not elementary. □

Finally, we want to comment on the converse.

Theorem 3.4.9. *Let K be a henselian field and assume that G_K is non-universal. If K is model complete, then K carries a $\emptyset$-definable non-trivial henselian valuation v such that (K, v) is model complete. In case* $\mathrm{char}(Kv) = 0$*, then vK and Kv are also model complete.*

Proof: As K is henselian and G_K is non-universal, K admits a $\emptyset$-definable non-trivial henselian valuation v by Theorem 3.2.3. Since K is model complete, v is both existentially and universally definable and therefore (K, v) is also model complete. Model completeness of Kv and vK now follow from Corollary 3.4.4. □

Chapter 4

Elementary Characterization

4.1 Fields Elementarily Characterized by their Absolute Galois Group

The main reference for the following chapter is Jochen Koenigsmann's Habilitationsschrift ([Koe04]), where in particular all proofs omitted here can be found. Our aim is to find out when G_K 'determines' K.

4.1.1 Notions of Determination

We start by defining several notions of determination. When we write elementary equivalence, we consider fields in $\mathcal{L}_{ring}$ and profinite groups as multi-sorted structures as described in the first chapter.

Definition. *A field K is* elementarily characterized by G_K *if for all fields F the following holds:*

$$G_K \cong G_F \Longleftrightarrow F \equiv K.$$

Examples. *1. $\mathbb{R}$ is elementarily characterized by its absolute Galois group,*

2. *$\mathbb{C}$ is not elementarily characterized by $G_{\mathbb{C}} \cong \{id\}$, as this is the absolute Galois group of every separably closed field (and thus occurs as the abolute Galois group of fields of any characteristic and imperfection degree),*

3. *no finite field is elementarily characterized by its absolute Galois group as $G_{\mathbb{F}} \cong \hat{\mathbb{Z}}$ for every finite field $\mathbb{F}$.*

Observation 4.1.1. *If K is elementarily characterized by G_K, then G_K is small.*

Proof: Assume G_K is not small, and suppose G_K has $\kappa \geq \aleph_0$ many quotients of order n for some $n \in \mathbb{N}$. Then there exists an elementary extension $F \succ K$ with $\lambda > \kappa$ many such extensions, e.g. a λ-saturated F. □

It makes no sense to ask for isomorphy on the field side: Every infinite field with a small absolute Galois group has arbitrarily large (i.e. in particular non-isomorphic) elementary extensions with the same absolute Galois group, so no such field K can ever be determined by G_K up to isomorphism. A variant of the definition which is preserved under elementary equivalence – and thus more natural – is the following

Definition. *A field K is* elementarily characterized by $\mathrm{Th}(G_K)$ *if for all fields F the following holds:*

$$G_K \equiv G_L \Longleftrightarrow K \equiv L.$$

Note that the direction from right to left always holds. We will also consider a third version.

Definition. *A field K is* elementarily determined by G_K *if for all fields F the following holds:*

$$G_K \cong G_F \Longrightarrow F \equiv K.$$

Clearly, if a field is elementarily characterized by G_K, then it is also elementarily characterized by $\mathrm{Th}(G_K)$ and elementarily determined by G_K. Also, being elementarily characterized by $\mathrm{Th}(G_K)$ implies being elementarily determined by G_K.

Furthermore, these notions coincide when G_K is small.

Observation 4.1.2. *If G_K is small and K is elementarily determined by G_K, then K is elementarily characterized by G_K.*

Proof: Realizability of a certain Galois group over a field is first-order expressible by 1.1.3, so we can always express the fact that there are only m many Galois extensions of degree n. Hence, $K \equiv L$ implies that G_L is small, and therefore $G_K \cong G_L$. □

Note that there are no known examples of fields which are elementarily characterized by $\mathrm{Th}(G_K)$ or elementarily determined by G_K but not elementarily characterized by G_K.

4.1.2 The Classification Theorem

Now we are in the position to state the main theorem of [Koe04]. The aim of this chapter is to give an improved version of it.

Theorem 4.1.3 (Koenigsmann). *A field K is elementarily characterized by G_K iff K is elementarily equivalent to one of the following:*

(A) $\mathbb{R}$.

(B) $L \supseteq \mathbb{Q}_p$ *finite with* $\gcd\left([L : L^{ab}], \frac{p}{p-1}\,\#\mu_L\right) = 1$, *where p is some prime, L^{ab} is the maximal abelian subextension of $\mathbb{Q}_p \subseteq L$, and μ_L is the group of roots of unity in L.*

(C) $L((\mathbb{Z}_{(q)}))$, *the generalized power series field over a field L with exponents from $\mathbb{Z}_{(q)}$ with L as in (B) and q an arbitrary prime (not necessarily $p \neq q$).*

(D) $L((\mathbb{Z}_{(p)}))$, *where L is a field such that*

- $\mathrm{char}(L) = 0$,
- $\mathrm{trdeg}[L : \mathbb{Q}] < \infty$,
- *L admits no proper abelian extensions,*
- *L admits a henselian valuation with residue characteristic p,*

- *L is elementarily characterized by G_L only in the class of all fields of non-p characteristic (i.e. not elementarily characterized by G_L in the class of all fields),*
- $\mathrm{cd}_p G_L = 1$.

(E) A field L elementarily characterized by G_L such that

- *G_L is not pro-soluble and*
- *all fields with isomorphic absolute Galois group have characteristic zero and infinite transcendence degree over $\mathbb{Q}$.*

Questions 4.1.4. *1. Are the classes defined in (D) and (E) empty?*

2. If K admits no proper abelian extensions, does this already imply that G_K is projective?

3. Does the theorem still hold if we replace the notion of being elementarily characterized by G_K with being elementarily characterized by $\mathrm{Th}(G_K)$ or with being elementarily determined by G_K?

Note that if the answer to the second question in 4.1.4 is yes, then the class defined in (D) is empty. Furthermore, by Proposition 3.1.5 in [Ser97], this question is equivalent to the same question restricted to fields of finite transcendence degree.

We will now give an example of a field L which satisfies all the properties described in class (E) apart from being elementarily characterized by G_L.

Example. *Let K be a PAC field of characteristic 0 with $G_K \cong \hat{\mathbb{Z}} * \hat{\mathbb{Z}}$, and consider the group $\Gamma = \bigoplus_{p \,\mathrm{prime}} \mathbb{Z}_{(p)}^p$ with lexicographical order. Define $H := \prod_{p \,\mathrm{prime}} \mathbb{Z}_p^p$. Then we have*

$$G_{K((\Gamma))} \cong H \rtimes (\hat{\mathbb{Z}} * \hat{\mathbb{Z}})$$

(see section 3.1.1 in [Koe04]). In particular, $G_{K((\Gamma))}$ is small and universal. Using Proposition I.3.3.14 in [Ser97], we conclude from $\mathrm{cd}_p(H) = p$ that $\mathrm{cd}_p(G_{K((\Gamma))}) \geq p$ holds for all primes p. Hence, any field L with $G_L \cong G_{K((\Gamma))}$ has characteristic 0 by Proposition II.2.2.3 in [Ser97].

Furthermore, the inequality

$$\mathrm{cd}_p(G_L) \leq \mathrm{cd}_p(\mathbb{Q}) + \mathrm{trdeg}[L : \mathbb{Q}]$$

holds (Proposition II.4.2.14 in [Ser97]). As $\mathrm{cd}_p(\mathbb{Q}) = 2$ *for* $p > 2$ *by [Ser97], Proposition II.4.4.13, this implies that* $\mathrm{trdeg}[L : \mathbb{Q}] = \infty$*. Using Theorem* 1.3.6*, every such field* L *also carries a henselian valuation.*

However, $K((\Gamma))$ *is not elementarily characterized by* $G_{K((\Gamma))}$*. Order* Γ *in a different way, and call the resulting group* Γ'*. If the orders are sufficiently different (say* Γ' *is also ordered lexicographically, but the sum is rearranged sufficiently), then* $\Gamma \not\equiv \Gamma'$*. Nevertheless,* $G_{K((\Gamma))} \cong G_{K((\Gamma'))}$*. On both* $K((\Gamma))$ *and* $K((\Gamma'))$*, the power series valuation is* $\emptyset$*-definable by Corollary* 3.1.9*. Hence by Theorem* 1.3.2*, they are not elementarily equivalent.*

To show that all the fields in the theorem are elementarily characterized by their absolute Galois group, Koenigsmann shows first that in each case the absolute Galois group is small and then uses very involved valuation theory and Theorem 1.3.2. For the other direction, he proves the following

Proposition 4.1.5. *Let* K *be elementarily determined by* G_K*. Then*

1. $\mathrm{char}(K) = 0$,
2. K *is t-henselian,*
3. $\mathrm{Gal}(K)$ *is not projective.*

Proof: [Koe04], Proposition 5.1.

4.2 Redefining Classes (D) and (E)

4.2.1 Defining Henselian Valuations

The aim of this section is to give a different characterization in the eventual hope of showing that the inconvenient classes (D) and (E) are empty. The first step is to show that non-real closed fields elementarily characterized by their absolute Galois group are essentially ($\emptyset$-definably) henselian. ¿From what we know so far, we obtain the following

Proposition 4.2.1. *Let K be a field elementarily characterized by its absolute Galois group, and assume that K is in class (B), (C) or (D). Then K admits a non-trivial $\emptyset$-definable henselian valuation.*

Proof: As K is elementarily characterized by G_K, K is t-henselian (Proposition 4.1.5). If K is in classes (B) or (C), then G_K is pro-soluble, thus in particular non-universal. Hence K admits a non-trivial $\emptyset$-definable valuation by Theorem 3.2.3.
If K is in class (D), then $K \equiv L((\mathbb{Z}_{(p)}))$ for some field L which admits no abelian extensions. By Lemma 3.1.5, $L((\mathbb{Z}_{(p)}))$ carries a $\emptyset$-definable non-trivial henselian valuation. The same formula defines a non-trivial henselian valuation on K. □

We would like to conclude the same for class (E), however, this is not quite as straightforward. Following the proof of Koenigsmann's characterization theorem closely, we will prove a partial result:

Proposition 4.2.2. *Let K be elementarily characterized by G_K, and assume K is not real closed. Then either K is henselian with respect to a non-trivial $\emptyset$-definable valuation, or for some saturated $L \equiv K$, the field Lv_L is t-henselian but not henselian.*

Note that all fields elementarily characterized by their absolute Galois group have small absolute Galois group by Observation 4.1.1. In the only known construction of fields which are t-henselian but not henselian (see p. 338 in [PZ78]), the corresponding absolute Galois group is very large and in particular not small.

Question 4.2.3. *Is there a field K which is t-henselian but not henselian such that G_K is small?*

The answer to this question should be negative, although we have not yet been able to prove it.

We now spend the remainder of this subsection to give a proof of Proposition 4.2.2. For the convenience of the reader, we quote (most of) the statements from [Koe04] which we will be using. The first one gives us a condition for a field not to be characterized by its absolute Galois group.

Theorem 4.2.4. *Let (K, v) be a henselian valued field of mixed characteristic $(0, p)$ with $\mathcal{O}_v\left[\frac{1}{p}\right] = K$. Assume that G_K is small. Then the following are equivalent:*

1. $\Gamma_v = p \cdot \Gamma_v$.
2. $\mathrm{cd}_p G_K \leq 1$.
3. $G_K \cong G_F$ *for some henselian valued field F with* $\mathrm{char}(F) = p$.

Proof: [Koe04], Theorem 3.3.

Next, we note that if a field has proper abelian extensions, it suffices to check elementary determination in characteristic 0.

Lemma 4.2.5. *Let K be a field, $K \neq K^{ab}$. Then K is elementarily determined by G_K iff* $\mathrm{char}(K) = 0$ *and K is elementarily determined by G_K among all fields of characteristic* 0.

Proof: [Koe04], Lemma 5.2.

Finally, we learn about the value group of henselian valuations on fields which are elementarily determined by their absolute Galois group:

Proposition 4.2.6. *Let K be elementarily determined by G_K and let v be a henselian valu-*

ation on K with $\mathrm{char}(Kv) = 0$*. Then* Γ_v *is divisible, or there is a prime* p *such that*

$$[\Gamma_v : q \cdot \Gamma_v] = \begin{cases} 1 & \text{for all primes } q \neq p \\ p & \text{for } q = p \end{cases}$$

and in this case Kv *is elementarily determined by* G_{Kv} *and* $\mu_{p^\infty}(Kv)$ *among all fields of characteristic* $\neq p$*.*

Proof: [Koe04], Proposition 5.4.

Combining these results, we can now give a proof of Proposition 4.2.2.

Proof of Proposition 4.2.2*:* Assume that K is not real closed, elementarily characterized by G_K and sufficiently saturated. In particular, K is not separably closed. Hence it follows from Proposition 4.1.5 that K is t-henselian and thus – by saturation – henselian. Let v_K be the canonical henselian valuation on K. In case that $v_K \in H^2(K)$, Kv_K is separably closed and – just like in the proof of Theorem 3.1.3 – we conclude that G_K is pro-soluble. Then K is in one of the classes (B) or (C), but none of the fields in these classes admits a henselian valuation with separably closed residue field. Thus, we get $v_K \in H^1(K)$, so v_K is the finest henselian valuation on K. Therefore, Kv_K is not henselian.

For the remainder of the proof, we assume that Kv_K is also not t-henselian, and then show that K admits a non-trivial $\emptyset$-definable henselian valuation. We divide the proof into two parts. First, we follow the proof of Corollary 5.5 in [Koe04].

Claim 1: If $\mathrm{char}(Kv_K) = 0$, then $v_K K$ is not divisible.

Proof of Claim 1: Assume that $v_K K$ is divisible. Then $G_{Kv_K} \cong G_K$, and hence $K \equiv Kv_K$. As K is neither real nor separably closed, this contradicts our assumption on Kv_K to be not t-henselian.

Claim 2: The canonical henselian valuation v_K has mixed characteristic.

Proof of Claim 2: By Proposition 4.1.5, we get $\mathrm{char}(K) = 0$. Assume $\mathrm{char}(Kv_K) = 0$. It follows from the previous claim that then there is some prime p with $v_K K \neq p \cdot v_K K$. Using

Proposition 4.2.6, we conclude that Kv_K is elementarily determined by G_{Kv_K} and μ_{p^∞} among all fields of characteristic not equal to p. Therefore $Kv_K \equiv Kv_K((\mathbb{Q}))$, and hence Kv_K is either t-henselian or real closed. The first contradicts our assumption on Kv_K, and the latter is not possible by [Koe04], Proposition 5.6. This proves the claim.

The case distinction in the second part of the proof follows section 5.3 in [Koe04]. If all henselian valuations on K with residue characteristic 0 have divisible value group, then K is of type (B) (see Proposition 3.2 in [Koe04]). Thus, K admits a non-trivial $\emptyset$-definable henselian valuation by Proposition 4.2.1.

Otherwise, there is some henselian valuation w on K with $\mathrm{char}(Kw) = 0$ and $wK \neq q \cdot wK$ for some prime q. By choosing w to be the coarsest such valuation, we may assume by Proposition 4.2.6 that $wK \equiv \mathbb{Z}_{(q)}$. Just like in [Koe04], we can then apply Theorem 1.3.2 to reduce to the case that the valuation induced by v_K on Kw also satisfies $\mathcal{O}_{v_K/w}\left[\frac{1}{p}\right] = Kw$.

If $\mathrm{cd}_p G_{Kw} > 1$, then, by Proposition 3.2 in [Koe04] and Proposition 4.2.6, K is of type (C) and once more admits a non-trivial $\emptyset$-definable henselian valuation by Proposition 4.2.1.

Otherwise $\mathrm{cd}_p G_{Kw} \leq 1$, and G_{Kw} occurs as an absolute Galois group of a field of characteristic p by Theorem 4.2.4. Using Lemma 4.2.5, we conclude that Kw has no proper abelian extensions. It therefore follows from Proposition 4.2.6 that $p = q$.

As $v_K K$ is not p-divisible, K admits a non-trivial $\emptyset$-definable henselian valuation – namely v_K^p – by Lemma 3.1.5. By the definition of v_K^p, we get in particular $w \subseteq v_K^p$. □

4.2.2 Defining a New Class (D*)

The proof of Proposition 4.2.2 allows us to conclude that if a field is elementary characterized it should be in classes (A)–(C), or at least look like a class (D) field:

Corollary 4.2.7. *Let K be elementarily characterized by G_K, and assume that K is not in one of the classes (A)–(C). Then either for some saturated $L \equiv K$, Lv_L is t-henselian but not henselian, or $K \equiv L((\mathbb{Z}_{(p)}))$ for some field L such that*

1. *the power series valuation on* $L((\mathbb{Z}_{(p)}))$ *is* $\emptyset$*-definable,*

2. $\mathrm{char}(L) = 0$*,*

3. *L admits no proper abelian extensions,*

4. *L has a henselian valuation with residue characteristic p,*

5. *L is elementarily characterized by* G_L *among all fields of characteristic* $\neq p$*,*

6. $\mathrm{cd}_p G_L = 1$*.*

Proof: Let v_K^p be the canonical p-henselian valuation on K. The proof of the above proposition shows that we may assume that $v_K^p K \equiv \mathbb{Z}_{(p)}$ and that Kv_K^p satisfies all the properties which are listed above for L. Note that $\mathrm{cd}_p G_{Kv_K^p} \neq 0$, as else G_K would be realized in characteristic p (see the construction in [Koe04], p. 27). Since the canonical p-henselian valuation is $\emptyset$-definable and henselian, the statement follows from Theorem 1.3.2. □

Definition. *We say that a field K satisfies* condition (D*) *if* $K \equiv L((\mathbb{Z}_{(p)}))$ *for some field L such that*

1. *the power series valuation on* $L((\mathbb{Z}_{(p)}))$ *is* $\emptyset$*-definable,*

2. $\mathrm{char}(L) = 0$*,*

3. *L admits no proper abelian extensions,*

4. *L has a henselian valuation with residue characteristic p,*

5. *L is elementarily characterized by* G_L *among all fields of characteristic* $\neq p$ *and*

6. $\mathrm{cd}_p G_L = 1$*.*

Clearly, every field in class (D) satisfies condition (D*). Unfortunately, we cannot prove the converse, that any (D*)-field K is indeed elementarily characterized by G_K. We will however prove a partial converse. Let us first note some properties of (D*)-fields.

Lemma 4.2.8. *Let* $K \equiv L((\mathbb{Z}_{(p)}))$ *satisfy condition (D*). Then the following properties hold:*

1. *L contains all roots of unity and any valuation on L has divisible value group.*

2. *No henselian valuation on L has separably closed residue field.*

3. *For any henselian valuation v on L, $I_v = R_v$.*

Proof: The first two statements follow immediately from $L = L^{ab}$. The third statement is a consequence of 1 and Proposition 1.2.4. □

Proposition 4.2.9. *Let $K \equiv L((\mathbb{Z}_{(p)}))$ be a field satisfying condition (D*) and assume further that v_L is $\emptyset$-definable on L. Then K is elementarily characterized by G_K.*

Proof: Same as the proof in section 4.3 in [Koe04]. □

Note that by [Koe04], Lemma 3.7, all fields which are elementary characterized by their absolute Galois group and in class (D) indeed satisfy the assumption of the above proposition. Furthermore by Theorem 3.2.5, the proposition also applies to any (D*)-field with non-universal absolute Galois group. This gives us the following characterization:

Corollary 4.2.10. *A field K is elementarily characterized by G_K iff K is elementarily equivalent to one of the following:*

((A)) $\mathbb{R}$.

((B)) $L \supseteq \mathbb{Q}_p$ *finite with* $\gcd\left([L : L^{ab}], \frac{p}{p-1}\#\mu_L\right) = 1$, *where p is some prime, L^{ab} is the maximal abelian subextension of $\mathbb{Q}_p \subseteq L$, and μ_L is the group of roots of unity in L.*

((C)) $L((\mathbb{Z}_{(q)}))$, *the generalized power series field over a field L with exponents from $\mathbb{Z}_{(q)}$ with L as in ((B)) and q an arbitrary prime (not necessarily $p \neq q$).*

((D)) $L((\mathbb{Z}_{(p)}))$, *where L is a field such that*

- $\mathrm{char}(L) = 0$,
- $\mathrm{trdeg}[L : \mathbb{Q}] < \infty$ *or G_L non-universal,*
- *L admits no proper abelian extensions,*
- *L admits a henselian valuation with residue characteristic p,*

- *L is elementarily characterized by G_L only in the class of all fields of non-p characteristic (i.e. not elementarily characterized by G_L in the class of all fields),*
- $\mathrm{cd}_p G_L = 1$.

((E)) A field L elementarily characterized by G_L such that

- *G_L is universal and*
- *all fields with isomorphic absolute Galois group have characteristic zero and infinite transcendence degree over $\mathbb{Q}$.*

4.2.3 Further Remarks

Finally, we prove that for class ((D)) fields, Lv_L is almost characterized by its absolute Galois group:

Proposition 4.2.11. *Let $K \equiv L((\mathbb{Z}_{(p)}))$ be a field as in class ((D)) in Corollary* 4.2.10*. If F is a field of positive characteristic with $G_F \cong G_L$, then either $F^{perf} \equiv Lv_L$ or F carries a henselian valuation w with $Fw \equiv Lv_L$.*

Proof: Assume that $G_F \cong G_L$. By a standard construction (see for example Corollary 2.22 in [Kuh09]), there is a henselian valued defectless field (F', w') of characteristic 0 with $F'w' = F^{\mathrm{perf}}$ and $G_F \cong G_{F'}$. Then $F' \equiv L$ and carries a henselian valuation v defined by the same formula as v_L. We have shown in the proof of Proposition 4.2.2 that the residue field of v_L is neither separably nor real closed. Thus, as Lv_L has either finite transcendence degree or a non-universal absolute Galois group, it is not t-henselian – otherwise it would already be henselian (by Proposition 2.2.6 and Theorem 3.2.3 respectively).

Therefore, w' is a coarsening of v and hence v induces a henselian valuation w on F with $Fw \equiv Lv_L$. If $F^{\mathrm{perf}} \not\equiv Lv_L$, then this valuation is non-trivial. □

The above proof does not generalize to all (D*)-fields because even when v_L is $\emptyset$-definable, we cannot conclude that Lv_L is not t-henselian. Finding a negative answer to Question 4.2.3 seems the right way forward here, too.

4.3 An Excursion: Abstract Elementary Classes

Model theory is mainly concerned with first-order theories. Unfortunately, not all structures studied in mathematics have a natural first order theory associated with them which encodes all of their 'interesting' information. Hence, Shelah introduced the notion of an abstract elementary class which permits the application of model-theoretic results to more classes of structures while still ensuring that these structures are similar 'enough' to the first-order context. Our aim is to point out different ways in which one can see fields with a given absolute Galois group as abstract elementary classes and show some of their basic properties. The main reference for this section is [Bal09]. This section contains more questions then answers.

4.3.1 Definitions

Let $\mathcal{L}$ be some language throughout this chapter.

Definition. *Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures and $\prec_{\mathcal{K}}$ a binary relation on $\mathcal{K}$. Then $(\mathcal{K}, \prec_{\mathcal{K}})$ is said to be an* abstract elementary class (AEC) *if the class $\mathcal{K}$ and the class of pairs satisfying $\prec_{\mathcal{K}}$ are each closed under isomorphism and satisfy the following conditions.*

(Axiom 1) If $M \prec_{\mathcal{K}} N$ then M is an $\mathcal{L}$-substructure of N.

(Axiom 2) $\prec_{\mathcal{K}}$ is a partial order (i.e. a reflexive and transitive binary operation) on $\mathcal{K}$.

(Axiom 3) Let $\langle M_i \mid i < \delta \rangle$ be a continuous $\prec_{\mathcal{K}}$-increasing chain, i.e. a sequence of members of $\mathcal{K}$ such that $M_i \prec_{\mathcal{K}} M_j$ for $i \leq j < \delta$ and $\bigcup_{i<\lambda} M_i = M_\lambda$ for any limit ordinal $\lambda < \delta$ hold. Then

(3.1) $\bigcup_{i<\delta} M_i \in \mathcal{K}$,

(3.2) for each $j < \delta$ we have $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$,

(3.3) if there is some $N \in \mathcal{K}$ with $M_i \prec_{\mathcal{K}} N$ for all $i < \delta$ then $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} N$.

(Axiom 4) Given $K, L, M \in \mathcal{K}$ with $K \prec_{\mathcal{K}} M$, $L \prec_{\mathcal{K}} M$ and $K \subseteq L$ then we have $K \prec_{\mathcal{K}} L$.

(Axiom 5) There exists a Löwenheim-Skolem number $LS(\mathcal{K})$ *such that if* $A \subseteq N \in \mathcal{K}$ *then there is an* $M \in \mathcal{K}$ *with* $A \subseteq M \prec_{\mathcal{K}} N$ *and* $|M| \leq |A| + LS(\mathcal{K})$.

When the class $\mathcal{K}$ is clear from the context, we often drop the subscript and write $\prec$ rather than $\prec_{\mathcal{K}}$. We will only consider non-empty AECs.

Examples. *1. The most natural specimen of an AEC is the class of models of a first-order theory with* $\prec_{\mathcal{K}}$ *interpreted as first-order elementary substructure.*

2. The class of all groups $\mathcal{K}$*, where* $\prec_{\mathcal{K}}$ *is given by the subgroup relation, forms an AEC.*

Following [Bal09], we call the members of $\mathcal{K}$ models and the substructures of a model (in the sense of $\prec_{\mathcal{K}}$) submodels. Naturally, the notion of a submodel gives rise to a notion of embedding.

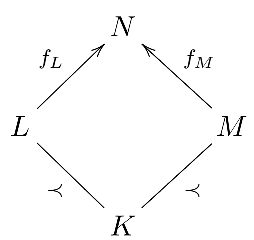
Definition. *Let* $(\mathcal{K}, \prec)$ *be an AEC,* $M, N \in \mathcal{K}$ *and* $f : M \to N$ *a monomorphism. We say that* f *is a* $\prec_{\mathcal{K}}$-embedding *if* $f(M) \prec_{\mathcal{K}} N$.

Often we identify structures with their image under an embedding and write, somewhat abusing notation, $M \prec_{\mathcal{K}} N$ instead of M $\prec_{\mathcal{K}}$-embeds into N.

We now give two important definitions for AECs. The first one gets us a bit closer to the first-order context, the second ensures a kind of completeness.

Definition. *Let* $(\mathcal{K}, \prec)$ *be an AEC.*

1. *We say that* $\mathcal{K}$ *has the* amalgamation property (AP) *if for any* $K, L, M \in \mathcal{K}$ *with* $K \prec L$ *and* $K \prec M$ *there exist some* $N \in \mathcal{K}$ *and* $\prec$*-embeddings* f_L *and* f_M *respectively such that the diagram*



commutes.

2. *We say that $\mathcal{K}$ has the* joint embedding property (JEP) *if for any $K, L \in \mathcal{K}$ there exists some $N \in \mathcal{K}$ such that both K and L $\prec$-embed into N.*

Examples. *1. For any first-order theory, the model class satisfies AP (this is a straightforward construction using compactness). The class has JEP iff the theory is complete.*

2. The class of groups with the subgroup relation satisfies both AP and JEP (via the free product).

4.3.2 The Class $\mathcal{K}_G$

Note that the class of fields with a given absolute Galois group is not always first-order axiomatizable, though it is first-order axiomatizable if the group is small. We prove now that in any case this class gives rise to an abstract elementary class.

Definition. *Let G be a profinite group. We define*

$$\mathcal{K}_G = \{\, K \mid K \textit{ is a field}, \, G_K \cong G \,\} .$$

Furthermore, we define a notion of embedding $\prec_G$ on $\mathcal{K}_G$ given by $K \prec_G L$ just in case the extension $K \subseteq L$ is regular and the canonical projection $\mathrm{pr}_{L/K} : G_L \to G_K$ is an isomorphism.

Proposition 4.3.1. *The class $(\mathcal{K}_G, \prec_G)$ is an abstract elementary class.*

Proof: Clearly, the class and the relation are closed under isomorphisms. We check the axioms successively:

(Axiom 1) Satisfied by definition.

(Axiom 2) We have $K \prec K$ for any $K \in \mathcal{K}_G$. Furthermore, given fields K, L and M satisfying $K \prec L \prec M$, the extension $K \subset M$ is regular. Since the canonical projection map $\mathrm{pr}_{M/K} : G_M \to G_K$ is just the composition of the projections $\mathrm{pr}_{M/L}$ and $\mathrm{pr}_{L/K}$, i.e. the diagram

$$\underbrace{G_M \xrightarrow{\mathrm{pr}_{M/L}} G_L \xrightarrow{\mathrm{pr}_{L/K}} G_K}_{\mathrm{pr}_{M/K}}$$

commutes, it is an isomorphism. Hence $K \prec M$.

(Axiom 3) Let $\langle K_i \,|\, i < \delta \rangle$ be a continuous $\prec$-increasing chain. Then the union $L \coloneqq \bigcup_{i<\delta} K_i$ is a field and the extension $K_i \subseteq L$ is regular for all $i < \delta$. Thus, all the canonical projections pr_{L/K_i} are surjective. Suppose there is some $i < \delta$ such that pr_{L/K_i} is not injective. Hence, there is some $\sigma \in G_L$, $\sigma \neq id_{L^{sep}}$, such that the restriction $\sigma|_{{K_i}^{sep}}$ is the identity map. Say σ permutes the zeroes of some polynomial $f \in L[X]$ non-trivially. Take $i < j < \delta$ such that $f \in K_j[X]$. Then σ restricted to K_j^{sep} is non-trivial, so pr_{K_j/K_i} is not an isomorphism. This is a contradiction. Thus, (3.1) and (3.2) both follow.

Assume that there is some $M \in \mathcal{K}$ such that $K_i \prec M$ for all $i < \delta$. Then $L \subseteq M$ is regular. As all the projections pr_{M/K_i} factor through G_L, i.e. once more the diagram

$$\underbrace{G_M \xrightarrow{\mathrm{pr}_{M/L}} G_L \xrightarrow{\mathrm{pr}_{L/K_i}} G_{K_i}}_{\mathrm{pr}_{M/K_i}}$$

commutes, $\mathrm{pr}_{M/L}$ is also an isomorphism.

(Axiom 4) Assume $K, L, M \in \mathcal{K}_G$ are such that $K \prec M$, $L \prec M$ and $K \subset L$. Then the extension $K \subseteq L$ is regular and since $\mathrm{pr}_{M/K} = \mathrm{pr}_{L/K} \circ \mathrm{pr}_{M/L}$ as above, the projection $\mathrm{pr}_{L/K}$ is an isomorphism.

(Axiom 5) We claim that

$$LS(\mathcal{K}_G) \coloneqq \max\{\,\aleph_0, \mathrm{card}(G)\,\}$$

is a Löwenheim-Skolem number for $\mathcal{K}_G$.

Take $K \in \mathcal{K}_G$, $A \subset K$. We need to show that there is some field $K_A \prec K$ containing A satisfying $|K_A| \leq LS(\mathcal{K}_G) + |A|$. Let K_1 be the field generated by A inside K, so $|K_1| \leq |A| + \aleph_0$. We may assume that the extension K/K_1 is regular, since this does not increase the cardinality of K_1 by more than $\aleph_0$.

Thus, pr_{K/K_1} is surjective. The idea is now to increase K_1 to K_2 by adjoining coefficients of minimal polynomials until pr_{K/K_2} becomes an isomorphism. For any $\sigma \in \ker(\mathrm{pr}_{K/K_1})$ take some $\alpha \in K^{sep} \setminus K$ with $\sigma(\alpha) \neq \alpha$. Then adjoin the coefficients of the minimal polynomial $\min(\alpha/K)$ to K_1. After at most $|G|$-many steps, we obtain a field $K_2 \supset K_1$ such that the projection pr_{K/K_2} is injective. Once more, we may assume that $K_2 = K_A$ is a regular subfield of K without increasing the cardinality of K_A. Now pr_{K/K_A} is indeed an isomorphism, so $K_A \prec K$. □

4.3.3 The Class $\mathcal{K}_G^{elem}$ and another Notion of Embedding

There also is another closely related class of fields, namely the elementary class

$$\mathcal{K}_G^{elem} := \{\, K \mid K \text{ is a field, } G_K \equiv G \,\} .$$

Furthermore, there is the natural embedding notion $\prec_G^{elem}$ on both of the classes $\mathcal{K}_G$ and $\mathcal{K}_G^{elem}$, interpreted as elementary substructure. Since it is an elementary class, $(\mathcal{K}_G^{elem}, \prec_G^{elem})$ is also an AEC.

Observation 4.3.2. *For small profinite groups G, we have $\mathcal{K}_G = \mathcal{K}_G^{elem}$. Thus, the class $(\mathcal{K}_G^{elem}, \prec_G)$ is abstract elementary iff G is small.*

Proof: If G is small, then $(\mathcal{K}_G^{elem}, \prec_G) = (\mathcal{K}_G, \prec_G)$ is abstract elementary by Proposition 4.3.1. If G is not small, then there is some $n \in \mathbb{N}$ such that G has infinitely many quotients of index n. If G occurs as the absolute Galois group of some field K, then for any λ there are elementary extensions G_λ of G which have more than λ-many quotients of index n and which occur as the absolute Galois group of an elementary extension of K. For $\lambda > \aleph_0$, all fields which have G_λ as an absolute Galois group have size $\geq \lambda$. Hence, there is no Löwenheim-Skolem number for the class $(\mathcal{K}_G^{elem}, \prec_G)$. □

Let G be a small profinite group. Then we get for any $K, L \in \mathcal{K}_G$

$$K \prec_G^{elem} L \Longrightarrow K \prec_G L, \tag{4.1}$$

as any surjection of a small profinite group onto itself is an isomorphism. The reverse implication to (4.1) does not hold, not even for small profinite groups. Take for example any (non-trivial) projective profinite group G, and let K be a perfect PAC field of characteristic 0 with $G_K \cong G$ (see Theorem 1.1.6). Consider $L = K((\mathbb{Q}))$. Then L is not PAC (Theorem 2.3.3) and $G_L \cong G_K$. On the other hand, there is some PAC field $M \supset L$ such that the projection $\mathrm{pr}_{M/L} : G_M \to G_L$ is an isomorphism (Theorem 1.1.6). Hence, we have $M, L \in \mathcal{K}_G$ and $L \prec_G M$ but M is not an elementary extension of L.

If the theory of fields induced by G is model complete, then clearly

$$K \prec_G L \Longrightarrow K \prec_G^{elem} L \tag{4.2}$$

holds for all $K, L \in \mathcal{K}_G$. Examples for this are given by the group $G = \mathbb{Z}/2\mathbb{Z}$ and the absolute Galois group of any field appearing in Theorem 4.1.3, class (B) (see [PR84], Theorem 5.1), i.e. in particular $G = G_{\mathbb{Q}_p}$.

Question 4.3.3. *For which profinite groups G does*

$$K \prec_G^{elem} L \Longleftrightarrow K \prec_G L$$

hold for all $K, L \in \mathcal{K}_G$?

In this context, the question of whether there is a field K elementarily characterized by $\mathrm{Th}(G_K)$ but not elementarily characterized by G_K (see Questions 4.1.4) comes down to whether there is a non-small profinite group G with $(\mathcal{K}_G^{elem}, \prec_G^{elem})$ non-empty and the model class of some complete theory.

4.3.4 Amalgamation for $(\mathcal{K}_G, \prec_G)$

We end by making some remarks about the amalgamation property for classes $(\mathcal{K}_G, \prec_G)$.

Proposition 4.3.4. *The class $(\mathcal{K}_G, \prec_G)$ has AP if*

1. $G \cong \{id\}$,
2. $G \cong \mathbb{Z}/2\mathbb{Z}$,
3. $G \cong G_L$, *for L as in Theorem* 4.1.3*, class (B),*
4. *G is projective.*

Proof: As the first three are elementary classes, it is obvious that they have AP. So let G be projective, and consider $K, L, M \in \mathcal{K}_G$ such that $K \prec L$ and $K \prec M$. We may assume L and M to be linearly disjoint over K, so the compositum LM is a regular extension of K. Thus, the projection map

$$\mathrm{pr}_{LM/K} : G_{LM} \longrightarrow G_K$$

is surjective. As G_K is projective, this map has a section $\pi : G_K \to G_{LM}$ (see Observation 1.1.3). The fixed field F of the image of π satisfies $G_F \cong G_K$ and all projection maps are isomorphisms. Hence F completes the amalgamation diagram. □

It seems more than likely that there should be profinite groups G such that $(\mathcal{K}_G, \prec_G)$ does not have AP. However, we have not been able to construct any counterexamples yet.

Question 4.3.5. *For which profinite groups G does $(\mathcal{K}_G, \prec_G)$ have AP?*

Note that if G is a small profinite group, then $(\mathcal{K}_G, \prec_G^{elem})$ has AP, so the question is closely connected to Question 4.3.3.

Outlook

This thesis presents some results on definable henselian valuations and their connections to absolute Galois groups. However, there are many more open questions and problems related to these matters. We will first look at some further issues around the existence of definable henselian valuations, and then discuss problems concerning applications of definable henselian valuations to other areas of model theory and to arithmetic geometry.

Definable Henselian Valuations

There are no known examples of henselian fields which do not carry a definable henselian valuation and which are neither separably nor real closed. We give several conditions for a henselian field to admit a $\emptyset$-definable non-trivial henselian valuation, but we also know that not every henselian valued field admits a $\emptyset$-definable non-trivial henselian valuation. All our definitions exploit the uniform definability of canonical p-henselian valuations, so a natural first question is: Are there more classes of henselian fields where similar techniques lead to definitions of non-trivial henselian valuations? Secondly, what happens if we allow parameters? Is there a henselian valued field which is not separably or real closed and on which no non-trivial henselian valuation is definable with parameters?

Furthermore, all the definitions of henselian valuations we give are obtained using Beth's Definability Theorem. Another matter for future research is to give more explicit definitions for (some) non-trivial henselian valuations.

At the moment, we also have no clear understanding of fields which are t-henselian but not henselian. It would be very desirable to give even a partial characterization of these fields. In particular, is there a field with small absolute Galois group which is t-henselian, but not henselian (Question 4.2.3)? On the other hand, there are fields for which henselianity is an $\mathcal{L}_{ring}$-elementary property, i.e. those fields such that any elementarily equivalent field admits a non-trivial henselian valuation. For example, this holds for any field which has a small absolute Galois group and admits a tamely branching henselian valuation (Corollary 2.2.3). Do all of these fields admit $\emptyset$-definable non-trivial henselian valuations? Is there an example of a field for which henselianity is elementary and which does not carry a $\emptyset$-definable non-trivial henselian valuation?

Applications

In the last chapter of this thesis, we give some applications of our results on definable henselian valuations. We use them to work towards a new classification of fields elementarily characterized by their absolute Galois group. Answering some of the above questions would give an improved characterization, but of course the ultimate aim is to develop new tools to prove the conjectured empty classes to actually be empty. Possibly, a more in-depth analysis of the abstract elementary classes discussed in Section 4.3 would help in achieving this.

Another possible application, already to some extent discussed in Subsection 3.2.4, is to NIP fields. Recall the conjecture of Saharon Shelah and Assaf Hasson:

Conjecture. *Let K be an NIP field which is neither separably nor real closed. If $K^\times/(K^\times)^n$ is finite for all $n \in \mathbb{N}$, then K admits a non-trivial definable valuation.*

It follows from Theorem 2.2.2 that any t-henselian field admits a $\emptyset$-definable non-trivial henselian valuation. Thus, clearly the conjecture is true for t-henselian NIP fields. However, it might be possible to use our methods more generally. We know that any p-henselian field K with $K \neq K(p)$ admits a non-trivial definable valuation. In order to show the conjecture, one could try to show the following: Let K be an NIP field such that K is neither separably nor real closed and such that $K^\times/(K^\times)^n$ is finite for all $n \in \mathbb{N}$. Assume that K also admits a Galois extension of prime degree p. Does this imply that K is p-henselian?

An indication that this might be true, at least under some additional assumptions, is our analysis in Subsection 2.3.3. Note that by a result of Duret, PAC fields which are not separably closed have the independence property ([Dur80]). Can we interpret a random graph in a similar fashion to Duret whenever K is not p-henselian and G_K is small? This amounts to gaining a clear understanding of what happens when one of the axioms of being the base of a V-topology fails for the sets described in Theorem 2.3.6.

A third application of the present results could be Pop's conjecture. Recall the conjecture about finitely generated fields from the introduction:

Conjecture ([Pop02]). *Any two finitely generated fields are elementarily equivalent if and only if they are isomorphic.*

The conjecture has been solved for Kronecker degree 1 and 2 (see [Rum80] and [Pop11]). By the results from [Sca08] and [Sca11], it suffices to show that for any infinite finitely generated field K and any function field F in one variable over K, the valuations on F coming from rational points of a fixed degree are uniformly definable. Note that all infinite finitely generated fields are hilbertian. Thus, we can conclude from Corollary 3.1.7 that for any such function field F with valuation v, the henselization v^h is uniformly definable on F^h. By the Key Lemma in [Koe07], these henselizations can be found Galois-theoretically as fixed fields of certain subgroups of G_F. The next step is now to check whether the arguments in [Sca08] can be adapted such that the results outlined above suffice to prove the conjecture. Another possibility is to follow the Galois-theoretic path further in order to conclude that the valuations coming from rational points, rather than only their henselizations, are indeed uniformly definable.

I hope that I will be able to make some progress on the problems outlined above in my future research.

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