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Profinite and pseudofinite groups
Sheet 11

Exercise 1. Let G and H be sets, $\mathcal{A}$ a collection of subsets of G, and $\mathcal{B}$ a collection of subsets of H. Let $\mathcal{A}'$ be the σ-algebra of G generated by $\mathcal{A}$, $\mathcal{B}'$ the σ-algebra of H generated by $\mathcal{B}$, and $\mathcal{C}$ a σ-algebra of $G \times H$. Suppose $A \times H$ and $G \times B$ are in $\mathcal{C}$ for all $(A, B) \in \mathcal{A} \times \mathcal{B}$. Use $\mathcal{A} \otimes \mathcal{B}$ to we denote the σ-algebra of $G \times H$ generated by all of the sets $A \times B$ with $(A, B) \in \mathcal{A} \times \mathcal{B}$. Show $\mathcal{A} \otimes \mathcal{B} \subseteq \mathcal{C}$.

The direct product $G \times H$ of profinite groups G and H is a profinite group. Recall that $G \times H$ has a unique Haar measure $\mu_{G\times H}$. Consider also the product measure $\mu_G \times \mu_H$ of $G \times H$. It is first defined on sets ('rectangles') of the form $A \times B$ with A a μ_G-measurable set and B a μ_H-measurable set by the rule $(\mu_G \times \mu_H)(A \times B) = \mu_G(A)\mu_H(B)$. Then $\mu_G \times \mu_H$ extends to a σ-additive measure on the σ-algebra generated by these rectangles. Finally, one completes $\mu_G \times \mu_H$ by adding all zero sets.
The aim of this sheet is to prove that $\mu_{G\times H}$ coincides with $\mu_G \times \mu_H$.

Exercise 2. Let G and H be profinite groups. Let $\mathcal{B}_0(G)$ be the Boolean algebra of all open-closed subsets of G, $\mathcal{B}_1(G)$ the σ-algebra generated by $\mathcal{B}_0(G)$, $\mathcal{B}(G)$ the Borel field of G, and $\hat{\mathcal{B}}(G)$ the family of all measurable subsets of G (corresponding notation for H and for $G \times H$). In addition let $\mathcal{R}_0$ be the Boolean algebra of $G \times H$ generated by all of the sets $A \times B$ with $(A, B) \in \mathcal{B}_0(G) \times \mathcal{B}_0(H)$, $\mathcal{R}_1 = \mathcal{B}_0(G) \otimes \mathcal{B}_0(H)$, $\mathcal{R} = \mathcal{B}(G) \otimes \mathcal{B}(H)$, and $\hat{\mathcal{R}}$ be the completion of $\mathcal{R}$ with respect to $\mu_G \times \mu_H$.

a) Show that we have $\mathcal{R}_0 = \mathcal{B}_0(G \times H)$, $\mathcal{R}_1 = \mathcal{B}_1(G \times H)$, and $\mathcal{B}_1(G) \otimes \mathcal{B}_1(H) \subseteq \mathcal{R}_1$.
b) Show that $\mu_{G\times H}$ and $\mu_G \times \mu_H$ coincide on $\mathcal{R}_1$.
c) Define $\mathcal{S} = \{C \in \hat{\mathcal{R}} \cap \hat{\mathcal{B}}(G \times H) \mid (\mu_G \times \mu_H)(C) = \mu_{G\times H}(C)\}$. Observe that $\mathcal{S}$ is closed under countable unions and taking complements and thus $\mathcal{R}_1 \subseteq \mathcal{S}$. If $E_0 \subseteq E_1$, $E_1 \in \mathcal{S}$, and $(\mu_G \times \mu_H)(E_1) = 0$, then $E_0 \in \mathcal{S}$. Show that $\mathcal{S}$ contains $\mathcal{R}$.
d) Conclude that $\hat{\mathcal{R}} = \mathcal{S} = \hat{\mathcal{B}}(G \times H)$.

Hand-in by Monday, 11.7., 09:00 am
The exercise sheets should be solved and handed in in groups of two.
Web page: `http://wwwmath.uni-muenster.de/u/franziska.jahnke/pp/`