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Universität Münster
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Profinite and pseudofinite groups
Sheet 5

Let G be a profinite group and denote by $\mathrm{Aut}(G)$ the group of all continuous automorphisms of G. For a closed normal subgroup K of G, define

$$A_G(K) = \{\varphi \in \mathrm{Aut}(G) : \varphi(g)g^{-1} \in K \text{ for all } g \in G\}.$$

We make $\mathrm{Aut}(G)$ into a topological group by letting the sets $A_G(U)$ serve as a fundamental system of neighborhoods of the identity, where U ranges over the set of all open normal subgroups of G.
We call the corresponding topology the *congruence subgroup topology* of $\mathrm{Aut}(G)$.

Exercise 1. Let G be a profinite group and U be an open normal subgroup of G.

a) Show that $A_G(U)$ is the subgroup of $\mathrm{Aut}(G)$ consisting of those automorphisms of G that leave U invariant and induce the trivial automorphism on G/U.
b) Show that $\mathrm{Aut}(G)$ is a Hausdorff and totally disconnected as a topological group endowed with the congruence subgroup topology.
c) Give an example of a profinite group G such that $\mathrm{Aut}(G)$ is not profinite.
Hint: Consider the group G given as an infinite direct product of cyclic groups of order 2.

Exercise 2. Let G be a profinite group and consider $\mathrm{Aut}(G)$ as a topological group with the congruence subgroup topology. Show that this is the coarsest topology that makes the natural action of $\mathrm{Aut}(G)$ on G continuous.

Let G be a profinite group. For a compact set K of G and an open set V of G, define

$$B(K, V) = \{\varphi \in \mathrm{Aut}(G) : \varphi(K) \subseteq V\}.$$

Then the collection of all subsets of the form $B(K, U)$ form a subbase for a topology on $\mathrm{Aut}(G)$; this topology is called the *compact-open topology* on $\mathrm{Aut}(G)$.

Exercise 3. Let G be a profinite group. Show that the congruence subgroup topology on $\mathrm{Aut}(G)$ coincides with the compact-open topology of $\mathrm{Aut}(G)$.

Exercise 4. Assume that a profinite group G admits a fundamental system $\mathcal{U}$ of open neighborhoods of the identity such that each $U \in \mathcal{U}$ is a characteristic subgroup of G. Then $\mathrm{Aut}(G)$ is a profinite group.

Hand-in by Monday, 30.5., 09:00 am
The exercise sheets should be solved and handed-in in groups of two.
Web page: `http://wwwmath.uni-muenster.de/u/franziska.jahnke/pp/`