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Universität Münster
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Profinite and pseudofinite groups
Sheet 3

Recall that the normalizer $N_G(H)$ of a subgroup H of an abstract group G is defined by $N_G(H) = \{g \in G : g^{-1}Hg = H\}$. It is a subgroup, and if G is profinite and H is closed then $N_G(H)$ is closed.

Exercise 1. Let G be a profinite group, K a normal subgroup and P a p-Sylow subgroup of G. Then show that:

a) $K \cap P$ is a p-Sylow subgroup of K;

b) KP/K is a p-Sylow subgroup of G/K;

c) $G = N_G(Q)K$ for each p-Sylow subgroup Q of K;

d) $H = N_G(H)$ whenever H is a subgroup which contains $N_G(Q)$ for some p-Sylow subgroup Q of K.

Let p be a prime. A subgroup H of a profinite group G is called a p-complement if p does not divide $|H|$ but it is the unique prime dividing $[G : H]$.

Exercise 2. Show that if G is a profinite group having p-complements for all primes p, then G is prosoluble.

Exercise 3. Let $\{H_i\}_{i\in I}$ be a family of normal subgroups of a profinite group G. Suppose that G is the closure of the abstract subgroup generated by $\bigcup_{i\in I} H_i$ and for each i let K_i be the closure of the abstract subgroup generated by $\bigcup_{j\neq i} H_j$. If $\bigcap_{i\in I} K_i = \{1\}$ then G is isomorphic to the cartesian product $\prod_{i\in I} H_i$.

Exercise 4. Let G be a profinite group. Show that the following conditions are equivalent:

a) G is pronilpotent;

b) each Sylow subgroup of G is normal in G;

c) G is (isomorphic to) the cartesian product of its Sylow subgroups;

d) each proper open subgroup U of G is strictly contained in $N_G(U)$.

Hint: Use the corresponding version of this statement for finite groups, and to prove *b)* $\Rightarrow$ *c)* use Exercise 3.

Hand-in by Monday, 9.5., 09:00 am
The exercise sheets should be solved and handed-in in groups of two.
Web page: `http://wwwmath.uni-muenster.de/u/franziska.jahnke/pp/`