Pf: Suffices to show: if $x \in H$ has prime order q, then $q = p$. Let $\{x_1, \dots x_d\}$ be a generating set for G, put

$$\Omega = \{(u_1 x_1, \dots, u_d x_d) \mid u_i \in \Phi(G)\} \leq G^d$$

Then x acts on Ω and has no fixed pts by Lemma 2.1 (ii). Thus any orbit of x has length q and so $q \mid |\Omega| = |\Phi(G)|^d$, and so $q = p$.

In order to show that $\mathrm{Aut}(G)$ for G fin. gen pro-p is again pro-p, we need

Lemma 4.6 Let G be profin., $H \trianglelefteq_o G$, $Z = Z(H)$. Let $\Delta \leq \mathrm{Aut}(G)$ s.th. Δ acts trivially on H and $[G, \Delta] \leq H$.

Then there is an inj. cont. homom.

$$\delta : \Delta \to Z^{tm} \quad \text{where } m = d(G/H).$$

If Δ is cp, then δ is a top. isom onto a closed subgp of Z^m.

Pf: Let $\{x_1 H, \dots x_m H\} = G/H$ and for $\gamma \in \Delta$ put

$$\gamma^\delta = ([x_1, \gamma], \dots [x_m, \gamma]) \in G^m$$

(Note that $[x, \gamma] = x^{-1} x^\gamma$.)

Then γ^δ determines the action of γ on G, so δ is inj.

For any $x \in G$, $h \in H$ we have

$$[x, \gamma]^h = x^{-h} (x^h)^\gamma = [x^h, \gamma] = [[h, x^{-1}] x, \gamma] = [x, \gamma]$$

since $h^\gamma = h$ and $[[h, x^{-1}], \gamma] \in [H, \gamma] = 1$.

Thus $[G, \gamma] \leq Z$ and so $\gamma^\delta \in Z^m$.

For $\alpha, \beta \in \Delta$, $x \in G$, we have

$$[x, \alpha\beta] = [x, \beta]\,[x, \alpha]^\beta = [x, \alpha]\,[x, \beta]$$

since Z is abelian and β acts trivially on Z. Thus

$\delta : \Delta \to \mathbb{Z}^m$ is a homom.

To see that δ is cont., it suffices to see that for any open nbhd U of $1 \in \mathbb{Z}^m$ there is $N \trianglelefteq_o G$ s.th. $(\Delta \cap \Gamma(N))^\delta \subseteq U$. Since there is $N \trianglelefteq_o G$ s.th. $(N \cap \mathbb{Z})^m \subseteq U$, N is as required.

If Δ is cp, then so is Δ^δ and the last claim follows from general facts on cp. Hausd. spaces.

Thm 4.7 Let G be a fin. gen. profin. gp. If G is virtually pro-p, then $\mathrm{Aut}(G)$ is virt. pro-p.

Pf: By Prop. 1.31, G has a top. char. open pro-p subgp H, and H is fin. gen by Prop. 1.32. Now $\Phi(H)$ is open and top. char in G. Let $\Delta = \Gamma(\Phi(H))$ be the kernel of the action of $\mathrm{Aut}(G)$ on $G/\Phi(H)$.

Then $\Delta \trianglelefteq_{\text{open}} \mathrm{Aut}(G)$.

Claim: Δ is pro-p.

Let $\pi : \Delta \to \mathrm{Aut}(H)$ be the restn. map, put $\Delta = \ker \pi$. Clearly π is contin. since for any $N \trianglelefteq_o H$ we have $(\Delta \cap \Gamma(N))^\pi \subseteq \Gamma_H(N)$, where $\Gamma_H(N) = \{\gamma \in \mathrm{Aut}(H) \mid [H, \gamma] \in N\}$.

Thus $\Delta \trianglelefteq_{\text{closed}} \Pi$ so Δ is cp and by Lemma 4.6 Δ is top. isom. to a closed subgp of $\mathbb{Z}(H)^m$ where $m = d(G/H)$ is fin. Hence Δ is pro-p.

Since Ξ is cp and π cont. Ξ/Δ is top isom. to the closed subgp $\Xi^\pi \subseteq \mathrm{Aut}(H)$. Now $\Xi^\pi \subseteq \Gamma_H(\Phi(H))$ so Ξ^π is pro-p by Prop. 4.5.

By Prop. 1.19 (should have) Ξ is pro-p.