§4 Automorphism gps of profin. gps

Aim: Aut gp of pro-p gp of fin. rk is pro-p.

First study $GL_d(\mathbb{Z}_p) =: \Gamma$

Then $\Gamma \subseteq_c M_{d\times d}(\mathbb{Z}_p)$ with the induced topol. from $\mathbb{Z}_p^{d\times d}$

Γ is clopen in $M_d(\mathbb{Z}_p)$: For $a \in M_d(\mathbb{Z}_p)$ we have $a \in \Gamma$ iff $\det a \not\equiv 0 \pmod p$. So for $b \equiv a \pmod p$ we have $a \in \Gamma \Leftrightarrow b \in \Gamma$.

Thus Γ is cp and a bbled basis for 1 is given by congruence subgps

$$\Gamma_i = \{ \gamma \in \Gamma \mid \gamma \equiv 1_d \pmod{p^i} \} = \{ 1 + a$$

$$\left(\text{cp.} = \left\{ \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix} \in \Gamma \;\middle|\; a,b,c,d \in p^{i}\mathbb{Z}_p \right\}, \quad i \geq 0. \right.$$

where M is the max ideal.

Since $\Gamma/\Gamma_i \cong GL_d(\mathbb{Z}/p^i\mathbb{Z})$, $i \geq 1$, we have

$$|\Gamma : \Gamma_1| = \prod_{j=0}^{d-1} (p^d - p^j) \quad \text{and}$$

$$|\Gamma_1 : \Gamma_i| = p^{d^2(i-1)}, \quad i \geq 1.$$

Thus Γ is profin. and Γ_1 is pro-p.

Clearly, Γ is p-adic anal. and we verify directly that Γ contains an open (unif.) powerful fin. gen. pro-p subgp.

Lemma 4.1 (Hensel's lemma) If p is odd, $n \geq 2$ ($p = 2$, $n \geq 3$) every $g \in \Gamma_n$ is p^{th} power of some $h \in \Gamma_{n-1}$.

Pf: We have to show that for $a \in \Pi_d(\mathbb{Z}_p)$ we can solve

$$1 + p^n a = (1 + p^{n-1} x)^p \qquad (*)$$

with $x \in \Pi_d(\mathbb{Z}_p)$.

By succ. approx. We have $(1 + p^{n-1} a)^p \equiv 1 + p^n a \pmod{p^{n+1}}$

Put $x_1 = a$ and assume induct. that for some $r \geq 1$ we have x_r s.th. $(1 + p^{n-1} x_r)^p \equiv 1 + p^n a \pmod{p^{n+r}}$

so $(1 + p^{n-1} x_r)^p = 1 + p^n a + p^{n+r} c$.

Put $z = (1 + p^{n-1} x_r)^{-(p-1)} c$ and

$$x_{r+1} = x_r - p^r z.$$

Then $(1 + p^{n-1} x_{r+1})^p \equiv 1 + p^n a \pmod{p^{n+r+1}}$

Clearly $x = \varprojlim x_r$ satisfies $(*)$.

Thm 4.2 For $i \geq 1$ put $\Gamma_i = \{\gamma \in GL_d(\mathbb{Z}_p) \mid \gamma \equiv 1_d \pmod{p^i}\}$

Put $G = \Gamma_1$ if p is odd, ($G = \Gamma_2$ if $p = 2$). Then G is a unif. powerful pro-p gp and $\dim(G) = rk(G) = d(G) = d^2$. Furthermore $P_i(G) = \Gamma_{i+\varepsilon}$ where $\varepsilon = 0$ if $p \neq 2$, $\varepsilon = 1$ if $p = 2$.

Pf. By def. $P_1(G) = G = \Gamma_{1+\varepsilon}$. Suppose $r \geq 1$ and $P_r(G) = \Gamma_{r+\varepsilon}$

Then $P_r(G)^p [P_r(G), G] \leq \Gamma_{r+1+\varepsilon}$ and by Lemma 4.1 we have $\Gamma_{r+1+\varepsilon} \leq \Gamma_{r+\varepsilon}^p = P_r(G)^p$, so $\Gamma_{r+1+\varepsilon} = P_{r+1}(G)$.

By ind, $P_i(G) = \Gamma_{i+\varepsilon}$ and $P_i(G)^p = P_{i+1}(G)$ for all i.

For $i = 1$ it follows that G is powerful and since $P_2(G) = \Gamma_{2+\varepsilon}$ is open in G, by Prop 2.4 we see that G is fin. gen. Since $|\Gamma_i : \Gamma_{i+1}| = p^{d^2}$ for all i, G is unif. powerful and since $G/\Phi(G) = \Gamma_{1+\varepsilon}/\Gamma_{2+\varepsilon}$ is

elem. ab. of order p^{d^2}, we see that

$$\dim(G) = \mathrm{rk}(G) = d(G) = d^2.$$

We thus see that G has fin. rk w/o any matrix calculations. One can show:

Thm: Any p-adic p gp of fin rk has a faithful lin. repres. over $\mathbb{Z}_p$, and hence is p-adic analytic. (and hence has an open unif. powerful subgp)

Thm 4.3 If G is a fin. gen. profin. gp, then $\mathrm{Aut}(G)$ is a profin. gp w.r.t. the topol. given by clopen subgps $\Gamma(N) = \{\gamma \in \mathrm{Aut}(G) \mid [G, \gamma] \subseteq N\}$, $N \trianglelefteq_o G$.

Rem (i) We have $\gamma \in \Gamma(N)$ iff γ is cont. and γ leaves N invariant and induces the identity on Γ/N.

(ii) For $G = \mathbb{Z}_p^d$, this induces the topol. given in Lemma 4.1.

Pf: Let $\Gamma = \mathrm{Aut}(G)$. Since every open $N \trianglelefteq_o G$ contains an open top. char. subgp $N_0 \trianglelefteq G$ by 1.31. Then $\Gamma(N_0) \leq \Gamma(N)$ and $\Gamma(N_0)$ is the kernel of the induced action of Γ on G/N_0. Thus $|\Gamma : \Gamma(N_0)|$ is finite. This shows that Γ has a nbhd basis for 1 consisting of subgps of fin. index.

By Thm 1.14 need to show $\mathrm{Gal}\,\Gamma = \varprojlim_{N_0 \text{ char } G} \Gamma/\Gamma(N_0)$.

Let N_0 char in G let

$\omega_{N_0} : \mathrm{Aut}(G) = \Gamma \longrightarrow \Gamma(N_0)$ given by induced action. Clearly, the ω_{N_0} form a compatible system.

Hence they induce a cont. system homom.

$$\omega : \Gamma \to \varprojlim \Gamma/\Gamma(N_0).$$

This map is injective since $\ker(\omega) = \bigcap_{N_0} \Gamma(N_0) = 1$.

So it suffices to show that ω is epi.

Let $(f_{N_0}) \in \varprojlim \Gamma/\Gamma(N_0)$. Then for $N, M \overset{\text{char open}}{\leq} G$, $M \leq N$

$$\begin{array}{ccc} \Gamma(M) & \xrightarrow{f_N} & \Gamma(N) \\ \downarrow & & \downarrow \\ \Gamma(M) & \xrightarrow[f_M]{} & \Gamma(N) \end{array}$$

commutes, so $\{f_N\} : G/N \to G/N$ is a map of inv. systems

Put $f = \varprojlim f_N$. Since each f_N is contin. autom.

the map $f = \lim f_N : G = \varprojlim G/N \to G = \varprojlim G/N$

is also a contin. autom. and clearly $\omega(f) = (f_N)$.

Prop 4.4 If G is a fin. gen. pro-p gp, then $\Gamma(\Phi(G))$ is pro-p.

Pf: Let $G_n = P_n(G)$. Then (G_n) is a nbhd basis for 1 in G consisting of char. subgps. Also $G_2 = \Phi(G)$.

Hence $\Gamma(G_n)$, $n \geq 2$, are normal in $\Gamma(G_2) = \Gamma(\Phi(G))$ and form a nbhd basis for 1 in $\Gamma(G_2)$ (by 4.3)

Hence it suffices by Thm 4.3 to show that $\Gamma(G_2)/\Gamma(G_n)$ is a p-gp. Note that $\Gamma(G_2)/\Gamma(G_n)$ acts faithfully on G/G_n, which is a fin. p-gp, and induces the trio. action on $G/G_2 = (G/G_n)/\Phi(G/G_n)$.

Now the result follows from the following lemma:

Lemma 4.5 Let G be a fin. p-gp, $H = \{\alpha \in \mathrm{Aut}(G) \mid [G, \alpha] \subseteq \Phi(G)\}$

Then H is a fin. p-gp.