and similarly

$$N = \prod_{j} P_j \quad , \text{ where } j \text{ is finite.}$$

Thus we have $N = \prod_{i=1}^{t} P_i$, $G/N = \prod_{i=1}^{t} Q_i \times U$,

U prod. of fin. many p-Sylows, $p \neq p_i$, $i = 1, .. t$.

By Schur-Zassenhaus $UN = N \rtimes V$, hence

V induces finite gp of autom., so wlog $UN = N \times U$

Etc.

In general we can only show

Prop: Let $\mathcal{G} = (G, I)$ be a NIP prof. gp. Then G has a prosolub. open subgp (of fin. index).

Pseudofiniteness and NIP

For the pf need the following def. and preparation:

Def 3.1 (i) Let M be an L-str., $\varphi(\bar{x}, \bar{y})$ an L-form. w/o $\exists^\infty$.
Then there is a max el $\in \mathbb{N}$ s.th. there are $\bar{a}_1, .. \bar{a}_d \in M$ s.th. for each $S \subseteq \{1, .. d\}$ there is $\bar{b}_S$ s.th.

$$M \models \varphi(\bar{a}_i, \bar{b}_S) \iff i \in S.$$

Such a el is called the Vapnik-Chervonenkis-(VC-) dimension of φ in M.

(ii) If $\mathcal{C}$ is a class of L-str., we call $\mathcal{C}$ a NIP-class if for any formula $\varphi(\bar{x}, \bar{y})$ there is $d_\varphi \in \mathbb{N}$ s.th. for any $M \in \mathcal{C}$ the VC-dim of φ in M is bounded by d_φ.

Lemma 3.2 (i) Let $\mathcal{C}$ be a NIP class of fin. L-str.
Then $\mathcal{C}$ is a NIP class iff any ultraprod.
$\prod M_i/\mathcal{U}$, $M_i \in \mathcal{C}$ has NIP theory.

(ii) Let L, L' be first order languages, M an L-str. with NIP-theory. Suppose that $\mathcal{C} = \{M_i \mid i \in I\}$ is a class of L'-str. which is unif. interpret. in M where I is an interpret. set in M. Then $\mathcal{C}$ is a NIP class

Pf: Follows from Los's Thm.

Recall For a fin. gp G, the solv. radical, $R(G)$, is the max sol. normal subgp of G

Prop 3.3 Let $\mathcal{C}$ be a NIP class of fin. gps. Then there is $d = d(\mathcal{C}) \in \mathbb{N}$ s.th. $|G : R(G)| \leq d$ for all $G \in \mathcal{C}$.

For the pf need:

Thm 3.4 (Wilson) There is a formula $\psi(x)$ in the lang. of gps s.th. for any fin. gp. G we have
$$R(G) = \{x \in G \mid G \models \psi(x)\}.$$

Pf of 3.3 Let $\mathcal{C} = \{G_i \mid i \in \mathbb{N}\}$ be a NIP class of fin. gps.
If G is an ultrapower on $\mathcal{C}$, then by Los's theorem $\psi(G)$ is an ultrapower of the $R(G_i) \trianglelefteq G_i$; so $\psi(G) \trianglelefteq G$. Then $\psi(G)$ need not be solv. Claim $[G : \psi(G)]$ fin.
Let $\bar{G}_i = G_i/R(G_i)$, $S_i = \mathrm{Soc}(\bar{G}_i)$, the dir. prod. of min. normal subgps of $\bar{G}_i$. By the max. of $R(G_i)$ each min. normal subgp of $\bar{G}_i$ is nonab., and hence each S_i can be written as a dir. prod. of non-ab. simple gps.

Claim 1 There is $t \in \mathbb{N}$, s.th. for all i, S_i is dir. prod. of at most t non-ab. simple gps.

Pf: Ow for each k there are (inf. many) gps G_k s.th. S_i contains at least k non-ab. factors $T_1 \times \dots \times T_k$. Pick $x_i, y_i \in T_i$ with $[x_i, y_i] \neq 1$. For $S \subseteq \{1, \dots k\}$ let $z_\omega = \prod_{i \notin \omega} y_i$. Then $[x_j, z_\omega] = 1 \iff j \in \omega$.

Then we find an ultraprod. $\prod G_k/\mathcal{U}$ s.th. $[x,y] = 1$ has IP $\{$.

Hence we may assume each $S_i = T_{i,1} \times \dots \times T_{i,c}$ for some fixed c and T_{ij} non ab. simple.

Claim 2 There is $e \in \mathbb{N}$ s.th. each T_{ij} has Lie rk $\leq e$

(rk $A_n = n$, rk of sporadics $= 1$)

Pf: If the rk is unbdd, then in A_n we find many commuting copies of A_5, in the gps of Lie type commuting copies of $PSL_2(q)$, some q.

Then argue as in Claim 1

Claim 3 If $\mathcal{F}$ is a family of fin. simple gps of Lie rk $\leq e$, then (there is $d = d(e) \in \mathbb{N}$ s.th.) $\prod \mathcal{F}/\mathcal{U}$ is boldly simple, i.e. for $g, h \in K$, $K \in \mathcal{F}$, g is a prod. of $\leq d$ conj. of h, h^{-1}.

By Claim 2 + 3, there is $b \in \mathbb{N}$ s.th. for i, j and $x_{ij} \in T_{ij} \setminus \{1\}$ any elt. of T_{ij} is a prod. of $\leq b$ (T_{ij}-)conj. of x_{ij}, x_{ij}^{-1}. Hence the T_{ij} are unif. def. in G_i, using that the S_i are def. in G_i using the x_{ij}, $j \leq c$.

Claim 4 There is a bd on $|S_i|$, $i \in J$

Pf: Ow. since c, e are bdd, there is some (twisted) Lie type of fixed rk s.th. T_{ij} includes arbitr. large fin. simple gps of this type.

By relabelling if nec., may assume that for some subsequence $n_i \to \infty$, the gps $T_{n_i,1}$ are of this type and have unbdd card.

Since the $T_{n_i,1}$ are unif. def., let $\mathcal{U}$ be a non-princ. ultrafilter on $\mathbb{N}$ containing $\{n_i : i \in J\}$

Put $G = \prod G_i/\mathcal{U}$, $\bar{G} = G/H$, where $H = \psi(G)$.

Then there is $\bar{a} \in \bar{G}$ s.th. $\varphi(x, a)$ def. the ultrapower of the $T_{n_i,1}$, which is of the same Lie type over ps fin. field.

By Ryten, this ps. fin field is interpret. in $\prod T_{n_i,1}/\mathcal{U}$ and hence in $\bar{G}$ $\lightning$ NIP.

Now the claim follows: We have seen that there is bdd k s.th. $|S_i| \leq k$, all $i \in J$.

Then $C_i = C_{\bar{G}_i}(S_i) \trianglelefteq \bar{G}_i$. Since $C_i \cap S_i = 1$ we have $C_i = 1$. Thus $\bar{G}_i \hookrightarrow \mathrm{Aut}(S_i)$, so $|\bar{G}_i| \leq k!$ □

Now use Prop 3.3 to prove

Prop 3.4 Let $\mathcal{G} = (G, I)$ be a NIP prof gp.

Then G has a prosol. open subgp of fin. index.

Pf: Let $\{K_i \mid i \in J\}$ be the open subgps of G given by I, $N_i = \bigcap K_i^g \trianglelefteq_0 G$. The N_i are unif.

def. in $\mathcal{L}_1$ so the fin. gps $H_i = G/N_i$ are unif. interpret. Now $\mathcal{C} = \{H_i \mid i \in J\}$ is a NIP class of fin. gps. By Prop 3.3 there is $d \in \mathbb{N}$ s.th. $[H_i : R(H_i)] \leq d$. Now $\varprojlim R(H_i)$ is prosol. and of index d. □

Thm 3.5 Let G be a pro fin. gp with NIP theory and suppose G satisfies the central. chain condition, (i.e. there do not exist arbitr. long chains $C_G(B_1) \geq \dots \geq C_G(B_{n+1})$.)
Then G has a solv. def. normal subgp of fin. index.

Rem: Any stable gp satisfies the assumption or a gp with tosy NIP theory.
For stable this was known [H-P.]

For the pf we use

Thm 3.6 (Khukhro) There is $f : \mathbb{N} \to \mathbb{N}$ s.th for $d \in \mathbb{N}$ if G is a fin. sol. gp with no strictly desc. chain of centralizers of length $d+1$, then G has derived length $\leq f(d)$.

Pf of 3.5 Let G be as in the thm, let τ_e be a sentence s.th $H \models \tau_e$ if H is a gp with no prop. desc. chain of centralizers of length $\geq e+1$.
Suppose $G \models \tau_e$.
Let $\mathcal{C} = \{G_i \mid i \in J\}$ be a class of fin. gps. s.th. $G \equiv \prod G_i/\mathcal{U} = \tilde{G}$. Then $\mathcal{C}$ is a NIP class, so there is $d \in \mathbb{N}$ s.th. $[\tilde{G} : R] \leq d$ for $R = \{x \in \tilde{G} \mid \tilde{G} \models \psi(x)\}$. Now by 3.6 $\psi(G_i)$ has

derived length $\leq f(e)$. Then $\prod_{\mathcal{U}} (G_i)/R$ is a sol. normal subgp of $\tilde{G}$ and the result follows.

Cor 3.7 Any stable ps fin. gp is sol. by finite.

Thm 3.8 There are ps fin NIP gps which are not sol. by finite (but pro solo. by fin.)

Pf: Let $H = SL_2(\mathbb{Z}_p)$. Then H has NIP theory and is not solo. by finite. H is def. in $\mathbb{Z}_p$.

Let $M = p\mathbb{Z}_p$ the max ideal.

For $k > 0$ let $H_k = \{ \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix} \in H \mid a,b,c,d \in p^k M \}$

Then $H_k \trianglelefteq_0 H$, $\bar{H}_k = H/H_k$. The $\bar{H}_k$ are unif. def. in $\mathbb{Z}_p$. Now $\mathcal{C} = \{ \bar{H}_k \mid k \in \mathbb{N} \}$ is a NIP class.

Let $G = \prod \bar{H}_k / \mathcal{U}$. Then G is NIP.

Claim G is not solo by finite:

There is $N \trianglelefteq G$ s. th. $G/N \cong SL_2(\mathbb{Z}_p)$

We expand the language of gps by unary predic. $\{P_i : i < \omega\}$. In $\bar{H}_k$, let P_i denote H_i/H_k for $i \leq k$, $1 = H_k/H_k$ for $i > k$. Thus the P_i form an desc. chain of normal subgps in the $\bar{H}_k$. In $G = \prod \bar{H}_k/\mathcal{U}$ the $P_0 > P_1 > \ldots >$ form normal subgps of G.

Put $N = \bigcap P_i$. Then $G/N \cong SL_2(\mathbb{Z}_p)$ □.

Finally

Rem: (Khelif) Any ps fin. ω-stable gp of fin MR is abelian-by-finite.