Thm: If G is a top. gp containing an open subgp N which is a uniform pro-p gp. Then G is p-adic analytic.

This really follows from

Thm: Let G be a unif. pro-p gp of rk d, $G = \overline{\langle a_1, .. a_d \rangle}$.

Let $u_1, .. u_r \in G$ and define

$g = (g_1, .. g_d) : \mathbb{Z}_p^r \to \mathbb{Z}_p^d$ by

$g_i(\lambda_1, .. \lambda_r) = \mu_i$, $i \le d$ where $u_1^{\lambda_1} .. u_r^{\lambda_r} = a_1^{\mu_1} .. a_d^{\mu_d}$.

Then g is analytic on $\mathbb{Z}_p^r$.

We summarize this as

Thm Let G be unif. powerful pro-p gp of rk d, $G = \overline{\langle a_1, .. a_d \rangle}$.

(i) For each $g \in G$ there are unique $\lambda_i \in \mathbb{Z}_p$ s.th. $g = a(\lambda) = a_1^{\lambda_1} .. a_d^{\lambda_d}$.

(ii) $f : \mathbb{Z}_p^d \times \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ s.th $a(\lambda) \cdot (a(\mu))^{-1} = a(f(\lambda, \mu))$ is analytic.

(iii) If $g = a(\lambda) \in G$, then $w(g) = \min \{ v(\lambda_i) + 1 \mid i \le d \}$ where $w(g) = n$ if $g \in P_n(G) \setminus P_{n+1}(G)$.

Cor: A unif. powerful pro-p gp G of rk d is a gp def. in $\mathbb{Z}_p^{an}$ with domain $\mathbb{Z}_p^d$, and $w : G \to \mathbb{N}$ is def.

For Prop I need to show that the family of all open subgps is unif. def. in $\mathbb{Z}_p^{an}$.

Lemma Let G be a unif. powerful pro-p gp with $d(G)=d$, $H \leq_o G$. Then there is a tuple $(h_1, .. h_d) \in G^d$ s.th.

(i) $w(h_i) \leq w(h_j)$, $i \leq j$

(ii) $h_i \neq 1$, $i = 1, .. d$.

(iii) $H = \{ h_1^{\lambda_1} .. h_d^{\lambda_d} \mid \lambda_i \in \mathbb{Z}_p \}$

(iv) for all $\lambda_1, .. \lambda_d \in \mathbb{Z}_p$ we have $w(h_1^{\lambda_1} .. h_d^{\lambda_d}) = \min \{ w(h_i) + v(\lambda_i) \}$

Note: The set of d-tuples s.th. (i), (ii), (iv) holds and $\{ h_1^{\lambda_1} .. h_d^{\lambda_d} \mid \lambda_i \in \mathbb{Z}_p \}$ is a subgp. is a def. set and there is a def. equiv. rel on these tuples given by $(h_1, .. h_d) \sim (h'_1, .. h'_d) \Leftrightarrow$
$\langle h_1, .. h_d \rangle = \langle h'_1, .. h'_d \rangle$.

This equiv. rel is def. by (iii) in $\mathbb{Z}_p^{an}$.

Hence the family of open subgps of G can be identified with equiv. cl. of d-tuples with this properties.

This shows that G as a full gp is interpret. $\mathbb{Z}_p^{an}$.

Now we can show (iii) $\Rightarrow$ (i) <24.6

Main Thm If $\mathcal{G} = (G, \mathcal{T})$ is a full profin. gp,

G has an open normal subgp $N \trianglelefteq G$

s.th. $N = P_1 \times .. \times P_s$ for cp p_i-adic Lie gps P_i

then $\mathcal{G}$ is NIP.

We use

Thm (Lazard, Lubotzky, Mann, Segal) G pro-p gp. TFAE

(i) G cp. p-adic analytic

(ii) G is top. fin. gen. and virt. unif. powerful

(iii) G has fin. rk.

Let $\mathbb{Z}_p^{an}$ be $\mathbb{Z}_p$ with Denef - v.d. Dries language containing m-ary fct symbols $F(X_1,..X_m) \in \mathbb{Z}_p\langle X\rangle$ ring of formal power series

$$\sum_{\nu \in \mathbb{N}^m} a_\nu X^\nu, \quad a_\nu \in \mathbb{Z}_p, |a_\nu|_p \xrightarrow[|\nu|\to\infty]{} 0.$$

Such power series converge on $\mathbb{Z}_p^m$.

Denef, v.d Dries: $\mathbb{Z}_p^{an}$ has QE in $\mathcal{L}^{an}$

with - bin fct. symbol $D(x,y) = \begin{cases} x/y, & |x|_p \le |y|_p \\ 0 & \text{o.w.} \end{cases}$

- unary pred. P_n, $n>0$, n^{th} powers

v.d Dries, Haskell, Macph T_p^{an} is NIP

Cor Hence any unif. powerful pro-p gp is NIP.

Pf of Main thm (iii) => (i)

Let $N = P_1 \times \ldots \times P_t \triangleleft_o G$. Then N is interpr. in disj. $\mathbb{Z}_p^{an}$

$G/N = F$ is a finite gp, the gp ext.

$$0 \to N \to G \to F \to 1$$

is also def. in $\mathbb{Z}_p^{an}$,

essentially because autom. of P_i are again p_i-analytic.

$\square$

For the proof of the Main thm (i) => (ii) is clear.

For (ii) => (iii) have to show.

Main Thm (ii) => (iii) If G is full NTP_2 profin, then G has an open normal subgp $N \trianglelefteq G$ s.th.

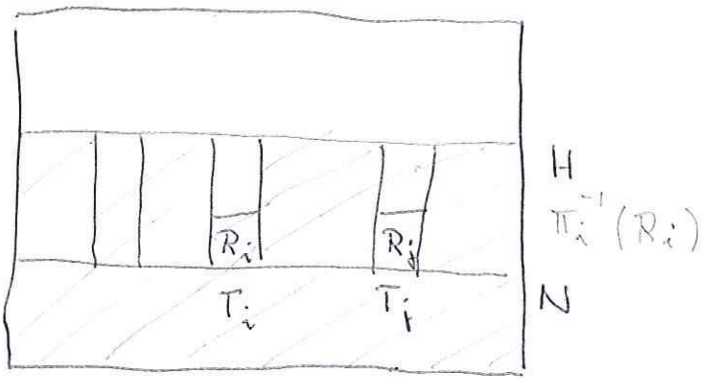
$N = P_1 \times \ldots \times P_s$ for (cp) p_i-adic analyt. gps P_i.

The proof is essentially the same for NIP and NTP_2 and relies on the following easy

Lemma: Let G be NIP (NTP_2), $H \leq G$, $N \trianglelefteq H$ s.th. $H/N \simeq \prod_{i \in J} T_i$, $\pi_i : H \to T_i$ proj.

Suppose $\forall i \in J$ $\exists$ $1 \leq R_i \lneq T_i$, $\tilde{R}_i \leq G$ where (finite inters. of) the $\tilde{R}_i$ are unif. def. by $\varphi(x, a_i)$ s.th. $\tilde{R}_i \cap H = \pi_i^{-1}(R_i)$

Then $|J| \in k_\varphi$ for some k_φ depending only on φ.



Pf: Pick $h_i \in H$ s.th $\pi_j(h_i) \in R_j \iff j \neq i$.

Then $h_i \in \tilde{R}_j \setminus \tilde{R}_i$ for all $j \neq i$.

Let $S \subseteq J$ fin., $h_S = \prod_{i \in S} h_i$

Then $h_S \in \tilde{R}_i \iff i \in S$ ↯ NIP.

This uses the fullness to apply! If H has fin. index in G, this is

We also need

Thm If G is a fin. gen profin gp, virtually pro-p, then $\mathrm{Aut}(G)$ is profin. and virt pro-p.

Pf: Later...

We apply the lemma when $\tilde{R}_i$ have fin. index (and so also fin. intrs. are unif. def.)

For NTP_2, if $|J|$ is infinite, consider $J \geq \ell^2$, divide index set accordingly, consider intersections of ℓ such $\tilde{R}_i$. These (cosets) are pw incons., but they all intersect all cosets of the other intersections.

We apply this several times:

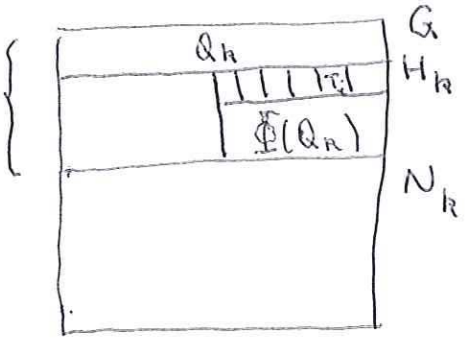
Step 1 $\mathcal{C}$ full profinite NIP/NTP$_2$ $\Longrightarrow$ G has fin. rk.

Use $rk(G) = \sup\{d(H) \mid H \leq_c G\} = \sup\{rk(G/N) \mid N \trianglelefteq_o G\}$.

and: if G is finite and $d(P) \leq d$ for all Sylow subgp of G, then $d(G) \leq d+1$.

Recall If P is fin. p-gp, then $P/\Phi(P)$ is elem. ab. of rk $d(P)$.

Thus if G has inf. rk, we can find $N_k \trianglelefteq_o G$ s.th. $rk(G/N_k) \geq k+1$, so there is $N_k \trianglelefteq H_k \leq G$ s.th. $d(H_k/N_k) \geq k+1$, so there is p-Sylow $R_k \leq H_k/N_k$ s.th. $d(Q_k) \geq k$.



$\pi: R_k \longrightarrow Q_k/\Phi(Q_k) \simeq \mathbb{F}_p^k$

Then $N_k \pi_i^{-1}(T_{\pi_i})$ has fin. index, so they are unif. def.

Step 2: G profin. of fin rk $\Rightarrow$ there are $N \trianglelefteq A \leq G$ s.th. N prosolp., A/N fin. gen. ab.

G/A fin. wlog $G = A$ is prosol.

Now $G/N = \prod_{i \in I} P_i$, P_i p_i-Sylow of G/N abelian

By the lemma, I is fin., $[P_i : \Pi_i] = p$.