Def 2.6 (i) A fin. p-gp G is powerful if p is odd and G/G^p is ab., or $p=2$ and G/G^4 ab.

(ii) If $N \leq G$, G fin. p-gp, then N is powerfully embedded in G, N p.e. G, if p odd and $[N,G] \leq N^p$, or $p=2$, $[N,G] \leq N^4$

Rem: G is powerful iff G p.e. G.

N p.e $G \Rightarrow N \trianglelefteq G$ and N powerful.

If p is odd, then G is powerful iff $\Phi(G) = G^p$.

i.e. powerful is generalis. of abelian.

Lemma 2.7 Let G be finite p-gp, $N, K, W \trianglelefteq G$ s.th. $N \leq W$.

(i) If N p.e. G, then NK/K p.e. G/K

(ii) If p is odd, $K \leq N^p$
$p=2$ $K \leq N^4$, then N p.e. G iff N/K p.e. G/K

(iii) If N p.e. G, $x \in G$, then $\langle N, x \rangle$ is powerful.

(iv) If N is not p.e. in W, then there is $J \trianglelefteq G$ s.th

$$N^p [N, W, W] \leq J < N^p [N, W]$$

$$[N^p [N, W] : J] = p.$$

Pf (i), (ii) obvious

(iii) Put $H = \langle N, x \rangle$. Then $[H,H] = [N,H]$ since $N \trianglelefteq H$.

So if N p.e. G, then $[H,H] \leq N^p \leq H^p$.

(iv) p odd: $[N,W] \not\leq N^p$. Then $N^p < N^p [N,W] = M$, say.

Since G is p-gp, $M, N \trianglelefteq G$, there is $J \trianglelefteq G$ s.th.

$N^p \leq J < M$ and $[M:J] = p$. Then $M/J \leq Z(G/J)$

since any normal subgp meets the centre of a nilp. gp.

p odd

N^4 for $p=2$.

(iv) shows that for N p.c. W, we may factor out $\exists$ s.th $N^p = 1$ and $[N, W]$ has order p and is central in G.

Prop 2.8 If G is fin. p-gp., N p.c. G, then N^p p.c. G.

Pf: See [D, dS, M, S] Prop. 2.3

Lemma 2.9 Let G be powerful p-gp, $P_{i+1}(G) = P_i(G)^p [P_i(G), G]$ with $G_i = P_i(G)$

(i) For each i, G_i p.c. G and $G_{i+1} = G_i^p = \Phi(G_i)$

(ii) For each i, $\Theta: G_i/G_{i+1} \to G_{i+1}/G_{i+2}$, $x \mapsto x^p$ is a homom.

Pf (i) By Since $G = G_1$ is powerful, G_1 p.c. G. Suppose G_i p.c. G for some $i \geq 1$. Then $G_{i+1} = G_i^p [G_i, G] = G_i^p$ and by Prop. 2.8 now G_{i+1} p.c. G.

Since $G_i^p \leq \Phi(G_i) = G_i^p [G_i, G_i] \leq G_{i+1}$, we have equal.

(ii) By (i), G_i is powerful, $G_{i+1} = P_2(G_i)$, $G_{i+2} = P_3(G_i)$

So changing notation, assume $i = 1$ and replace G by G/G_3, so that $G_3 = 1$.

Then $[G, G] \leq G_2 \leq Z(G)$, so for $x, y \in G$ we have

$$(xy)^p = x^p y^p [y, x]^{p(p-1)/2}.$$

If p is odd, then $p \mid p(p-1)/2$, so $[y, x]^{p(p-1)/2} \in G_2^p = G_3 = 1$

Lemma 2.10 If $G = \langle a_1, \ldots a_d \rangle$ is a powerful p-gp, then

$$G^p = \langle a_1^p, \ldots a_d^p \rangle.$$

Pf: Let $\Theta: G/G_2 \to G_2/G_3$ be the homom. from 2.9

Then G_2/G_3 is generated by $\{(a_i G_1)^\Theta \mid i = 1, \ldots d\}$

so $G_2 = \langle a_1^p, \ldots a_d^p \rangle G_3$. Since $G_3 = \Phi(G_2)$ and $G_2 = G^p$ by prev. lemma, the claim follows.

Prop 2.11 If G is a powerful p-gp, then every elt of G^p is a p^{th} power

Pf By ind. on $|G|$. Let $g \in G^p$. By Lemma 2.9 there is $x \in G$, $y \in G_3$ s.th. $g = x^p y$. Put $H = \langle G^p, x \rangle$. Since $G^p = G_2$ p.c G, also H is powerful and $g \in H^p$ since $y \in G_3 = G_2^p$. If $H \neq G$, we can use induction to see that g is a pth power in H.
Otherwise $G = \langle x \rangle = \langle G^p, x \rangle = \Phi(G)\langle x \rangle$ and hence Abel.

We can summerize everything in Thm 2.12

Thm 2.12 Let $G = \langle a_1, \dots a_d \rangle$ be a powerful p-gp. $G_i = P_i(G)$. Then

(i) G_i p.c. G

(ii) $G_{i+k} = P_{k+1}(G_i) = G_i^{p^k}$, $k \geq 0$

(iii) $G_i = G^{p^{i-1}} = \{x^{p^{i-1}} \mid x \in G\} = \langle a_1^{p^{i-1}}, \dots a_d^{p^{i-1}} \rangle$

Cor 2.13 If $G = \langle a_1, \dots a_d \rangle$ is a powerful p-gp, then $G = \langle a_1 \rangle \dots \langle a_d \rangle$, i.e. G is prod. of cycl. gps

Pf: Say $G_e > G_{e+1} = 1$. By ind. one suppose $G = \langle a_1 \rangle \dots \langle a_d \rangle G_e$. Since $G_e = \langle a_1^{p^{e-1}}, \dots a_d^{p^{e-1}} \rangle \leq Z(G)$ the result follows.

Thm 2.14 If G is a powerful p-gp, $H \leq G$, then $d(H) \leq d(G)$

Def 2.15 The rk of a fin. gp G is

$$rk(G) = \sup\{d(H) \mid H \leq G\}.$$

Cor: If G is a powerful p-gp, then $d(G) = rk(G)$.

Now:

Def 2.16 Let G be a pro-p gp

(i) G is powerful if p odd and $G/G^{\overline{P}}$ ab.

(ii) $p = 2$ G/G^4 so $\Phi(G) = G^{\overline{P}}$ p odd

(iii) Recall $N \trianglelefteq_0 G$ is p.c G if $[N, G] \leq N^{\overline{P}}$

Prop 2.17 G pro-p gp, $N \trianglelefteq_0 G$. Then N p.c. G iff NK/K p.c. G/K for ever $K \trianglelefteq_0 G$

Pf By 1.18

Cor 2.18 (i) A top. gp G is powerful pro-p gp iff $G = \varprojlim G_i$, G_i powerful p-gp

(ii) If G is fin. gen. powerful pro-p gp, then every elt of G^p is a pth power in G and G^p is open in G

(iii) $G^{p^i} = \{x^{p^i} \mid x \in G\}$ p.e. $G^{p^{i-1}}$

Prop 2.19 If $G = \overline{\langle a_1, \dots a_d \rangle}$ is powerful pro-p gp, then $G = \overline{\langle a_1 \rangle} \cdots \overline{\langle a_d \rangle}$.

Pf. Put $A = \overline{\langle a_1 \rangle} \cdots \overline{\langle a_d \rangle}$. Then $A \leq_c G$, so $A = \bigcap_{N \trianglelefteq_o G} AN$

By 2.13 $AN/N = G/N$ for all $N \trianglelefteq_o G$, so $A = G$.

Def 2.20 The rank $rk(G)$ of a profin. gp G is def. as $rk(G) = \min \{\sup d(H) \mid H \leq_c G\}$

where $d(H)$ = size of min. top. gen. set.

Cor 2.21 If G is a pro-p-gp, then TFAE

(i) G is prod. of fin. many procyclic gps

(ii) G has finite rk

(iii) G is fin. gen. and $\mathbb{Z}_p$-powered, i.e. there is finite $X \subseteq G$ s.th. any $g \in G$ can be written as $g = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$, $x_i \in X$, $\lambda_i \in \mathbb{Z}_p$.

Def 2.22 A pro-p gp G is unif. powerful if G is fin. gen., powerful and $|P_i(G) : P_{i+1}(G)| = |G : P_2(G)|$ $\forall i$.

Thm 2.23 If G is fin. gen. powerful, then $P_k(G)$ is unif. powerful for k suff. large.

Any pro-p gp of fin. rk has char. open unif. powerful subgp.

We will show that any fin. gen unif. powerful pro-p gp is NIP. We have just seen that any such gp 'lives' on $\mathbb{Z}_p^d$, $d = rk(G) = d(G)$.

Prop I (Macph. T.) If $\mathcal{G} = (G, I)$ is a full unif. powerful pro-p gp, then $\mathcal{G}$ is interpretable in $\mathbb{Z}_p^{an}$, so $Th(\mathcal{G})$ is NIP.

Excursion: p-adic analytic gps

Def 2.24 (i) If $V \subseteq \mathbb{Z}_p^r$ is open, $f = (f_1, .. f_s) : V \to \mathbb{Z}_p^s$, $y \in V$. Then f is analytic at y if there ex. formal power series $F_i(X) \in \mathbb{Q}_p[[X_1, .. X_r]]$, $i \leq s$ and $h \in \mathbb{N}$ s.th

$$f_i(y + p^h x) = F_i(x) \quad \text{for all } x \in \mathbb{Z}_p^r.$$

We call f analytic on V if anal. at every $y \in V$.

(ii) For a top. space X, $U \subseteq_o X$, $\varphi : U \to V \subseteq_o \mathbb{Z}_p^n$ homeom we call (U, φ, n) a chart on X, n = dim.

(iii) We call charts (U, φ, n) and (V, ψ, m) on X compatible if for $W = U \cap V$ we have $\psi \circ \varphi^{-1}|_{\varphi(W)}$, $\varphi \circ \psi^{-1}|_{\psi(W)}$ analytic.

(iv) An atlas on top. space X is given by a family of charts $A = \{(U_i, \varphi_i, n_i) \mid i \in I\}$ s.th. any two charts are comp. and $X = \bigcup U_i$.

Rem: Compatibility of atlases is an equiv. rel.

Def 2.25 (i) For a top space X an analytic p-adic manifold str. is given by an equiv. class of compat. atlases on X.

(ii) A top. gp G is called p-adic analytic if G has an p-adic anal. mfd str. s.th. $G \times G \to G$, $(x, y) \mapsto xy^{-1}$ is analytic.